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LEFT-SYMMETRIC ALGEBRAS FOR $\mathfrak{gl}(n)$

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Abstract. We study the classification problem for left-symmetric algebras with commutation Lie algebra $\mathfrak{gl}(n)$ in characteristic 0. The problem is equivalent to the classification of étale affine representations of $\mathfrak{gl}(n)$. Algebraic invariant theory is used to characterize those modules for the algebraic group $\mathrm{SL}(n)$ which belong to affine étale representations of $\mathfrak{gl}(n)$. From the classification of these modules we obtain the solution of the classification problem for $\mathfrak{gl}(n)$. As another application of our approach, we exhibit left-symmetric algebra structures on certain reductive Lie algebras with a one-dimensional center and a non-simple semisimple ideal.

1. Introduction

Let k be a field, and V a k-vector space. A *left-symmetric algebra* structure on V is a k-bilinear product $\cdot : V \times V \to V$ satisfying the condition

$$x \cdot (y \cdot z) - (x \cdot y) \cdot z = y \cdot (x \cdot z) - (y \cdot x) \cdot z \tag{1}$$

for all $x, y, z \in V$. The condition implies that the commutators

$$[\, x, y \,] = x \cdot y - y \cdot x \tag{2}$$

satisfy the Jacobi identity. Accordingly, each left-symmetric product has an associated *commutation Lie algebra*.

Clearly, each *associative* algebra product is par forte a left-symmetric product. Among the *Lie-admissible*, *non-associative algebras*, left-symmetric algebras are a natural class, generalizing associative algebras. Over the real and complex numbers left-symmetric algebras are of special interest because of their role in the differential geometry of *flat manifolds* and in the representation theory of Lie groups, see for example [Mi, Me, Se1, Vi, GS].

In this paper, we explain how left-symmetric algebras show up in the invariant theory of reductive groups.

We will assume from now on that k is of characteristic 0 and V is finite-dimensional. If $\mathfrak{g}$ is the commutation Lie algebra of a left-symmetric product, we will say that $\mathfrak{g}$ *admits* the left-symmetric product. Given a Lie algebra $\mathfrak{g}$, it is a fundamental problem *to decide whether* $\mathfrak{g}$ *admits a left symmetric product and to give a classification of such products* [Se2].

This problem is also of geometric and topological importance. Assume $k = \mathbf{R}$, and let G be the connected, simply connected Lie group which belongs to the Lie algebra $\mathfrak{g}$. It is well known that *left invariant flat affine structures* on G correspond to isomorphism classes of left-symmetric products for $\mathfrak{g}$. Assume further, that there

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exists a discrete uniform subgroup Γ of G. Then, the *compact* manifold $\Gamma\backslash G$ inherits a flat affine structure from G. In some cases, the existence of a *flat affine structure* on the compact manifold $\Gamma\backslash G$ is even equivalent to the existence of a *left invariant* flat affine structure on G. See [Be2, Mi].

There are many examples of Lie algebras which do not admit a left-symmetric product. For example, it is easy to see that there are no left-symmetric algebras with semisimple Lie algebra $\mathfrak{g}$, [Me]. It was believed for some time that every solvable Lie algebra admits a left-symmetric product. Recently, Benoist [Be] gave a striking example of an 11-dimensional *nilpotent* Lie algebra without a left-symmetric product. This was generalized in [BG, Bu2].

The problem for $\mathfrak{gl}(n)$. In what follows, we discuss the case of the Lie algebra $\mathfrak{gl}(n)$, the commutation Lie algebra of the *associative* algebra of $n \times n$ matrices. It was noticed by Helmstetter [He] that there exist *non-associative* left-symmetric products for $\mathfrak{gl}(n)$. Burde classified the left-symmetric products for $\mathfrak{gl}(2)$ over the complex numbers by explicit computation of the structure constants [Bu]. Similiar work had also been accomplished previously by Boyom in his thesis [By]. The question how to determine the structure of the variety of left-symmetric algebra products for $\mathfrak{gl}(n)$, $n > 2$, remained open.

The problem for $\mathfrak{gl}(n)$ is part of the more general problem to determine the left-symmetric algebras with *reductive Lie algebra*, or, as a first step, the products which have a reductive Lie algebra with one-dimensional center. This is still an interesting unsolved problem. In this note, we present a general approach, and apply our method to describe the isomorphism classes of left-symmetric algebras for $\mathfrak{gl}(n)$. Two other applications are immediate: Over an algebraically closed ground field, $\mathfrak{gl}(n)$ is the only reductive Lie algebra with one-dimensional center and a *simple* semisimple ideal which admits left-symmetric algebras. (This was already observed in [Bu], but for the complete proof of this fact, Burde refers to the results explained in section 3 of this paper) As a second application, we can show that there do exist reductive Lie algebras with *one-dimensional* center and a *non-simple* semisimple ideal that admit left-symmetric algebras (see section 5).

Description of results. To explain our results in the context of the work in [Bu, He], we first introduce some additional notation: Let $(\mathfrak{g}, \cdot)$ denote a left-symmetric product with commutation Lie algebra $\mathfrak{g}$. A left-symmetric algebra $(\mathfrak{g}, \cdot)$ gives rise to a Lie algebra homomorphism $L : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ defined by

$$L(a)\, b = a \cdot b \quad (a, b \in \mathfrak{g}).$$

L is called the *left regular representation* of $(\mathfrak{g}, \cdot)$. We also define the right multiplication R by $R(a)\, b = b \cdot a$, for $a, b \in \mathfrak{g}$. But, in general, $R : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is not a Lie algebra homomorphism.

The associative kernel (or nucleus) of $(\mathfrak{g}, \cdot)$ is the subalgebra

$$\mathfrak{K}(\cdot) = \{u \in \mathfrak{g} \mid (a \cdot b) \cdot u = a \cdot (b \cdot u)\ \forall a, b \in \mathfrak{g}\}\ .$$

Note, that $\mathfrak{K}(\cdot)$ is an associative subalgebra of $(\mathfrak{g}, \cdot)$, and that the right multiplications R_u give $\mathfrak{g}$ the structure of a module for $\mathfrak{K}(\cdot)$.

The following definition is due to Helmstetter [He]. Let h be an endomorphism of $\mathfrak{g}$, such that $h(\mathfrak{g}) \subset \mathfrak{K}(\cdot)$, and such that $(id - h)$ is bijective. Let $\varphi = (id - h)^{-1}$, then

$$a * b = \varphi(\ a \cdot \varphi^{-1}(b) - \varphi^{-1}(b) \cdot h(a)\ )$$

defines a left-symmetric product for $\mathfrak{g}$, which Helmstetter called the *h-transformation* of $\mathfrak{g}$. The classification of left-symmetric algebras for $\mathfrak{gl}(n,k)$ may now be stated as follows:

Theorem 1. *Every left-symmetric algebra for $\mathfrak{gl}(n,k)$ is isomorphic to an h-transformation of the associative product, for all $n \geq 3$. For $\mathfrak{gl}(2,k)$, there exists, besides the h-transformations of the associative product, one exceptional isomorphism class of left-symmetric products.*

The theorem is supplemented by the following proposition:

Proposition 1. *The isomorphism classes of h-transformations of the associative product are parametrized by the conjugacy classes of matrices $\varphi \in \mathfrak{gl}(n,k)$ with* trace $\varphi = n$.

Geometric interpretation. It should be mentioned that the results formulated here in pure algebraic terms admit interpretation in terms of the geometry of the associated flat affine structures on real reductive Lie groups. These structures are all *incomplete* and, as a consequence of the considerations in section 3, the semisimple invariant subgroup S of such a reductive group G is endowed with a left invariant flat *projective* structure. The group G, as an affine manifold, may be viewed as covering a cone over S. The description of flat affine structures on reductive Lie groups with a one-dimensional center is thus reduced to the description of flat projective structures on semisimple Lie groups. Such structures are of independent interest, see [NS].

Overview of the paper. As a starting point, we show in section 2 how left-symmetric algebras are related to *affine étale representations* of Lie algebras. (An affine representation of a Lie group on a vector space V is called *étale*, if G has an open orbit with a discrete stabilizer. However, this notion may be expressed via the differential of the action in purely Lie algebraic terms.)

In the case of a reductive Lie algebra $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{a}$, where $\mathfrak{s}$ is the semisimple ideal of $\mathfrak{g}$ and $\mathfrak{a}$ the center, an affine action of $\mathfrak{g}$ induces a linear representation of $\mathfrak{s}$. Over an algebraically closed ground field k, this representation integrates to an action of a connected, simply connected, semisimple algebraic group S. We call the S-modules corresponding to such actions *special modules for S*. In section 3, we show that these modules are *coregular* with a one-dimensional ring of invariants and discuss their properties from the point of view of invariant theory. Special modules may also be viewed as a subclass of *prehomogeneous* vector spaces.

In section 4, we present the classification of special modules for $\mathrm{SL}(n,k)$ and sketch a proof. The result is also contained in Schwarz's tables of coregular representations of simple groups (see [S1]). The determination of coregular representations for general semisimple groups is a difficult problem, and even the classification of the special modules for such groups seems to be open.

Finally, in section 5, we pull together the considerations in sections 2 and 3 to derive a structure theory for left-symmetric algebras with a reductive Lie algebra and one-dimensional center. The classification result of section 4 implies our result on left-symmetric algebras for $\mathfrak{gl}(n)$.

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I am indebted to Fritz Grunewald for introducing me to the subject of left-symmetric algebras.

2. Étale affine representations of Lie algebras

In this section, we work out the correspondence between left-symmetric products for $\mathfrak{g}$ and étale affine representations of $\mathfrak{g}$. Equivalence classes of étale affine representations are called *affine structures* for $\mathfrak{g}$. Affine structures correspond to isomorphism classes of left-symmetric algebras. An affine structure is *linear*, precisely if the left-symmetric algebras in the corresponding isomorphism class contain a right identity.

2.1. **Preliminaries.** Let V be a vector space over the ground field k. We identify V with the hyperplane $V \times \{1\}$ in the vector space $V \oplus k$. The group of affine transformations $\mathrm{Aff}(V)$ is identified with the subgroup

$$\mathrm{Aff}(V) = \left\{ A = \begin{pmatrix} g & v \\ 0 & 1 \end{pmatrix} \mid g \in \mathrm{GL}(V),\, v \in V \right\} \subset \mathrm{GL}(V \oplus k)\ .$$

Given $A \in \mathrm{Aff}(V)$, we call $l(A) = g$ the *linear part* and v the *translational part* of A. Let $\mathfrak{aff}(V)$ be the Lie algebra of $\mathrm{Aff}(V)$. Then we have the embedding

$$\mathfrak{aff}(V) = \left\{ \phi = \begin{pmatrix} \varphi & v_0 \\ 0 & 0 \end{pmatrix} \mid \varphi \in \mathfrak{gl}(V),\, v_0 \in V \right\} \subset \mathfrak{gl}(V \oplus k)\ .$$

From this we have a direct sum decomposition $\mathfrak{aff}(V) = \mathfrak{gl}(V) \oplus V$. Again, given $\phi \in \mathfrak{aff}(V)$ we call $l(\phi) = \varphi$ the *linear part* and $t_0(\phi) = v_0$ the *translational part* of ϕ.

The *affine action* of $\phi \in \mathfrak{aff}(V)$ on V is defined by

$$\phi \cdot x = \varphi(x) + v_0 \ \text{ for } x \in V.$$

For all $x \in V$ we have a linear map

$$t_x : \mathfrak{aff}(V) \to V,\ \phi \mapsto \phi \cdot x\ ,$$

which we call the evaluation at x. Let $H \in \mathrm{Aff}(V)$ and $\mathrm{Ad}\ H : \mathfrak{aff}(V) \to \mathfrak{aff}(V)$ conjugation with H. The evaluations t_x and t_{Hx} are related by

$$t_{Hx}\, \mathrm{Ad}\ H = l(H) t_x\ . \tag{3}$$

The embedding $\mathfrak{aff}(V) \subset \mathfrak{gl}(V \oplus k)$ gives us a composition $\phi\psi$ of elements ϕ, ψ in $\mathfrak{aff}(V)$. Then

$$t_x(\phi\psi) = l(\phi) t_x(\psi)\ . \tag{4}$$

2.1.1. *Affine representations.* Let $\mathfrak{g}$ be a Lie algebra and $\rho : \mathfrak{g} \to \mathfrak{aff}(V)$ be a homomorphism. The evaluation mapping $ev_x : \mathfrak{g} \to V$ of ρ at the point x is defined by

$$ev_x = t_x \rho\ .$$

Let $\rho' = H\rho H^{-1}$ be a representation conjugate to ρ. The evaluation ev'_{Hx} of ρ' satisfies

$$ev'_{Hx} = l(H) ev_x\ . \tag{5}$$

A homomorphism $\rho : \mathfrak{g} \to \mathfrak{aff}(V)$ is called an *étale affine representation*, if there exists a base point $x_0 \in V$ such that the evaluation mapping

$$ev_{x_0} : \mathfrak{g} \to V\ ,\ a \mapsto \rho(a) \cdot x_0$$

is an isomorphism. We say that ρ is *étale at* x_0.

Definition. Two affine étale representations (ρ, x) and (ρ', x') of $\mathfrak{g}$ on V are called *isomorphic*, if there exists a $g \in \mathrm{Aff}(V)$, such that $\rho' = g\rho g^{-1}$ and $x' = gx$. We call them *equivalent*, if there exists an automorphism h of $\mathfrak{g}$ such that (ρ', x') and $(\rho h, x)$ are isomorphic.

An equivalence class of étale representations is called an *affine structure* for $\mathfrak{g}$. An affine structure is called *linear* if it has a representing representation (ρ, x_0) which is linear, i.e. $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. A point $x \in V$ is called a *fixed point for* ρ if $ev_x(\mathfrak{g}) = \rho(\mathfrak{g}) \cdot x = 0$. Then it is clear that an affine structure is linear iff every representing representation (ρ, x_0) has a fixed point.

Example 2.1. A typical example of a (linear) affine structure for $\mathfrak{gl}(n, k)$ is given as follows. Let $\mathfrak{gl}(n, k)$ act on itself via left multiplication of matrices. The corresponding linear representation is étale in all matrices with nonvanishing *determinant*, and the zero matrix is the only fixed point.

2.2. **Correspondence with left-symmetric algebras.** Let V be a k-vector space and $\dim V = \dim \mathfrak{g}$. We fix a linear isomorphism $\alpha : \mathfrak{g} \to V$. Let $(\mathfrak{g}, \cdot)$ be a left-symmetric product and L the left regular representation. Then, the map

$$\mathfrak{g} \ni a \mapsto \begin{pmatrix} \alpha L(a)\alpha^{-1} & \alpha(a) \\ 0 & 0 \end{pmatrix} \in \mathfrak{aff}(V) \tag{6}$$

defines an étale affine representation of $\mathfrak{g}$ with base $0 \in V$ and evaluation mapping $ev_0 = \alpha$. Conversely, let $\rho : \mathfrak{g} \to \mathfrak{aff}(V)$ be an étale affine representation of $\mathfrak{g}$ with base $0 \in V$ and evaluation mapping $t = t_{x_0}\rho : \mathfrak{g} \to V$. The equation

$$a \cdot b = t^{-1}(l(\rho(a))t(b)) \tag{7}$$

defines a left-symmetric product $(\mathfrak{g}, \cdot)$ which will be called *the left-symmetric algebra corresponding to* ρ. We have the following relation between the left-symmetric product and the (Lie algebra) homomorphism ρ:

$$t_{x_0}(\rho(a \cdot b)) = t_{x_0}(\rho(a)\rho(b)) \ . \tag{8}$$

Via (6) and (7) we have defined maps

$$\begin{aligned} \Phi : \{ \text{ left-symmetric products } \} &\longrightarrow \{ \text{ étale representations with base } 0 \}, \\ \Psi : \{ \text{ étale representations } \} &\longrightarrow \{ \text{ left-symmetric products } \} \end{aligned}$$

and we obtain the following basic result:

Proposition 2.1. *The maps Φ and Ψ induce a bijection between the set of left-symmetric products for $\mathfrak{g}$ and the isomorphism classes of étale affine representations with base point. Under this correspondence isomorphic left-symmetric products are mapped to equivalent étale representations and vice versa.*

Proof. Let (ρ, x_0) be an étale affine representation of $\mathfrak{g}$ and $H \in \mathrm{Aff}(V)$. $(\rho^H = H\rho H^{-1}, Hx_0)$ is isomorphic to (ρ, x_0). We have to show that their corresponding left-symmetric algebras coincide. Let $t' = ev_{Hx_0}$ be the evaluation mapping for ρ^H. According to (3) we have $t' = l(H)t$, and we compute the left-symmetric product $*$ corresponding to ρ^H as follows: $a * b = t'^{-1}l(\rho(a)^H)t'(b) = t'^{-1}l(\rho(a))^{l(H)}t'(b) = t^{-1}l(\rho(a))t(b) = a \cdot b$. Now the map Ψ is defined on the set of isomorphism classes of étale affine representations.

It is immediate from the definitions that $\Psi\Phi(\cdot) = \cdot$. Now we show that $\Phi\Psi(\rho, x_0)$ is an étale affine representation isomorphic to (ρ, x_0) and the correspondence is

proved. Let L be the left regular representation of the product $\cdot = \Psi(\rho, x_0)$ and $t_0\rho$ the translational part of ρ. We have

$$L(a) = t^{-1}l(\rho(a))t \ , \ t_0\rho(a) = t(a) - l(\rho(a))x_0 \ . \tag{9}$$

The representation $\Phi(\Psi(\rho, x_0))$ with base point $0 \in V$ is given as follows:

$$\Phi(\Psi(\rho))(a) = \begin{pmatrix} \alpha t^{-1}l(\rho(a))t\alpha^{-1} & \alpha(a) \\ 0 & 0 \end{pmatrix} \ .$$

Let H be the affine transformation with linear part $l(H) = t\alpha^{-1}$ and translational part x_0. Then we have $((\Phi\Psi(\rho))^H, H \cdot 0) = (\rho, x_0)$.

Now we prove that isomorphic left-symmetric algebras correspond to equivalent representations. Let h be an automorphism of $\mathfrak{g}$. The product

$$a * b = h\,(h^{-1}(a) \cdot h^{-1}(b))$$

is isomorphic to $\cdot$. Let $H = \alpha h \alpha^{-1} \in \mathrm{GL}(V)$. Then the representation

$$\Phi(*) : a \mapsto \begin{pmatrix} \alpha h L(h^{-1}(a))h^{-1}\alpha^{-1} & \alpha(a) \\ 0 & 0 \end{pmatrix} = H(\Phi(\cdot)(h^{-1}(a))H^{-1}$$

is an étale representation equivalent to $\Phi(\cdot)$. Conversely, let (ρ, x_0) and (ρ', x_0') be equivalent. We can assume that $(\rho', x_0') = (\rho h, x_0)$. Then, obviously $t' = th$ and $l(\rho'(a)) = l(\rho(ha))$. It follows that the product $\Psi(\rho', x_0)$ is isomorphic to $\Psi(\rho)$ via h^{-1}. □

The next proposition gives us a simple criterion of linearity for an étale affine representation in terms of the corresponding left-symmetric algebra.

Proposition 2.2. *An étale affine representation is isomorphic to a linear representation, iff the corresponding left-symmetric algebra has a right-identity.*

Proof. We use (7). Then it is clear that $v \in V$ is a fixed point for ρ, iff $t^{-1}(x_0 - v)$ is a right identity of the product $\cdot$. □

2.3. **Centralizer and associative kernel.** The centralizer and normalizer of an étale affine representation also have a left-symmetric algebra interpretation.

Let $\rho : \mathfrak{g} \to \mathfrak{aff}(V)$ be an affine representation. The normalizer of ρ in $\mathrm{Aff}(V)$ is defined as

$$\mathrm{N}(\rho) = \{H \in \mathrm{Aff}(V) \mid H\rho(\mathfrak{g})H^{-1} \subset \rho(\mathfrak{g})\} \ .$$

Every $H \in \mathrm{N}(\rho)$ defines an automorphism Ad H of $\rho(\mathfrak{g})$ by restriction of conjugation. The centralizer of ρ in $\mathrm{Aff}(V)$ is defined as

$$Z(\rho) = \{H \in \mathrm{N}(\rho) \mid \mathrm{Ad}\ H = id\} \ .$$

The normalizer and the centralizer of ρ are *algebraic* subgroups of $\mathrm{Aff}(V)$. Their corresponding Lie algebras are the following subalgebras of $\mathfrak{aff}(V)$:

$$\begin{aligned} \mathfrak{n}(\rho) &= \{\phi \in \mathfrak{aff}(V) \mid [\phi, \rho(\mathfrak{g})] \subset \rho(\mathfrak{g})\} \text{ and} \\ \mathfrak{z}(\rho) &= \{\phi \in \mathfrak{aff}(V) \mid [\phi, \rho(\mathfrak{g})] = 0\} \ . \end{aligned}$$

Let

$$\mathfrak{n}(\rho, x_0) = \{\phi \in \mathfrak{n}(\rho) \mid \phi \cdot x_0 = 0\} \ .$$

The structure of the normalizer of an étale affine representation is described by

Lemma 2.3. *Let (ρ, x_0) be an étale affine representation. Then*

$$\mathfrak{n}(\rho) = \rho(\mathfrak{g}) \oplus \mathfrak{n}(\rho, x_0) \ .$$

Now, we want to study the Lie centralizer $\mathfrak{z}(\rho)$ of the étale representation ρ. From the imbedding $\mathfrak{aff}(V) \subset \mathfrak{gl}(V \oplus k)$ the Lie algebra $\mathfrak{aff}(V)$ inherits a compatible associative product. (Of course, we do not assume that an associative algebra must have a unity element.) We observe that $\mathfrak{z}(\rho) \subset \mathfrak{aff}(V)$ is an associative subalgebra of $\mathfrak{aff}(V)$.

Lemma 2.4. $\mathfrak{z}(\rho) \cap \mathfrak{n}(\rho, x_0) = \{0\}$.

Proof. Let $\psi \in \mathfrak{z}(\rho) \cap \mathfrak{n}(\rho, x_0)$ and $v \in V$. Since $v = \rho(a) \cdot x_0$ for some $a \in \mathfrak{g}$, $l(\psi)v = l(\psi)(\rho(a) \cdot x_0) = l(\rho(a))(\psi \cdot x_0) = 0$. Hence $\psi = 0$. □

Let $(\mathfrak{g}, \cdot)$ be the left-symmetric algebra corresponding to (ρ, x_0). For $\phi \in \mathfrak{z}(\rho)$, $\phi = \rho(a) + n$, where $n \in \mathfrak{n}(\rho, x_0)$ and $a \in \mathfrak{g}$, we define

$$I(\phi) = a \ . \tag{10}$$

Lemma 2.5. $I(\phi) \in \mathfrak{K}(\cdot)$.

Proof. Let $\varphi = l(\phi)$ be the linear part and v the translational part of ϕ. Then

$$I(\phi) = t^{-1}(\phi \cdot x_0) \ . \tag{11}$$

Since ϕ commutes with $\rho(a)$, we have $[\varphi, l(\rho(a))] = 0$ and $l(\rho(a))v = \varphi t_0(\rho(a))$, for all $a \in \mathfrak{g}$. Now let $c = I(\phi)$. The definition of the left-symmetric product corresponding to ρ gives $t(a \cdot c) = l(\rho(a))(\varphi x_0 + v) = \varphi(\rho(a))x_0 + l(\rho(a))v = \varphi t(a)$. It follows that for all $a, b \in \mathfrak{g}$: $t(a \cdot (b \cdot c)) = l(\rho(a))\varphi t(b) = \varphi t(a \cdot b) = t((a \cdot b) \cdot c)$. □

Thus, we have a map $I : \mathfrak{z}(\rho) \to \mathfrak{K}(\cdot)$.

Proposition 2.6. *The map I is an anti-isomorphism of associative algebras from the centralizer $\mathfrak{z}$ of $\rho(\mathfrak{g})$ in $\mathfrak{aff}(V)$ onto the associative kernel of the left-symmetric product corresponding to ρ.*

Proof. To show that I respects the associative products, an easy calculation using (11) suffices. Injectivity follows from Lemma 2.4. We have to show that I is surjective. So let $u \in \mathfrak{K}(\cdot)$. We consider the right multiplication $R_u : \mathfrak{g} \to \mathfrak{g}$. The map $R : \mathfrak{K}(\cdot) \to \mathfrak{gl}(\mathfrak{g})$ is an anti-homomorphism of associative algebras. Since $[L(a), R_u] = 0$ for all $a \in \mathfrak{g}$, we conclude, using (9), that

$$\phi(u) = \begin{pmatrix} tR_u t^{-1} & t(u) - tR_u t^{-1} x_0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{z}(\rho) \ .$$

Using (11), we have $I(\phi(u)) = u$. □

2.4. **Automorphisms.** Now we study the normalizer of (ρ, x_0) in $\mathrm{Aff}(V)$. Let

$$\mathrm{N}(\rho, x_0) = \{H \in \mathrm{N}(\rho) \mid Hx_0 = x_0\} \ .$$

Since ρ is faithful, the automorphism Ad H of $\rho(\mathfrak{g})$ induces an automorphism h of $\mathfrak{g}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{h} & \mathfrak{g} \\ \downarrow & & \downarrow \\ \rho(\mathfrak{g}) & \xrightarrow{\mathrm{Ad}\ H} & \rho(\mathfrak{g}) \end{array}$$

Proposition 2.7. *Let* $\mathrm{Aut}(\cdot)$ *be the group of automorphisms of the left-symmetric product* $(\mathfrak{g},\cdot)$ *corresponding to* (ρ,x_0)*. The map* $H\mapsto h$ *is an isomorphism*

$$\mathrm{N}(\rho,x_0)\xrightarrow{\cong}\mathrm{Aut}(\cdot)\ .$$

Proof. We show first that h is an automorphism of $(\mathfrak{g},\cdot)=\Psi(\rho)$. It follows from the proof of 2.1 that h is an automorphism from $\Psi(\rho)$ to $\Psi(\rho h)$. But it is also proved there that $\Psi(\rho)=\Psi(H\rho H^{-1})$. Since $H\rho H^{-1}=\rho h$, our claim follows. Let $l(H)$ be the linear part of H. Then (3) implies that the diagram

$$\begin{array}{ccc}\rho(\mathfrak{g}) & \xrightarrow{\mathrm{Ad}\ H} & \rho(\mathfrak{g})\\ \downarrow & & \downarrow\\ V & \xrightarrow{l(H)} & V\end{array}$$

is commutative, where the arrows $\rho(\mathfrak{g})\to V$ equal the evaluation t_{x_0}. Since all maps are isomorphisms, Ad H determines the linear part of H. The translational part is determined by the condition $Hx_0=x_0$. From this follows the injectivity of $H\mapsto h$. To show surjectivity, we can use the same reasoning. Let $t=t_{x_0}\rho$ be the evaluation of (ρ,x_0). The unique transformation $H\in\mathrm{N}(\rho,x_0)$ with $l(H)=tht^{-1}$ induces via conjugation $h\in\mathrm{Aut}(\cdot)$. □

The group $\mathrm{N}(\rho,x_0)$ also operates via conjugation on the centralizer $\mathfrak{z}(\rho)\subset\mathfrak{aff}(V)$. We have the isomorphism

$$I:\mathfrak{z}(\rho)\xrightarrow{\cong}\mathfrak{K}(\cdot)\ .$$

Clearly, $\mathrm{Aut}(\cdot)$ operates on $\mathfrak{K}(\cdot)$. We leave the proof of the following lemma to the reader.

Lemma 2.8. *Let* $H\in\mathrm{N}(\rho,x_0)$ *and* $h\in\mathrm{Aut}(\cdot)$ *be induced by* Ad H*. Then we have* $I\,\mathrm{Ad}\ H=hI$*.*

3. Special modules for semisimple groups

We assume the ground field k to be algebraically closed now. Let $\mathfrak{g}=\mathfrak{s}\oplus\mathfrak{a}$ be a reductive Lie algebra, $\mathfrak{a}$ the center of $\mathfrak{g}$ and $\mathfrak{s}$ the semisimple ideal. Let (ρ,x_0) be an étale affine representation of $\mathfrak{g}$. Let S be the connected, simply-connected, semisimple algebraic group with Lie algebra $\mathfrak{s}$. The representation of $\mathfrak{s}$ induced by ρ is the differential of a *rational representation* $\sigma:S\to\mathrm{Aff}(V)\subset\mathrm{GL}(V\oplus k)$. Thus, we view the affine space V as an *algebraic S-variety*, and we have the machinery of algebraic invariant theory available. For an introduction into this theory, see for example [Po, Kr].

Since, as is proved in [Mi], the reductive group S has a fixed point in V, we can assume that S acts linearly on V.

Definition. We call the modules V for S, arising in the manner just described, *special modules* for S.

The restriction of the evaluation mapping $ev_{x_0}:\mathfrak{g}\to V$ of (ρ,x_0) to $\mathfrak{s}$ is the differential $(d\pi)_e$ of the orbit mapping

$$\pi:s\mapsto\sigma(s)x_0\ .$$

The orbit $\sigma(S)x_0$ is a smooth algebraic variety, and the dimension of $\sigma(S)x_0$ is the dimension of $(d\pi)_e(\mathfrak{s})$ ([Bo, §6.3]). Since the representation of $\mathfrak{g}$ is étale at x_0, the

orbit $\sigma(S)x_0$ is an orbit of maximal dimension. It follows that S has orbits of the same dimension on a Zariski open subset of V.

In the following, we restrict ourselves to the case that $\mathfrak{a}$ is one-dimensional. In this case, the S-module V has the following properties:

1. $\dim V = \dim S + 1$, *and*
2. *S has open orbits of codimension* 1 *on a Zariski open subset of* V.

We will see in this section that these properties give a characterisation of special S-modules as coregular modules with a one-dimensional ring of invariants.

3.1. **Algebraic quotients.** Let G be a reductive, affine algebraic group, and V an affine G-variety. Let $k[V]^G$ be the ring of G-invariant polynomials. We want to study the *algebraic quotient* $V\tilde{/}G$. $V\tilde{/}G$ is the affine algebraic variety with coordinate ring $k[V]^G$. The surjective quotient mapping

$$\pi : V \to V\tilde{/}G$$

parametrizes the *closed* G-orbits. The existence of the algebraic quotient for G is the fundamental result of Hilbert on the finite generation of invariants. Recall that the variety V is called *normal* if V is irreducible and if the coordinate ring $k[V]$ is integrally closed. If V is normal, then so is the quotient $V\tilde{/}G$.

Let V be a vector space. We need the following well known fact:

Lemma 3.1. *If $V\tilde{/}G$ is one-dimensional, then $V\tilde{/}G$ is isomorphic to the affine line $\mathbf{A}^1 k$.*

Proof. As remarked above, the quotient $Z = V\tilde{/}G$ is a normal variety. Since Z is one-dimensional, this implies that Z is smooth. Z is in fact isomorphic to $\mathbf{A}^1 k$: The coordinate ring $k[Z] = k[V]^G$ is a graded subring of $k[V]$. Write $k[V]^G = k + m$, where m is the homogeneous ideal of invariants vanishing at 0. Z is smooth at $\pi(0)$, hence $\dim_k m/m^2 = 1$. Let $0 \neq F \in m$ be a homogenous element of smallest degree, then $k[V]^G = k[F]$. □

From now on, V is a special S-module for a connected, semisimple algebraic group S. The structure of the quotient $V\tilde{/}S$ is described by the next proposition:

Proposition 3.2. *The quotient variety $V\tilde{/}S$ is isomorphic to the affine line $\mathbf{A}^1 k$. The ring of invariants $k[V]^S$ is a polynomial ring over k, generated by an irreducible, non-constant, homogeneous polynomial $F \in k[V]^S$.*

Proof. Let $Z = V\tilde{/}S$ and $\pi : V \to Z$ be the S-equivariant quotient mapping. The usual dimension formula for the fibres of an algebraic morphism gives us

$$\dim \pi^{-1}(y) \geq \dim V - \dim Z \ ,$$

for any $y \in Z$, and equality holds for a non-empty open subset of Z. Hence, $\dim Z \leq 1$. The fact, that S is semisimple allows us to conclude that $\dim Z = 1$, using the following formula of Rosenlicht [Ro]:

$$\text{tr deg}_k k(V)^G = \dim V - \max\{\dim Gv \mid v \in V\} \ . \tag{12}$$

(This formula eludes the geometric meaning of the field of invariant rational functions for the action of a reductive group G.) The semisimplicity of S implies that $k(V)^S$ is the quotient field of $k[V]^S$. The proposition now follows from the previous lemma. □

Definition. The quotient mapping $\pi : V \to V\tilde{/}S$ may be identified with $F : V \to k$. We call the polynomial F *the invariant polynomial* of the special S-module V.

Remark. S-modules which have a free ring of invariants are called *coregular* in the literature. It follows from above, that the quotient mapping has equidimensional fibres. Such modules are called *equidimensional*. In the next proposition, we will show that special modules are *stable* in the sense of invariant theory. A G-module is called stable, if the generic orbit is closed.

We study now the S-orbits in the fibres of F. The one-dimensional torus k^* acts by scalar multiplication on V. Let $V_0 = F^{-1}(0)$ be the zero fibre. V_0 is invariant under the k^*-action, thus V_0 is a closed cone of codimension *1*. $\{0\}$ is the only closed S-orbit in V_0, since the quotient variety parametrizes the closed orbits of S. In the complement of V_0 the k^*-action permutes the fibres $F(x) = c$, $c \neq 0$.

Proposition 3.3. *Every fibre $F^{-1}(c)$, $c \neq 0$, consists of a single, closed S-orbit of codimension* 1.

We deduce the following consequence

Corollary 3.4. *S has exactly one fixed point on V.*

Proof of the proposition. First, we note that S has an orbit of codimension 1 in every non-zero fibre $V_c = F^{-1}(c)$. Since the stabilizer of such an orbit is a *finite* subgroup of S, the orbit is an affine variety. Thus, the generic orbit of S is affine. It follows from a theorem of Popov [Po2] that the generic orbit is closed. Since every orbit contains a closed orbit in its closure, we conclude that the generic fibre of F consists of a single closed orbit, and hence the result. □

Corollary 3.5. *The semisimple group S has no étale representation on a vector space.*

Proof. If S has an étale representation, the result of Popov shows that the open S-orbit is closed. Hence, S acts transitively on V. However, the point 0 is a fixed point for S. □

3.2. **Prehomogenous vector spaces.** Vector spaces with an action of the algebraic group G, such that G has a Zariski dense (hence open) orbit, are called *prehomogeneous vector spaces*. In the preceding paragraph we showed that a special S-module is a prehomogeneous vector space for the action of the reductive group $G = k^* \times S$. Since the generic stabilizer of G is finite in our case, the action of G on V is in fact a linear étale representation. Thus, we have proved the following theorem:

Theorem 3.6. *The S-module V is special, iff the action of S on V gives V the structure of an étale prehomogeneous vector space for $G = k^* \times S$.*

A prehomogeneous vector space is *strongly prehomogeneous*, if the complement of the open orbit has codimension greater than one. We will need the following lemma, which gives us a restriction on the submodules of a special module.

Lemma 3.7. *Let V be a special S-module. If $W \subsetneq V$ is a non-trivial S-submodule, then W is a strongly prehomogeneous vector space for S.*

Proof. We have a direct sum decomposition $V = W \oplus W'$, W' another S-submodule of V. If $\dim W = \dim S$, then W' is one-dimensional, hence trivial. It follows, that W has an open S-orbit. This is impossible according to Corollary 3.5. So we must have $\dim W < \dim S$. But, clearly, then $W \subset V_0 = F^{-1}(0)$, and it follows that $k[W]^S = k$. The Rosenlicht formula implies that $\dim W = \max\{\dim Sv \mid v \in W\}$. □

3.3. **Classification of étale representations.** We start with an affine étale action $\rho : \mathfrak{g} \to \mathfrak{aff}(V)$ of a reductive Lie algebra $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{a}$ with one-dimensional center. We can assume that $\rho(\mathfrak{s}) \subset \mathfrak{gl}(V)$. The induced representation of $\mathfrak{s}$ is the differential of a representation $\sigma : S \to \mathrm{GL}(V)$, and V is a special module for S. Let

$$\mathrm{N}_{\mathrm{Aff}(V)}(\sigma(S))$$

be the normalizer of $\sigma(S)$ in the affine group $\mathrm{Aff}(V)$.

Proposition 3.8. *The normalizer of S is linear, i.e.*

$$\mathrm{N}_{\mathrm{Aff}(V)}(\sigma(S)) \subset \mathrm{GL}(V) \ .$$

Proof. The special S-module V has the unique fixed point 0. Hence, the normalizer of S fixes 0. □

Corollary 3.9. *The affine étale action $\rho : \mathfrak{g} \to \mathfrak{aff}(V)$ is linear, i.e. we can assume that $\rho(\mathfrak{g}) \subset \mathfrak{gl}(V)$.*

Let x_0 be the base point of ρ.

Lemma 3.10. *x_0 is contained in the complement of the zero fibre V_0 of the quotient $V/\!\!\tilde{}\,S$.*

Proof. Assume that $x_0 \in V_0$. Then the S orbit through x_0 is open in V_0, and x_0 is a smooth point of V_0. Hence, the tangent space T_{x_0} along the fibre is a subspace of V of dimension $\dim T_{x_0} = \dim V_0 < \dim V$.

Let $H = Z_{\mathrm{GL}(V)}(\sigma(S))$ be the centralizer of S. H stabilizes V_0. Let $\mathfrak{h}$ be the Lie algebra of H. It follows that $\mathfrak{h}x_0 \subset T_{x_0}$. Since $\rho(\mathfrak{s})x_0 \subset T_{x_0}$, this gives $\rho(\mathfrak{g})x_0 \subset T_{x_0}$, a contradiction. □

Definition 3.11. Two representations σ, σ' are called *isomorphic*, if they are conjugate by some $g \in \mathrm{GL}(V)$. σ, σ' are called *equivalent* if there exists an automorphism h of S such that σh and σ' are isomorphic.

We describe now the set of affine structures for $\mathfrak{g}$ which induce equivalent S-module structures on V. Let F be the invariant polynomial of a special S-module V. Let $x_0 \in V$, $F(x_0) \neq 0$, and $dF_{x_0} : V \to K$ be the differential of F at x_0.

Lemma 3.12. *The representation ρ of $\mathfrak{g}$ is étale at x_0, iff $dF_{x_0}(\rho(\mathfrak{a})\, x_0) \neq \{0\}$.*

Proof. We show that the evaluation $t : \mathfrak{g} \to V$, $X \mapsto \rho(X) \cdot x_0$ is an isomorphism. t restricted to $\mathfrak{s}$ is the differential of the orbit mapping $s \to \sigma(s)x_0$ of S. Hence, t is injective on $\mathfrak{s}$. Now, $t(\mathfrak{s}) = dF_{x_0}^{-1}(0)$. □

Since ρ is linear, $\rho(\mathfrak{a})$ is contained in the centralizer

$$\mathfrak{z} = \{\phi \in \mathfrak{gl}(V) \mid [\phi, \rho(\mathfrak{s})] = 0\} \ .$$

The normalizer $N = \mathrm{N}_{\mathrm{GL}(V)}(\sigma(S))$ acts via conjugation on $\mathfrak{z}$. Let

$$N_{x_0} = \{n \in N \mid nx_0 = x_0\} .$$

We can write $N = \sigma(S)k^* N_{x_0}$. $\sigma(S)k^*$ acts trivially on $\mathfrak{z}$. Hence, the functional $\varphi \mapsto dF_{x_0}(\varphi(x_0))$ is not changed under the action of N.

Theorem 3.13. *Let V be a special S-module with invariant polynomial F, $x_0 \in V$ and $F(x_0) \neq 0$. The set of affine structures for $\mathfrak{g}$ which induce the special S-module V (up to equivalence) is parametrized by the N-conjugacy classes of elements $\varphi \in \mathfrak{z}$ which satisfy $dF_{x_0}(\varphi(x_0)) = 1$.*

Proof. Let (ρ, y_0), $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, be an étale representation which induces the representation σ of the S-module V. It follows from the above Lemma 3.10 that (ρ, y_0) and (ρ, x_0) are equivalent.

Let $0 \neq Z \in \mathfrak{a}$ be a fixed element in the center of $\mathfrak{g}$. Then $\varphi = \rho(Z) \in \mathfrak{z}$ and $dF_{x_0}(\varphi(x_0)) \neq 0$. Let $n \in N_{x_0}$ and h be the automorphism of $\mathfrak{s}$ induced by conjugation with n. For every $X \in \mathfrak{s}$ we have

$$n\rho(X)n^{-1} = \sigma h\,(X) .$$

h may be extended to an automorphism $\tilde{h}$ of $\mathfrak{g}$, such that $\tilde{h}(Z) = cZ$, for some $c \in k^*$. The étale representation $\rho' = n(\rho\tilde{h}^{-1})n^{-1}$ of $\mathfrak{g}$ induces the same representation σ of S, and we have $\rho'(Z) = c\, n\varphi n^{-1}$. Thus, $\varphi = \rho(Z)$ is defined up to scalar multiplication and up to conjugation with $n \in N$. The claimed correspondence now follows easily. □

4. Special modules for SL(n)

Let k be algebraically closed. In this section, we determine the special modules for $\mathrm{SL}(n, k)$. As a first step, we give a list of the $\mathrm{SL}(n)$-modules W satisfying

$$\dim W = \dim \mathrm{SL}(n) + 1 . \tag{13}$$

$\mathrm{SL}(n)$ has orbits of maximal dimension on an open subset of W. Our task will then be to determine those modules which have a codimension 1 orbit.

Now, each representation of a semisimple algebraic group decomposes into a direct sum of irreducible submodules. The irreducible representations of a given group are parametrized by non-negative integer coefficients of the finitely many *fundamental representations*. The *character formula of Hermann Weyl* may be used to compute the dimension of an irreducible module as a polynomial in these coefficients. (For an introduction into representation theory, we refer the reader to [FH]) Weyl's formula implies in particular that the number of irreducible modules of a fixed dimension is finite. *Hence, for any semisimple group S, there are at most finitely many equivalence classes of special modules.*

In this context, various groups of mathematicians have given classification results for the generic orbits and for various properties of the ring of invariants of reductive group actions, cf. [AVE, El, Li, S1, S2]. The tables in [S1] contain the coregular representations of $\mathrm{SL}(n)$ and the subclass of special modules may be extracted. Nevertheless, we will consider the n^2-dimensional representations W of $\mathrm{SL}(n)$ in detail.

Lemma 3.7 gives us the necessary condition that all local factors of a special module for S are prehomogeneous. Sato and Kimura gave in [SK] a classification of all *irreducible* prehomogeneous vector spaces for reductive groups.

4.1. **Irreducible modules for** $\mathrm{SL}(n,k)$**.** Let V be the natural representation of $\mathrm{SL}(n)$ on k^n. The exterior powers $\bigwedge^i V$, $i = 1, \dots, n-1$, are the *fundamental representations* ϕ_i for $\mathrm{SL}(n)$. We have that $(\bigwedge^i V)^* \cong \bigwedge^i V^*$. We also have the following duality relation of $\mathrm{SL}(n)$-modules

$$\bigwedge^i V \cong (\bigwedge^{n-i} V)^* \ .$$

On the level of *highest weights* the duality is expressed in the formula

$$\phi^* = \sum_{i=1}^{n-1} a_{n-i}\phi_i,$$

where $\phi = \sum_{i=1}^{n-1} a_i\phi_i$ is any irreducible module for $\mathrm{SL}(n)$. The duality and the exterior product may be used to construct invariant polynomials on $\mathrm{SL}(n)$-modules. Note also that if $\mathrm{SL}(n)$-modules are dual, they are equivalent. The equivalence is given by the unique nontrivial outer automorphism of $\mathrm{SL}(n)$, $n > 2$. This automorphism is represented by the map $\nu : g \mapsto (g^{-1})^{\mathrm{t}}$, where $g \in \mathrm{SL}(n)$ and g^{t} is the transpose of g.

The Weyl formula for $\mathrm{SL}(n)$ is

$$\dim \phi = \prod_{1 \le i < j \le n} \frac{a_i + \ldots + a_{j-1} + j - i}{j - i} \ .$$

From this formula follows ([SK]):

Proposition 4.1. *The highest weights of the non-trivial irreducible* $\mathrm{SL}(n)$*-modules of dimension* $\le n^2$ *are*

$$\begin{aligned} \textit{for } n = 2: &\quad \phi_1, 2\phi_1, 3\phi_1 \ ; \\ \textit{for } n \ge 3: &\quad \phi_1, \phi_2, 2\phi_1, \phi_1 + \phi_{n-1}, 2\phi_{n-1}, \phi_{n-2}, \phi_{n-1} \ ; \\ \textit{for } 6 \le n \le 8 \textit{ also}: &\quad \phi_3, \phi_{n-3} \ . \end{aligned}$$

Regarding the question of open orbits, we need the following lemma, cf. [SK, §5], [El].

Lemma 4.2. *Let* $6 \le n \le 8$*. Then the* $\mathrm{SL}(n)$*-module* ϕ_3 *has a generic orbit of codimension* 1*.*

The remaining candidates are up to equivalence: ϕ_1, $\phi_1 + \phi_{n-1}$ (the adjoint representation), $2\phi_1$ (the symmetric power of the standard representation), ϕ_2. Since these are well known representations, we see

Proposition 4.3. *The irreducible prehomogeneous* $\mathrm{SL}(n)$*-modules are*

$$\begin{aligned} \textit{for } n = 2m: &\quad \phi_1, \phi_{n-1} \ , \\ \textit{for } n = 2m+1: &\quad \phi_1, \phi_2, \phi_{n-2}, \phi_{n-1} \ . \end{aligned}$$

Proposition 4.1 implies that the $\mathrm{SL}(2)$-module $3\phi_1$ is the only candidate for an irreducible special module.

Proposition 4.4. *The* $\mathrm{SL}(2)$*-module* $3\phi_1$ *is special. The invariant of* $3\phi_1$ *is a polynomial of degree* 4*.*

Proof. $3\phi_1$ is the third symmetric power of the standard module. This is the classical SL(2)-module of binary forms of degree 3. The *discriminant* D of such a form is a SL(2)-invariant polynomial, separating the closed orbits. On the set of forms f, $D(f) \neq 0$, SL(2) has closed orbits with a finite, *non-trivial* stabilizer. In the zero-fiber $D^{-1}(0)$, SL(2) has an open orbit with trivial stabilizer. See the literature cited above. □

4.2. **The classification theorem.** We have the following canonical example:

Example 4.1. Let $W = \mathrm{Mat}(n \times n, k)$. The $\mathrm{SL}(n)$-representation given by left-multiplication of matrices gives W the structure of a special module. The invariant polynomial of W is given by the *determinant* of a matrix.

We show now:

Theorem 4.5. *For $n > 2$ the module $\mathrm{Mat}(n \times n, k)$ is (up to equivalence) the only special module for $\mathrm{SL}(n)$. The group $\mathrm{SL}(2)$ has the special modules $\mathrm{Mat}(2 \times 2, k)$ and the module of binary forms of degree 3.*

Proof. The SL(2)-module of binary forms of degree 3 is the only irreducible special module for $SL(n)$. Let W be a reducible special module. All its local factors are contained in the list given in Proposition 4.3.

Let $V = k^n$ be the standard module. We consider direct sums with factors V and V^*. The modules $V^{\oplus n}$ and $(V^*)^{\oplus n}$ are special and they are equivalent to $\mathrm{Mat}(n \times n, k)$. For $n = 2$, $V \oplus V^*$ is isomorphic to $\mathrm{Mat}(2 \times 2, k)$. For $n > 3$, direct sums with mixed factors V and V^* have independent quadratic invariants given by the dual pairings. Hence they are not special, and the theorem is proved for even n and $n = 3$.

Now assume that n is odd, $n = 2m + 1 > 3$. Let W be a special module with a factor $\bigwedge^2 V$. The iterated exterior product

$$\phi \mapsto \phi^{\wedge m}$$

gives us a non-trivial, equivariant map $J : \bigwedge^2 V \to \bigwedge^{2m} V \cong V^*$. Using the duality, we get a polynomial invariant of $\bigwedge^2 V \oplus V$. Lemma 3.7 implies that V is not a local factor of W.

Suppose now that V^* is a local factor of multiplicity > 1. Then it follows that $W = \bigwedge^2 V \oplus (V^*)^{\oplus(m+1)}$. $W_1 = \bigwedge^2 V \oplus V^* \oplus V^*$ is a true submodule. From the dual pairing between $\bigwedge^2 V$ and $\bigwedge^2 V^*$ we have an invariant polynomial for this module, a contradiction. We conclude that W is isomorphic to $W = \bigwedge^2 V \oplus \bigwedge^2 V \oplus V^*$.

Using the map J, we construct a map $J' : W \to W_1$, and then an invariant of W. We have two possibilities to construct J' leading to independent homogeneous invariants of the same degree. Clearly W is not special. □

4.3. **The castling transformation.** The following proposition describes an important tool in the theory of invariants of semisimple groups. For a proof, see [El] or [SK].

Proposition 4.6. *Let S be a semisimple group, V a module for S, and V^* the dual module. Let $\dim V = n + m$ for natural numbers n, m. Then the ring of invariants of the $S \times \mathrm{SL}(n)$-module $V \otimes_K K^n$ and the $S \times \mathrm{SL}(m)$-module $V^* \otimes_K K^m$ are isomorphic.*

In the situation of the proposition, $V \otimes_K K^n$ and $V^* \otimes_K K^m$ are called *castling equivalent*. It is known that, for any semisimple group, V and V^* are equivalent modules (in the sense of Definition 3.11). For a special S-module W, we can use the decomposition $\dim W = 1 + q$ and get

Corollary 4.7. *Let W be a special module and let $q = \dim S$. Then the $S \times \mathrm{SL}(q)$-module $W \otimes_K K^q$ is special.*

5. Families of left-symmetric algebras

Let $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{a}$ be a reductive Lie algebra with one-dimensional center $\mathfrak{a}$ and semisimple ideal $\mathfrak{s}$. We are now in the position to apply the results of the preceding sections to left-symmetric algebras $(\mathfrak{g}, \cdot)$.

As an immediate consequence of 3.9 we have

Proposition 5.1. *The left-symmetric algebra $(\mathfrak{g}, \cdot)$ has a unique right identity.*

Let $(\mathfrak{g}, *)$ be another left-symmetric algebra. The left regular representations L and L' of $(\mathfrak{g}, \cdot)$ and $(\mathfrak{g}, *)$ give the vector space $\mathfrak{g}$ the structure of a module for the subalgebra $\mathfrak{s}$.

Definition. We say that $(\mathfrak{g}, \cdot)$ and $(\mathfrak{g}, *)$ belong to the same family, if the representations L and L' for $\mathfrak{s}$ are equivalent.

A unity for a left-symmetric algebra is an element $e \in \mathfrak{g}$ with $e \cdot a = a \cdot e = a$ for all $a \in \mathfrak{g}$.

Proposition 5.2. *There are only finitely many families of left-symmetric products for $\mathfrak{g}$. Up to isomorphism each family has a unique product with a unity. The associative kernel of this product is a semisimple associative algebra.*

Proof. We use the results of the preceding sections. From (6) and the correspondence Theorem 2.1 it follows that $(\mathfrak{g}, \cdot)$ and $(\mathfrak{g}, *)$ belong to the same family iff they induce the same special modules for S. From 3.13 we see that there exists a unique equivalence class of étale representations with $\rho(z) = id_V$, for a fixed $z \in \mathfrak{a}$. For every linear étale representation of $\mathfrak{g}$, the element $t^{-1}(x_0) = I(id_V)$ is the right-identity e_R. Now e_R is a unity iff e_R is central. That is, $I(id_V) = z$ for a $z \in \mathfrak{a}$. If $\rho(z) = id_V$, the representation of $\mathfrak{g}$ is completely reducible and its centralizer is a semisimple associative algebra. This centralizer is isomorphic to $\mathfrak{K}(\cdot)$. □

We call the distinguished product in the preceding proposition the *canonical representative* of the family. The family may be derived from the canonical representative. Let

$$\mathcal{F}(\cdot) = \{u \in \mathfrak{K}(\cdot) \,|\, u \notin \mathfrak{s}\} \ .$$

Proposition 5.3. *Let $(\mathfrak{g}, \cdot)$ be a left-symmetric algebra with a unity. The isomorphism classes of left-symmetric algebras in the family of $(\mathfrak{g}, \cdot)$ correspond to the orbits of the group $k^* \operatorname{Aut}(\cdot)$ in the set $\mathcal{F}(\cdot)$.*

Proof. This proposition is simply a reformulation of Theorem 3.13 under the correspondence described in §1. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be an étale linear representation with base point x_0 corresponding to $(\mathfrak{g}, \cdot)$. Then the centralizer $\mathfrak{z}(\rho)$ coincides with $\mathfrak{z}(\rho(\mathfrak{s}))$.

Under the isomorphism $I : \mathfrak{z}(\rho) \to \mathfrak{K}(\cdot)$ the set $\{\varphi \in \mathfrak{z} \,|\, dF_{x_0}(\varphi(x_0)) = 0\}$ corresponds to $\mathfrak{K}(\cdot) \setminus \mathcal{F}$. According to Lemma 2.8 the action of the normalizer N on $\mathfrak{K}(\cdot)$ corresponds to the action of $\operatorname{Aut}(\cdot)$. □

As an application of the castling transformation we have the following corollary to Proposition 4.7.

Proposition 5.4. *Let $(\mathfrak{g}, \cdot)$ be a left-symmetric algebra with unity and associative kernel $\mathfrak{K}(\cdot)$. Let $q = \dim \mathfrak{g} - 1$. There exists a left-symmetric algebra $(\mathfrak{g} \oplus \mathfrak{sl}(q), *)$ with unity, such that the associative kernels $\mathfrak{K}(\cdot)$ and $\mathfrak{K}(*)$ are isomorphic.*

5.1. **The classification for $\mathfrak{gl}(n, k)$.**

Family I. $\mathfrak{gl}(n, k) = \mathrm{Mat}(n \times n, k)$ with the natural associative product represents a family of left-symmetric algebras. The $\mathfrak{sl}(n, k)$-representation of family I is the $\mathfrak{sl}(n, k)$-action on $\mathrm{Mat}(n \times n, k)$ via left multiplication.

We describe now the elements of this family in the spirit of Proposition 5.3. Let $\circ$ denote the associative product. Then we have $\mathfrak{K}(\circ) = \mathfrak{gl}(n)$ and

$$\mathfrak{F}(\circ) = \{\varphi \in \mathfrak{gl}(n) \mid \text{ trace } \varphi \neq 0\} .$$

It follows from the Skolem-Noether theorem on simple associative algebras that the action of $\mathrm{Aut}(\circ)$ on $\mathrm{Mat}(n \times n, k)$ coincides with the $\mathrm{GL}(n)$-action via conjugation.

Proposition 5.5. *The isomorphism classes of left-symmetric algebras in family I are parametrized by the the conjugacy classes of elements $\varphi \in \mathfrak{gl}(n)$ with* trace $\varphi = n$.

Example 5.1. Let $n = 2$ and k be algebraically closed. The Jordan canonical forms

$$\left\{ A_\alpha = \begin{pmatrix} 1+\alpha & 0 \\ 0 & 1-\alpha \end{pmatrix} \mid \alpha \in k \right\} , \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

represent the isomorphism classes of left-symmetric algebras in the family of the associative product on $\mathfrak{gl}(2)$. Two matrices A_α and A_β define isomorphic products, iff $\alpha^2 = \beta^2$. The identity matrix represents the associative product.

Family II. Let $\mathrm{S}^3\, k^2$ denote the third symmetric power of the $\mathfrak{sl}(2, k)$-module k^2. It follows from §4 that the natural extension to a $\mathfrak{gl}(2, k)$-module, gives an étale representation of $\mathfrak{gl}(2, k)$ on $\mathrm{S}^3\, k^2$. This representation corresponds to a left-symmetric algebra product on $\mathfrak{gl}(2, k)$, which is the representative of a family of left-symmetric products for $\mathfrak{gl}(2)$.

Proposition 5.6. *Family II is defined for $\mathfrak{gl}(2, k)$ and contains a single isomorphism class of left-symmetric products. The center of $\mathfrak{gl}(2, k)$ coincides with the associative kernel of this product.*

Proof. Let $\mathfrak{a}$ be the center of $\mathfrak{gl}(2, k)$. Since the action on $\mathrm{S}^3\, k^2$ is irreducible the centralizer coincides with the scalar matrices in $\mathfrak{gl}(\mathrm{S}^3\, k^2)$. It follows that $\mathfrak{K}(\cdot) = \mathfrak{a}$. □

Summary. As a consequence of Proposition 4.5 we have

Theorem 5.7. *Family I and family II comprise all left-symmetric algebras for $\mathfrak{gl}(n, k)$.*

5.2. **The h-transformation.** We clarify now the role of the h-transformation (see the introduction for a definition).

Proposition 5.8. *Let $\mathfrak{g}$ be reductive with one-dimensional center and let $(\mathfrak{g}, \cdot)$ be the canonical representative of a family $\mathcal{F}$ of left-symmetric algebras. Every element of $\mathcal{F}$ is a h-transformation of $(\mathfrak{g}, \cdot)$.*

Proof. We define $h = 0$ on $\mathfrak{s}$. h is determined now by the image of a fixed vector $z \in \mathfrak{a}$. To ensure that $id - h$ is bijective, it suffices that $z - h(z) \notin \mathfrak{s}$. We can choose freely $z - h(z) \in \mathcal{F}(\cdot) = \mathfrak{K}(\cdot) \setminus \mathfrak{s}$. Since h vanishes on $\mathfrak{s}$, the left-regular representations of $\cdot$ and the h-transformation $*$ are isomorphic, hence $*$ belongs to the family $\mathcal{F}$. Let $z \in \mathfrak{a}$ be the unity of the canonical representative, then $\varphi(z)$ is the right identity of the h-transformation $*$. We compare this to the construction of the family $\mathcal{F}$, and our claim is obvious. □

Remark. The associative kernel $\mathfrak{K}(*)$ of the h-transformation of $(\mathfrak{g}, \cdot)$ is isomorphic to the centralizer of $h(z)$ in $\mathfrak{K}(\cdot)$.

Corollary 5.9. *The left-symmetric products in family I are precisely the h-transformations of the associative product. The canonical family II product does not admit a non-trivial h-transformation.*

Proof. For $n > 2$, the corollary is proved. For $n = 2$ we have to show that the families do not transform into each other. It suffices to show that h vanishes on the simple ideal $\mathfrak{sl}(2)$. Suppose h does not vanish on $\mathfrak{sl}(2)$, then h is an automorphism of $\mathfrak{sl}(2)$. Since $\mathfrak{sl}(2)$ has only inner automorphims, h is an inner automorphism, and obviously has a fixed point. This is a contradiction to the requirement that $(id - h)$ is bijective. □

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