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CHAOTIC SOLUTIONS IN DIFFERENTIAL INCLUSIONS: CHAOS IN DRY FRICTION PROBLEMS

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ABSTRACT. The existence of a continuum of many chaotic solutions is shown for certain differential inclusions which are small periodic multivalued perturbations of ordinary differential equations possessing homoclinic solutions to hyperbolic fixed points. Applications are given to dry friction problems. Singularly perturbed differential inclusions are investigated as well.

1. INTRODUCTION

Consider a mass attached to a spring and sliding on a horizontal surface. When there is a friction between the surface and the mass, and a periodic external force as well, the following equation is studied:

$$\ddot{x} + q(x) + \mu_1 \operatorname{sgn} \dot{x} = \mu_2 \psi(t)\,, \tag{1.1}$$

where $q \in C^2(\mathbb{R}, \mathbb{R})$, $\mu_{1,2} > 0$ are small constants, $\psi \in C^1(\mathbb{R}, \mathbb{R})$ is 2–periodic and $\operatorname{sgn} r = r/|r|$ for $r \in \mathbb{R} \setminus \{0\}$. For more physical background we refer the reader to [1], [3] and [10].

By introducing the multivalued mapping

$$\operatorname{Sgn} r = \begin{cases} \operatorname{sgn} r & \text{for } r \neq 0\,, \\ [-1, 1] & \text{for } r = 0\,, \end{cases}$$

(1.1) is rewritten as

$$\ddot{x} + q(x) - \mu_2 \psi(t) \in -\mu_1 \operatorname{Sgn} \dot{x}\,.$$

To deal with much more general equations like (1.1), in Section 3 we consider differential inclusions which take the following form

$$\dot{x}(t) \in f(x(t)) + \sum_{i=1}^{k} \mu_i f_i(x(t), \mu, t) \quad \text{a.e. on} \quad \mathbb{R} \tag{1.2}$$

with $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}^k$, $\mu = (\mu_1, \cdots, \mu_k)$. In this paper, by a solution of any differential inclusion we mean a function which is absolutely continuous and satisfies the differential inclusion almost everywhere.

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We put

$$q(x,\mu,t) \equiv \sum_{i=1}^{k} \mu_i f_i(x,\mu,t)$$

and make the following assumptions about (1.2):

(i) $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^2 and $f_i : \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R} \to 2^{\mathbb{R}^n} \setminus \emptyset$, $i = 1, \cdots, k$, are upper–semicontinuous ([2], [12]) with compact and convex values.

(ii) $f(0) = 0$ and the eigenvalues of $Df(0)$ lie off the imaginary axis.

(iii) The unperturbed equation has a homoclinic solution. That is, there exists a differentiable function $t \to \gamma(t)$ such that $\lim_{t\to+\infty} \gamma(t) = \lim_{t\to-\infty} \gamma(t) = 0$ and $\dot{\gamma}(t) = f(\gamma(t))$.

(iv) $q(x,\mu,t+2) = q(x,\mu,t)$ for $t \in \mathbb{R}$.

The purpose of this paper is to find conditions ensuring that for any sequence $\{e_j\}_{j\in\mathbb{Z}}$, $e_j \in \{0,1\}$ and μ small, there is a solution of (1.2) on $\mathbb{R}$. Moreover, we have different solutions for different sequences and in addition, sequences characterize (or count) turnings of corresponding solutions around γ. Consequently, we extend the deterministic chaos of [9] and [11] to (1.2).

This paper is a continuation of [4]–[6] where the existence of subharmonics of (1.2) are studied. Hence we prove in this paper the chaotic behaviour of (1.2) conjectured in [4] and [5]. Proofs of results in this paper are very similar to proofs in [4]–[6].

In Section 4, we extend results of Section 3 to the singular differential inclusions studied in [4] and [5].

Finally, we note that all examples of nonautonomous differential inclusions presented in [4] and [5] satisfy the conditions of the theorems proved in this paper. So they exhibit chaotic solutions and we refer the reader to these papers for application to concrete examples of dry friction problems.

2. The Variational Equation

Let W^s, W^u be the stable and unstable manifolds, respectively, of the origin of the unperturbed equation

$$\dot{x} = f(x)\,. \tag{2.1}$$

By the variational equation of (2.1) along γ we mean the linear differential equation

$$\dot{u}(t) = Df(\gamma(t))u(t)\,. \tag{2.2}$$

We need the following result.

Theorem 2.1. ([8]) *Let $d_s = \dim(W^s)$, $d_u = \dim(W^u)$ for (2.1) and let I_s, I_u denote the identity matrices of order d_s, d_u respectively. There exists a fundamental solution U for (2.2) along with a non–singular matrix C, constants $M > 0$, $K_0 > 0$ and four projections P_{ss}, P_{su}, P_{us}, P_{uu} such that $P_{ss} + P_{su} + P_{us} + P_{uu} = I$ and the following hold:*

(i) $|U(t)(P_{ss} + P_{us})U(s)^{-1}| \le K_0 \mathrm{e}^{2M(s-t)}$ *for* $0 \le s \le t$,

(ii) $|U(t)(P_{su} + P_{uu})U(s)^{-1}| \le K_0 \mathrm{e}^{2M(t-s)}$ *for* $0 \le t \le s$,

(iii) $|U(t)(P_{ss} + P_{su})U(s)^{-1}| \le K_0 \mathrm{e}^{2M(t-s)}$ *for* $t \le s \le 0$,

(iv) $|U(t)(P_{us} + P_{uu})U(s)^{-1}| \le K_0 \mathrm{e}^{2M(s-t)}$ *for* $s \le t \le 0$,

(v) $\lim\limits_{t\to+\infty} U(t)(P_{ss} + P_{us})U(t)^{-1} = C \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} C^{-1}$,

(vi) $\lim_{t\to+\infty} U(t)(P_{su}+P_{uu})U(t)^{-1} = C\begin{pmatrix}0 & 0\\ 0 & I_u\end{pmatrix}C^{-1}\,,$

(vii) $\lim_{t\to-\infty} U(t)(P_{ss}+P_{su})U(t)^{-1} = C\begin{pmatrix}0 & 0\\ 0 & I_u\end{pmatrix}C^{-1}\,,$

(viii) $\lim_{t\to-\infty} U(t)(P_{us}+P_{uu})U(t)^{-1} = C\begin{pmatrix}I_s & 0\\ 0 & 0\end{pmatrix}C^{-1}\,.$

Also, there exists an integer d with $\operatorname{rank} P_{ss} = \operatorname{rank} P_{uu} = d$.

Let u_j denote column j of U and assume these are numbered so that

$$P_{uu} = \begin{pmatrix} I_d & 0_d & 0\\ 0_d & 0_d & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad P_{ss} = \begin{pmatrix} 0_d & 0_d & 0\\ 0_d & I_d & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Here, I_d denotes the $d\times d$ identity matrix and 0_d denotes the $d\times d$ zero matrix.

For each $i=1,\cdots,n$ we define $u_i^\perp(t)$ by $\langle u_i^\perp(t), u_j(t)\rangle = \delta_{ij}$, where $\langle\cdot,\cdot\rangle$ is the inner product. The vectors $u_i^\perp$ can be computed from the formula $U^{\perp t} = U^{-1}$ where $U^\perp$ denotes the matrix with $u_j^\perp$ as column j. Differentiating $UU^{\perp t} = I$ we obtain $\dot{U}U^{\perp t} + U\dot{U}^{\perp t} = 0$ so that $\dot{U}^\perp = -(U^{-1}\dot{U}U^{\perp t})^t = -Df(\gamma)^t U^\perp$. Thus, $U^\perp$ is the adjoint of U.

The function $\dot\gamma$ is always a solution to the variational equation (2.2). If $\dot\gamma$ appears as a column of U we shall assume $u_{2d} = \dot\gamma$. In general we shall always assume $\langle u_{2d}^\perp, \dot\gamma\rangle \neq 0$.

Fix $m\in\mathbb{N}$ and define the following Banach spaces:

$$Z_m = C\left([-m,m],\mathbb{R}^n\right)\,,$$
$$Y_m = L_\infty\left([-m,m],\mathbb{R}^n\right)\,.$$

We take as norm $\|\cdot\|_m$, the supremum for Z_m, respectively $|\cdot|_m$, the usual L_∞ norm for Y_m.

Integration of the inequalities in Theorem 2.1 yields the following result.

Theorem 2.2. ([7]) *Let U be the solution to* (2.2) *along with the projections P_{ss}, P_{su}, P_{us}, P_{uu} as in Theorem* 2.1. *Then there exists a constant $A>0$ such that for any $m>0$ and any $z\in Y_m$ the following hold:*

(i) $\displaystyle\int_0^t |U(t)(P_{ss}+P_{us})U(s)^{-1}z(s)|\,ds \le A|z|_m \quad \textit{for } t\in[0,m]\,,$

(ii) $\displaystyle\int_t^m |U(t)(P_{su}+P_{uu})U(s)^{-1}z(s)|\,ds \le A|z|_m \quad \textit{for } t\in[0,m]\,,$

(iii) $\displaystyle\int_t^0 |U(t)(P_{ss}+P_{su})U(s)^{-1}z(s)|\,ds \le A|z|_m \quad \textit{for } t\in[-m,0]\,,$

(iv) $\displaystyle\int_{-m}^t |U(t)(P_{us}+P_{uu})U(s)^{-1}z(s)|\,ds \le A|z|_m \quad \textit{for } t\in[-m,0]\,.$

Now we consider the linear equation

$$\dot{v}(t) = Df(0)v(t)\,, \tag{2.3}$$

and put

$$V(t)=\mathrm{e}^{tDf(0)},\quad Q_{us}=C\begin{pmatrix}I_s & 0\\ 0 & 0\end{pmatrix}C^{-1},$$
$$Q_{ss}=0,\quad Q_{uu}=0,\quad Q_{su}=C\begin{pmatrix}0 & 0\\ 0 & I_u\end{pmatrix}C^{-1}\,,$$

where C is from Theorem 2.1.

Theorem 2.3. *By considering in Theorems* 2.1 *and* 2.2 *the exchanges*

$$U(t)\leftrightarrow V(t),\quad P_{ss}\leftrightarrow Q_{ss},\quad P_{su}\leftrightarrow Q_{su},\quad P_{us}\leftrightarrow Q_{us},\quad P_{uu}\leftrightarrow Q_{uu}\,,$$

Theorems 2.1 *and* 2.2 *are valid for* (2.3).

Proof. In the language of dichotomies [11] we see that Theorem 2.1 provides a two–sided exponential dichotomy for (2.2). For $t\to-\infty$ an exponential dichotomy is given by the fundamental solution U and the projection $P_{ss}+P_{su}$ while for $t\to+\infty$ such is given by U and $P_{ss}+P_{us}$. Since $\lim_{t\to\pm\infty}Df(\gamma(t))=Df(0)$ and $Df(0)$ is hyperbolic, the results follow from [11, Lemma 3.4]. □

For any finite sequence $E=\{e_j\}_{j=1}^p\in\{0,1\}^p$, $p\in\mathbb{N}$ we say that if $e_j=1$, then

$$A_j(t)=Df(\gamma(t)),\quad U_j=U,\quad P_{ss}^j=P_{ss},$$
$$P_{su}^j=P_{su},\quad P_{us}^j=P_{us},\quad P_{uu}^j=P_{uu}\,;$$

and if $e_j=0$, then

$$A_j(t)=Df(0),\quad U_j=V,\quad P_{ss}^j=Q_{ss},$$
$$P_{su}^j=Q_{su},\quad P_{us}^j=Q_{us},\quad P_{uu}^j=Q_{uu}\,.$$

We put $I_p=\{1,2,\cdots,p\}$. For any $\alpha\in\mathcal{A}_E$, where the set $\mathcal{A}_E$ is defined by

$$\mathcal{A}_E=\Big\{(\alpha_1,\cdots,\alpha_p)\in\mathbb{R}^p\ :\ \alpha_j\in\mathbb{R}\quad\text{if}\quad e_j=1\quad\text{and}\quad\alpha_j=0\quad\text{if}\quad e_j=0\Big\},$$

we consider the non–homogeneous linear equations

$$\begin{aligned}\dot z_j&=A_j(t-\alpha_j)z_j+h_j,\quad j\in I_p,\quad h_j\in Y_m,\\ z_j(m)&=z_{j+1}(-m)\quad\text{for}\quad 1\le j\le p-1,\\ z_p(m)&=z_1(-m)\,,\end{aligned}\tag{2.4}$$

and we prove a Fredholm–like alternative result for (2.4).

Theorem 2.4. *For any $K>0$, there exist $m_0>0$, $A>0$, $B>0$ such that for every $j\in I_p$, $m>m_0$, $m\in\mathbb{N}$ and $\alpha\in\mathcal{A}_E$ such that $|\alpha|\le K$, there exist linear functions $\mathcal{L}_{m,\alpha,j}:Y_m\to\mathbb{R}^n$ with $\|P_{uu}^j\mathcal{L}_{m,\alpha,j}\|\le A\mathrm{e}^{-2Mm}$ and with the property that if $h_j\in Y_m$, $j\in I_p$ satisfy*

$$\int_{-m}^m P_{uu}^jU_j(t-\alpha_j)^{-1}h_j(t)\,dt+P_{uu}^j\mathcal{L}_{m,\alpha,j}h_j=0,$$

then (2.4) *has solutions in $z_j\in Z_m$ satisfying*

$$P_{ss}^jU_j(-\alpha_j)^{-1}z_j(0)=0\quad\textit{and}\quad\max_{j\in I_p}\|z_j\|_m\le B\max_{j\in I_p}|h_j|_m\,.$$

Moreover, these solutions z_j depend linearly on h_j and continuously on α as well.

Proof. We follow [7] by putting $\tilde{U}_j(t) = U_j(t-\alpha_j)$, $j \in I_p$. Given $h_j \in Y_m$, $j \in I_p$ we use variation of constants to construct the following two solutions to (2.4):

$$\begin{aligned} z_{1,j}(t) = \tilde{U}_j(t)P^j_{su}\xi_{1,j} &+ \tilde{U}_j(t)(P^j_{us}+P^j_{uu})\tilde{U}_j(-m)^{-1}\varphi_{1,j} \\ &+ \tilde{U}_j(t)\int_0^t (P^j_{ss}+P^j_{su})\tilde{U}_j(s)^{-1}h_j(s)\,ds \\ &+ \tilde{U}_j(t)\int_{-m}^t (P^j_{us}+P^j_{uu})\tilde{U}_j(s)^{-1}h_j(s)\,ds\,, \\ z_{2,j}(t) = \tilde{U}_j(t)P^j_{us}\xi_{2,j} &+ \tilde{U}_j(t)(P^j_{su}+P^j_{uu})\tilde{U}_j(m)^{-1}\varphi_{2,j} \\ &+ \tilde{U}_j(t)\int_0^t (P^j_{ss}+P^j_{us})\tilde{U}_j(s)^{-1}h_j(s)\,ds \\ &- \tilde{U}_j(t)\int_t^m (P^j_{su}+P^j_{uu})\tilde{U}_j(s)^{-1}h_j(s)\,ds\,. \end{aligned}$$

Here $\xi_{1,j}$, $\xi_{2,j}$, $\varphi_{1,j}$, $\varphi_{2,j}$ are arbitrary. Each of these represents the general solution to (2.4) satisfying $P^j_{ss}\tilde{U}_j(0)^{-1}z_j(0) = 0$. We consider $z_{1,j}(t)$ for $t \in [-m,0]$ and $z_{2,j}(t)$ for $t \in [0,m]$.

The conditions $z_{1,j}(0) - z_{2,j}(0) = 0$ decompose into the following three families of equations:

$$P^j_{su}\xi_{1,j} - P^j_{su}\tilde{U}_j(m)^{-1}\varphi_{2,j} + \int_0^m P^j_{su}\tilde{U}_j(s)^{-1}h_j(s)\,ds = 0\,, \tag{2.5a}$$

$$P^j_{us}\tilde{U}_j(-m)^{-1}\varphi_{1,j} + \int_{-m}^0 P^j_{us}\tilde{U}_j(s)^{-1}h_j(s)\,ds - P^j_{us}\xi_{2,j} = 0\,, \tag{2.5b}$$

$$P^j_{uu}\tilde{U}_j(-m)^{-1}\varphi_{1,j} - P^j_{uu}\tilde{U}_j(m)^{-1}\varphi_{2,j} + \int_{-m}^m P^j_{uu}\tilde{U}_j(s)^{-1}h_j(s)\,ds = 0\,. \tag{2.5c}$$

We can solve (2.5a), (2.5b) for $\xi_{1,j}$, $\xi_{2,j}$ respectively. If we write out the equations $z_{1,j+1}(-m) - z_{2,j}(m) = 0$, substitute the formulas for $\xi_{1,j+1}$, $\xi_{2,j}$ and rearrange terms we get the equations

$$\begin{aligned} &\left[\begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} + R_{1,j}(m,\alpha)\right] C^{-1}\varphi_{1,j+1} + \left[-\begin{pmatrix} 0 & 0 \\ 0 & I_u \end{pmatrix} + R_{2,j}(m,\alpha)\right] C^{-1}\varphi_{2,j} \\ &\quad + R_{3,j}(m,\alpha)C^{-1}\varphi_{2,j+1} + R_{4,j}(m,\alpha)C^{-1}\varphi_{1,j} = \Psi_j(m,\alpha,h)\,, \end{aligned} \tag{2.6}$$

where the matrix C is taken from Theorem 2.1 and

$$\begin{aligned} R_{1,j}(m,\alpha) &= C^{-1}\left[\tilde{U}_{j+1}(-m)(P^{j+1}_{us}+P^{j+1}_{uu})\tilde{U}_{j+1}(-m)^{-1} - C\begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} C^{-1}\right] C\,, \\ R_{2,j}(m,\alpha) &= C^{-1}\left[-\tilde{U}_j(m)(P^j_{su}+P^j_{uu})\tilde{U}_j(m)^{-1} + C\begin{pmatrix} 0 & 0 \\ 0 & I_u \end{pmatrix} C^{-1}\right] C, \\ R_{3,j}(m,\alpha) &= C^{-1}\tilde{U}_{j+1}(-m)P^{j+1}_{su}\tilde{U}_{j+1}(m)^{-1}C \\ R_{4,j}(m,\alpha) &= -C^{-1}\tilde{U}_j(m)P^j_{us}\tilde{U}_j(-m)^{-1}C, \end{aligned}$$

$$
\begin{aligned}
C\Psi_j(m,\alpha,h) &= \tilde{U}_{j+1}(-m)\int_{-m}^{0}(P_{ss}^{j+1}+P_{su}^{j+1})\tilde{U}_{j+1}(s)^{-1}h_{j+1}(s)\,ds \\
&+\tilde{U}_j(m)\int_0^m (P_{ss}^j+P_{us}^j)\tilde{U}_j(s)^{-1}h_j(s)\,ds+\tilde{U}_j(m)\int_{-m}^0 P_{us}^j\tilde{U}_j(s)^{-1}h_j(s)\,ds \\
&+\tilde{U}_{j+1}(-m)\int_0^m P_{su}^{j+1}\tilde{U}_{j+1}(s)^{-1}h_{j+1}(s)\,ds\,.
\end{aligned}
$$

Using Theorems 2.1–2.3 we see that each $|R_{i,j}(m,\alpha)|\to 0$ as $m\to\infty$ and we get $|\Psi_j(m,\alpha,h)|=\max\limits_{j\in I_p}|h_j|_m O(1)$ as well. If we write

$$
C^{-1}\varphi_{1,j}=\begin{bmatrix}u_{1,j}\\0\end{bmatrix},\qquad C^{-1}\varphi_{2,j}=-\begin{bmatrix}0\\u_{2,j}\end{bmatrix},\qquad \tilde{u}_j=\begin{bmatrix}u_{1,j}\\u_{2,j}\end{bmatrix},
$$

where $u_{1,j}$, $u_{2,j}$ are of order s, u respectively, (2.6) become

$$
\begin{aligned}
&\left[I+R_{1,j}(m,\alpha)\begin{pmatrix}I_s&0\\0&0\end{pmatrix}-R_{2,j}(m,\alpha)\begin{pmatrix}0&0\\0&I_u\end{pmatrix}\right]\tilde{u}_j \\
&-R_{3,j}(m,\alpha)\begin{bmatrix}0\\u_{2,j+1}\end{bmatrix}+R_{4,j}(m,\alpha)\begin{bmatrix}u_{1,j}\\0\end{bmatrix}=\Psi_j(m,\alpha,h)\,.
\end{aligned}
$$

There exists $m_0>0$ so that the coefficient matrices of $\tilde{u}_j$ in the preceding equations are invertible whenever $m\ge m_0$. In this case the equations can be solved for $\tilde{u}_j$ which leads to functions $\varphi_{i,j}(m,\alpha,h)$ such that $\max\limits_{j\in I_p}|\varphi_{i,j}|=\max\limits_{j\in I_p}|h_j|_m O(1)$.

By (2.5c) we obtain

$$
\int_{-m}^m P_{uu}^j\tilde{U}_j(s)^{-1}h_j(s)\,ds+P_{uu}^j\mathcal{L}_{m,\alpha,j}h_j=0\,,
$$

where

$$
\mathcal{L}_{m,\alpha,j}h_j=\tilde{U}_j(-m)^{-1}\varphi_{1,j}(m,\alpha,h)-\tilde{U}_j(m)^{-1}\varphi_{2,j}(m,\alpha,h)\,.
$$

It follows from the properties of the $\varphi_{i,j}$ and Theorems 2.1–2.3 that

$$
\|P_{uu}^j\mathcal{L}_{m,\alpha,j}h_j\|=\max_{j\in I_p}|h_j|_m O\left(\mathrm{e}^{-2Mm}\right).
$$

□

Using the preceding result we define closed linear subspaces $Y_{m,\alpha,j,E}\subset Y_m$ by

$$
Y_{m,\alpha,j,E}=\left\{z\in Y_m\ :\ \int_{-m}^m P_{uu}^jU_j(t-\alpha_j)^{-1}z(t)\,dt+P_{uu}^j\mathcal{L}_{m,\alpha,j}z=0\right\}.
$$

We can define a variation of constants map $K_{m,\alpha,E}\,:\,Y_{m,\alpha,E}=\bigtimes_{j\in I_p}Y_{m,\alpha,j,E}\to\bigtimes_{j\in I_p}Z_m=Z_{m,p}$ by taking $K_{m,\alpha,E}(h)$, $h=(h_1,\cdots,h_p)$ to be the solution in $Z_{m,p}$ to (2.4) from Theorem 2.4 satisfying $P_{ss}^jU_j(-\alpha_j)^{-1}K_{m,\alpha,E}(h)_j(0)=0$. Furthermore, the norm $\|K_{m,\alpha,E}\|$ is uniformly bounded with respect to m, E and $\alpha\in\mathcal{A}_E$ bounded. According to (2.4), we have moreover that $K_{m,\alpha,E}$ maps any bounded subset of $Y_{m,\alpha,E}$ into a bounded one of the Banach space $\bigtimes_p W^{1,2}([-m,m],\mathbb{R}^n)$ which is compactly embedded into $Z_{m,p}$. Hence $K_{m,\alpha,E}$ is a compact linear operator.

Remark 2.5. We note that if $e_j=0$, then $Y_{m,\alpha,j,E}=Y_m$ and $P_{uu}^j=0$.

3. Chaotic Solutions

We find chaotic solutions of (1.2) in this section. To this end, we first find periodic solutions of (1.2) associated to any $E = \{e_j\}_{j=1}^p \in \{0,1\}^p$, $p \in \mathbb{N}$ and then by passing to the limit $p \to \infty$, we show the desired solutions.

We consider $\alpha \in \mathcal{A}_E$ and say that if $e_j = 1$, then

$$\begin{aligned}&\gamma_j(t) = \gamma(t-\alpha_j), \quad \beta_j = (\beta_{1,j}, \cdots, \beta_{d-1,j}) \in \mathbb{R}^{d-1},\\ &u_{i+d,j}(t) = u_{i+d}(t-\alpha_j),\ i \in \{1, \cdots, d-1\}\end{aligned}$$

and if $e_j = 0$, then

$$\gamma_j(t) = 0, \quad \beta_j = (0, \cdots, 0) \in \mathbb{R}^{d-1}, \quad u_{i+d,j} = 0,\ i \in \{1, \cdots, d-1\}$$

We define the functions b_j, $j \in I_p$ by

$$\begin{aligned}&b_j(\alpha_j, \alpha_{j+1}, \beta_j, \beta_{j+1}, r)\\ &= \gamma_{j+1}(-r) - \gamma_j(r) + \sum_{i=1}^{d-1} \Big(\beta_{i,j+1} u_{i+d,j+1}(-r) - \beta_{i,j} u_{i+d,j}(r)\Big).\end{aligned}$$

Note that

$$\begin{aligned}&|b_j(\alpha_j, \alpha_{j+1}, \beta_j, \beta_{j+1}, r)| = O(\mathrm{e}^{-Mr}) \quad \text{as} \quad r \to \infty \quad \text{uniformly}\\ &\qquad \text{with respect to bounded} \quad \alpha_j, \alpha_{j+1}, \beta_j, \beta_{j+1}.\end{aligned}$$

In (1.2) we now make the changes of variables

$$\begin{aligned}x\,(t + 2(m+S)j) &= \gamma_j(t) + s^2 z_j(t) + \sum_{i=1}^{d-1} s\beta_{i,j} u_{i+d,j}(t)\\ &+ \frac{1}{2(m+S)} b_j(\alpha_j, \alpha_{j+1}, s\beta_j, s\beta_{j+1}, m+S)(t+m+S), \quad z_j \in Z_{m+S},\end{aligned} \tag{3.1}$$

where $1 > s > 0$, $m \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $S = [1/s]$ and $[\tilde{s}]$ is the integer part of $\tilde{s}$. The functions b_j are constructed so that if $z_j(m+S) = z_{j+1}(-m-S)$ and z_j are continuous, then x is continuously extended on $\mathbb{R}$ and $2(m+S)p$–periodic.

The differential inclusions for z_j, $j \in I_p$ are

$$\begin{aligned}&\dot{z}_j(t) - Df(\gamma_j(t)) z_j(t) \in g_{m,s,j}(z_j(t), \alpha_j, \alpha_{j+1}, \beta_j, \beta_{j+1}, \mu, t)\\ &\qquad \text{a.e. on} \quad [-m-S, m+S], \quad j \in I_p,\end{aligned} \tag{3.2}$$

where

$$
\begin{aligned}
&g_{m,s,j}(x,\alpha_j,\alpha_{j+1},\beta_j,\beta_{j+1},\mu,t)\\
&=\Bigg\{v\in\mathbb{R}^n\ :\ v\in\frac{1}{s^2}\Big\{f\Big(s^2x+\gamma_j(t)+s\sum_{i=1}^{d-1}\beta_{i,j}u_{i+d,j}(t)\\
&+\frac{1}{2(m+S)}b_j(\alpha_j,\alpha_{j+1},s\beta_j,s\beta_{j+1},m+S)(t+m+S)\Big)-f(\gamma_j(t))\\
&-s\sum_{i=1}^{d-1}\beta_{i,j}\dot u_{i+d,j}(t)-\frac{1}{2(m+S)}b_j(\alpha_j,\alpha_{j+1},s\beta_j,s\beta_{j+1},m+S)-Df(\gamma_j(t))s^2x\Big\}\\
&+\sum_{r=1}^{k}\mu_rf_r\Big(s^2x+\gamma_j(t)+s\sum_{i=1}^{d-1}\beta_{i,j}u_{i+d,j}(t)\\
&+\frac{1}{2(m+S)}b_j(\alpha_j,\alpha_{j+1},s\beta_j,s\beta_{j+1},m+S)(t+m+S),s^2\mu,t\Big)\Bigg\}.
\end{aligned}
$$

We note that $Df(\gamma_j(t))=A_j(t-\alpha_j)$ in the notation of Section 2. Now we can repeat with the help of Theorem 2.4 the arguments of Section 3 in [4] to solve (3.2) in $Z_{m+S,p}$. We omit details, since we can directly modify the proofs without any changes. We point out that in this way, (3.2) is solvable uniformly for E, $m\in\mathbb{Z}_+$ and $1>s>0$ sufficiently small. For $j\in I_p$ such that $e_j=1$, according to Section 3 of [4] (see (3.6) of [4]), (3.2) is homotopically associated to a mapping

$$
\begin{aligned}
&M_\mu\,:\,\mathbb{R}^d\to 2^{\mathbb{R}^d}\setminus\emptyset\,,\quad M_\mu=(M_{\mu 1},\cdots,M_{\mu d}),\\
&M_{\mu l}(\alpha_j,\beta_j)=\Bigg\{\int_{-\infty}^{\infty}\langle h(s),u_l^\perp(s)\rangle\,ds\ :\ h\in L_{2,\mathrm{loc}}(\mathbb{R},\mathbb{R}^n)\text{ satisfying a.e. on }\mathbb{R}\\
&h(t)\in\Big(\frac12\sum_{i,r=1}^{d-1}\beta_{i,j}\beta_{r,j}D^2f(\gamma(t))(u_{d+i}(t),u_{d+r}(t))+\sum_{r=1}^{k}\mu_rf_r(\gamma(t),0,t+\alpha_j)\Big)\Bigg\}.
\end{aligned}
$$

While for $j\in I_p$ such that $e_j=0$, according to Section 3 of [4] and Remark 2.5 (see again (3.6) of [4]), (3.2) is homotopically solvable. We note that the mapping M_μ is upper–semicontinuous with compact convex values and maps bounded sets into bounded ones [4].

Consequently, the solvability of (3.2) is reduced to the solvability of the system of p_E equal inclusions

$$\underbrace{0\in M_\mu,\quad\cdots,\quad 0\in M_\mu}_{p_E\text{ times}},$$

where p_E is the number of 1's in E. Summarizing we have the following generalizations of Theorems 3.2 and 2.3 of [4].

Theorem 3.1. *Let $d>1$. If there is a non–empty open bounded set $\mathcal{B}\subset\mathbb{R}^d$ and $\mu_0\in S^{k-1}$ such that*

(i) $0\notin M_{\mu_0}(\partial\mathcal{B})$,
(ii) $deg\,(M_{\mu_0},\mathcal{B},0)\neq 0$,

where deg is the topological degree in the sense of [2, pp. 154-155] *and* [12, p. 20], *then there is a constant $K > 0$ and a wedge–shaped region in $\mathbb{R}^k$ for μ of the form*

$$\mathcal{R} = \Big\{ s^2\tilde{\mu} \ : \ s > 0, \textit{ respectively } \tilde{\mu}, \textit{ is from an open small connected neighborhood } U_1, \textit{ respectively } U_2 \subset S^{k-1}, \textit{ of } 0 \in \mathbb{R}, \textit{ respectively of } \mu_0 \Big\}$$

such that for any $\mu \in \mathcal{R}$ of the form $\mu = s^2\tilde{\mu}$, $0 < s \in U_1$, $\tilde{\mu} \in U_2$ and any finite sequence $E = \{e_j\}_{j=1}^p \in \{0,1\}^p$, $p \in \mathbb{N}$, the differential inclusion (1.2) *possesses a subharmonic solution $x_{m,E}$ of period $2mp$ for any $m \in \mathbb{N}$, $m \ge [1/s]$ satisfying, according to* (3.1),

$$\begin{aligned} &\sup_{(2j-1)m \le t \le (2j+1)m} \big| x_{m,E}(t) - \gamma(t - 2mj - \alpha_{m,j,E}) \big| \le Ks \quad \textit{when} \quad e_j = 1\,, \\ &\sup_{(2j-1)m \le t \le (2j+1)m} \big| x_{m,E}(t) \big| \le Ks^2 \quad \textit{when} \quad e_j = 0\,, \end{aligned} \tag{3.3}$$

where $\alpha_{m,j,E} \in \mathbb{R}$ and $|\alpha_{m,j,E}| \le K$.

Theorem 3.2. *Let $d = 1$. If there are constants $a < b$ and $\mu_0 \in S^{k-1}$ such that*

$M_{\mu_0}(a)$ contains only positive (respectively negative) numbers
and $M_{\mu_0}(b)$ contains only negative (respectively positive) ones,

then there is a constant $K > 0$ and a wedge–shaped region in $\mathbb{R}^k$ for μ of the form

$$\mathcal{R} = \Big\{ \pm s^2\tilde{\mu} \ : \ s > 0, \textit{ respectively } \tilde{\mu}, \textit{ is from an open small connected neighborhood } U_1, \textit{ respectively } U_2 \subset S^{k-1}, \textit{ of } 0 \in \mathbb{R}, \textit{ respectively of } \mu_0 \Big\}$$

such that for any $\mu \in \mathcal{R}$ of the form $\mu = \pm s^2\tilde{\mu}$, $0 < s \in U_1$, $\tilde{\mu} \in U_2$ and any finite sequence $E = \{e_j\}_{j=1}^p \in \{0,1\}^p$, $p \in \mathbb{N}$, the differential inclusion (1.2) *possesses a subharmonic solution $x_{m,E}$ of period $2mp$ for any $m \in \mathbb{N}$, $m \ge [1/s]$ satisfying, according to* (3.1),

$$\begin{aligned} &\sup_{(2j-1)m \le t \le (2j+1)m} \big| x_{m,E}(t) - \gamma(t - 2mj - \alpha_{m,j,E}) \big| \le Ks^2 \quad \textit{when} \quad e_j = 1\,, \\ &\sup_{(2j-1)m \le t \le (2j+1)m} \big| x_{m,E}(t) \big| \le Ks^2 \quad \textit{when} \quad e_j = 0\,, \end{aligned} \tag{3.4}$$

where $\alpha_{m,j,E} \in \mathbb{R}$ and $|\alpha_{m,j,E}| \le K$.

Now let us take any infinite sequence $E = \{e_j\}_{j=-\infty}^{\infty} \in \{0,1\}^{\mathbb{Z}}$ and apply Theorems 3.1 and 3.2 to $E_p = \{e_j\}_{j=-p}^{p}$, $p \in \mathbb{N}$ to obtain the solution x_{m,E_p}. By passing to the limit $p \to \infty$, then the proof of Theorem 7.1 of [4] immediately implies the following main result of this paper.

Theorem 3.3. *Theorems* 3.1 *and* 3.2 *are valid for any infinite sequence $E = \{e_j\}_{j=-\infty}^{\infty} \in \{0,1\}^{\mathbb{Z}}$, where of course $x_{m,E}$ are generally non–periodic.*

Hence for any $\mu \in \mathcal{R}$, $m \in \mathbb{N}$ sufficiently large and $E \in \{0,1\}^{\mathbb{Z}}$, (1.2) possesses a solution $z_{m,E}$ satisfying either (3.3) or (3.4). These estimates (3.3), (3.4) give the injectivity of the mapping $E \to z_{m,E}$ for $s > 0$ small. Let $J : \mathcal{E} = \big\{E \in \{0,1\}^{\mathbb{Z}}\big\} \to \mathcal{E}$ be the Bernoulli shift defined by $J\big(\{e_j\}_{j\in\mathbb{Z}}\big) = \{\tilde{e}_j\}_{j\in\mathbb{Z}}$, $\tilde{e}_j = e_{j+1}$. Now the estimates (3.3)–(3.4) imply that $x_{m,J(E)}(t)$ is orbitally close to $x_{m,E}(t+2m)$. Hence we can extend the deterministic chaos of [9] and [11] to (1.2) as follows.

Theorem 3.4. *Under the assumptions of either Theorem* 3.1 *or Theorem* 3.2*, for any $\mu \in \mathcal{R}$ and $m \in \mathbb{N}$ sufficiently large,* (1.2) *possesses a family of solutions $\{z_{m,E}\}_{E\in\mathcal{E}}$ such that*

(i) *$E \to x_{m,E}$ is injective;*
(ii) *$x_{m,J(E)}(t)$ is orbitally close to $x_{m,E}(t+2m)$.*

4. Singular Differential Inclusions

In Section 3, Theorem 3.3 was obtained by extending Theorems 3.2 and 3.3 of [4]. In this section we use the same method to extend Theorems 4.1 and 4.2 of [4]. We consider singularly perturbed differential inclusions of the form

$$\epsilon\dot{x}(t) \in f(x(t)) + \epsilon h(x(t),\epsilon,t) \quad \text{a.e. on} \quad \mathbb{R}, \quad \epsilon \neq 0\,, \tag{4.1}$$

where $\epsilon \in \mathbb{R}$ is small, f has the properties (i)–(iv) of the Introduction and $h : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}^n} \setminus \emptyset$ is 2–periodic in t as well as upper–semicontinuous with compact convex values. We also introduce the multivalued mappings (see (4.3), (4.4) of [4]) of the forms

(4.2)

$$\begin{aligned} &M_\pm : \mathbb{R}^d \to 2^{\mathbb{R}^d} \setminus \emptyset\,, \quad M_\pm = (M_{\pm 1}, \cdots, M_{\pm d}),\\ &M_{\pm l}(\alpha,\beta) = \Bigg\{ \int_{-\infty}^{\infty} \langle p(s), u_l^\perp(s)\rangle\, ds \ : \ p \in L_{2,\mathrm{loc}}(\mathbb{R},\mathbb{R}^n);\\ &p(t) \in \Big(\frac{1}{2}\sum_{i,j=1}^{d-1} \beta_i\beta_j D^2 f(\gamma(t))(u_{d+i}(t),u_{d+j}(t)) \pm h(\gamma(t),0,\alpha)\Big) \quad \text{a.e. on} \quad \mathbb{R} \Bigg\}, \end{aligned}$$

and

$$\begin{aligned} &M : \mathbb{R} \to 2^{\mathbb{R}} \setminus \emptyset,\\ &M(\alpha) = \Bigg\{ \int_{-\infty}^{\infty} \langle p(s), u_1^\perp(s)\rangle\, ds \ : \ p \in L_{2,\mathrm{loc}}(\mathbb{R},\mathbb{R});\\ &\qquad\qquad p(t) \in h(\gamma(t),0,\alpha) \quad \text{a.e. on} \quad \mathbb{R} \Bigg\}. \end{aligned} \tag{4.3}$$

Then Theorems 4.1 and 4.2 of [4] have the following extensions.

Theorem 4.1. *Let $d > 1$. If there is a non–empty open bounded set $\mathcal{B} \subset \mathbb{R}^d$ and $* \in \{-,+\}$ such that*

(i) $0 \notin M_*(\partial\mathcal{B})\,,$
(ii) $deg\,(M_*,\mathcal{B},0) \neq 0\,,$

where M_* is given by (4.2)*, then there is a constant $K > 0$ such that for any sufficiently small $\epsilon \neq 0, \operatorname{sgn}\epsilon = *1$ and any $E = \{e_j\}_{j\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$, the differential inclusion* (4.1) *possesses a solution $x_{m,\epsilon,E}$ for any $m \in \mathbb{N}$ satisfying*

$$\sup_{(2j-1)m\leq t\leq(2j+1)m} \Big|x_{m,\epsilon,E}(t) - \gamma\Big(\frac{t-2mj-\alpha_{m,j,\epsilon,E}}{\epsilon}\Big)\Big| \leq K\sqrt{|\epsilon|} \quad \textit{when} \quad e_j = 1\,,$$

$$\sup_{(2j-1)m\leq t\leq(2j+1)m} \big|x_{m,\epsilon,E}(t)\big| \leq K|\epsilon| \quad \textit{when} \quad e_j = 0\,,$$

where $\alpha_{m,j,\epsilon,E} \in \mathbb{R}$ and $|\alpha_{m,j,\epsilon,E}| \leq K$.

Theorem 4.2. *Let $d = 1$. If there are constants $a < b$ such that*

$$\begin{aligned}&M(a) \text{ contains only positive (respectively negative) numbers}\\ &\text{and } M(b) \text{ contains only negative (respectively positive) ones},\end{aligned}$$

where M is given by (4.3), then there is a constant $K > 0$ such that for any sufficiently small $\epsilon \neq 0$ and any $E = \{e_j\}_{j\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$, the differential inclusion (4.1) possesses a solution $x_{m,\epsilon,E}$ for any $m \in \mathbb{N}$ satisfying

$$\begin{aligned}&\sup_{(2j-1)m\le t\le(2j+1)m} \Big|x_{m,\epsilon,E}(t) - \gamma\Big(\frac{t-2mj-\alpha_{m,j,\epsilon,E}}{\epsilon}\Big)\Big| \le K|\epsilon| \quad \text{when} \quad e_j = 1\,,\\ &\sup_{(2j-1)m\le t\le(2j+1)m} \big|x_{m,\epsilon,E}(t)\big| \le K|\epsilon| \quad \text{when} \quad e_j = 0\,,\end{aligned}$$

where $\alpha_{m,j,\epsilon,E} \in \mathbb{R}$ and $|\alpha_{m,j,\epsilon,E}| \le K$.

Finally, we consider the singularly perturbed differential inclusions of [5] which take the following form

$$\begin{aligned}\dot{x}(t) &\in f(x(t),y(t)) + \epsilon h_1(x(t),y(t),t) \quad \text{a.e. on} \quad \mathbb{R},\\ \epsilon\dot{y}(t) &\in g(x(t),y(t)) + \epsilon h_2(x(t),y(t),t) \quad \text{a.e. on} \quad \mathbb{R},\end{aligned} \tag{4.4}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^k$ and $\epsilon > 0$ is small, such that

(i) $f \in C^2(\mathbb{R}^n \times \mathbb{R}^k, \mathbb{R}^n)$, $g \in C^2(\mathbb{R}^n \times \mathbb{R}^k, \mathbb{R}^k)$ and $h_1 : \mathbb{R}^n \times \mathbb{R}^k \to 2^{\mathbb{R}^n} \setminus \emptyset$, $h_2 : \mathbb{R}^n \times \mathbb{R}^k \to 2^{\mathbb{R}^k} \setminus \emptyset$ are upper–semicontinuous with compact and convex values.

(ii) $f(0,0) = 0$ and $\Re\tau \neq 0$ for any $\tau \in \sigma(D_x f(0,0))$.

(iii) $g(\cdot,0) = 0$, $g(x,y) = A(x)y + o(|y|)$ for $A(x) \in \mathcal{L}(\mathbb{R}^k)$ satisfying

$$B(x)A(x)B^{-1}(x) = (D_1(x), D_2(x)) \quad \forall\, x \in \mathbb{R}^n\,,$$

where $B : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^k)$, $D_1 : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^{k_1})$, $D_2 : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^{k_2})$ are C^1–smooth mappings, $k = k_1 + k_2$ and

$$\begin{gathered}\langle D_1(x)v, v\rangle > a|v|^2, \quad \langle D_2(x)w, w\rangle < -a|w|^2\\ \forall\, x \in \mathbb{R}^n,\, \forall\, v \in \mathbb{R}^{k_1},\, \forall\, w \in \mathbb{R}^{k_2}\,,\end{gathered}$$

where $a > 0$ is a constant and $\langle\cdot,\cdot\rangle$ are inner products.

(iv) The reduced equation of (4.4), of the form $\dot{x} = f(x,0)$, has a homoclinic solution. That is, there exists a differentiable function $t \to \gamma(t)$ such that $\lim_{t\to+\infty}\gamma(t) = \lim_{t\to-\infty}\gamma(t) = 0$ and $\dot{\gamma}(t) = f(\gamma(t),0)$.

(v) $h_i(x,y,t+2) = h_i(x,y,t)$ for $t \in \mathbb{R}$ and $i = 1,\, 2$.

We assume in addition (see [5]) that the following condition holds:

(H) There is an upper–semicontinuous mapping $C : \mathbb{R}\times\mathbb{R} \to 2^{\mathbb{R}^k}\setminus\emptyset$ with compact convex values such that $C(\mathbb{R}\times\mathbb{R})$ is bounded and $C(t,\alpha+2) = C(t,\alpha)$. Moreover, for any $\delta > 0$, $l \in \mathbb{N}$, $l > l_0$, where $l_0 \in \mathbb{N}$ is fixed, and $\alpha \in \mathbb{R}$, there are $\epsilon_{\delta,l} > 0$, $l < m_{\delta,l} \in \mathbb{N}$, $\zeta_{\delta,l} > 0$ such that for any $\mathbb{N} \ni m \ge m_{\delta,l}$ and

$$h \in Y_m \text{ satisfying}$$
$$|h|_m \le 1 + \sup_{s,t\in\mathbb{R}} \Big\{ \max \big\{ |v| \ : \ v \in h_2(\gamma(t),0,s) \cup h_2(0,0,s) \big\} \Big\}$$
$$(t,h(t)) \in \Big\{ u \in \mathbb{R}^{k+1} \ : \ \exists\, s \in \mathbb{R};\ \mathrm{dist}\, \big\{ u, \big(s, h_2(\gamma(s),0,s+\alpha)\big) \big\} < \zeta_{\delta,l} \Big\}$$
$$\text{a.e. on} \quad [-l-1, l+1]\,,$$
the solution y of $\epsilon\dot y = A(\gamma(t))y + h(t)$ with $0 < \epsilon < \epsilon_{\delta,l}$ satisfies
$$(t,y(t)) \in \Big\{ u \in \mathbb{R}^{n+1} \ : \ \exists\, s \in \mathbb{R};\ \mathrm{dist}\, \big\{ u, \big(s, C(s,\alpha)\big) \big\} < \delta \Big\} \quad \text{on} \quad [-l,l]\,.$$

Now we introduce the multivalued mappings (see (3.9), (3.10) of [5])

(4.5)
$$M \,:\, \mathbb{R}^d \to 2^{\mathbb{R}^d} \setminus \emptyset\,, \quad M = (M_1, \cdots, M_d),$$
$$M_l(\alpha,\beta) = \Bigg\{ \int_{-\infty}^{\infty} \langle h(s), u_l^\perp(s)\rangle\, ds \ : \ h \in L_{2,\mathrm{loc}}(\mathbb{R},\mathbb{R}^n) \text{ satisfying a.e. on } \mathbb{R}$$
$$\text{the relation} \quad h(t) \in \big(\tfrac{1}{2} \textstyle\sum_{i,j=1}^{d-1} \beta_i\beta_j D_x^2 f(\gamma(t),0)(u_{d+i}(t), u_{d+j}(t))$$
$$+ D_y f(\gamma(t),0)(C(t,\alpha)) + h_1(\gamma(t),0,t+\alpha)\big) \Bigg\}\,,$$

and

$$M \,:\, \mathbb{R} \to 2^{\mathbb{R}} \setminus \emptyset, \tag{4.6}$$
$$M(\alpha) = \Bigg\{ \int_{-\infty}^{\infty} \langle h(s), u_1^\perp(s)\rangle\, ds \ : \ h \in L_{2,\mathrm{loc}}(\mathbb{R},\mathbb{R});$$
$$h(t) \in D_y f(\gamma(t),0)(C(t,\alpha)) + h_1(\gamma(t),0,t+\alpha) \quad \text{a.e. on} \quad \mathbb{R} \Bigg\}\,.$$

By combining the arguments of Section 3 of this paper with those of [5], we obtain the following extensions of Theorems 3.1 and 3.2 of [5].

Theorem 4.3. *Let $d > 1$. If there is a non–empty open bounded set $\mathcal{B} \subset \mathbb{R}^d$ such that*

(i) $0 \notin M(\partial\mathcal{B})\,,$
(ii) $\mathit{deg}\,(M, \mathcal{B}, 0) \neq 0\,,$

where M is given by (4.5), then there are constants $K > 0$ and $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$ and any $E = \{e_j\}_{j\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$, the differential inclusion (4.4) possesses a solution $(x_{m,\epsilon,E}, y_{m,\epsilon,E})$ for any $m \in \mathbb{N}$, $m \ge [1/\sqrt{\epsilon}\,]$ satisfying
$$\sup_{t\in\mathbb{R}} |y_{m,\epsilon,E}(t)| \le K\epsilon$$
and
$$\sup_{(2j-1)m \le t \le (2j+1)m} \big| x_{m,\epsilon,E}(t) - \gamma(t - 2mj - \alpha_{m,j,\epsilon,E}) \big| \le K\sqrt{\epsilon} \quad \textit{when} \quad e_j = 1\,,$$
$$\sup_{(2j-1)m \le t \le (2j+1)m} \big| x_{m,\epsilon,E}(t) \big| \le K\epsilon \quad \textit{when} \quad e_j = 0\,,$$

where $\alpha_{m,j,\epsilon,E} \in \mathbb{R}$ and $|\alpha_{m,j,\epsilon,E}| \le K$.

Theorem 4.4. *Let $d = 1$. If there are constants $a < b$ such that*

$$M(a) \text{ contains only positive (respectively negative) numbers}$$
$$\text{and } M(b) \text{ contains only negative (respectively positive) ones},$$

where M is given by (4.6), then there are constants $K > 0$ and $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$ and any $E = \{e_j\}_{j\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$, the differential inclusion (4.4) possesses a solution $(x_{m,\epsilon,E}, y_{m,\epsilon,E})$ for any $m \in \mathbb{N}$, $m \geq [1/\sqrt{\epsilon}\,]$ satisfying

$$\sup_{t\in\mathbb{R}} |y_{m,\epsilon,E}(t)| \leq K\epsilon$$

and

$$\sup_{(2j-1)m\leq t\leq(2j+1)m} \left|x_{m,\epsilon,E}(t) - \gamma(t - 2mj - \alpha_{m,j,\epsilon,E})\right| \leq K\epsilon \quad \text{when} \quad e_j = 1\,,$$
$$\sup_{(2j-1)m\leq t\leq(2j+1)m} \left|x_{m,\epsilon,E}(t)\right| \leq K\epsilon \quad \text{when} \quad e_j = 0\,,$$

where $\alpha_{m,j,\epsilon,E} \in \mathbb{R}$ and $|\alpha_{m,j,\epsilon,E}| \leq K$.

References

1. A. A. Andronow, A. A. Witt & S. E. Chaikin, *"Theorie der Schwingungen I"*, Akademie Verlag, Berlin, 1965. MR **39:**2373
2. K. Deimling, *"Multivalued Differential Equations"*, W. De Gruyter, Berlin, 1992. MR **94b:**34026
3. J.P. Den Hartog, *"Mechanische Schwingungen"*, 2nd ed., Springer–Verlag, Berlin, 1952.
4. M. Fečkan, *Bifurcation from homoclinic to periodic solutions in ordinary differential equations with multivalued perturbations*, J. Differential Equations **130** (1996), 415-450. CMP 97:01
5. M. Fečkan, *Bifurcation from homoclinic to periodic solutions in singularly perturbed differential inclusions*, Proc. Royal Soc. Edinburgh **127A** (1997), 727-753. CMP 97:17
6. M. Fečkan, *Bifurcation of periodic solutions in differential inclusions*, Appl. Math. **42** (1997), 369–393. CMP 97:17
7. M. Fečkan & J. Gruendler, *Bifurcation from homoclinic to periodic solutions in ordinary differential equations with singular perturbations*, submitted.
8. J. Gruendler, *Homoclinic solutions for autonomous ordinary differential equations with nonautonomous perturbations*, J. Differential Equations **122** (1995), 1-26. MR **96j:**58123
9. J. Guckenheimer & P. Holmes, *"Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields"*, Springer–Verlag, New York, 1983. MR **85f:**58002
10. H. Kauderer, *"Nichtlineare Mechanik"*, Springer–Verlag, Berlin, 1958. MR **26:**3238
11. K. J. Palmer, *Exponential dichotomies and transversal homoclinic points*, J. Differential Equations **55** (1984), 225-256. MR **86d:**58088
12. T. Pruszko, *Some applications of the topological degree theory to multi–valued boundary value problems*, Dissertationes Math. **229** (1984). MR **85j:**34030

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