TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 351, Number 8, Pages 3433–3440
S 0002-9947(99)02092-9
Article electronically published on March 29, 1999

GORENSTEIN SPACE WITH NONZERO EVALUATION MAP

H. GAMMELIN

Abstract. Let (A, d) be a differential graded algebra of finite type, if $H^*(A)$ is a Gorenstein graded algebra, then so is A. The purpose of this paper is to prove the converse under some mild hypotheses. We deduce a new characterization of Poincaré duality spaces as well as spaces with a nonzero evaluation map.

1. Introduction

A *Gorenstein* ring [3] is a particularly nice Cohen-Macaulay ring which is defined as follows:

Let (A, m) be a local commutative noetherian ring with residue field $\mathbf{k} = A/m$ and let d be the Krull dimension, then A is a *Gorenstein* ring if:

$$\dim Ext^i_A(\mathbf{k}, A) = \begin{cases} 0, & i \neq d, \\ 1, & i = d. \end{cases}$$

By analogy [5], a connected differential graded algebra (dga for short), A over a field $\mathbf{k}$, is *Gorenstein* if

$$\dim \mathcal{E}xt_A(\mathbf{k}, A) = 1$$

where $\mathcal{E}xt_A(-, -)$ denotes the differential ext functor [11]. This is a graded vector space which coincides with the usual ext, denoted by $Ext_A(-, -)$, when the differential is zero. In general, we have the Milnor-Moore spectral sequence:

$$E_2^{p,q} = Ext^{p,q}_{H^*(A;\mathbf{k})}(\mathbf{k}, H^*(A; \mathbf{k})) \Rightarrow \mathcal{E}xt^{p+q}_A(\mathbf{k}, A)$$

which converges if we suppose that A is isomorphic to the graded dual of a given coalgebra. This allows us to prove ([5], Proposition 3.2(ii) and Theorem 3.6):

Proposition 1. *Let A be a 1-connected dga such that $\dim H^i(A; \mathbf{k}) < \infty$ for every i.*

(a) *($H^*(A; \mathbf{k})$ Gorenstein) $\Rightarrow$ (A is Gorenstein).*

(b) *($\dim H^*(A; \mathbf{k}) < \infty$ and A is Gorenstein) $\Leftrightarrow$ ($H^*(A; \mathbf{k})$ satisfies the Poincaré duality).*

The converse to part a) is false in general ([5], Example 3.3.1) even if we assume that $H^*(A; \mathbf{k})$ is a noetherian ring (see Section 6).

Received by the editors December 2, 1996.
1991 *Mathematics Subject Classification.* Primary 55P62.
Key words and phrases. Gorenstein space, Cohen-Macaulay ring, Poincaré duality space, evaluation map.

In this paper, we work over the field of the rational numbers denoted by $\mathbf{Q}$. We prove the converse under some mild hypotheses:

Theorem 1. *If $H^*(A;\mathbf{Q})$ is Cohen-Macaulay and A a Gorenstein dga, then $H^*(A;\mathbf{Q})$ is a Gorenstein graded algebra.*

A new invariant has been introduced by Félix, Halperin and Thomas ([5], (1.4)) called the *evaluation map*:

$$ev_A : \mathcal{E}xt_A(\mathbf{Q}, A) \to H^*(A).$$

Theorem 2. *Let (A,d) be a 1-connected, Gorenstein commutative dga, such that $H^*(A;\mathbf{Q})$ is noetherian. If $ev_A \neq 0$, then $H^*(A;\mathbf{Q})$ satisfies the Poincaré duality.*

We can translate this result from algebra to topology making use of the dga $A = C^*(S;\mathbf{k})$ where S is a simply connected space Theorem 2 gives a new characterization of the spaces satisfying Poincaré duality over $\mathbf{Q}$:

Corollary 2. *The following statements are equivalent*:
(i) *S is a Poincaré duality space over $\mathbf{Q}$.*
(ii) *S is a Gorenstein space with a nonzero evaluation map and $H^*(S;\mathbf{Q})$ is noetherian.*

Finally, we prove

Theorem 3. *Let S be a 1-connected, pointed, Gorenstein space. Then*

$$ev_S \neq 0 \Rightarrow \begin{cases} S \textit{ is a Poincaré duality space over } \mathbf{Q}, \\ \textit{or} \\ H^*(S;\mathbf{Q}) \textit{ is not noetherian and} \\ \textit{not a Gorenstein graded algebra.} \end{cases}$$

In fact, it is conjectured that only the first alternative occurs.

2. Algebraic preliminaries

2.1. **Definitions.** Henceforth, we assume that all objects, vector spaces, algebras, tensor products,... are over $\mathbf{Q}$. For definitions and facts about Sullivan models and their connections with rational homotopy theory, standard references are [8] and [14].

A *KS-complex* is a commutative differential graded algebra (cdga) $(\Lambda X, d)$, where $\Lambda X = \text{Exterior}(X^{odd}) \otimes \text{Symmetric}(X^{even})$ is a free commutative graded algebra. The vector space $X = X^{\geq 0}$ has a well-ordered basis $\{x_\alpha\}$ satisfying $dx_\alpha \in \Lambda(X_{<\alpha})$.

A *KS-extension* of an augmented cdga $A \xrightarrow{\varepsilon} \mathbf{Q}$ is a sequence of dga maps of the form $(A,\delta) \xrightarrow{\text{id}\otimes 1} (A\otimes\Lambda X, d) \xrightarrow{\varepsilon\otimes\text{id}} (\Lambda X, \bar{d})$ where $(\Lambda X, \bar{d})$ is a KS-complex and $dx_\alpha \in A \otimes \Lambda(X_{<\alpha})$.

Let (A,d) be a dga with underlying graded algebra $A^\#$ and let M be an A-module.

(i) M is A-*semifree* [2] if there exists a filtration by A-submodules $0 = F_{-1} \subset F_0 \subset F_1 \subset \cdots$ such that $M = \bigcup_i F_i$ and for each i, F_{i+1}/F_i is an $A^\#$-free module admitting a basis of cycles.

(ii) A quism (quasi-isomorphism, in other words: a map inducing a cohomology isomorphism) of A-modules $P \xrightarrow{\simeq} M$ is called an A-*semifree resolution* of M if P is A-semifree.

For each A-module M, there exists an A-semifree resolution. Let M and N be two A-modules and $P \xrightarrow{\simeq} M$ be an A-semifree resolution of M, then, by definition,

$$\mathcal{E}xt_A(M,N) = H^*(\mathrm{Hom}_A(P,N))$$

A *Gorenstein* cdga (over $\mathbf{Q}$) is a simply connected cdga A such that

$$\dim \mathcal{E}xt_A(\mathbf{Q}, A) = 1.$$

A *Gorenstein* space (over $\mathbf{Q}$) is a simply connected space X such that $C^*(X;\mathbf{Q})$ is a Gorenstein dga. The *formal dimension* of A ([5], Section 5) is defined by

$$\mathrm{fd}(A) = \sup\{r \in \mathbf{Z} | [\mathcal{E}xt_A(\mathbf{Q}, A)]^r \neq 0\}.$$

If $\mathcal{E}xt_A(\mathbf{Q}, A) = 0$, we put $\mathrm{fd}(A) = \infty$. By ([5], Proposition 5.1), if $H^*(A)$ is finite dimensional, then $\mathrm{fd}(A) \geq 0$ and

$$\mathrm{fd}(A) = \sup\{r \in \mathbf{Z} | H^r(A) \neq 0\}.$$

The *socle* of $H^*(A)$ is defined by

$$\mathrm{socle}(H^*(A)) = \{\alpha \in H^*(A) | \alpha.H^+(A) = 0\}.$$

The *evaluation map* ([5], (1.4)) is a natural vector space map:

$$ev_A \colon \mathcal{E}xt_A(\mathbf{Q}, A) \to H^*(A)$$

defined as follows: let $[f]$ be an element of $\mathcal{E}xt_A(\mathbf{Q}, A)$ represented by a cycle $f : P \to A$ (P is an A-semifree resolution of $\mathbf{Q}$), $ev_A([f]) = [f(p)]$ where p is a cycle in P representing 1. We have [7]: $\mathrm{im}\, ev_A \subset \mathrm{socle}(H^*(A))$. For a 1-connected topological space X, ev_X denotes $ev_{C^*(X;\mathbf{Q})}$.

2.2. **Preliminary results.** Recall first two fundamental results about Gorenstein spaces and the evaluation map.

Theorem 4 ([5], Theorem 4.3). *If $F \to E \to B$ is a fibration such that $H^1(F;\mathbf{Q}) = 0 = H^1(B;\mathbf{Q})$, $H^*(F;\mathbf{Q}) < \infty$ and $\dim H^i(B;\mathbf{Q}) < \infty$ for each i, then*:

$$\begin{aligned}&\mathcal{E}xt_{C^*(E;\mathbf{Q})}(\mathbf{Q}, C^*(E;\mathbf{Q}))\\ &\quad = \mathcal{E}xt_{C^*(B;\mathbf{Q})}(\mathbf{Q}, C^*(B;\mathbf{Q})) \otimes \mathcal{E}xt_{C^*(F;\mathbf{Q})}(\mathbf{Q}, C^*(F;\mathbf{Q})).\end{aligned}$$

In particular, E is a Gorenstein space over $\mathbf{Q}$ if and only if B is a Gorenstein space over $\mathbf{Q}$ and $H^*(F;\mathbf{Q})$ satisfies Poincaré duality.

Theorem 5 ([13], Theorem B). *Let $F \to E \xrightarrow{\rho} B$ be a fibration of simply connected spaces. If $H^*(F;\mathbf{Q})$ is finite dimensional, then $ev_B \neq 0$ implies $ev_E \neq 0$.*

From Theorems 4 and 5, we deduce immediately the following lemma which will be used in the proof of Theorem 2:

Lemma 3. *Let $(A,d) \to (A \otimes \Lambda(x), D) \to (\Lambda(x), 0)$ be a KS-extension, with x of odd degree. If A is a Gorenstein dga and if $ev_A \neq 0$, then $A \otimes \Lambda(x)$ is a Gorenstein dga and $ev_{A\otimes\Lambda(x)} \neq 0$.*

We also need the next results for the proof of Lemma 6.

Lemma 4. *If A is Gorenstein and if $ev_A \neq 0$, then there exists an element ω in the socle of $H^*(A)$ whose degree is exactly the formal dimension of A.*

Indeed, $\mathrm{fd}(A) = |\omega|$, where $\omega = [f(1)] = ev_A([f])$, $\mathcal{E}xt_A(\mathbf{Q}, A) = \mathbf{Q}[f]$. □

Theorem 4 implies immediately:

Proposition 5. *If $(A,\delta) \to (A \otimes \Lambda X, d) \to (\Lambda X \bar{d})$ is a KS-extension of simply connected dga with $H^*(\Lambda X)$ finite dimensional and $H^*(A)$ of finite type, then*

$$\mathrm{fd}(A \otimes \Lambda X) = \mathrm{fd}(A) + \mathrm{fd}(\Lambda X).$$

3. Proof of Theorem 1

3.1. **Some graded commutative algebra.** Standard references are [4] and [10]. Let H be a finitely generated commutative graded algebra. $f \in H$ is *regular* if $\mathrm{Ann}(f)$ is zero, it is clear that f has an even degree.

A sequence $(f_i)_{i=1\ldots n}$ is *regular* if f_i is regular in $H/_{(f_1,\ldots,f_{i-1})}$. The *Krull dimension* of H is n if H contains $\mathbf{Q}[x_1,\ldots,x_n]$ and H is a finitely generated module over $\mathbf{Q}[x_1,\ldots,x_n]$. H is *Cohen-Macaulay* if there exists a regular sequence $(f_1,\ldots,f_n)$ such that $H/_{(f_1,\ldots,f_n)}$ is finite dimensional. In this case, we have $\mathrm{K}\dim H = n$. H is *Gorenstein* if there exists a regular sequence $(f_1,\ldots,f_n)$ such that $H/_{(f_1,\ldots,f_n)}$ is a Poincaré duality algebra.

Remark. If H is a Gorenstein commutative graded algebra with a nonzero socle, then H is a Poincaré duality algebra.

3.2. **Proof.** Let $(f_0,\ldots,f_n)$ be a regular sequence such that $H^*(A)/_{(f_0,\ldots,f_n)}$ is finite dimensional. Choose some elements $x_0,\ldots,x_n$ such that $\mathrm{cl}\, dx_i = f_i$ ($\mathrm{cl}\, x$ denotes the cohomology class of x). As in 3.1, there is a finite sequence of KS-extensions $(A_i,d_i) \to (A_{i+1},d_{i+1}) \to (\Lambda(x_i),0)$, $i=0,\ldots,n$, where, by Theorem 4, (A_i,d_i) are Gorenstein dga. Here $H^*(A_{n+1})$ is finite dimensional. Indeed, since f_i is regular in $H^*(A_i)$, $H^*(A_i) = H^*(A_{i-1})/_{(f_i)}$. Hence $H^*(A_{n+1}) = H^*(A)/_{(f_1,\ldots,f_n)}$.

Finally, we conclude that (A_{n+1},d_{n+1}) is a Poincaré duality algebra and this shows that $H^*(A)$ is a Gorenstein graded algebra.

4. Proof of Theorem 2

4.1. **Proof.** Notice that it is sufficient to prove that $H^*(A)$ is finite dimensional (Proposition 1, b). We admit for the moment the following lemma:

Lemma 6. *Let $(A,d) \to (A \otimes \Lambda(x), D) \to (\Lambda(x),0)$ be a KS-extension such that*:
- *A is a Gorenstein cdga,*
- *there exists an element ω in the socle of $H^*(A)$ whose degree is exactly the formal dimension of A,*
- *$H^*(A \otimes \Lambda(x))$ is finite dimensional, $\mathrm{cl}\, Dx \neq 0$ in $H^*(A)$ and x has an odd degree.*

Then $H^*(A)$ is finite dimensional.

Proof of the theorem. Since $H^*(A)$ is noetherian, there exists a KS-extension:

$$(A,d) \to (A \otimes \Lambda(x_0,\ldots,x_N), D) \to (\Lambda(x_0,\ldots,x_N), \bar{D})$$

such that $H^*(A \otimes \Lambda(x_0,\ldots,x_N))$ is finite dimensional. Indeed, fix a system of generators for $H^*(A)$ and let $f_0,\ldots,f_N$ be the ones of even degree. They are finite in number since $H^*(A)$ is noetherian. Define the x_i such that $\mathrm{cl}(Dx_i) = f_i$. The E_2-term of the Serre spectral sequence of the KS-extension is a $H^*(A)$-module of finite type as $H^*(\Lambda(x_0,\ldots,x_N))$ is finite dimensional. Therefore, the E_∞-term is a $H^*(A)_{/(f_0,\ldots,f_N)}$-module of finite type. As $H^*(A)_{/(f_0,\ldots,f_N)}$ is finite dimensional, we conclude that $H^*(A \otimes \Lambda(x_0,\ldots,x_N))$ is finite dimensional also. From the KS-extension above, we define a finite sequence of KS-extensions $(A_i,d_i) \to$

$(A_{i+1}, d_{i+1}) = A_i \otimes \Lambda(x_i) \to (\Lambda(x_i), 0)$, $i = 0, \dots, N$, where $(A_0, d_0) = (A, d)$ and $H^*(A_{N+1})$ is finite dimensional. Applying inductively Lemma 3, we show that each (A_i, d_i) is Gorenstein with a nonzero evaluation map. Lemma 4 implies that the KS-extension $(A_N, d_N) \to (A_{N+1}, d_{N+1}) \to (\Lambda(x_N), 0)$ satisfies the hypotheses of Lemma 6. Applying inductively Lemma 6, we prove the theorem. □

4.2. **Proof of Lemma 6.** The proof breaks down into five steps:

First step. Let f be an element of even degree of $H^*(A)$ and denote by (f) the ideal that it generates. If $H^*(A)/(f)$ is finite dimensional, then $\operatorname{Ann}(f)$ is also finite dimensional.

Proof. If $H^*(A)/(f)$ is finite dimensional, $H^*(A)$ is a finite type $\mathbf{Q}[f]$-module and as well $\operatorname{Ann}(f)$, but $f.\operatorname{Ann}(f) = 0$, thus $\operatorname{Ann}(f)$ is finite dimensional.

Second step. $H^*(A \otimes \Lambda(x))$ is finite dimensional if and only if $H^*(A)/(f)$ is finite dimensional where $f = \operatorname{cl} Dx$.

Proof. The previous step together with $H^*(A \otimes \Lambda(x)) = H^*(A)/(f) \oplus \operatorname{Ann}(f) \otimes x\mathbf{Q}$ implies the statement.

Third step. Let $(A, d) \to (A \otimes \Lambda(x), D) \to (\Lambda(x), 0)$ and $(A, d) \to (A \otimes \Lambda(y), D) \to (\Lambda(y), 0)$ be KS-extensions such that $H^*(A \otimes \Lambda(x))$ is finite dimensional, $f = \operatorname{cl} Dx \neq 0$ in $H^*(A)$, x has an odd degree and $\operatorname{cl} Dy = f^j$, then $H^*(A \otimes \Lambda(y))$ is also finite dimensional.

Proof Write $H^*(A) = B_0 \oplus (f)$. Notice that $B_0 \cong H^*(A)/(f)$ and $H^*(A) = \sum_{i \geq 0} B_0 f^i$. As $H^*(A \otimes \Lambda(x))$ is finite dimensional, so are $H^*(A)/(f)$, B_0 and $H^*(A)/(f^j) = \sum_{i=0}^{i=j-1} B_0 f^i$. By the previous step, we conclude.

Fourth step. Let $\omega \in \operatorname{socle}(H^*(A))$ such that $\operatorname{fd}(A) = |\omega|$, we observe that $|\omega| \geq |f|$.

Proof. $A \otimes \Lambda(x)$ is a Gorenstein dga with finite dimensional cohomology, so that $H^*(A \otimes \Lambda(x))$ is a Poincaré duality algebra. Moreover $\operatorname{fd} H^*(A \otimes \Lambda(x)) = \operatorname{fd}(A \otimes \Lambda(x)) = \operatorname{fd}(A) + \operatorname{fd}(\Lambda(x)) = |\omega| + |x|$.

If we suppose that $|\omega| < |f|$, then ω projects on a nontrivial element in $H^*(A)/(f)$. If we call α the Poincaré dual class of ω in $H^*(A \otimes \Lambda(x))$, then $|\alpha| = |x|$. For degree reason and because $\operatorname{cl} Dx \neq 0$, we obtain that α is in $H^*(A)/(f)$. This is impossible because ω is in the socle.

Last step. If we suppose that, for each i, $f^i \neq 0$, then there exists j such that $|f^j| \geq |\omega|$. By step 3, the KS-extension $(A, d) \to (A \otimes \Lambda(y), D) \to (\Lambda(y), 0)$ with $\operatorname{cl}(Dy) = f^j$ satisfies also the hypotheses of the lemma, but $|f^j| \geq |\omega|$ which is impossible by step 4. This proves that, for some j, $f^j = 0$ and that $H^*(A) = \sum_{i=0}^{i=j-1} B_0 f^k$, we conclude that $H^*(A)$ is finite dimensional.

5. Proof of Corollary 2 and of Theorem 3

5.1. **Proof of Corollary 2.** Proposition 1 b) and Theorem 2 give us (ii)⇒(i).

For the converse, it is clear that S is a Gorenstein space and $H^*(S)$ is noetherian. By ([7], Theorem 1 and the remark after Proposition 2), we have $T_a(S; \mathbf{Q}) \subset E(S; \mathbf{Q})$ where $E(S; \mathbf{Q}) = \operatorname{Im} ev_S$ and $T_a(S; \mathbf{Q})$ is the set of the algebraic terminal classes. It is enough to prove that $T_a(S; \mathbf{Q})$ is not empty. For this we take the minimal Adams-Hilton model [1] of S: $(TW, \delta) \xrightarrow{\simeq} C_*(\Omega S; \mathbf{Q})$ with $W = s^{-1} H_+(S; \mathbf{Q})$. As S is a 1-connected Poincaré duality space, $H_*(S; \mathbf{Q}) = H_{\leq n-2}(S; \mathbf{Q}) \oplus \mathbf{Q}\omega$ where $|\omega| = n$. Now, write $W = V \oplus s^{-1}\omega$ and remark that, for degree reason, $\delta s^{-1}\omega \in T(V)$. Hence, $s^{-1}\omega \in T_a(S; \mathbf{Q})$ and finally $ev_S \neq 0$. □

5.2. **Proof of Theorem 3 and remarks.** Theorem 3 is a corollary of Theorem 2 and of the following proposition:

Proposition 7. *If $H^*(A;\mathbf{Q})$ is not noetherian with nonzero socle, then $H^*(A;\mathbf{Q})$ is not a Gorenstein graded algebra.*

Proof of the theorem. Indeed, let S be a Gorenstein space such that $ev_S \neq 0$: If $H^*(S)$ is noetherian, then $H^*(S)$ is finite dimensional by Theorem 2, and, by the remark at the beginning of the previous section, S is a Poincaré duality space. If $H^*(S)$ is not noetherian, the socle is not zero since $\operatorname{im} ev_S \subset \operatorname{socle}(H^*(S))$ and by Proposition 7, $H^*(S)$ is not a Gorenstein graded algebra. $\square$

If fact, we do not know any Gorenstein spaces S such that $ev_S \neq 0$ and with $H^*(S;\mathbf{Q})$ not noetherian. Therefore, we can conjecture that

$$(S \text{ Gorenstein and } ev_S \neq 0) \Leftrightarrow (H^*(S;\mathbf{Q}) \text{ satisfies the Poincaré duality}).$$

This conjecture has been proved in a particular case by A. Murillo.

Theorem 6 ([13], Theorem A)**.** *Let S be a 1-connected pointed space with $\pi_*(S)\otimes \mathbf{Q}$ finite dimensional. Then the following statements are equivalent*:
(i) $H^*(S;\mathbf{Q})$ *is finite dimensional.*
(ii) ev_S *is nonzero.*

Indeed, if $\pi_*(S)\otimes\mathbf{Q}$ is finite dimensional, then S is a Gorenstein space ([5], Proposition 3.4). Notice also that Corollary 2 implies the following:

Let X be a 1-connected space such that $H_*(\Omega X;\mathbf{Q})$ has polynomial growth, $H^*(X)$ is noetherian and $ev_X \neq 0$, then X is an elliptic space. Indeed, by [6], the polynomial growth of $H_*(\Omega X;\mathbf{Q})$ implies that $\pi_*(\Omega X)\otimes\mathbf{Q}$ is finite dimensional, so that X is a Gorenstein space ([5], Proposition 3.4), finally $H^*(X;\mathbf{Q})$ is finite dimensional by Theorem 2, so X is an elliptic space.

5.3. **Proof of Proposition 7.** We use the bigraded model of $H^*(A) = H$ ([9], Section 3): $(\Lambda Z.d) \to (H,0)$. Consider the projective resolution of $\mathbf{Q}$ by H-modules:

$$\cdots \xrightarrow{d} H\otimes(\Lambda\bar{Z})_2 \xrightarrow{d} H\otimes(\Lambda\bar{Z})_1 \xrightarrow{d} H \xrightarrow{d} \mathbf{Q} \xrightarrow{\varepsilon} 0$$

where $\bar{Z}_i^j = Z_{i-1}^{j+1}$. If H is not noetherian, then Z_0 is infinite ([9], 3.8.2). Hence, $(\Lambda\bar{Z})_1 = \bar{Z}_1$ is also infinite. Let $\omega \in \operatorname{socle}(H)$, $\omega \in \Lambda(x_1,\dots,x_n), x_i \in Z_0$. We define for $k>0$

$$\begin{aligned} f_k\colon H\otimes(\Lambda\bar{Z})_1 &\to H\\ 1\otimes\bar{x}_{n+k} &\mapsto \omega\\ 1\otimes\bar{x}_j &\mapsto 0\end{aligned}$$

when $j\neq n+k$. This map satisfies $f_k\circ d = 0$ and there is no map $g\colon H\mapsto H$ such that $f_k = g\circ d$. Therefore, $Ext_H(\mathbf{Q},H)$ is infinite dimensional and H is not a Gorenstein graded algebra.

6. Example

In the following, we give two examples of Gorenstein cdga whose cohomologies are not Gorenstein graded algebras. In each case, we show by different methods that $H^*(A)$ is not a Gorenstein graded algebra.

First example. The assumption "socle is different from zero" is not sufficient: Let $(A,d) = (\Lambda(u,v,w,t),d)$ with $du = dv = dw = 0$, $dt = uvw$, $|u| = 2$, $|v| =$

$|w| = 3$ and $|t| = 7$. $H^*(A) = \Lambda(u,v,w,a,b)_{/\{uvw, aw+bv\}}$ where $a = [vt]$ and $b = [wt]$. The graded algebra $H^*(A)$ is noetherian. By ([5], Proposition 5.2), $\mathrm{fd}(A) = 1 - |u| + |v| + |w| + |t| = 12$ so there exist an element of degree 12 in $Ext_{H^*(A)}(\mathbf{Q}, H^*(A))$. But, $Ext^{0,-}_{H^*(A)}(\mathbf{Q}, H^*(A)) = \mathrm{socle}(H^*(A))$ and $[vw]$ is in the socle, $|vw| = 6$ so $Ext_{H^*(A)}(\mathbf{Q}, H^*(A))$ contains at least two elements (one of degree 12 and one of degree 6). Thus, $H^*(A)$ is not a Gorenstein graded algebra.

Second example, related to Proposition 7. Let $(A,d) = (\Lambda(u,v,w,t), d)$ with $du = dv = dw = 0$, $dt = uvw$, $|u| = |v| = 2$, $|w| = 3$ and $|t| = 6$. In this case, $H^*(A)$ is not noetherian and the socle is zero. $H^*(A)$ is not a Gorenstein graded algebra. Indeed, we prove that $Ext_{H^*(A)}(\mathbf{Q}, H^*(A))$ contains an infinite number of elements. $H^*(A) = H = \Lambda(u,v,t_i)_{/\{uvt_i, t_it_j\}}$ where $t_i = [wt^i]$. As in the proof of Proposition 7, we use the projective resolution of $\mathbf{Q}$ by H-modules obtained from the bigraded model of H. Here $(\Lambda\bar{Z})_1 = \bar{Z}_1 = \{\bar{u}, \bar{v}, \bar{t}_i\}$ and $\bar{Z}_2 = \{\bar{w}_i, \bar{t}_{ij}\}$ with $d\bar{w}_i = -\bar{u}vt_i$ and $d\bar{t}_{ij} = -\bar{t}_it_j$. The H-linear maps

$$\begin{aligned} f_i \colon H \otimes (\Lambda\bar{Z})_1 &\to H \\ 1 \otimes \bar{u} &\mapsto ut_i \\ 1 \otimes \bar{v} &\mapsto 0 \\ 1 \otimes \bar{t}_i &\mapsto 0 \end{aligned}$$

satisfy $f_i \circ d = 0$ and, for each i, there is no map $g\colon H \mapsto H$ such that $f_i = g \circ d$. Hence, $Ext_H(\mathbf{Q}, H)$ is infinite dimensional.

Acknowledgments

This paper is a part of my thesis, and I'm grateful to my supervisor, Professor Jean-Claude Thomas of the University of Angers, for his patience, help and encouragement. I also thank the Fields Institute for a congenial atmosphere that allowed me to have helpful conversations with Yves Félix, Stephen Halperin and Aniceto Murillo.

References

1. J. F. Adams and P. J. Hilton, *On the chain algebra of a loop space*, Comment. Math. Helv. **30** (1956), 305–330. MR **17:**1119b
2. L. Avaramov and S. Halperin, *Through the looking glass: A dictionary between rational homotopy theory and local algebra*, Algebra, Algebraic Topology and their Interactions, Lecture Notes in Math., vol. 1183, Springer-Verlag, Berlin, 1986. MR **87k:**55015
3. H. Bass, *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963), 8–28. MR **27:**3669
4. D. Eisenbud, *Commutative algebra with view toward algebraic geometry*, Graduate Texts in Mathematics, 150, Springer-Verlag, 1995. MR **97a:**13001
5. Y. Félix, S. Halperin and J. C. Thomas, *Gorenstein spaces*, Adv. in Math. **71** (1988), 92–112. MR **89k:**55019
6. ———, *The homotopy Lie algebra for finite complexes*, Inst. Hautes Études Sci. Publ. Math **56** (1982), 179–202. MR **85c:**55010
7. Y. Félix, J.-M. Lemaire and J. C. Thomas, *L'application d'évaluation, les groupes de Gottlieb duaux et les cellules terminales*, J. Pure and Applied Algebra **91** (1994), 143–164. MR **94j:**55004
8. S. Halperin, *Lectures on minimal models*, Mém. Soc. Math. France **244** (1978), 199–224. MR **85i:**55009
9. S. Halperin and J. Stasheff, *Obstructions to homotopy equivalences*, Adv. in Math. **32** (1979), 233–279. MR **80j:**55016
10. H. Matsumura, *Commutative algebra*, Benjamin, New York, 1970. MR **42:**1813

11. J. C. Moore, *Algèbre homologique et homologie des espaces classifants*, Séminaire Cartan 1959/1960, exposé 7.
12. A. Murillo, *Rational fibrations in differential homological algebra*, Trans. Amer. Math. Soc. **332** (1992), 79–91. MR **92j:**55019
13. ———, *On the evaluation map*, Trans. Amer. Math. Soc. **339** (1993), 611–622. MR **93m:**55021
14. D. Sullivan, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. **47** (1978), 269–331. MR **58:**31119

Département de Mathématiques, Université des Sciences et Technologies de Lille, 59655 Villeneuve D'Ascq, France

E-mail address: gammelin@gat.univ-lille1.fr