GORENSTEIN SPACE
WITH NONZERO EVALUATION MAP

H. GAMMELIN

Abstract. Let \((A, d)\) be a differential graded algebra of finite type, if \(H^\ast(A)\) is a Gorenstein graded algebra, then so is \(A\). The purpose of this paper is to prove the converse under some mild hypotheses. We deduce a new characterization of Poincaré duality spaces as well as spaces with a nonzero evaluation map.

1. Introduction

A Gorenstein ring [3] is a particularly nice Cohen-Macaulay ring which is defined as follows:

Let \((A, m)\) be a local commutative noetherian ring with residue field \(k = A/m\) and let \(d\) be the Krull dimension, then \(A\) is a Gorenstein ring if:

\[
\dim \text{Ext}^i_A(k, A) = \begin{cases} 
0, & i \neq d, \\
1, & i = d. 
\end{cases}
\]

By analogy [5], a connected differential graded algebra (dga for short), \(A\) over a field \(k\), is Gorenstein if

\[
\dim \text{Ext}^1_A(k, A) = 1
\]

where \(\text{Ext}^n_A(-, -)\) denotes the differential ext functor [11]. This is a graded vector space which coincides with the usual ext, denoted by \(\text{Ext}^n_A(-, -)\), when the differential is zero. In general, we have the Milnor-Moore spectral sequence:

\[
E^{p,q}_2 = \text{Ext}^{p+q}_{H^\ast(A; k)}(k, H^\ast(A; k)) \Rightarrow \text{Ext}^{p+q}_A(k, A)
\]

which converges if we suppose that \(A\) is isomorphic to the graded dual of a given coalgebra. This allows us to prove ([5], Proposition 3.2(ii) and Theorem 3.6):

**Proposition 1.** Let \(A\) be a 1-connected dga such that \(\dim H^i(A; k) < \infty\) for every \(i\).

(a) \((H^\ast(A; k) \text{ Gorenstein}) \Rightarrow (A \text{ is Gorenstein}).\)

(b) \((\dim H^\ast(A; k) < \infty \text{ and } A \text{ is Gorenstein}) \Leftrightarrow (H^\ast(A; k) \text{ satisfies the Poincaré duality}).\)

The converse to part a) is false in general ([5], Example 3.3.1) even if we assume that \(H^\ast(A; k)\) is a noetherian ring (see Section 6).
In this paper, we work over the field of the rational numbers denoted by \( \mathbb{Q} \). We prove the converse under some mild hypotheses:

**Theorem 1.** If \( H^*(A; \mathbb{Q}) \) is Cohen-Macaulay and \( A \) a Gorenstein dga, then \( H^*(A; \mathbb{Q}) \) is a Gorenstein graded algebra.

A new invariant has been introduced by Félix, Halperin and Thomas ([5], (1.4)) called the evaluation map:

\[
ev_A : \mathbb{E}xt_A(\mathbb{Q}, A) \to H^*(A).
\]

**Theorem 2.** Let \((A, d)\) be a 1-connected, Gorenstein commutative dga, such that \( H^*(A; \mathbb{Q}) \) is noetherian. If \( \ev_A \neq 0 \), then \( H^*(A; \mathbb{Q}) \) satisfies the Poincaré duality.

We can translate this result from algebra to topology making use of the dga \( A = C^*\left(S; \mathbb{k}\right) \) where \( S \) is a simply connected space Theorem 2 gives a new characterization of the spaces satisfying Poincaré duality over \( \mathbb{Q} \):

**Corollary 2.** The following statements are equivalent:

(i) \( S \) is a Poincaré duality space over \( \mathbb{Q} \).

(ii) \( S \) is a Gorenstein space with a nonzero evaluation map and \( H^*(S; \mathbb{Q}) \) is noetherian.

Finally, we prove

**Theorem 3.** Let \( S \) be a 1-connected, pointed, Gorenstein space. Then

\[
ev_S \neq 0 \Rightarrow \begin{cases} S \text{ is a Poincaré duality space over } \mathbb{Q}, \\
\text{or} \\
H^*(S; \mathbb{Q}) \text{ is not noetherian and} \\
\text{not a Gorenstein graded algebra.}
\end{cases}
\]

In fact, it is conjectured that only the first alternative occurs.

## 2. Algebraic preliminaries

### 2.1. Definitions.

Henceforth, we assume that all objects, vector spaces, algebras, tensor products, ... are over \( \mathbb{Q} \). For definitions and facts about Sullivan models and their connections with rational homotopy theory, standard references are [8] and [14].

A **KS-complex** is a commutative differential graded algebra (cdga) \((\Lambda X, d)\), where \( \Lambda X = \text{Exterior}(X^{\text{odd}}) \otimes \text{Symmetric}(X^{\text{even}}) \) is a free commutative graded algebra. The vector space \( X = X_{\geq 0} \) has a well-ordered basis \( \{x_\alpha\} \) satisfying \( dx_\alpha \in \Lambda(X_{<\alpha}) \).

A **KS-extension** of an augmented cdga \( A \xrightarrow{\delta} Q \) is a sequence of dga maps of the form \((A, d) \xrightarrow{id \otimes 1} (A \otimes \Lambda X, d) \xrightarrow{\bar{e} \otimes id} (\Lambda X, \bar{d})\) where \((\Lambda X, \bar{d})\) is a KS-complex and \( dx_\alpha \in A \otimes \Lambda(X_{<\alpha}) \).

Let \((A, d)\) be a dga with underlying graded algebra \( A^\# \) and let \( M \) be an \( A \)-module.

(i) \( M \) is **A-semifree** [2] if there exists a filtration by \( A \)-submodules \( 0 = F_{-1} \subset F_0 \subset F_1 \subset \cdots \) such that \( M = \bigcup_i F_i \) and for each \( i, F_{i+1}/F_i \) is an \( A^\# \)-free module admitting a basis of cycles.

(ii) A quism (quasi-isomorphism, in other words: a map inducing a cohomology isomorphism) of \( A \)-modules \( P \xrightarrow{\cong} M \) is called an **A-semifree resolution** of \( M \) if \( P \) is \( A \)-semifree.
For each $A$-module $M$, there exists an $A$-semifree resolution. Let $M$ and $N$ be two $A$-modules and $P \xrightarrow{\sim} M$ be an $A$-semifree resolution of $M$, then, by definition,
\[ \mathcal{E}xt_A(M, N) = H^*(\text{Hom}_A(P, N)) \]
A Gorenstein cdga (over $Q$) is a simply connected cdga $A$ such that
\[ \dim \mathcal{E}xt_A(Q, A) = 1. \]
A Gorenstein space (over $Q$) is a simply connected space $X$ such that $C^*(X; Q)$ is a Gorenstein dga. The formal dimension of $A$ ([5], Section 5) is defined by
\[ \text{fd}(A) = \sup\{r \in \mathbb{Z}|\mathcal{E}xt_A(Q, A)[r] \neq 0\}. \]
If $\mathcal{E}xt_A(Q, A) = 0$, we put $\text{fd}(A) = \infty$. By ([5], Proposition 5.1), if $H^*(A)$ is finite dimensional, then $\text{fd}(A) \geq 0$ and
\[ \text{fd}(A) = \sup\{r \in \mathbb{Z}|H^r(A) \neq 0\}. \]
The socle of $H^*(A)$ is defined by
\[ \text{socle}(H^*(A)) = \{\alpha \in H^*(A)|\alpha.H^+(A) = 0\}. \]
The evaluation map ([5], (1.4)) is a natural vector space map:
\[ ev_A : \mathcal{E}xt_A(Q, A) \to H^*(A) \]
defined as follows: let $[f]$ be an element of $\mathcal{E}xt_A(Q, A)$ represented by a cycle $f : P \to A$ ($P$ is an $A$-semifree resolution of $Q$), $ev_A([f]) = [f(p)]$ where $p$ is a cycle in $P$ representing 1. We have [7]: $\text{im} ev_A \subset \text{socle}(H^*(A))$. For a 1-connected topological space $X$, $ev_X$ denotes $ev_{C^*(X; Q)}$.

2.2. Preliminary results. Recall first two fundamental results about Gorenstein spaces and the evaluation map.

**Theorem 4** ([5], Theorem 4.3). If $F \to E \to B$ is a fibration such that $H^1(F; Q) = 0 = H^1(B; Q)$, $H^*(F; Q) < \infty$ and $\dim H^i(B; Q) < \infty$ for each $i$, then:
\[ \mathcal{E}xt_{C^*(E; Q)}(Q, C^*(E; Q)) = \mathcal{E}xt_{C^*(B; Q)}(Q, C^*(B; Q)) \otimes \mathcal{E}xt_{C^*(F; Q)}(Q, C^*(F; Q)). \]
In particular, $E$ is a Gorenstein space over $Q$ if and only if $B$ is a Gorenstein space over $Q$ and $H^*(F; Q)$ satisfies Poincaré duality.

**Theorem 5** ([13], Theorem B). Let $F \to E \xrightarrow{p} B$ be a fibration of simply connected spaces. If $H^*(F; Q)$ is finite dimensional, then $ev_B \neq 0$ implies $ev_E \neq 0$.

From Theorems 4 and 5, we deduce immediately the following lemma which will be used in the proof of Theorem 2:

**Lemma 3.** Let $(A, d) \to (A \otimes \Lambda(x), D) \to (\Lambda(x), 0)$ be a KS-extension, with $x$ of odd degree. If $A$ is a Gorenstein dga and if $ev_A \neq 0$, then $A \otimes \Lambda(x)$ is a Gorenstein dga and $ev_{A \otimes \Lambda(x)} \neq 0$.

We also need the next results for the proof of Lemma 6.

**Lemma 4.** If $A$ is Gorenstein and if $ev_A \neq 0$, then there exists an element $\omega$ in the socle of $H^*(A)$ whose degree is exactly the formal dimension of $A$.

Indeed, $\text{fd}(A) = |\omega|$, where $\omega = [f(1)] = ev_A([f])$, $\mathcal{E}xt_A(Q, A) = Q[f]$. □

Theorem 4 implies immediately:
Proposition 5. If \((A, \delta) \to (A \otimes \Lambda X, d) \to (\Lambda Xd)\) is a KS-extension of simply connected dga with \(H^*(\Lambda X)\) finite dimensional and \(H^*(A)\) of finite type, then
\[
\text{fd}(A \otimes \Lambda X) = \text{fd}(A) + \text{fd}(\Lambda X).
\]

3. Proof of Theorem 1

3.1. Some graded commutative algebra. Standard references are [4] and [10]. Let \(H\) be a finitely generated commutative graded algebra. \(f \in H\) is regular if \(\text{Ann}(f)\) is zero, it is clear that \(f\) has an even degree.

A sequence \((f_i)_{i=1}^n\) is regular if \(f_i\) is regular in \(H/(f_1, \ldots, f_{i-1})\). The Krull dimension of \(H\) is \(n\) if \(H\) contains \(Q[x_1, \ldots, x_n]\) and \(H\) is a finitely generated module over \(Q[x_1, \ldots, x_n]\). \(H\) is Cohen-Macaulay if there exists a regular sequence \((f_1, \ldots, f_n)\) such that \(H/(f_1, \ldots, f_n)\) is finite dimensional. In this case, we have \(\text{Kdim } H = n\). \(H\) is Gorenstein if there exists a regular sequence \((f_1, \ldots, f_n)\) such that \(H/(f_1, \ldots, f_n)\) is a Poincaré duality algebra.

Remark. If \(H\) is a Gorenstein commutative graded algebra with a nonzero socle, then \(H\) is a Poincaré duality algebra.

3.2. Proof. Let \((f_0, \ldots, f_n)\) be a regular sequence such that \(H^*(A)/(f_0, \ldots, f_n)\) is finite dimensional. Choose some elements \(x_0, \ldots, x_n\) such that \(\text{cl } dx_i = f_i\) (\(\text{cl } x\) denotes the cohomology class of \(x\)). As in 3.1, there is a finite sequence of KS-extensions \((A_i, d_i) \to (A_{i+1}, d_{i+1}) \to (\Lambda(x_i), 0), i = 0, \ldots, n\), where, by Theorem 4, \((A_i, d_i)\) are Gorenstein dga. Here \(H^*(A_{n+1})\) is finite dimensional. Indeed, since \(f_i\) is regular in \(H^*(A_i)\), \(H^*(A_i) = H^*(A_{i-1})/(f_i)\). Hence \(H^*(A_{n+1}) = H^*(A)/(f_1, \ldots, f_n)\).

Finally, we conclude that \((A_{n+1}, d_{n+1})\) is a Poincaré duality algebra and this shows that \(H^*(A)\) is a Gorenstein graded algebra.

4. Proof of Theorem 2

4.1. Proof. Notice that it is sufficient to prove that \(H^*(A)\) is finite dimensional (Proposition 1, b). We admit for the moment the following lemma:

Lemma 6. Let \((A, d) \to (A \otimes \Lambda(x), D) \to (\Lambda(x), 0)\) be a KS-extension such that:
- \(A\) is a Gorenstein cdga,
- there exists an element \(\omega\) in the socle of \(H^*(A)\) whose degree is exactly the formal dimension of \(A\),
- \(H^*(A \otimes \Lambda(x))\) is finite dimensional, \(\text{cl } Dx \neq 0\) in \(H^*(A)\) and \(x\) has an odd degree.

Then \(H^*(A)\) is finite dimensional.

Proof of the theorem. Since \(H^*(A)\) is noetherian, there exists a KS-extension:
\[(A, d) \to (A \otimes \Lambda(x_0, \ldots, x_N), D) \to (\Lambda(x_0, \ldots, x_N), D)\]
such that \(H^*(A \otimes \Lambda(x_0, \ldots, x_N))\) is finite dimensional. Indeed, fix a system of generators for \(H^*(A)\) and let \(f_0, \ldots, f_N\) be the ones of even degree. They are finite in number since \(H^*(A)\) is noetherian. Define the \(x_i\) such that \(\text{cl } (Dx_i) = f_i\).

Here \(E_2\)-term of the Serre spectral sequence of the KS-extension is a \(H^*(A)\)-module of finite type as \(H^*(A \otimes \Lambda(x_0, \ldots, x_N))\) is finite dimensional. Therefore, the \(E_\infty\)-term is a \(H^*(A)/(f_0, \ldots, f_N)\)-module of finite type. As \(H^*(A)/(f_0, \ldots, f_N)\) is finite dimensional, we conclude that \(H^*(A \otimes \Lambda(x_0, \ldots, x_N))\) is finite dimensional also. From the KS-extension above, we define a finite sequence of KS-extensions \((A_i, d_i) \to\)
\((A_{i+1}, d_{i+1}) = A_i \otimes \Lambda(x_i) \to (\Lambda(x_i), 0), i = 0, \ldots, N,\) where \((A_0, d_0) = (A, d)\) and \(H^*(A_{N+1})\) is finite dimensional. Applying inductively Lemma 3, we show that each \((A_i, d_i)\) is Gorenstein with a nonzero evaluation map. Lemma 4 implies that the KS-extension \((A_N, d_N) \to (A_{N+1}, d_{N+1}) \to (\Lambda(x_N), 0)\) satisfies the hypotheses of Lemma 6. Applying inductively Lemma 6, we prove the theorem.

4.2. Proof of Lemma 6. The proof breaks down into five steps:

First step. Let \(f\) be an element of even degree of \(H^*(A)\) and denote by \((f)\) the ideal that it generates. If \(H^*(A)/(f)\) is finite dimensional, then \(\text{Ann}(f)\) is also finite dimensional.

Proof. If \(H^*(A)/(f)\) is finite dimensional, \(H^*(A)\) is a finite type \(\mathbb{Q}[f]\)-module and as well \(\text{Ann}(f)\), but \(f, \text{Ann}(f) = 0\), thus \(\text{Ann}(f)\) is finite dimensional.

Second step. \(H^*(A \otimes \Lambda(x))\) is finite dimensional if and only if \(H^*(A)/(f)\) is finite dimensional where \(f = \text{cl}Dx\).

Proof. The previous step together with \(H^*(A \otimes \Lambda(x)) = H^*(A)/(f) \oplus \text{Ann}(f) \otimes x \mathbb{Q}\) implies the statement.

Third step. Let \((A, d) \to (A \otimes \Lambda(x), D) \to (\Lambda(x), 0)\) and \((A, d) \to (A \otimes \Lambda(y), D) \to (\Lambda(y), 0)\) be KS-extensions such that \(H^*(A \otimes \Lambda(x))\) is finite dimensional, \(f = \text{cl}Dx \neq 0\) in \(H^*(A), x\) has an odd degree and \(\text{cl}Dy = f^j\), then \(H^*(A \otimes \Lambda(y))\) is also finite dimensional.

Proof. Write \(H^*(A) = B_0 \oplus (f)\). Notice that \(B_0 \cong H^*(A)/(f)\) and \(H^*(A) = \sum_{i \geq 0} B_0 f^i\). As \(H^*(A \otimes \Lambda(x))\) is finite dimensional, so are \(H^*(A)/(f)\), \(B_0\) and \(H^*(A)/(f^j) = \sum_{i=j}^{i=j-1} B_0 f^j\). By the previous step, we conclude.

Fourth step. Let \(\omega \in \text{socle}(H^*(A))\) such that \(\text{fd}(A) = |\omega|\), we observe that \(|\omega| \geq |f|\).

Proof. \(A \otimes \Lambda(x)\) is a Gorenstein dga with finite dimensional cohomology, so that \(H^*(A \otimes \Lambda(x))\) is a Poincaré duality algebra. Moreover \(\text{fd}H^*(A \otimes \Lambda(x)) = \text{fd}(A \otimes \Lambda(x)) = \text{fd}(A) + \text{fd}(\Lambda(x)) = |\omega| + |x|\).

If we suppose that \(|\omega| < |f|\), then \(\omega\) projects on a nontrivial element in \(H^*(A)/(f)\). If we call \(\alpha\) the Poincaré dual class of \(\omega\) in \(H^*(A \otimes \Lambda(x))\), then \(|\alpha| = |x|\). For degree reason and because \(\text{cl}Dx \neq 0\), we obtain that \(\alpha\) is in \(H^*(A)/(f)\). This is impossible because \(\omega\) is in the socle.

Last step. If we suppose that, for each \(i, f^i \neq 0\), then there exists \(j\) such that \(|f^j| \geq |\omega|\). By step 3, the KS-extension \((A, d) \to (A \otimes \Lambda(y), D) \to (\Lambda(y), 0)\) with \(\text{cl}(Dy) = f^j\) satisfies also the hypotheses of the lemma, but \(|f^j| \geq |\omega|\) which is impossible by step 4. This proves that, for some \(j, f^j = 0\) and that \(H^*(A) = \sum_{i=0}^{i=j-1} B_0 f^k\), we conclude that \(H^*(A)\) is finite dimensional.

5. Proof of Corollary 2 and of Theorem 3

5.1. Proof of Corollary 2. Proposition 1 b) and Theorem 2 give us \((ii) \Rightarrow (i)\).

For the converse, it is clear that \(S\) is a Gorenstein space and \(H^*(S)\) is noetherian. By (7), Theorem 1 and the remark after Proposition 2, we have \(T_a(S; \mathbb{Q}) \subset E(S; \mathbb{Q})\) where \(E(S; \mathbb{Q}) = \text{Im}ev_S\) and \(T_a(S; \mathbb{Q})\) is the set of the algebraic terminal classes. It is enough to prove that \(T_a(S; \mathbb{Q})\) is not empty. For this we take the minimal Adams-Hilton model \([1]\) of \(S\): \((TW, \delta) \xrightarrow{\sim} C_\ast(\Omega S; \mathbb{Q})\) with \(W = s^{-1}H_+\langle S; \mathbb{Q}\rangle\). As \(S\) is a 1-connected Poincaré duality space, \(H_+(S; \mathbb{Q}) = H_{\leq n-2}(S; \mathbb{Q}) \oplus \mathbb{Q}_\omega\) where \(|\omega| = n\). Now, write \(W = V \oplus s^{-1} \omega\) and remark that, for degree reason, \(\delta s^{-1} \omega \in T(V)\). Hence, \(s^{-1} \omega \in T_a(S; \mathbb{Q})\) and finally \(ev_S \neq 0\).
5.2. **Proof of Theorem 3 and remarks.** Theorem 3 is a corollary of Theorem 2 and of the following proposition:

**Proposition 7.** If \( H^*(A; Q) \) is not noetherian with nonzero socle, then \( H^*(A; Q) \) is not a Gorenstein graded algebra.

**Proof of the theorem.** Indeed, let \( S \) be a Gorenstein space such that \( ev_S \neq 0 \): If \( H^*(S) \) is noetherian, then \( H^*(S) \) is finite dimensional by Theorem 2, and, by the remark at the beginning of the previous section, \( S \) is a Poincaré duality space. If \( H^*(S) \) is not noetherian, the socle is not zero since \( im ev_S \subset \text{socle}(H^*(S)) \) and by Proposition 7, \( H^*(S) \) is not a Gorenstein graded algebra. \( \square \)

If fact, we do not know any Gorenstein spaces \( S \) such that \( ev_S \neq 0 \) and with \( H^*(S; Q) \) not noetherian. Therefore, we can conjecture that

\[(S \text{ Gorenstein and } ev_S \neq 0) \Leftrightarrow (H^*(S; Q) \text{ satisfies the Poincaré duality}).\]

This conjecture has been proved in a particular case by A. Murillo.

**Theorem 6** ([13], Theorem A). Let \( S \) be a 1-connected pointed space with \( \pi_*(S) \otimes Q \) finite dimensional. Then the following statements are equivalent:

(i) \( H^*(S; Q) \) is finite dimensional.

(ii) \( ev_S \) is nonzero.

Indeed, if \( \pi_*(S) \otimes Q \) is finite dimensional, then \( S \) is a Gorenstein space ([5], Proposition 3.4). Notice also that Corollary 2 implies the following:

Let \( X \) be a 1-connected space such that \( H_*(\Omega X; Q) \) has polynomial growth, \( H^*(X) \) is noetherian and \( ev_X \neq 0 \), then \( X \) is an elliptic space. Indeed, by [6], the polynomial growth of \( H_*(\Omega X; Q) \) implies that \( \pi_*(\Omega X) \otimes Q \) is finite dimensional, so that \( X \) is a Gorenstein space ([5], Proposition 3.4), finally \( H^*(X; Q) \) is finite dimensional by Theorem 2, so \( X \) is an elliptic space.

5.3. **Proof of Proposition 7.** We use the bigraded model of \( H^*(A) = H \) ([9], Section 3): \((\Lambda(Z),d) \to (H,0)\). Consider the projective resolution of \( Q \) by \( H \)-modules:

\[
\cdots \xrightarrow{d} H \otimes (\Lambda Z)_2 \xrightarrow{d} H \otimes (\Lambda Z)_1 \xrightarrow{d} H \otimes Q \xrightarrow{e} 0
\]

where \( \Lambda Z_i = Z_{i+1} \). If \( H \) is not noetherian, then \( Z_0 \) is infinite ([9], 3.8.2). Hence, \((\Lambda Z)_1 = Z_1 \) is also infinite. Let \( \omega \in \text{socle}(H) \), \( \omega \in \Lambda(x_1, \ldots, x_n), x_i \in Z_0 \). We define for \( k > 0 \)

\[
f_k : H \otimes (\Lambda Z)_1 \to H
\]

\[
1 \otimes \bar{x}_{n+k} \mapsto \omega
\]

\[
1 \otimes \bar{x}_j \mapsto 0
\]

when \( j \neq n + k \). This map satisfies \( f_k \circ d = 0 \) and there is no map \( g : H \to H \) such that \( f_k = g \circ d \). Therefore, \( \text{Ext}_H(Q,H) \) is infinite dimensional and \( H \) is not a Gorenstein graded algebra.

6. **Example**

In the following, we give two examples of Gorenstein cdga whose cohomologies are not Gorenstein graded algebras. In each case, we show by different methods that \( H^*(A) \) is not a Gorenstein graded algebra.

**First example.** The assumption “socle is different from zero” is not sufficient: Let \((A,d) = (\Lambda(u,v,w,t),d)\) with \( du = dv = dw = 0, dt = uvw, |u| = 2, |v| = \)
\(|w| = 3\) and \(|t| = 7\). \(H^*(A) = \Lambda(u, v, w, a, b)/(uvw, aw + bv)\) where \(a = [vt]\) and \(b = [wt]\). The graded algebra \(H^*(A)\) is noetherian. By ([5], Proposition 5.2), \(\text{fd}(A) = 1 - |u| + |v| + |w| + |t| = 12\) so there exist an element of degree 12 in \(\text{Ext}^*_{H^*(A)}(\mathbb{Q}, H^*(A))\). But, \(\text{Ext}^0_{H^*(A)}(\mathbb{Q}, H^*(A)) = \text{socle}(H^*(A))\) and \([vw]\) is in the socle, \(|vw| = 6\) so \(\text{Ext}^*_{H^*(A)}(\mathbb{Q}, H^*(A))\) contains at least two elements (one of degree 12 and one of degree 6). Thus, \(H^*(A)\) is not a Gorenstein graded algebra.

**Second example**, related to Proposition 7. Let \((A, d) = (\Lambda(u, v, w, t), d)\) with \(du = dw = dt = 0, |u| = |v| = 2, |w| = 3\) and \(|t| = 6\). In this case, \(H^*(A)\) is not noetherian and the socle is zero. \(H^*(A)\) is not a Gorenstein graded algebra. Indeed, we prove that \(\text{Ext}^*_{H^*(A)}(\mathbb{Q}, H^*(A))\) contains an infinite number of elements. \(H^*(A) = H = \Lambda(u, v, t)/\langle ut_i, t_j\rangle\) where \(t_i = [ut^i]\). As in the proof of Proposition 7, we use the projective resolution of \(\mathbb{Q}\) by \(H\)-modules obtained from the bigraded model of \(H\). Here \((\Lambda\bar{Z})_1 = Z_1 = \{\bar{u}, \bar{v}, \bar{t}_i\}\) and \((\Lambda\bar{Z})_2 = \{\bar{w}_i, \bar{t}_{ij}\}\) with \(d\bar{w}_i = -\bar{u}t_i\) and \(d\bar{t}_{ij} = -\bar{t}_{i}t_j\). The \(H\)-linear maps

\[
f_i: H \otimes (\Lambda\bar{Z})_1 \to H
\]

\[
1 \otimes \bar{u} \mapsto ut_i
\]

\[
1 \otimes \bar{v} \mapsto 0
\]

\[
1 \otimes \bar{t}_i \mapsto 0
\]

satisfy \(f_i \circ d = 0\) and, for each \(i\), there is no map \(g: H \to H\) such that \(f_i = g \circ d\). Hence, \(\text{Ext}_H(\mathbb{Q}, H)\) is infinite dimensional.

**Acknowledgments**

This paper is a part of my thesis, and I’m grateful to my supervisor, Professor Jean-Claude Thomas of the University of Angers, for his patience, help and encouragement. I also thank the Fields Institute for a congenial atmosphere that allowed me to have helpful conversations with Yves Félix, Stephen Halperin and Aniceto Murillo.

**References**


DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DES SCIENCES ET TECHNOLOGIES DE LILLE, 59655 VILLENEUVE D’ASCQ, FRANCE

E-mail address: gammelin@gat.univ-lille1.fr