TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 351, Number 9, Pages 3725–3741
S 0002-9947(99)01982-0
Article electronically published on January 26, 1999

EXACT HAUSDORFF MEASURE AND INTERVALS OF MAXIMUM DENSITY FOR CANTOR SETS

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Abstract. Consider a linear Cantor set K, which is the attractor of a linear iterated function system (i.f.s.) $S_j x = \rho_j x + b_j$, $j = 1, \dots, m$, on the line satisfying the open set condition (where the open set is an interval). It is known that K has Hausdorff dimension α given by the equation $\sum_{j=1}^m \rho_j^\alpha = 1$, and that $\mathcal{H}_\alpha(K)$ is finite and positive, where $\mathcal{H}_\alpha$ denotes Hausdorff measure of dimension α. We give an algorithm for computing $\mathcal{H}_\alpha(K)$ exactly as the maximum of a finite set of elementary functions of the parameters of the i.f.s. When $\rho_1 = \rho_m$ (or more generally, if $\log \rho_1$ and $\log \rho_m$ are commensurable), the algorithm also gives an interval I that maximizes the density $d(I) = \mathcal{H}_\alpha(K \cap I)/|I|^\alpha$. The Hausdorff measure $\mathcal{H}_\alpha(K)$ is not a continuous function of the i.f.s. parameters. We also show that given the contraction parameters ρ_j, it is possible to choose the translation parameters b_j in such a way that $\mathcal{H}_\alpha(K) = |K|^\alpha$, so the maximum density is one. Most of the results presented here were discovered through computer experiments, but we give traditional mathematical proofs.

1. Introduction

Let $S_j x = \rho_j x + b_j$, $j = 1, \dots, m$, be a linear iterated function system on the line, with contraction ratios satisfying $0 < \rho_j < 1$. We assume the following form of the open set condition: there exists an open interval I such that $S_j I \subseteq I$ and the images $S_j I$ are disjoint. (There are examples where the open set condition holds, but not with an interval.) Without loss of generality we take $I = (0, 1)$, and we assume the images $S_j I$ are in increasing order, with $S_1(0) = 0$ and $S_1(1) = 1$. Let K denote the attractor of the i.f.s. Figure 1.1 shows a typical example. It is a generalized Cantor set with diameter equal to one and Hausdorff dimension α, where α satisfies

$$\sum_{j=1}^m \rho_j^\alpha = 1. \tag{1.1}$$

We are interested in the exact computation of $\mathcal{H}_\alpha(K)$, where $\mathcal{H}_\alpha$ denotes Hausdorff measure in dimension α. To avoid triviality we always assume $m \geq 2$ and $\alpha < 1$. It is easy to see that $\mathcal{H}_\alpha(K) \leq 1$, since K can be covered by its iterated images under the i.f.s., but it is not always true that we have equality (for $m \geq 3$). See Falconer [F] for some examples with $\mathcal{H}_\alpha(K) = 1$.

Received by the editors July 28, 1995 and, in revised form, November 13, 1996.
1991 *Mathematics Subject Classification.* Primary 28A80, 28A78.
Research supported by the National Science Foundation through the REU program (Ayer) and through Grant DMS–9303718 (Strichartz).

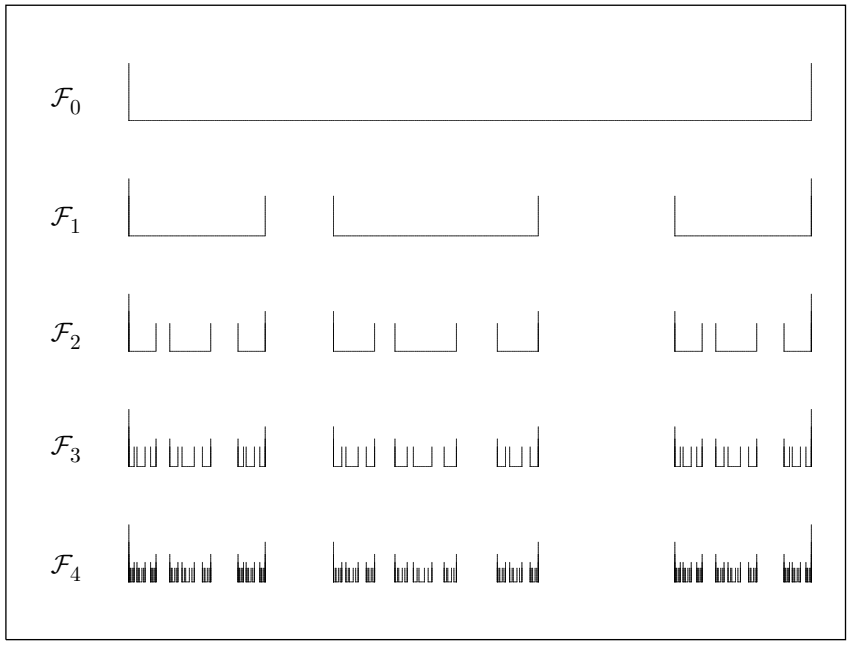

FIGURE 1.1. The first five levels of iteration of the generalized Cantor set with parameters $\rho_1 = 0.2, b_1 = 0, \rho_2 = 0.3, b_2 = 0.3, \rho_3 = 0.2, b_3 = 0.8$. α for this example is approximately 0.7524. Note that vertical bars have been added at island-lake boundaries.

Let μ denote the unique probability measure satisfying

$$\mu = \sum_{j=1}^{m} \rho_j^\alpha \mu \circ S_j^{-1}. \tag{1.2}$$

Then

$$\mu = c\mathcal{H}_\alpha\big|_K \tag{1.3}$$

for some constant c, and clearly $\mathcal{H}_\alpha(K)$ is just the reciprocal c^{-1}. Furthermore, the constant c in (1.3) can be characterized as the maximum density of μ, since $\mathcal{H}_\alpha\big|_K$ has maximum density equal to one [F]. For any interval J we define the density

$$d(J) = \mu(J)/|J|^\alpha, \tag{1.4}$$

where $|J|$ denotes the diameter of J. Then

$$c = \sup\{d(J) : J \subseteq [0,1]\}. \tag{1.5}$$

Our goal is not only to find c, but to find an interval J which achieves the supremum, if it exists. Since the density is preserved under the action of the i.f.s., there is no uniqueness for intervals of maximum density; in fact, we can essentially cover K by images of J (up to a set of measure zero), and this gives a simple proof that $\mathcal{H}_\alpha(K) = c^{-1}$.

The generation of the attractor K from the i.f.s. leads naturally to an increasing sequence of finite fields $\mathcal{F}_n$ of subsets of $[0,1]$, where $\mathcal{F}_0$ is the trivial field ($\emptyset$ and $[0,1]$), and $\mathcal{F}_n$ is generated by intervals of the form $S_J([0,1]) = S_{j_1}S_{j_2}\cdots S_{j_n}([0,1])$.

We call the intervals $S_1([0,1]), \dots, S_m([0,1])$ the *islands* of the first generation, and the intervals in between we call *lakes*. Let $\ell_j, \dots, \ell_{m-1}$ denote the lengths of these lakes. Note that we allow $\ell_j = 0$ in the case of touching islands, and indeed this case leads to some of the most interesting phenomena. The identity

$$\rho_1 + \cdots + \rho_m + \ell_1 + \cdots + \ell_{m-1} = 1 \tag{1.6}$$

and the non–negativity of the ρ's and ℓ's are the only restrictions on these parameters. From now on we will use these parameters to describe the i.f.s.

The defining self–similar identity (1.2) enables us to compute

$$\mu(S_J([0,1]) = \rho_{j_1}^{\alpha} \cdots \rho_{j_n}^{\alpha}, \tag{1.7}$$

and so the density of any interval in $\mathcal{F}_n$ is expressible as an elementary function of the parameters. There is an obvious algorithm for finding the maximum density of intervals in $\mathcal{F}_n$. We say that the i.f.s. has the *finiteness* property if the supremum in (1.5) is attained for an interval in $\mathcal{F}_n$, for some n. In Section 2 we show that the finiteness property holds in many cases: if the lakes are all non–zero, or if $\log \rho_1$ and $\log \rho_m$ are commensurable numbers. We also give an estimate for the size of n (in the second case it is independent of the lake parameters, but in the first case it increases without bound if one of the lake parameters tends to zero). In Section 3 we show that the finiteness property does not hold (generically) if one of the lake parameters is zero and if $\log \rho_1$ and $\log \rho_m$ are incommensurable. In that case the supremum is not attained, and we show how to obtain a sequence of intervals of length tending to zero whose densities approximate the supremum from below. These intervals all contain the point where the two first generation islands touch. Combining the two results, we have an algorithm for computing the maximum density in all cases. However, the maximum density is not a continuous function of the parameters. (In contrast, the probability measure determined by (1.2) does depend continuously on the parameters [CV].) Also, we can conclude that if the maximum density is attained, then the finiteness property holds.

In Section 4 we investigate the question of whether we can always choose the lake parameters, once the ρ's are chosen, to make the maximum density equal to one. We show that this is always possible, and it follows from the interesting observation that the maximum density is one if the maximum density of intervals in $\mathcal{F}_1$ is one. It is not true in general that if the maximum density of intervals in $\mathcal{F}_n$ is the same as for intervals in $\mathcal{F}_{n-1}$, then the maximum density is attained in $\mathcal{F}_{n-1}$. This invalidates the seductive "dumb search" algorithm: compute the maximum density of intervals in $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \dots$ and stops when there is no increase.

In Section 5 we consider the slightly more general case of i.f.s.'s that contain orientation reversing similarities, so we allow negative ρ's. The results are quite similar, except the finiteness property can occur without commensurability.

In Section 6 we discuss briefly the situation in the plane, but our results are all negative: all the obvious generalizations of our results are false.

Most of the results in this paper were first conjectured on the basis of extensive numerical experiments. We will not report in detail about these experiments, since the proofs stand on their own. The basic experimental procedure is described in [STZ]. There is one conjecture suggested by the experimental evidence that we have not succeeded in proving: when $m = 3$, the maximum density of an interval of the form $[0, x]$ occurs when $x = S_2^k(1)$ for some k. It is not clear what the analogous conjecture should be for $m > 3$.

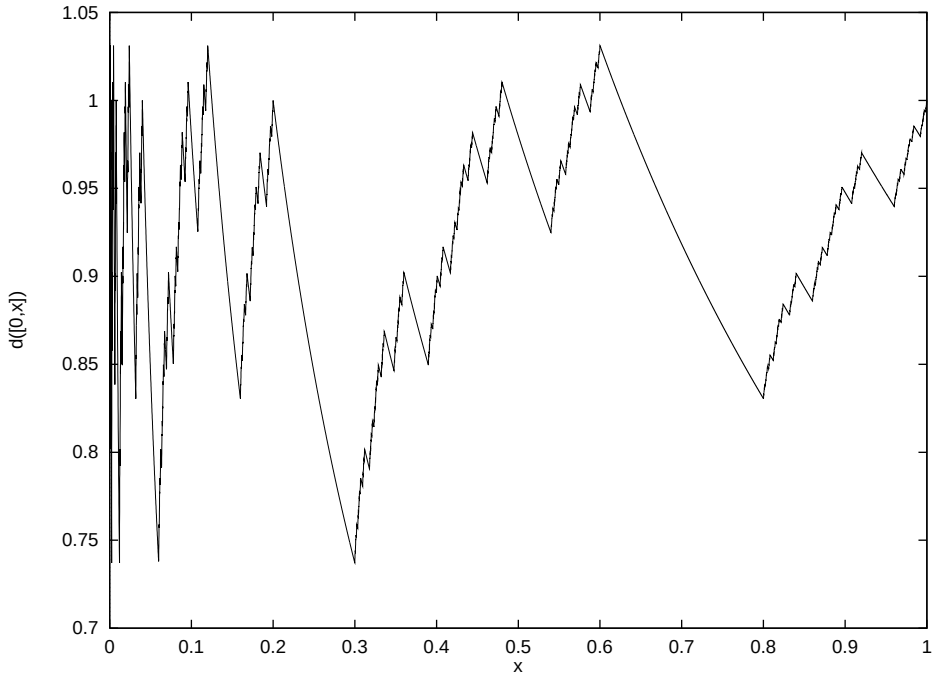

FIGURE 1.2. The density of intervals of the form $[0, x]$ as a function of x of the i.f.s. given by $\rho_1 = 0.2, b_1 = 0, \rho_2 = 0.3, b_2 = 0.3, \rho_3 = 0.2, b_3 = 0.8$, the same i.f.s. as in Figure 1.1.

The proofs of our theorems are based on calculus, specifically Lemmas 2.2 and 3.1. However, the functions we are maximizing are highly non–differentiable. For example, Figure 1.2 shows the graph of the density $d([0, x])$ of intervals of the form $[0, x]$ as a function of x for a typical example. The secret of success is to find a differentiable upper bound for the non–differentiable function with equality holding at the point where the derivative is zero.

After this work was completed, we became aware of earlier work of Marion [M1], [M2], on these problems (we are grateful to Rolf Riedi for pointing out these references). Our Theorems 2.4 and 4.2 were first proved in [M1], but the proofs we give are considerably shorter.

2. FINITENESS RESULTS

Let $\mathcal{F}_k$ denote the finite field of sets generated by k applications of the i.f.s. to $[0, 1]$. We call $\mathcal{F}_k$ the *k–th generation field*. Thus $\mathcal{F}_1$ is generated by the sets $S_j([0, 1])$, which we call the *first generation islands*. In this section we give results that say that an interval of maximum density will occur in $\mathcal{F}_k$, with estimates on the value of k. Since the density is preserved under the action of S_j, and also S_j^{-1} provided the interval lies in $S_j([0, 1])$, we have the following *blow–up principle*:

Lemma 2.1. *If I is any interval, there exists another interval I', not contained in a first generation island, with the same density. Moreover, if one endpoint of I is 0 or 1, then I' can be chosen with the same endpoint.*

Proof. If $I \subseteq S_j([0,1])$, then $S_j^{-1}I$ is a larger interval of the same density. We iterate this procedure until we obtain I' not lying in any first generation island. If 0 is the left endpoint of I, then we must have $j = 1$ in all stages of this procedure, so 0 is also the left endpoint of I'. Similarly if 1 is the right endpoint. Q.E.D.

Thus we need only consider intervals not contained in first generation islands. The following elementary calculus lemma is the key tool in studying densities of intervals whose length is bounded from below.

Lemma 2.2. *Suppose* $0 < \alpha < 1$, $p \le p_0$, $a \ge a_0$ *and* $y \ge \lambda x^\alpha$. *Then*

$$0 < x \le (a_0\lambda/p_0)^{1/(1-\alpha)} \tag{2.1}$$

implies

$$\frac{p-y}{(a-x)^\alpha} < \frac{p}{a^\alpha}. \tag{2.2}$$

Proof. Define $f(x) = (p - \lambda x^\alpha)/(a-x)^\alpha$. Note that $(p-y)/(a-x)^\alpha \le f(x)$ by the assumption $y \ge \lambda x^\alpha$, and $p/a^\alpha = f(0)$, so it suffices to show $f'(x) < 0$ on the interval (2.1). A direct computation gives $f'(x) = \alpha(a-x)^{-\alpha-1}(p - a\lambda x^{\alpha-1})$, which is negative if $x < (a\lambda/p)^{1/(1-\alpha)}$ (this uses $0 < \alpha < 1$). This holds on the interval (2.1) because $a \ge a_0$ and $p \le p_0$. Q.E.D.

Let $\rho_{\max} = \max(\rho_1, \dots, \rho_m)$ and $\rho_{\min} = \min(\rho_1, \dots, \rho_m)$. Our first finiteness result concerns the maximum density of intervals of the form $[0, x]$. It will be useful for obtaining non–finiteness results in Section 3.

Theorem 2.3. *Let k be the smallest integer such that*

$$\rho_{\max}^k \le (\rho_1 \rho_m^\alpha)^{1/(1-\alpha)}. \tag{2.3}$$

Then the maximum density of intervals of the form $[0, x]$ is attained by an interval in $\mathcal{F}_k$.

Proof. By the blow–up principle we can take $x_0 \ge \rho_1$ in the interval $[0, x_0]$ of maximum density (by compactness, the maximum is attained). Let $[0, a]$ be the smallest interval in $\mathcal{F}_k$ that contains $[0, x]$. Then $x_0 = a - x$ for some $x \le \rho_{\max}^k$, because $\rho_{\max}^k$ is the length of the largest island generating $\mathcal{F}_k$ (the point x_0 cannot fall in a lake of $\mathcal{F}_k$ because then $[0, x_0]$ would not have maximum density). Set $\rho = \mu([0,a])$ and $y = \mu([a-x, a])$. Then $d([0, a-x]) = (p-y)/(a-x)^\alpha$ and $d([0,a]) = p/a^\alpha$. Thus the conclusion (2.2) of Lemma 2.2 would give $d([0, a-x]) < d([0,a])$ unless $x = 0$, which implies that $[0, a]$ attains the maximum density.

To complete the proof we will verify the hypotheses of Lemma 2.2 with $p_0 = 1$, $a_0 = \rho_1$ and $\lambda = \rho_m^\alpha$. We already know $a \ge \rho_1$, and $p \le 1$ is trivial. To verify $y \ge \lambda x^\alpha$ we observe that $y/x^\alpha = d([a-x, a])$, and by the blow–up principle this is equal to the density of an interval of the form $[b, 1]$ which contains $S_m([0,1])$. An obvious lower bound for the density of such an interval is ρ_m^α (this is a lower bound for the measure, and 1 is an upper bound for the length). The hypotheses of Lemma 2.2 are verified, and condition (2.1) follows from $x \le \rho_{\max}^k$ and the hypothesis (2.3). Q.E.D.

Let $\ell_{\min} = \min(\ell_1, \dots, \ell_{m-1})$ be the minimum length of the lakes separating the first generation islands. We have $\ell_{\min} = 0$ if two islands touch, and this is the case when finiteness may fail. With a minimum separation $\ell_{\min} > 0$ we have the following quantitative finiteness result. The same result, with a slightly different

estimate for k, was proved in [M1] (Corollaire 6.4). In fact this work treats self–similar sets in $\mathbb{R}^n$ of dimension less than one.

Theorem 2.4. *Assume $\ell_{\min} > 0$, and let k be the smallest integer such that*

$$2\rho_{\max}^k \leq (\ell_{\min})^{1/(1-\alpha)}(\min(\rho_1, \rho_m))^{\alpha/(1-\alpha)}. \tag{2.4}$$

Then the maximum density is attained for an interval in $\mathcal{F}_k$.

Proof. By the blow–up principle we may restrict attention to intervals containing at least one lake, so we have the lower bound $\ell_{\min}$ for the length of the interval, which implies by compactness that the maximum density is attained. If $[x_1, x_2]$ is an interval of maximum density we let $[z_1, z_2]$ be the smallest interval in $\mathcal{F}_k$ containing $[x_1, x_2]$. Write $a = z_2 - z_1$ for the length of the interval, $x = (z_2 - z_1) - (x_2 - x_1)$ for the difference of the lengths, $p = \mu([z_1, z_2])$ and $y = \mu([z_1, x_1]) + \mu([x_2, z_2])$, so that $d([x_1, x_2]) = (p - y)/(a - x)^\alpha$ and $d([z_1, z_2]) = p/a^\alpha$. Once again we will complete the proof by applying Lemma 2.2.

We take $p_0 = 1$ and $a_0 = \ell_{\min}$, so that $a \geq a_0$ and $p \leq p_0$. We choose $\lambda = (\min(\rho_1, \rho_m))^\alpha$. For the right side interval $[x_2, z_2]$ we have

$$\mu([x_2, z_2])/(z_2 - x_2)^\alpha \geq \rho_m^\alpha$$

as in the proof of Theorem 2.3, and similarly for the left side interval $[z_1, x_1]$ we have

$$\mu([z_1, x_1])/(x_1 - z_1)^\alpha \geq \rho_1^\alpha.$$

Thus we have

$$y \geq \lambda((z_2 - x_2)^\alpha + (x_1 - z_1)^\alpha) \geq \lambda x^\alpha$$

since $0 < \alpha < 1$. Thus the hypotheses of Lemma 2.2 are verified, and condition (2.1) follows from (2.4) since x is the sum of two terms, $x_1 - z_1$ and $z_2 - x_2$, each being at most $\rho_{\max}^k$. Q.E.D.

Even if we allow touching islands, we can still obtain a finiteness result if we assume a (logarithmic) arithmetic relation between ρ_1 and ρ_m. Of course this is not a generic condition, and in the next section we will show that it is close to being necessary. Notice that in the next theorem the condition on k depends only on the contraction ratios.

Theorem 2.5. *Suppose there exist positive integers n_1 and n_m such that $\rho_1^{n_1} = \rho_m^{n_m}$. Let k be the smallest integer satisfying*

$$2\rho_{\max}^k \leq (\rho_{\min}\rho_1^{n_1})^{1/(1-\alpha)}(\min(\rho_1, \rho_m))^{\alpha/(1-\alpha)}. \tag{2.5}$$

Then the maximum density is attained for an interval in $\mathcal{F}_k$.

Proof. We claim that it suffices to look at intervals of length at least $\rho_{\min}\rho_1^{n_1}$. To see this we need a variant of the blow–up principle that shows us how to replace smaller intervals with larger intervals of greater density. Start with any interval not contained in a first generation island. If it actually contains a first generation island its length is at least $\rho_{\min}$, and we are done. If not, it begins at a point in $S_j([0,1])$ and ends at a point in $S_{j+1}([0,1])$. Now consider the intervals $J = S_j S_m^{n_m}([0,1])$ and $J' = S_{j+1}S_1^{n_1}([0,1])$ which lie on the extreme ends of the lake L_j separating $S_j([0,1])$ and $S_{j+1}([0,1])$. These intervals have length $\rho_j\rho_m^{n_m} = \rho_j\rho_1^{n_1}$ and $\rho_{j+1}\rho_1^{n_1}$, so if our interval contains one of them we are done.

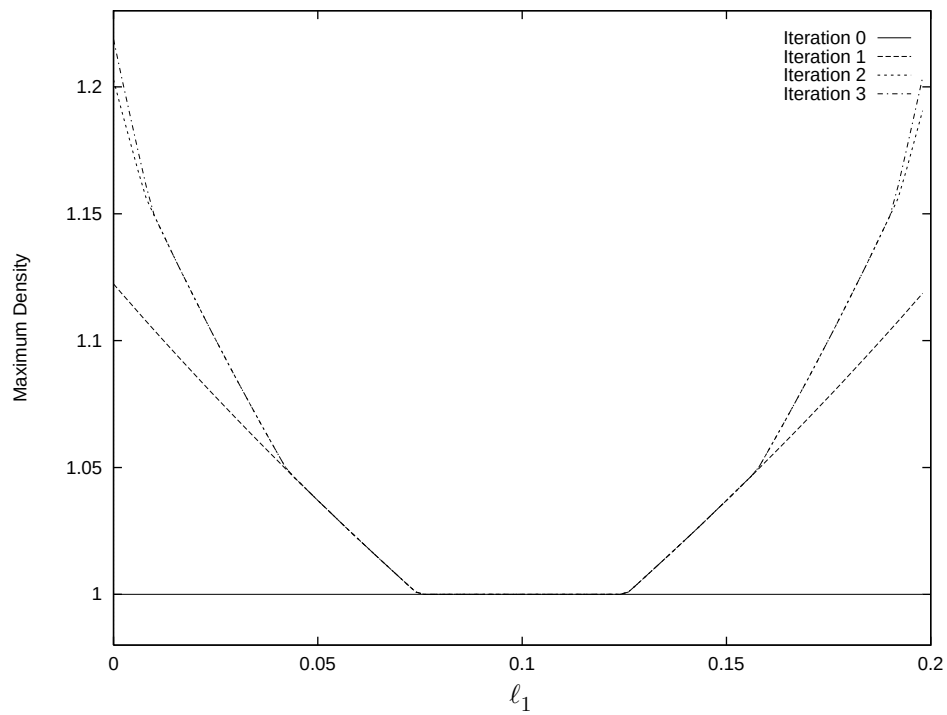

FIGURE 2.1. The maximum density versus ℓ_1 for the first three iterations of the i.f.s with contraction parameters $\rho_1 = 0.3, \rho_2 = 0.2, \rho_3 = 0.2$. The maximum density for this example occurs in $\mathcal{F}_3$.

Next suppose our interval begins with a point in J and ends with a point in J', say $I = J_0 \cup L_j \cup J_0'$ where $J_0 = I \cap J$ and $J_0' = I \cap J'$ and L_j is the lake separating J and J'. We generate another interval $I_1 = J_1 \cup L_j \cup J_1'$ by blowing up J_0 to J_1 and J_0' to J_1' by a factor $\rho_1^{-n_1} = \rho_m^{-n_m}$; specifically, we set $J_1 = S_j S_m^{-n_m} S_j^{-1} J_0$ and $J_1' = S_{j+1} S_1^{-n_1} S_{j+1}^{-1} J_0'$. Note that $S_j S_m^{n_m} S_j^{-1}$ maps J onto $S_j([0,1])$ and fixes the right endpoint, while $S_{j+1} S_1^{n_1} S_{j+1}^{-1}$ maps J' onto $S_{j+1}([0,1])$ and fixes the left end point, so I_1 is an interval. Also

$$d(I) = \frac{\mu(J_0) + \mu(J_0')}{(|J_0| + \ell_j + |J_0'|)^\alpha},$$

while

$$\begin{aligned} d(I_1) &= \frac{\rho_m^{-n_m\alpha}\mu(J_0) + \rho_1^{-n_1\alpha}\mu(J_0')}{(\rho_m^{-n_m}|J_0| + \ell_j + \rho_1^{-n_1}|J_0'|)^\alpha} \\ &= \frac{\mu(J_0) + \mu(J_0')}{(|J_0| + \rho_1^{-n_1}\ell_j + |J_0'|)^\alpha}, \end{aligned}$$

so $d(I_1) \geq d(I)$. By iterating this blow–up construction we eventually arrive at an interval containing either J or J', and we are done. This completes the proof of the existence of the lower bound $\rho_{\min}\rho_1^{n_1}$ for the length of the interval.

The rest of the argument is identical to the proof of Theorem 2.4, except we take $a_0 = \rho_{\min}\rho_1^{n_1}$. Q.E.D.

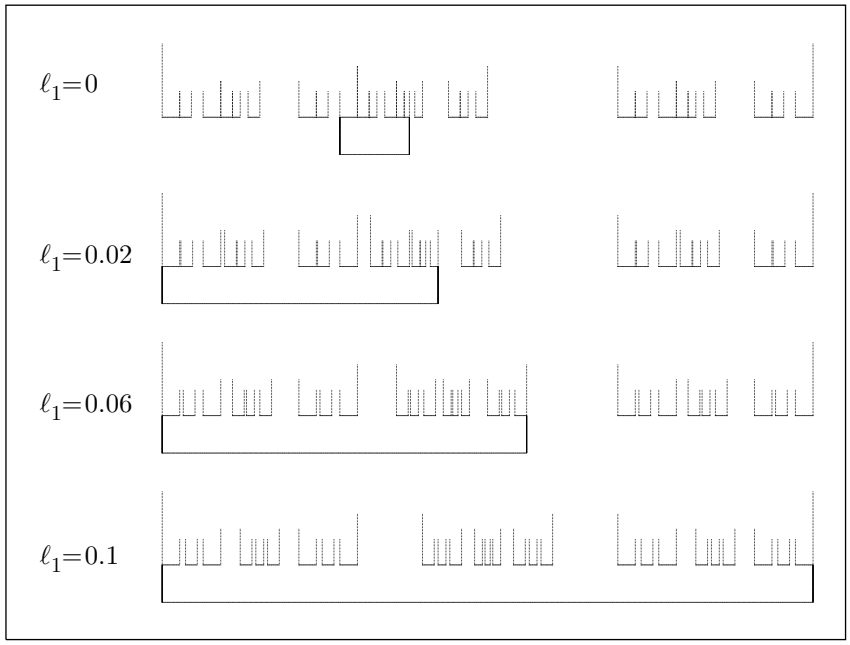

FIGURE 2.2. The island configurations which give rise to the maximum densities for the i.f.s. of Figure 2.1 for four representative values of ℓ_1.

Corollary 2.6. *If all ρ_j are equal to ρ, then the maximum density is attained by an interval in $\mathcal{F}_k$ where k is the smallest integer satisfying $k \geq (2+\alpha)/(1-\alpha) + \log 2/\log(1/\rho)$.*

Figure 2.1 shows the maximum densities of intervals in $\mathcal{F}_k$ for $k = 1, 2, 3$, as a function of the lake parameter ℓ_1 in a typical example (ρ_1, ρ_2, ρ_3 fixed) with $m = 3$ and $\rho_1 = \rho_3$. In this case, the maximum density occurs already in $\mathcal{F}_3$, for all values of ℓ_1 (when $m = 3$ the second lake parameter ℓ_2 is determined by (1.6)). Figure 2.2 shows the island configurations that give rise to these maximum densities for typical values of ℓ_1.

3. Non–finiteness results

We consider now the case when the contraction ratios ρ_1 and ρ_m do not satisfy the arithmetic condition of Theorem 2.5. We will need another elementary calculus lemma.

Lemma 3.1. *For positive constants a_1, a_2, q_1, q_2, and $0 < \alpha < 1$, consider the function*

$$F(x) = \frac{q_1 + q_2 x^\alpha}{(a_1 + a_2 x)^\alpha} \tag{3.1}$$

of a positive variable. Then F attains the maximum value of

$$\left((q_1/a_1^\alpha)^{1/(1-\alpha)} + (q_2/a_2^\alpha)^{1/(1-\alpha)}\right)^{1-\alpha} \tag{3.2}$$

at the point

$$x_0 = (a_1 q_2 / a_2 q_1)^{1/(1-\alpha)}. \tag{3.3}$$

Furthermore, $F(x)$ is strictly increasing on $0 \le x \le x_0$, and strictly decreasing on $x > x_0$.

Proof. We directly compute $F'(x) = \alpha(a_1+a_2x)^{-\alpha-1}(a_1q_2x^{\alpha-1}-a_2q_1)$, so $F'(x_0) = 0$ and $F'(x) > 0$ when $0 < x < x_0$, while $F'(x) < 0$ for $x > x_0$. Q.E.D.

Theorem 3.2. *Suppose $S_j([0,1])$ and $S_{j+1}([0,1])$ are touching islands, and ρ_1 and ρ_m are non–arithmetic in the sense that $\rho_1^{n_1} \neq \rho_m^{n_m}$ for any positive integers n_1 and n_m. Then the maximum density of intervals beginning in $S_j([0,1])$ and ending in $S_{j+1}([0,1])$ is*

$$(d_1^{1/(1-\alpha)} + d_2^{1/(1-\alpha)})^{1-\alpha}, \tag{3.4}$$

where d_1 denotes the maximum density of intervals of the form $[0,x]$, and d_2 denotes the maximum density of intervals of the form $[y,1]$. Furthermore, with the exception of a set of Lebesgue measure zero in the parameters $\rho_1, \dots, \rho_m, \ell_1, \dots, \ell_{m-1}$, the maximum value (3.4) is not attained.

Proof. Any such interval can be written $I = I_1 \cup I_2$ where $I_1 \subseteq S_j([0,1])$ extends to the right endpoint of $S_j([0,1])$, and $I_2 \subseteq S_{j+1}([0,1])$ begins at the left endpoint of $S_{j+1}([0,1])$. For any positive integers k_1 and k_2 we can form the interval

$$I(k_1,k_2) = S_j S_m^{k_1} S_j^{-1} I_1 \cup S_{j+1} S_1^{k_2} S_{j+1}^{-1} I_2,$$

which contracts I_1 by a factor of $\rho_m^{k_1}$ and I_2 by a factor of $\rho_1^{k_2}$, keeping their common endpoint fixed. Then

$$d(I(k_1,k_2)) = \frac{\rho_m^{\alpha k_1} q_1 + \rho_1^{\alpha k_2} q_2}{(\rho_m^{k_1} a_1 + \rho_1^{k_2} a_2)^\alpha}, \tag{3.5}$$

where $a_j = |I_j|$ and $q_j = \mu(I_j)$ for $j = 1,2$. Notice that this is exactly of the form (3.1) with $x = \rho_1^{k_2}\rho_m^{-k_1}$, and by the non–arithmetic hypothesis x takes on a dense set of values on the positive line. Thus (3.5) has maximum value

$$(d(I_1)^{1/(1-\alpha)} + d(I_2)^{1/(1-\alpha)})^{1-\alpha} \tag{3.6}$$

by Lemma 3.1, and the maximum is attained if and only if

$$(a_1 q_2 / a_2 q_1)^{1/(1-\alpha)} = \rho_1^{k_2}\rho_m^{-k_1} \tag{3.7}$$

for some integers k_1 and k_2.

Since (3.6) is an increasing function of $d(I_1)$ and $d(I_2)$, it is clear that its maximum is attained when $d(I_1)$ and $d(I_2)$ assume their maxima, and these are clearly d_1 and d_2, proving (3.4). Furthermore, the maximum of (3.5) for this choice of I_1 and I_2 will not be attained unless (3.7) holds. Now Theorem 2.3 (and its reverse analog) implies that I_1 and I_2 belong to $\mathcal{F}_k$ for some k. So the set of exceptional values for which the maximum is attained is contained in the set of solutions to (3.7) for all integer values of k_1 and k_2, where a_1 and a_2 are lengths and q_1 and q_2 are measures of sets in $\mathcal{F}_k$. For each fixed choice of k_1, k_2, a_1, a_2, q_1, q_2, the set of solutions to (3.7) clearly has measure zero, so the entire exceptional set is contained in a countable union of sets of measure zero. Q.E.D.

Theorem 3.3. *Suppose ρ_1 and ρ_m are non–arithmetic. Define k, k_1 and k_2 in terms of the i.f.s. parameters as the least integers such that*

$$2\rho_{\max}^k \le \lambda^{1/(1-\alpha)}(\min(\rho_1,\rho_2))^{\alpha/(1-\alpha)}, \tag{3.7}$$

$$\rho_{\max}^{k_1} \le (\rho_1\rho_m^\alpha)^{1/(1-\alpha)}, \tag{3.8}$$

$$\rho_{\max}^{k_2} \le (\rho_1^\alpha\rho_m)^{1/(1-\alpha)}, \tag{3.9}$$

where λ is the minimum of the ρ_j and the non–zero ℓ_j. Then the maximum density is equal to the maximum of the finite set of values $d(I)$ as I varies over all intervals in $\mathcal{F}_k$, and (if at least one $\ell_j = 0$) $(d(I_1)^{1/(1-\alpha)} + d(I_2)^{1/(1-\alpha)})^{(1-\alpha)}$ as I_1 varies over all intervals of the form $[0,x]$ in $\mathcal{F}_{k_1}$ and I_2 varies over all intervals of the form $[y,1]$ in $\mathcal{F}_{k_2}$.

Proof. By a minor variant of Theorem 2.4, the maximum density over all intervals that contain either a non–zero lake or a first generation island is attained by an interval in $\mathcal{F}_k$ (since the length of such intervals is bounded below by λ). The only other possibility is that one $\ell_j = 0$, in which case we apply Theorem 3.2, which means we have to consider also the values of (3.4). But by Theorem 2.3 the maximum value of d_1 is attained for an interval of the form $[0,x]$ in $\mathcal{F}_{k_1}$, and by the reverse analog the maximum value of d_2 is attained by an interval of the form $[y,1]$ in $\mathcal{F}_{k_2}$. Q.E.D.

Corollary 3.4. *In all cases, either the maximum density is attained for some interval in $\mathcal{F}_k$ for some k, or it is never attained.*

Proof. In the arithmetic case Theorem 2.5 says that the maximum density is attained in some $\mathcal{F}_k$, while in the non–arithmetic case Theorem 3.3 says either the maximum density is attained in some $\mathcal{F}_k$ or it is never attained. Q.E.D.

It is now easy to see that the maximum density is not a continuous function of the i.f.s. parameters. To be specific, take $m = 3$ and $\ell_1 = 0$. It is not difficult to see that an interval of maximum density cannot contain the second lake (this follows from Lemma 4.1 below), and so we need only consider intervals beginning in $S_1([0,1])$ and ending in $S_2([0,1])$. If ρ_1 and ρ_3 are non–arithmetic then Theorem 3.2 applies, and the maximum density is

$$(d_1^{1/(1-\alpha)} + 1)^{1-\alpha}, \tag{3.10}$$

where d_1 is the maximum density of intervals of the form $[0,x]$, since it is easy to see that one is the maximum density of intervals of the form $[y,1]$. Furthermore, the density d_1 is attained in $\mathcal{F}_k$ for suitable k, but the density (3.10) is not attained (with the exception of a set of parameter values of measure zero). In contrast, if ρ_1 and ρ_3 are arithmetic, then by Theorem 2.5 the maximum is attained in $\mathcal{F}_k$, and this maximum value is strictly less than (3.10). But (3.10) varies continuously with the parameters. Thus for any ρ_1 and ρ_3 that are arithmetic, the lim sup of the maximum density for neighboring values exceeds the value at the point.

Figure 3.1, similar to Figure 2.1, shows the maximum density of intervals in $\mathcal{F}_k$ for $0 \le k \le 5$ as a function of the lake parameter ℓ_1 in a typical example with $m = 3$, but this time ρ_1 and ρ_3 are non–arithmetic. In this case, for values of ℓ_1 close to zero, the maximum density increases with k. Figure 3.2 shows the corresponding island configurations for this example, with the size of the intervals tending to zero.

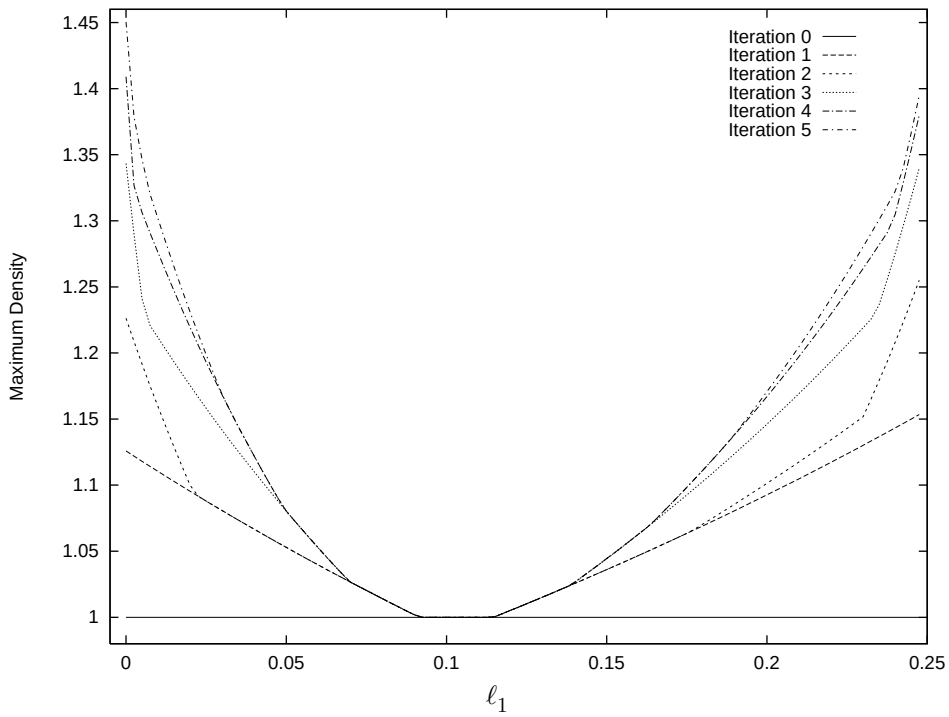

FIGURE 3.1. The maximum density versus ℓ_1 for $\mathcal{F}_0 - \mathcal{F}_5$ of the i.f.s. with contraction parameters $\rho_1 = 0.1, \rho_2 = 0.45, \rho_3 = 0.2$. The maximum density for this example does not occur in any $\mathcal{F}_k$.

4. When is the maximum density one?

The maximum density is never less than one, since $d([0,1]) = 1$. In this section we give a new proof that the maximum density is one if and only if the maximum density of all intervals in $\mathcal{F}_1$ is one ([M1], Theorem 7.1). In other words, if you are ever going to get off the ground, you have to do it right away! We will then show that, given any contraction ratios $\rho_1, \dots, \rho_m$ with $\rho_1 + \cdots + \rho_m < 1$, it is possible to choose the lake parameters $\ell_1, \dots, \ell_{m-1}$ so that the maximum density is one. This is clear in the examples shown in Figures 2.1 and 3.1.

Lemma 4.1. *Fix $j < k$, and assume $d(I_{jk}) \le 1$, where I_{jk} is the interval that extends from the beginning of $S_j([0,1))$ to the end of $S_k([0,1])$. For any interval I that begins in $S_j([0,1])$ and ends in $S_k([0,1])$,*

$$d(I) \le \max\big(1, d(I \cap S_j([0,1]), d(I \cap S_k([0,1]))\big). \tag{4.1}$$

Furthermore, the inequality is strict unless $d(I) = 1$.

Proof. Let M denote the right side of (4.1). We can write

$$d(I) = \frac{p + \rho_{j+1}^\alpha + \cdots + \rho_{k-1}^\alpha + q}{(a + \ell_j + \rho_{j+1} + \cdots + \rho_{k-1} + \ell_{k-1} + b)^\alpha}, \tag{4.2}$$

where $a = |I \cap S_j([0,1])|$, $p = \mu(I \cap S_j([0,1]))$, $b = |I \cap S_k([0,1])|$ and $q = \mu(I \cap S_k([0,1]))$. Note that $p \le Ma^\alpha$ and $q \le Mb^\alpha$ by the definition of M,

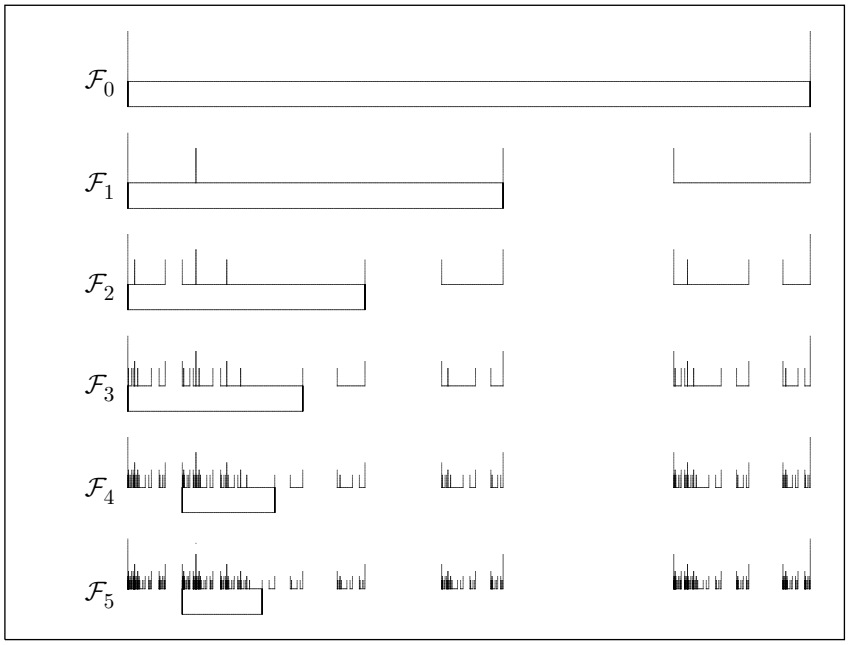

FIGURE 3.2. The island configurations which give rise to the maximum densities for the first five iterations of the i.f.s. of Figure 3.1 for $\ell_1 = 0.1$.

and also $a \le \rho_j$ and $b \le \rho_k$. Since $M \ge 1$ we obtain $d(I) \le MQ$, where

$$Q = \frac{a^\alpha + \rho_{j+1}^\alpha + \cdots + \rho_{k-1}^\alpha + b^\alpha}{(a + \ell_j + \rho_{j+1} + \cdots + \rho_{k-1} + \ell_{k-1} + b)^\alpha}\ , \tag{4.3}$$

so it remains to show that $Q \le 1$, with strict inequality unless $a = \rho_j$ and $b = \rho_k$.

Now $Q = Q(a, b)$ is a function of the form (3.1) in both variables, so we can apply Lemma 3.1. By hypothesis we have $Q(\rho_j, \rho_k) = d(I_{jk}) \le 1$, and we need to show that $Q(a, b) \le 1$ for all $a \le \rho_j$ and $b \le \rho_k$. First fix $a = \rho_j$ and look at $Q(\rho_j, b)$ as a function of b. By Lemma 3.1 it first increases until $b = x_0$, where it assumes a value > 1 (because $q_2 = a_2 = 1$ in (3.1)). But also $\lim_{b\to\infty} Q(\rho_j, b) = 1$, so $Q(\rho_j, \rho_k) \le 1$ implies $\rho_k < x_0$, since $Q(\rho_j, b) > 1$ for $b \ge x_0$. Thus $Q(\rho_j, b)$ is increasing for $b \le \rho_k$; hence $Q(\rho_j, b) \le 1$ for $b \le \rho_k$, with strict inequality unless $b = \rho_k$.

Now fix $b \le \rho_k$ and look at $Q(a, b)$ as a function of a. We can run the exact same argument as before because we know $Q(\rho_j, b) \le 1$. Thus $Q(a, b) \le 1$ for $a \le \rho_j$ and $b \le \rho_k$, with strict inequality unless $a = \rho_j$ and $b = \rho_k$. Q.E.D.

Theorem 4.2 ([M1], Theorem 7.1)**.** *If the maximum density of intervals in $\mathcal{F}_1$ is ≤ 1, then the maximum density of all intervals is ≤ 1.*

Proof. Since $I_{jk} \in \mathcal{F}_1$, the hypothesis of Lemma 4.1 holds for all $j < k$. If the maximum density is attained by an interval I, then strict inequality in (4.1) is impossible, so $d(I) = 1$. On the other hand for intervals of the form $I \cap S_j([0, 1])$ or $I \cap S_k([0, 1])$ in Lemma 4.1, the maximum density is attained, since by the blow–up

principle we can put a lower bound of ρ_m or ρ_1 on the length of an interval of equal density (in fact Theorem 2.3 shows the maximum is attained in some $\mathcal{F}_k$). But (4.1) implies that this gives the maximum density over all intervals. Q.E.D.

To verify the hypotheses of the theorem, it suffices to show that $d(I_{jk}) \leq 1$ for all $j < k$. In terms of the i.f.s. parameters this can be written

$$\ell_j + \cdots + \ell_{k-1} \geq (\rho_j^\alpha + \cdots + \rho_k^\alpha)^{1/\alpha} - (\rho_j + \cdots + \rho_k). \tag{4.4}$$

In particular, if we fix the contraction ratios $\rho_1, \ldots, \rho_m$, then the set of lake parameters $\ell_1, \ldots, \ell_{m-1}$ in $\mathbb{R}^{m-1}$ for which the maximum density is one forms a convex polytope given by the lower bounds (4.4) and the positivity conditions $\ell_j \geq 0$, intersected with the hyperplane

$$\ell_1 + \cdots + \ell_{m-1} = 1 - (\rho_1 + \cdots + \rho_m). \tag{4.5}$$

We will show that this polytope is always non–empty, and in fact (for $m \geq 3$) has non–empty interior in the hyperplane (4.5). To see this we introduce the abbreviation

$$c_{jk} = (\rho_j^\alpha + \cdots + \rho_k^\alpha)^{1/\alpha} - (\rho_j + \cdots + \rho_k). \tag{4.6}$$

Then the equations for the polytope are of the form

$$\ell_j + \cdots + \ell_{k-1} \geq c_{jk} \quad \text{for} \quad 1 \leq j < k \leq m, \tag{4.7}$$

$$\ell_1 + \cdots + \ell_{m-1} = c_{1m} \tag{4.8}$$

(this uses $\rho_1^\alpha + \cdots + \rho_m^\alpha = 1$).

Lemma 4.3. *If $1 \leq i < j < k \leq m$ then*

$$c_{ij} + c_{jk} < c_{ik}. \tag{4.9}$$

Proof. Let $x = \rho_i^\alpha + \cdots + \rho_{j-1}^\alpha$, $y = \rho_j^\alpha$, $z = \rho_{j+1}^\alpha + \cdots + \rho_k^\alpha$ and $p = 1/\alpha$. Note that x, y, z are positive and $p > 1$. Then (4.9) is just

$$(x+y)^p + (y+z)^p < y^p + (x+y+z)^p. \tag{4.10}$$

But we have equality when $z = 0$, and the z–derivatives of both sides of (4.10) clearly satisfy $p(y+z)^{p-1} < p(x+y+z)^{p-1}$. Q.E.D.

Theorem 4.4. *Let c_{jk} be positive constants for $1 \leq j < k \leq m$ and $m \geq 3$ satisfying (4.9). Then there exist solutions of (4.7) and (4.8); for example*

$$\begin{cases} \ell_{m-1} = c_{(m-1)m}, \\ \ell_j \ = c_{jm} - c_{(j+1)m}, \ 1 \leq j < m-1. \end{cases} \tag{4.11}$$

Furthermore, the set of solutions has non–empty interior in the hyperplane (4.8).

Proof. It is easy to verify that (4.11) gives a solution, for we have $\ell_j + \cdots + \ell_{m-1} = c_{jm}$, and $\ell_j + \cdots + \ell_{k-1} = c_{jm} - c_{km}$ if $k < m$. Thus (4.8) follows by setting $j = 1$ in the first equation, and (4.7) follows from (4.9).

To show that the solution set has non–empty interior we reason by induction on m. When $m = 3$ we have only three conditions, $\ell_1 + \ell_2 = c_{13}$, $\ell_1 \geq c_{12}$ and $\ell_2 \geq c_{23}$.

Thus we may take any ℓ_1 satisfying $c_{12} \le \ell_1 \le c_{13} - c_{23}$, and this interval has non–empty interior by the strict inequality in (4.9) (then $\ell_2 = c_{13} - \ell_1$ is determined). For the induction step, we choose ℓ_{m-1} to satisfy

$$c_{(m-1)m} \le \ell_{m-1} \le \min_{k<m-1} c_{km} - c_{k(m-1)}. \tag{4.12}$$

This interval has non–empty interior by the strict inequality in (4.9). We define new constants $\widetilde{c}_{jk}$ for $1 \le j < k \le m-1$ by

$$\begin{cases} \widetilde{c}_{jk} &= c_{jk} \quad \text{if} \quad k \le m-2, \\ \widetilde{c}_{j(m-1)} &= c_{jm} - \ell_{m-1}. \end{cases} \tag{4.13}$$

We verify that $\widetilde{c}_{jk}$ satisfy (4.9) for $1 \le i < j < k \le m-1$, since this follows from the analogous properties of c_{jk}. By the induction hypothesis, there is an open set of solutions to the inequalities

$$\ell_j + \cdots + \ell_{k-1} \ge \widetilde{c}_{jk} \quad \text{for} \quad 1 \le j < k \le m-1 \tag{4.14}$$

in the hyperplane (in $\mathbb{R}^{m-2}$)

$$\ell_1 + \cdots + \ell_{m-2} = \widetilde{c}_{1(m-1)}. \tag{4.15}$$

To complete the proof we need to show that if ℓ_{m-1} satisfies (4.12) and $\ell_1, \dots, \ell_{m-2}$ satisfy (4.14) and (4.15), then $\ell_1, \dots, \ell_{m-1}$ satisfy (4.7) and (4.8). Now (4.8) is just (4.15) in view of the second equation in (4.13), and (4.7) for $k < m-1$ is just (4.14) in view of the first equation in (4.13). Thus it remains to verify (4.7) for $k = m$ and $k = m-1$. Note that when $j = m-1$ and $k = m$ then (4.7) is the left inequality in (4.12). When $j < m-1$ and $k = m$ then (4.7) is $\ell_j + \cdots + \ell_{m-2} \ge c_{jm} - \ell_{m-1}$, which follows from (4.14) and the second equation in (4.13). Finally, when $k = m-1$ and $j < k$ then (4.14) and (4.13) yield $\ell_j + \cdots + \ell_{m-2} \ge \widetilde{c}_{j(m-1)} = c_{jm} - \ell_{m-1}$ and $c_{jm} - \ell_{m-1} \ge c_{j(m-1)}$ by the right inequality in (4.12). Q.E.D.

Corollary 4.5. *Given $\rho_1, \dots, \rho_m$ with $\alpha < 1$, the choice*

$$\begin{cases} \ell_j &= (\rho_j^\alpha + \cdots + \rho_m^\alpha)^{1/\alpha} - (\rho_{j+1}^\alpha + \cdots + \rho_m^\alpha)^{1/\alpha} - \rho_j \text{ for } j < m-1, \\ \ell_{m-1} &= (\rho_{m-1}^\alpha + \rho_m^\alpha)^{1/\alpha} - \rho_{m-1} - \rho_m \end{cases} \tag{4.16}$$

yields an i.f.s. for which the maximum density is one. For $m \ge 3$, we may increase ℓ_j slightly for $j < m-1$ and obtain the same conclusion.

5. Orientation reversing similarities

In this section we consider i.f.s.'s which contain orientation reversing similarities, which simply means that we allow some of the ρ_j to be negative. We again normalize by assuming that $S_j((0,1))$ are disjoint subintervals of $(0,1)$ in increasing order, with $S_1([0,1])$ containing 0 and $S_m([0,1])$ containing 1. This means either $\rho_1 > 0$ and $S_1 x = \rho_1 x$, or $\rho_1 < 0$ and $S_1 x = |\rho_1|(1-x)$; similarly, either $\rho_m > 0$ and $S_m x = \rho_m x + 1 - \rho_m$, or $\rho_m < 0$ and $S_m x = 1 - |\rho_m| x$.

We indicate briefly how to extend the results of Sections 2, 3 and 4 to this case. The results of Section 4 extend easily, with the same proof. For the results of Sections 2 and 3 we have to consider three cases, according to the signs of ρ_1 and ρ_m.

Case I. ρ_1 and ρ_m are both positive. The results are the same, and the proofs are essentially the same.

Case II. ρ_1 and ρ_m are both negative. In this case we can replace the i.f.s. with its iterated square, i.e., all compositions S_jS_k. When we do this, the transformation S_1S_m maps 0 to 0 with contraction ratio $\rho_1\rho_m$, and S_mS_1 maps 1 to 1 with the same contraction ratio. Thus Theorem 2.5 always applies. In terms of the original i.f.s., an interval of maximum density occurs in $\mathcal{F}_k$ as soon as

$$2\rho_{\max}^k \le \rho_{\min}^{2/(1-\alpha)}(\rho_1\rho_m)^{(1+\alpha)/(1-\alpha)} \tag{5.1}$$

(of course $\rho_{\max}$ and $\rho_{\min}$ are defined in terms of $|\rho_j|$). Since the finiteness property is always valid, the results of Section 3 are irrelevant.

Case III. Exactly one of ρ_1 and ρ_m is positive, and one is negative. For example, suppose $\rho_1 < 0$ and $\rho_m > 0$. In this case we will again have the finiteness property, with a maximum density interval in $\mathcal{F}_k$ provided

$$2\rho_{\max}^k \le (\rho_{\min}|\rho_1\rho_m|)^{1/(1-\alpha)}(\min|\rho_1|,|\rho_m|)^{\alpha/(1-\alpha)}. \tag{5.2}$$

This requires a slight modification of the proof of Theorem 2.5 as follows. We need to show that we can take $\rho_{\min}|\rho_1\rho_m|$ as a lower bound for an interval of maximum density. We take $J = S_jS_m([0,1])$ if S_j is orientation preserving, and $J = S_jS_1S_m([0,1])$ if S_j is orientation reversing; in either case J lies at the left end of the lake L_j. Similarly, we take $J' = S_{j+1}S_1S_m([0,1])$ if S_{j+1} is orientation preserving and $J' = S_{j+1}S_m([0,1])$ if S_{j+1} is orientation reversing; again J' lies at the right end of L_j. Note that $\rho_{\min}|\rho_1\rho_m|$ is a lower bound for the length of any of these intervals. For our blow–up procedure we set $J_1 = S_jS_m^{-1}S_jJ_0$ if S_j is orientation preserving, and $J_1 = S_jS_1S_m^{-1}S_1^{-1}S_j^{-1}J_0$ if S_j is orientation reversing. Also $J_1' = S_{j+1}S_1S_m^{-1}S_1^{-1}S_j^{-1}J_0'$ if S_{j+1} is orientation preserving, and $J_1' = S_{j+1}S_m^{-1}S_jJ_0$ if S_{j+1} is orientation reversing. The rest of the proof is the same.

6. Sierpinski gaskets

We indicate briefly in this section why the results of this paper will not generalize to linear self–similar sets in higher dimensional Euclidean spaces. One obvious obstacle is that our applications of calculus in Lemmas 2.2 and 3.1 require $\alpha < 1$. But even if we were to limit attention to self–similar sets of dimension $\alpha < 1$, it is unlikely that the same results would hold.

Consider the usual Sierpinski gasket on an equilateral triangle T of diameter one. The i.f.s. is given by $S_1(x,y) = \left(\frac{1}{2}x, \frac{1}{2}y\right)$, $S_2(x,y) = \left(\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y\right)$ and $S_3(x,y) = \left(\frac{1}{2}x + \frac{1}{4}, \frac{1}{2}y + \frac{\sqrt{3}}{4}\right)$. The dimension α is $\log 3/\log 2$, and the diameter is one. The analogues of the fields $\mathcal{F}_k$ are generated by the images of the equilateral triangle T under k iterations of the i.f.s. The maximum density (for the probability measure satisfying (1.2)) of sets in $\mathcal{F}_1$ is one, achieved by the triangle T and its images S_jT, because a union of two small triangles, say $S_1T \cup S_2T$, already has diameter one, and measure only 2/3. Nevertheless the maximum density of subsets of the gasket is greater than one. A search among all convex sets in $\mathcal{F}_3$ yielded the set pictured in Figure 6.1 with density 1.098405. (Any set can be replaced by its convex hull without decreasing the density, so it is reasonable to limit any search algorithm to convex sets. However, it does not follow that the maximum density among $\mathcal{F}_k$ sets is achieved by a convex set, since the convex hull of a set in $\mathcal{F}_k$ may not belong to $\mathcal{F}_k$. Also, the time involved in searching all sets in $\mathcal{F}_k$ rapidly becomes impractical, already when $k = 3$.)

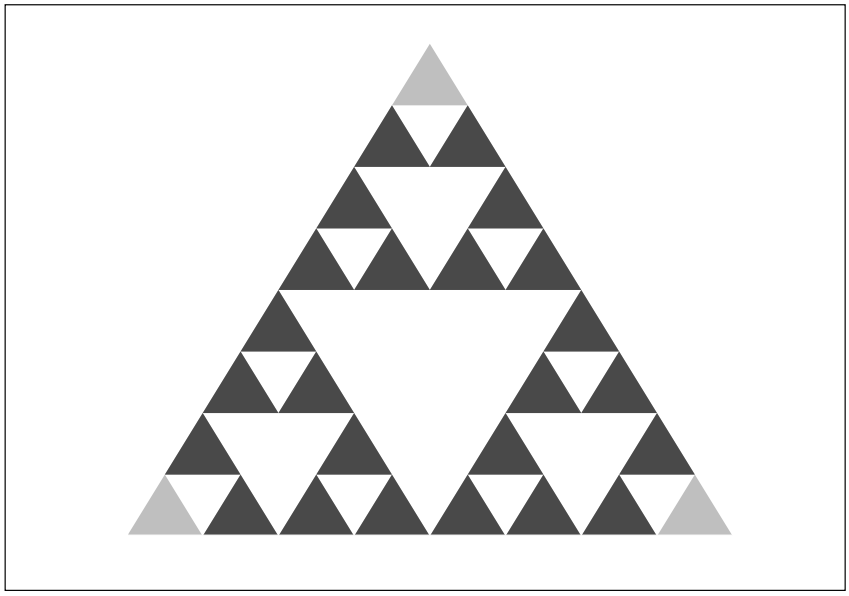

FIGURE 6.1. Set of maximum density (1.098405) in $\mathcal{F}_3$ of the Sierpinski gasket.

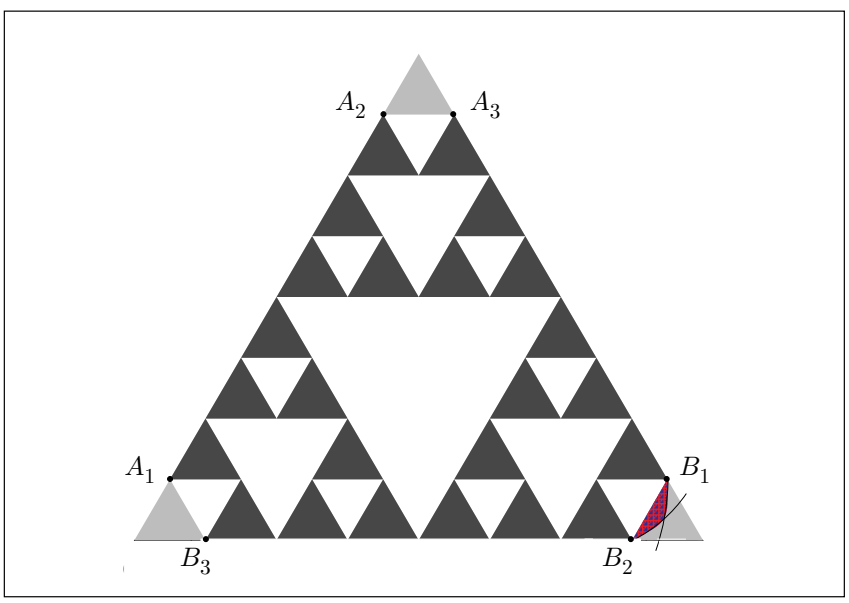

FIGURE 6.2. At every $\mathcal{F}_k$ more measure may be added to the set without increasing the diameter, so the maximum density is never reached.

But this set cannot have maximum density, because we can enlarge the set without increasing the diameter. There are exactly three pairs of points (labeled

A_j, B_j in Figure 6.1) in the set which achieve the diameter 7/8. If we take a circular arc centered at A_1 of radius 7/8 starting from B_1 and extending midway down the deleted lower right triangle, followed by a circular arc centered at A_2 of radius 7/8 continuing down to B_2, we trace the boundary of a slice of the deleted triangle that can be added to our set without increasing the diameter (see Figure 6.2). It is clear that this slice has positive measure, so the enlarged set has greater density.

It is clear that this enlarging process could be applied to any set in any $\mathcal{F}_k$. Thus the finiteness property does not hold for the Sierpinski gasket. But it fails for a reason different from the non–arithmetic case in Section 3, since it is not hard to show that the maximum density is attained. To see this we observe that the blow–up principle (the analog of Lemma 2.1) continues to hold, so we can restrict attention to sets not contained entirely in some triangle S_jT. If the set contains points in all three triangles, then we have a lower bound of 1/2 for its diameter. If it lies in the union of two triangles, say $S_1T \cup S_2T$, then we can blow up by a factor of 2^k about the intersection point $\left(\frac{1}{2}, 0\right)$ without changing the density, until the set extends beyond either S_1S_2T or S_2S_1T, which gives a lower bound of 1/4 for the diameter. Once we have a lower bound for the diameter, a compactness argument shows that the maximum density is attained.

Generally speaking, a set of maximal density should have constant breadth, as well as being convex, since otherwise we could enlarge the set without increasing the diameter. (Of course, this only increases the density if the enlargement increases the measure.) An intelligent search procedure for sets of maximal density should involve sets of constant breadth, yet this seems to rule out the sets in $\mathcal{F}_k$. Some new ideas will be needed here.

Acknowledgments

We would like to thank Phil Rzewski and Arthur Taylor for help with the numerical experiments. We are grateful to Rolf Riedi for the references to Marion's work.

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