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A CLASSIFICATION OF BAIRE-1 FUNCTIONS

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Abstract. In this paper we give some topological characterizations of bounded Baire-1 functions using some ranks. Kechris and Louveau classified the Baire-1 functions to the subclasses $\mathbb{B}_1^\xi(K)$ for every $\xi < \omega_1$ (where K is a compact metric space). The first basic result of this paper is that for $\xi < \omega$, $f \in \mathbb{B}_1^{\xi+1}(K)$ iff there exists a sequence (f_n) of differences of bounded semicontinuous functions on K with $f_n \to f$ pointwise and $\gamma((f_n)) \leq \omega^\xi$ (where "γ" denotes the convergence rank). This extends the work of Kechris and Louveau who obtained this result for $\xi = 1$. We also show that the result fails for $\xi \geq \omega$. The second basic result of the paper involves the introduction of a new ordinal-rank on sequences (f_n), called the δ-rank, which is smaller than the convergence rank γ. This result yields the following characterization of $\mathbb{B}_1^\xi(K)$: $f \in \mathbb{B}_1^\xi(K)$ iff there exists a sequence (f_n) of continuous functions with $f_n \to f$ pointwise and $\delta((f_n)) \leq \omega^{\xi-1}$ if $1 \leq \xi < \omega$, resp. $\delta((f_n)) \leq \omega^\xi$ if $\xi \geq \omega$.

Introduction

Let K be a compact metric space and $C(K)$ the set of continuous real-valued functions on K. A function $f : K \to \mathbb{R}$ is Baire-1 if there exists a sequence (f_n) in $C(K)$ that converges pointwise to f. Let $\mathbb{B}_1(K)$ be the set of bounded Baire-1 functions on K. Haydon, Odell and Rosenthal in [H-O-R] and Kechris and Louveau in [K-L] defined the oscillation rank $\beta(f)$ of a general function $f : K \to \mathbb{R}$ and proved that f is Baire-1 iff $\beta(f) < \omega_1$. Also, for every ordinal $\xi < \omega_1$ the subclass $\mathbb{B}_1^\xi(K)$ was defined by Kechris and Louveau in [K-L] to be the set of all f in $\mathbb{B}_1(K)$ such that $\beta(f) \leq \omega^\xi$, and it was proved that f in $\mathbb{B}_1^1(K)$ iff f is the uniform limit of differences of bounded semicontinuous functions on K (Theorem 3). Theorem 3 was originally proved in [H-O-R] (where $\mathbb{B}_1^1(K)$ is called $\mathbb{B}_{1/2}(K)$). This is in fact stated in [K-L], just before the statement of their Theorem 1, Section 3.

In this paper we give a general result for $\mathbb{B}_1^\xi(K)$ which is analogous to the above result for $\mathbb{B}_1^1(K)$.

In Theorem 7, we obtain the result that for $\xi < \omega$, $f \in \mathbb{B}_1^{\xi+1}(K)$ iff there exists a sequence (f_n) in DBSC(K) with $f_n \to f$ pointwise and $\gamma((f_n)) \leq \omega^\xi$ (where "γ" denotes the convergence rank, whose definition is recalled below). This extends the work of [K-L], who obtained this result for $\xi = 1$. We also show in Corollary 9 that the result fails for $\xi \geq \omega$; indeed we obtain there that if $f_n \to f$ pointwise and $\gamma((f_n)) \leq \omega^\xi$ with $(f_n) \subset$ DBSC(K), then also $\beta(f) \leq \omega^\xi$. Also Proposition 12 shows that Theorem 7 fails if we suppose in addition that $\sup_n |f_n|_D < \infty$. In Theorem 8 we obtain that if $f_n \to f$ pointwise, with f_n's Baire-1 functions, λ a

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limit ordinal, and $m < \omega$, with $\gamma((f_n)) \leq \omega^{\lambda+m}$ and $\sup_n \beta(f_n) < \omega^{\lambda}$, then f is Baire-1 with $\beta(f) \leq \omega^{\lambda+m}$. In Proposition 10 we show by example that this result fails, if we allow $\sup_n \beta(f_n) = \omega^{\lambda}$ instead (for $\lambda = \omega$).

The final result of the paper, Theorem 17, involves the introduction of a new ordinal-rank on sequences (f_n), called the δ-rank, which is smaller than the convergence rank γ. This is motivated by a characterization of $\mathbb{B}_{1/4}(K)$ given in [H-O-R]. Theorem 17 yields the following characterization of $\mathbb{B}_1^{\xi}(K)$, analogous to the $\mathbb{B}_{1/4}(K)$ characterization given in [H-O-R]: $f \in \mathbb{B}_1^{\xi}(K)$ iff there exists a sequence (f_n) of continuous functions with $f_n \to f$ pointwise and $\delta((f_n)) \leq \omega^{\xi-1}$ if $1 \leq \xi < \omega$, resp. $\delta((f_n)) \leq \omega^{\xi}$ if $\xi \geq \omega$. In fact, such a sequence (f_n) may be chosen as convex blocks of any sequence (g_n) of continuous functions converging pointwise to f; the analogous result for the γ-rank is due to Kechris and Louveau, and used in a fundamental way in the proof.

1. Definition. Let K be a compact metric space, $f : K \to \mathbb{R}, P \subset K$ and $\varepsilon > 0$. Let $P^0_{\varepsilon,f} = P$ and for any ordinal number a let $P^{a+1}_{\varepsilon,f}$ be the set of those $x \in P^a_{\varepsilon,f}$ such that for every open set U around x there are two points x_1 and x_2 in $P^a_{\varepsilon,f} \cap U$ such that $|f(x_1) - f(x_2)| \geq \varepsilon$.

At a limit ordinal a we set

$$P^a_{\varepsilon,f} = \bigcap_{\beta<a} P^{\beta}_{\varepsilon,f}.$$

Let

$$\beta(f,\varepsilon) = \begin{cases} \text{the least } a \text{ with } K^a_{\varepsilon,f} = \varnothing \text{ if such an } a \text{ exists,} \\ \omega_1, \quad \text{otherwise.} \end{cases}$$

Define the **oscillation rank** $\beta(f)$ of f by

$$\beta(f) = \sup\{\beta(f,\varepsilon) : \varepsilon > 0\}.$$

The above rank is defined by Haydon, Odell and Rosenthal in [H-O-R] and Kechris and Louveau in [K-L].

Let (f_n) be a sequence of real functions on $K, P \subset K$ and $\varepsilon > 0$. Let $P^0_{\varepsilon,(f_n)} = P$ and for any ordinal number a let $P^{a+1}_{\varepsilon,(f_n)}$ be the set of those $x \in P^a_{\varepsilon,f}$ such that for every open set U around x and any p in $\mathbb{N}$, there are n and m in $\mathbb{N}$ with $n > m > p$ and there is x' in $P \cap U$ with $|f_n(x') - f_m(x')| \geq \varepsilon$.

At a limit ordinal a we set

$$P^a_{\varepsilon,(f_n)} = \bigcap_{\beta<\alpha} P^{\beta}_{\varepsilon,(f_n)}.$$

Let

$$\gamma((f_n),\varepsilon) = \begin{cases} \text{the least } a \text{ with } K^a_{\varepsilon,(f_n)} = \varnothing \text{ if such an } a \text{ exists,} \\ \omega_1, \quad \text{otherwise.} \end{cases}$$

Define the **convergence rank** $\gamma((f_n))$ of (f_n) by

$$\gamma((f_n)) = \sup\{\gamma((f_n),\varepsilon) : \varepsilon > 0\}.$$

The derivative sets $P^1_{\varepsilon,(f_n)}$ are defined by Zalcwasser in [Z], Gillespie and Hurwicz in [G-H]. The convergence rank is defined by Kechris and Louveau in [K-L].

Remark 1. (i) By compactness of K it is easy to see that $\beta(f,\varepsilon)$ and $\gamma((f_n),\varepsilon)$ are isolated ordinals for all positive real number ε.

(ii) As in the proof of Corollary 4, section 2 of [K-L], it is easy to prove that $\beta(X_A) = \beta(X_A, 1/2)$ and hence $\beta(X_A)$ is an isolated ordinal.

2. Definition ([H-O-R], [K-L]). Let K be a compact metric space.

(a) $\mathrm{DBSC}(K)$ is the class of differences of two bounded semicontinuous real-valued functions on K. Without difficulty it can be shown that $\mathrm{DBSC}(K)$ coincides with the class of those $F : K \to \mathbb{R}$ for which there exist $(f_n) \subset C(K)$ and $C \in \mathbb{R}$ such that $f_0 = 0$, $f_n \to F$ pointwise, and $\sum_{n=0}^{\infty} |f_{n+1}(y) - f_n(y)| \leq C$ for all $y \in K$.

(b) We define $|\cdot|_D : \mathrm{DBSC}(K) \to \mathbb{R}$ using (a) as follows: $|F|_D$ is the infimum of all positive numbers C satisfying the condition in (a). Then $|\cdot|_D$ is a norm and $\mathrm{DBSC}(K)$ with $|\cdot|_D$ is a Banach space.

3. Theorem ([K-L], Theorem 1, Section 3). *$\mathbb{B}_1^1(K)$ is the sup-norm-closure of* $\mathrm{DBSC}(K)$.

4. Proposition ([K-L], Lemma 5, Section 2). *Let K be a compact metric space, $(f_n), (g_n)$ be the two sequences of functions on K, pointwise converging to f and g respectively. If $\xi < \omega_1$ is such that $\gamma((f_n)) \leq \omega^\xi$ and $\gamma((g_n)) \leq \omega^\xi$, then $\gamma((f_n + g_n)) \leq \omega^\xi$.*

5. Theorem ([K-L], Theorem 3, Section 2). *Let (f_n) be a bounded sequence of continuous functions on K, pointwise converging to some (bounded) Baire-1 function f.*

Then there exists a sequence (g_n) of convex blocks of (f_n) with $\gamma((g_n)) = \beta(f)$.

The following proposition is due to Kechris and Louveau, [K-L], Prop. 9, Section 2.

6. Proposition. *Let $f \in \mathbb{B}_1(K)$, $f \geq 0$ and $\xi < \omega_1$ with $\beta(f) \leq \omega^\xi$ and $n \in \mathbb{N}$, $n > 2$. Then there are $n-2$ sets $A_1, \ldots, A_{n-2}$ with $\beta(X_{A_k}) < \omega^\xi$, such that the function*

$$g = \frac{\|f\|_\infty}{n} \sum_{k=1}^{n-2} X_{A_k}$$

satisfies $0 \leq g \leq f \leq g + 2\|f\|_\infty/n$.

7. Theorem ([K-N]). *Let K be a compact metric space, $\xi < \omega$ an ordinal and $f \in \mathbb{B}_1(K)$. Then $f \in \mathbb{B}_1^{\xi+1}(K)$ if and only if there is a sequence $(f_n) \subset \mathrm{DBSC}(K)$ converging pointwise to f such that $\gamma((f_n)) \leq \omega^\xi$.*

Proof. Necessity. Let $f \in \mathbb{B}_1^{\xi+1}(K)$. Then $\beta(f) \leq \omega^{\xi+1}$.

Case 1. We assume that $f = X_A$. Then by Remark 1(ii) $\beta(X_A)$ is isolated and hence $\beta(X_A) < \omega^{\xi+1}$. Then there is $k < \omega$ such that $\beta(X_A) < k\omega^\xi$. Then there is a decreasing sequence $(F_\eta)_{\eta < k\omega^\xi}$ of closed subsets of K such that

$$A = \bigcup_{\substack{\eta < k\omega^\xi \\ \eta \text{ even}}} (F_\eta \setminus F_{\eta+1}).$$

We set

$$A_i = \bigcup \{(F_\eta \setminus F_{\eta+1}) : i\omega^\xi \leq \eta < (i+1)\omega^\xi, \eta \text{ even}\} \quad \forall i = 0, 1, \ldots, k.$$

Then $X_A = X_{A_1} + \cdots + X_{A_k}$. By Proposition 4 we shall show the conclusion for X_{A_i}, $i = 0, 1, \ldots, k$.

Without loss of generality we can assume that $k = 1$, that is,

$$A = \bigcup_{\substack{\eta < \omega^\xi \\ \eta \text{ even}}} (F_\eta \setminus F_{\eta+1}).$$

Let $\{\eta_1, \eta_2, \ldots, \eta_n, \ldots\}$ be an enumeration of the set $\{\eta : \eta \text{ even with } 0 \leq \eta < \omega^\xi\}$. For every $n \in \mathbb{N}$ we set:

$$A_n = \bigcup_{i=1}^{n} (F_{\eta_i} \setminus F_{\eta_i+1}).$$

Then $X_{A_n} \in \mathrm{DBSC}(K)$ for every $n \in \mathbb{N}$ and $X_{A_n} \to X_A$ pointwise.

We shall show that: $\gamma((X_{A_n})) < \omega^\xi$.

Let $0 < \varepsilon < 1$. We prove first that $K^1_{\varepsilon,(X_{A_n})} \subset \bigcap_{\eta<\omega} F_\eta$.

Let $x \in K^1_{\varepsilon,(X_{A_n})}$ such that $x \notin \bigcap_{\eta<\omega} F_\eta$.

Then there exists an open neighborhood V of x such that $\overline{V} \cap \bigcap_{\eta<\omega} F_\eta = \varnothing$.

Since K is compact we have that V intersects at most finitely many $(F_\eta)_{\eta<\omega}$. Then since $(F_\eta)_{\eta<\omega^\xi}$ is decreasing we have that V intersects at most finite many $F_{\eta_n} \setminus F_{\eta_n+1}$, $n = 1, 2, \ldots$. Hence there is $n_0 \in \mathbb{N}$ such that $X_{A_n|V} = X_{A_{n_0}|V}$ for every $n \geq n_0$ which is a contradiction, because $x \in K^1_{\varepsilon,(X_{A_n})}$. By induction $K^\eta_{\varepsilon,(x_{A_n})} \subset \bigcap_{\eta<n\omega} F_\eta$ for every $n < \omega$. Then $K^\omega_{\varepsilon,(X_{A_n})} \subset \bigcap_{\eta<\omega^2} F_\eta$.

Again by induction, we have: $K^{\omega^n}_{\varepsilon,(X_{A_n})} \subset \bigcap_{\eta<\omega^{n+1}} F_\eta$ for every $n < \omega$.

Hence $K^{\omega^{\xi-1}}_{\varepsilon,(X_{A_n})} \subset \bigcap_{\eta<\omega^\xi} F_\eta$ and since $X_{A_n}(y) = 0$ for every $y \in \bigcap_{\eta<\omega^\xi} F_\eta$ and $n \in \mathbb{N}$ we have $K^{\omega^{\xi-1}+1}_{\varepsilon,(X_{A_n})} = \varnothing$, that is, $\gamma((X_{A_n})) = \omega^{\xi-1} + 1 < \omega^\xi$.

Case 2. Suppose that $f \geq 0$. Then using Theorem 5 we find a sequence (g_n) where $0 \leq g_n = \sum_{i=1}^{k_n} a_i^n X_{A_i^n}$ with $\beta(X_{A_i^n}) < \omega^{\xi+1}$ for every $i = 1, 2, \ldots, k_n$, $n \in \mathbb{N}$, such that:

$$0 \leq g_1 + \cdots + g_n \leq f \leq g_1 + \cdots + g_n + \frac{\|f\|_\infty}{2^{n+2}} \quad \forall n \in \mathbb{N}.$$

Then for every $n > 1$ we have

$$\begin{aligned} 0 \leq g_n &= g_1 + \cdots + g_n + \frac{\|f\|_\infty}{2^{n+1}} - g_1 - \cdots - g_{n-1} - \frac{\|f\|_\infty}{2^{n+1}} \\ &\leq f + \frac{\|f\|_\infty}{2^{n+1}} - f \leq \frac{\|f\|_\infty}{2^n}. \end{aligned}$$

Hence $\|g_n\|_\infty \leq \|f\|_\infty 2^{-n}$ for any $n > 1$. Without loss of generality we can assume that $\|f\|_\infty \leq 1$. Then $\|g_n\|_\infty \leq 2^{-n}$ for every $n > 1$. Also $f = \sum_{n=1}^\infty g_n$ uniformly.

Since $\beta(X_{A_i^n}) < \omega^{\xi+1}$ for every $i = 1, 2, \ldots, k_n$, $n \in \mathbb{N}$, and by Case 1 and Proposition 4 we have that, for each $n \in \mathbb{N}$, there is $(g_n^p) \subset \mathrm{DBSC}(K)$ pointwise converging to g_n such that $g_n^p \geq 0$ for every $p \in \mathbb{N}$ and $\gamma((g_n^p)) \leq \omega^\xi$. For $\xi = 0$ this is proved by Kechris and Louveau (cf. [K-L]).

Since $\|g_n^p\|_\infty \leq \|g_n\|_\infty \leq 2^{-n}$ for every $n > 1$ and $\|g_1^p\|_\infty \leq \|g_1\|_\infty$ for every $p \in \mathbb{N}$, we have that for any $p \in \mathbb{N}$ $\sum_{n=1}^\infty g_n^p < \infty$ uniformly.

For any $p \in \mathbb{N}$ we set $g^p = \sum_{n=1}^\infty g_n^p$. Since $g_n^p \in \mathrm{DBSC}(K)$ for every $n \in \mathbb{N}$ and the convergence of the series is uniform we have $g^p \in \mathbb{B}_1^1(K)$ for every $p \in \mathbb{N}$.

Then, by Theorem 3 we have that for every $p \in \mathbb{N}$ there exists $f_p \in \mathrm{DBSC}(K)$ such that $\|g^p - f_p\|_\infty < \frac{1}{p}$. Then since (g^p) is pointwise converging to f we have that and the sequence (f_p) is also pointwise converging to f.

The proof of Case 2 can be finished by proving that $\gamma((f_n)) \leq \gamma((g^p)) \leq \omega^\xi$.

We see this, as follows:

Let $\varepsilon > 0$, P be a closed subset of K. We shall show that $P^1_{\varepsilon,(f_p)} \subset P^1_{\varepsilon/2,(g^p)}$.

Let $x \in P^1_{\varepsilon,(f_p)} \setminus P^1_{\varepsilon/2,(g^p)}$. Then there exists an open subset U of P with $x \in U$ and $p_0 \in \mathbb{N}$ such that

$$|g^p(x') - g^{p'}(x')| \leq \varepsilon/2 \quad \forall x' \in U, p, p' \geq p_0.$$

Let $p_1 \geq p_0$ with $\frac{2}{p_1} < \frac{\varepsilon}{2}$. Then for each $p, p' \geq p_1$ and $x' \in U$ we have

$$\begin{aligned}|f_p(x') - f_{p'}(x')| &\leq |f_p(x') - g^p(x')| + |g^p(x') - g^{p'}(x')| + |g^{p'}(x') - f_{p'}(x')| \\ &< \frac{1}{p} + \frac{\varepsilon}{2} + \frac{1}{p'} < \varepsilon,\end{aligned}$$

a contradiction since $x \in P^1_{\varepsilon,(f_p)}$. Hence $\gamma((f_p)) \leq \gamma((g^p))$.

Note that for $q, q' \geq p > 1$, we have

$$(*) \qquad \|g^q - g^{q'}\|_\infty \leq \left\| \sum_{n \leq p} g^q_n - \sum_{n \leq p} g^{q'}_n \right\|_\infty + 4.2^{-p}.$$

Also, $\gamma((g^q_n)) \leq \omega^\xi$ for all $n \in \mathbb{N}$ and by Proposition 4 we have that $\gamma((\sum_{n \leq p} g^q_n)) \leq \omega^\xi$ and hence by $(*)$ this implies that $\gamma((g^q)) \leq \omega^\xi$.

Case 3. (General case). If $f \in \mathbb{B}^{\xi+1}_1(K)$ then $f = f^+ - f^-$ where $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$. Then $0 \leq f^+, f^- \in \mathbb{B}^{\xi+1}_1(K)$ and from Case 2 there are sequences $(f^1_n), (f^2_n)$ in $\mathrm{DBSC}(K)$ with (f^1_n) converging pointwise to f^+, (f^2_n) converging pointwise to f^-, $\gamma((f^1_n)) \leq \omega^\xi$ and $\gamma((f^2_n)) \leq \omega^\xi$. Then $f^1_n - f^2_n \in \mathrm{DBSC}(K)$ for every $n \in \mathbb{N}$, $(f^1_n - f^2_n)$ converges pointwise to f and by Proposition 4 we have that $\gamma((f^1_n - f^2_n)) \leq \omega^\xi$.

Sufficiency. Let $(f_n) \subset \mathrm{DBSC}(K)$ be a sequence converging pointwise to f with $\gamma((f_n)) \leq \omega^\xi$. We prove that $\beta(f) \leq \omega^\xi$.

Claim. $P^\omega_{\varepsilon,f} \subset P^1_{\varepsilon/3,(f_n)}$ for all closed subsets P of K and $\varepsilon > 0$.

[Proof of claim: Let P be a closed subset of K and $x \in P^\omega_{\varepsilon,f} \setminus P^1_{\varepsilon/3,(f_n)}$. Then choose an open subset V of P with $x \in V$ and $n_0 \in \mathbb{N}$ such that

$$|f_m(y) - f_n(y)| \leq \varepsilon/3 \quad \forall y \in \overline{V}, n \geq n_0.$$

Then $|f_{n_0}(y) - f_n(y)| \leq \varepsilon/3$ for all $y \in \overline{V}$, all $n \geq n_0$ and since (f_n) converges pointwise to f we have that $|f_{n_0}(y) - f(y)| \leq \varepsilon/3$ for all $y \in \overline{V}$.

Then, $\overline{V}^\eta_{\varepsilon,f} \subset \overline{V}^\eta_{\varepsilon/3,f_{n_0}}$ for all $\eta < \omega$. Since $\beta(f_{n_0}) \leq \omega$ we have $\overline{V}^\omega_{\varepsilon/3,f_{n_0}} = \varnothing$.

Then $V \cap P^\omega_{\varepsilon,f} \subset \overline{V}^\omega_{\varepsilon,f} \subset \overline{V}^\omega_{\varepsilon/3,f_{n_0}} = \varnothing$, a contradiction, since $x \in V \cap P^\omega_{\varepsilon,f}$. Hence the proof of the claim is finished.]

By induction and applying the claim we get

$$K^{m\omega}_{\varepsilon,f} \subset K^m_{\varepsilon/3,(f_n)} \quad \forall m < \omega \Rightarrow K^{\omega^2}_{\varepsilon,f} \subset K^\omega_{\varepsilon/3,(f_n)}.$$

Also, by induction we have $K^{\omega^{n+1}}_{\varepsilon,f} \subset K^{\omega^n}_{\varepsilon/3,(f_n)}$ for all $n < \omega$.

Hence $K^{\omega^{\xi+1}}_{\varepsilon,f} \subset K^{\omega^{\xi}}_{\varepsilon/3,(f_n)} = \varnothing$ and hence $\beta(f) \leq \omega^{\xi+1}$. $\square$

Remark 2. In Theorem 7, the sequence (f_n) can in fact also be chosen uniformly bounded (as the proof shows).

For $\xi = 1$, Theorem 7 was proved by Kechris and Louveau in [K-L].

8. Theorem ([K-N]). *Let K be a compact metric space, $f, f_n \in \mathbb{B}_1(K)$, $n \in \mathbb{N}$, with (f_n) converging pointwise to f, $\lambda < \omega_1$ a limit ordinal and $m < \omega$ such that*

$$\sup\{\beta(f_n) : n \in \mathbb{N}\} < \omega^{\lambda} \quad \textit{and} \quad \gamma((f_n)) \leq \omega^{\lambda+m}.$$

Then $\beta(f) \leq \omega^{\lambda+m}$.

Proof. Since λ is a limit ordinal and $\sup\{\beta(f_n) : n \in \mathbb{N}\} < \omega^{\lambda}$ we choose a strictly increasing sequence (λ_n) such that $\sup_n \lambda_n = \lambda$ and $\sup\{\beta(f_n) : n \in \mathbb{N}\} < \omega^{\lambda_1}$.

Claim. $P^{\omega^{\lambda_1}}_{\varepsilon,f} \subset P^{1}_{\varepsilon/3,(f_n)}$ for all closed subsets P of K and $\varepsilon > 0$.

[Proof of claim: Let $P \subset K$ be closed, $\varepsilon > 0$ and $x \in P^{\omega^{\lambda_1}}_{\varepsilon,f} \setminus P^{1}_{\varepsilon/3,(f_n)}$.

Then there exists an open subset V of P with $x \in V$ and $n_0 \in \mathbb{N}$ such that

$$|f_m(y) - f_n(y)| \leq \varepsilon/3 \quad \forall y \in \overline{V}, n, m \geq n_0.$$

Then $|f_{n_0}(y) - f(y)| \leq \varepsilon/3 \forall y \in \overline{V}, n \geq n_0$ and hence $\overline{V}^{\eta}_{\varepsilon,f} \subset \overline{V}^{\eta}_{\varepsilon/3,f_{n_0}} \forall \eta < \omega^{\lambda_1}$.

Since $\beta(f_{n_0}) \leq \omega^{\lambda_1}$ implies that $\overline{V}^{\omega^{\lambda_1}}_{\varepsilon/3,f_{n_0}} = \varnothing$. Also $V \cap P^{\omega^{\lambda_1}}_{\varepsilon,f} \subset \overline{V}^{\omega^{\lambda_1}}_{\varepsilon/3,f_{n_0}}$.

Then $V \cap P^{\omega^{\lambda_1}}_{\varepsilon,f} = \varnothing$, a contradiction. Hence the proof of the claim is finished.]

By induction and applying the claim we get $K^{\theta\omega^{\lambda_1}}_{\varepsilon,f} \subset K^{\theta}_{\varepsilon/3,(f_n)} \forall \theta < \omega^{\lambda}$ and hence $K^{\omega^{\lambda}}_{\varepsilon,f} = \bigcap_{n=1}^{\infty} K^{\omega^{\lambda_n+\lambda_1}}_{\varepsilon,f} \subset \bigcap_{n=1}^{\infty} K^{\omega^{\lambda_n}}_{\varepsilon/3,(f_n)} = K^{\omega^{\lambda}}_{\varepsilon/3,(f_n)}$.

By induction we have that $K^{n\omega^{\lambda}}_{\varepsilon,f} \subset K^{n\omega^{\lambda}}_{\varepsilon/3,(f_n)} \forall n < \omega$ and hence $K^{\omega^{\lambda+1}}_{\varepsilon,f} \subset K^{\omega^{\lambda+1}}_{\varepsilon/3,(f_n)}$.

Also, by induction we get $K^{\omega^{\lambda+m}}_{\varepsilon,f} \subset K^{\omega^{\lambda+m}}_{\varepsilon/3,(f_n)} = \varnothing$ and hence $\beta(f) \leq \omega^{\lambda+m}$. $\square$

Note. Theorems 7 and 8 are due jointly to Professor Negrepontis (cf. [K-N]). I am grateful to Professor Negrepontis for his kind permission to present some of our joint work here.

In the following corollary it is proved that the conclusion of Theorem 7 is not true for $\xi \geq \omega$.

9. Corollary. *Le K be a compact metric space, $\omega \leq \xi < \omega_1, f \in \mathbb{B}_1(K)$ and $(f_n) \subset \mathrm{DBSC}(K)$ such that (f_n) is pointwise converging to f and $\gamma((f_n)) \leq \omega^{\xi}$.*

Then $\beta(f) \leq \omega^{\xi}$.

Proof. If $\xi \geq \omega$ there is a limit ordinal $\lambda \geq \omega$ and $m < \omega$ such that $\xi = \lambda + m$. Also $\sup\{\beta(f_n) : n \in \mathbb{N}\} = \omega < \omega^{\omega} \leq \omega^{\lambda}$. Hence by Theorem 8 we have $\beta(f) \leq \omega^{\lambda}$. $\square$

10. Proposition. *Let K be a scattered compact metric space with $K^{(\omega^{\omega+1})} \neq \varnothing$. Then there is a sequence $(f_n) \subset \mathbb{B}_1(K)$, $f \in \mathbb{B}_1(K)$ such that (f_n) is pointwise converging to f, $\sup\{\beta(f_n) : n \in \mathbb{N}\} = \omega^{\omega}$, $\gamma((f_n)) \leq \omega^{\omega+1}$ and $\beta(f) > \omega^{\omega+1}$.*

Proof. We set

$$A = \bigcup\{(K^{(\eta)} \setminus K^{(\eta+1)}) : \eta \text{ even and } \eta < \omega^{\omega+1}\}.$$

Then $\beta(X_A) = \omega^{\omega+1} + 1$. For every $n \in \mathbb{N}$ we set

$$A_n^k = \bigcup \{(K^{(\eta)} \setminus K^{(\eta+1)}) : \eta \text{ even and } (k-1)\omega^\omega \leq \eta < \omega^\omega + \omega^\eta\},$$
$$k = 1, 2, \ldots, n.$$

Then we have $\omega^n < \beta(X_{A_n^k}) \leq \omega^{n+1}$ $\forall k = 1, 2, \ldots, n$, $n \in \mathbb{N}$.

We set $A_n = \bigcup_{k=1}^n A_n^k$ $\forall n \in \mathbb{N}$. Then $X_{A_n} = X_{A_n^1} + \cdots + X_{A_n^n}$ and hence $\omega^n < \beta(X_{A_n}) \leq \omega^{n+1}$ for all $n \in \mathbb{N}$.

Then $\sup\{\beta(X_{A_n}) : n \in \mathbb{N}\} = \omega^\omega$. Also (X_{A_n}) is pointwise converging to X_A.

The proof will be finished by proving that $\gamma((X_{A_n})) \leq \omega + 1$.

To see this, if $\varepsilon > 0$ then $K^m_{\varepsilon,(X_{A_n})} \subset \bigcap_{\eta < m\omega} K_\omega^{(\eta)}$ for all $m < \omega$ and hence $K^\omega_{\varepsilon,(X_{A_n})} \subset \bigcap_{\eta<\omega} K^{(\eta)}_{\omega+1}$. Since the functions X_{A_n} are zero on $\bigcap_{\eta<\omega} K^{(\eta)}_{\omega+1}$ we have that $K^{\omega+1}_{\varepsilon,(X_{A_n})} = \varnothing$. □

Remark 3. Proposition 10 is an example, showing that one of the conditions in Theorem 7 is best possible. Also, there is surely no need to assume K scattered in the statement of the result. I thank the referee for this remark.

11. Proposition ([H-O-R])**.** *Let K be a compact metric space, $m \in \mathbb{N}$, $\delta > 0$ and a function $f : K \to \mathbb{R}$ is such that $K^m_{\varepsilon,f} \neq \varnothing$. Then $|f|_D \geq m\delta/4$.*

12. Proposition. *Let K be a compact metric space, $f \in \mathbb{B}_1(K), \xi < \omega, (f_n) \subset$ DBSC(K) pointwise converging to f, $\gamma((f_n)) \leq \omega^\xi$ and $\sup_n |f_n|_D < \infty$.*

Then $\beta(f) \leq \omega^\xi$.

Proof. Let $\varepsilon > 0$.

Claim 1. $\exists n_0 \in \mathbb{N} : \beta(f_n, \varepsilon/3) = \beta(f_{n_0}, \varepsilon/3)$ $\forall n \geq n_0$.

[Proof of Claim 1. Let then $\beta(f_n, \varepsilon/3) = m_n + 1$, where m_n, $n \in \mathbb{N}$. Then $K^{m_n}_{\varepsilon,f_n} \neq \varnothing$ and hence by Proposition 10 we have that $|f_n|_D \geq m_n\varepsilon/12$. If the sequence (m_n) is infinite then $\sup_n |f_n|_D = \infty$, a contradiction.

Thus there is $n_0 \in \mathbb{N}$ such that $m_n = m_{n_0}$ for all $n \geq n_0$.]

Claim 2. If $m = \beta(f_{n_0}, \varepsilon/3)$ then $P^m_{\varepsilon,f} \subset P'_{\varepsilon/3,(f_n)}$ for each closed subset P of K.

[Proof of Claim 2. Let $x \in P^m_{\varepsilon,f} \setminus P'_{\varepsilon/3,(f_n)}$. Then there are an open neighborhood V of x in P and $n_0 \in \mathbb{N}$ such that

$$|f_m(y) - f_n(y)| \leq \varepsilon/3 \quad \forall n, m \geq n_0, y \in \overline{V}.$$

Then, $|f_{n_0}(y) - f(y)| \leq \varepsilon/3$ for all $y \in \overline{V}$ and hence $\overline{V}'_{\varepsilon,f} \subset \overline{V}'_{\varepsilon/3,f_{n_0}}$.

Finally, by induction we get $\overline{V}^m_{\varepsilon,f} \subset \overline{V}^m_{\varepsilon/3,f_{n_0}} = \varnothing$. Since $V \cap P^m_{\varepsilon,f} \subset \overline{V}^m_{\varepsilon,f}$ we have $V \cap P^m_{\varepsilon,f} = \varnothing$, a contradiction since $x \in V \cap P^m_{\varepsilon,f}$.]

Since $\gamma((f_n)) \leq \omega^\xi$ we have that $\gamma((f_n), \varepsilon/3) < \omega^\xi$ and hence there is $k < \omega$ such that $\gamma((f_n), \varepsilon/3) < k\omega^{\xi-1}$. Applying Claim 1 we have

$$K^m_{\varepsilon,f} \subset K'_{\varepsilon/3,(f_n)}, K^{2m}_{\varepsilon,f} \subset K''_{\varepsilon/3,(f_n)}, \ldots, K^{mk\omega^{\xi-1}}_{\varepsilon,f} \subset K^{k\omega^{\xi-1}}_{\varepsilon/3,(f_n)} = \varnothing.$$

Then $\beta(f, \varepsilon) \leq km\omega^{\xi-1} < \omega^\xi$. Hence it is proved that $\beta(f) \leq \omega^\xi$. □

13. Definition ([H-O-R])**.** Define $\mathbb{B}_{1/4}(K)$ to be the set of those f in $\mathbb{B}_1(K)$ for which there is a sequence (f_n) in DBSC(K) that converges uniformly to f and is such that $\sup_n |f_n|_D < \infty$.

14. Theorem ([H-O-R], Th. 6.1)**.** *Let K be a compact metric space and let $f \in \mathbb{B}_1(K)$. Then $f \in \mathbb{B}_{1/4}(K)$ iff there exists a $C < \infty$ such that for all $\varepsilon > 0$ there exists a sequence $(s_n)_{n=0}^{\infty} \subset C(K)$, $s_0 = 0$, with (s_n) converging pointwise to f and such that for all subsequences (n_i) of $\{0\} \cup \mathbb{N}$ and $x \in K$,*

$$\sum_{j \in B((n_i),x)} |s_{n_{j+1}}(x) - s_{n_j}(x)| \leq C,$$

where $B((n_i),x) = \{j : |s_{n_{j+1}}(x) - s_{n_j}(x)| \geq \varepsilon\}$.

The above result gave the idea for the definition of the rank δ (cf. [K-N]). I am grateful to Professor Negrepontis who gave me this idea.

15. Definition. Let K be a compact metric space, $f, s_n : K \to \mathbb{R}$, $n \in \mathbb{N}$, real-valued functions with $s_0 = 0$ such that (s_n) is pointwise converging to f. For each closed subset P of K and $\varepsilon > 0$ we set:

$$P^0((s_n),\varepsilon) = P,$$

$$\begin{aligned} P'((s_n),\varepsilon) = \Big\{ & x \in P : \forall 0 < C < \infty,\ \forall m \in \mathbb{N},\ \forall U \subset K \text{ open neighborhood of } x \\ & \exists j_p > \cdots > j_1 \geq m \text{ and } x' \in U \cap P \text{ such that} \\ & |s_{j_{i+1}}(x') - s_{j_i}(x')| > \varepsilon \text{ for } i = 1, 2, \ldots, p \\ & \text{and } \sum_{i=1}^{p} |s_{j_{i+1}}(x') - s_{j_i}(x')| > C \Big\}. \end{aligned}$$

For each ordinal $a < \omega_1$ we set

$$P^{a+1}((s_n),\varepsilon) = (P^a((s_n),\varepsilon))'((s_n),\varepsilon).$$

If β is a limit ordinal, we set

$$P^{\beta}((s_n),\varepsilon) = \bigcap_{a<\beta} P^a((s_n),\varepsilon).$$

We set

$$\delta((s_n),\varepsilon) = \begin{cases} \text{the least ordinal } a < \omega_1 \text{ such that } K^a((s_n),\varepsilon) = \varnothing \\ \qquad \text{if such an } a \text{ exists,} \\ \omega_1, \quad \text{otherwise.} \end{cases}$$

and

$$\delta((s_n)) = \sup\{\delta((s_n),\varepsilon) : \varepsilon > 0\}.$$

Remark 4. $\delta((s_n)) \leq \gamma((s_n))$.

We see this as follows: Let P be a closed subset of K, $\varepsilon > 0$ and $x \in P \setminus P^1_{\varepsilon,(s_n)}$. Then there are an open neighborhood of x in P and $p \in \mathbb{N}$ such that for every $y \in U$ and $m, n \in \mathbb{N}$ with $m, n \geq p$ we have $|f_m(y) - f_n(y)| \leq \varepsilon$. By definition of $P'((s_n),\varepsilon)$ we have that $x \notin P'((s_n),\varepsilon)$. Hence $P'((s_n),\varepsilon) \subset P^1_{\varepsilon,(s_n)}$.

16. Proposition ([H-O-R])**.** *Let X be a Banach space and C, D be convex subsets of X. Then*

$$\inf\{\|c-d\| : c \in C, d \in D\} = \inf\{\|c-d\| : c \in \widetilde{C}, d \in \widetilde{D}\},$$

where $\widetilde{C}$ and $\widetilde{D}$ are the w^*-closure of C and D in X^{**}.

17. Theorem. *Let K be a compact metric space, $f \in \mathbb{B}_1(K)$, a sequence $(f_n) \subset C(K)$ pointwise converging to f and $\xi < \omega_1$.*

Then the following equivalences are satisfied:

(i) *If $1 \le \xi < \omega$, then $\beta(f) \le \omega^\xi$ if and only if there exists a sequence (s_n) of convex blocks of (f_n) with $\delta((s_n)) \le \omega^{\xi-1}$.*

(ii) *If $\xi \ge \omega$, then $\beta(f) \le \omega^\xi$ if and only if there exists a sequence (s_n) of convex blocks of (f_n) with $\delta((s_n)) \le \omega^\xi$.*

Proof. (i). *Necessity.* Let $1 \le \xi < \omega$ and $\beta(f) \le \omega^\xi$. Then by Theorem 7 we have there is a sequence $(F_n) \subset \mathrm{DBSC}(K)$ pointwise converging to f and $\gamma((F_n)) \le \omega^\xi$. Let $\varepsilon > 0$. Then $\gamma((F_n), \frac{\varepsilon}{4}) = \theta + 1 < \omega^\xi$.

For every $\eta \le \theta$ we set $K_\eta = K^\eta_{\varepsilon/4,(F_n)}$. Then for every $\eta \le \theta$ and $x \in K_\eta \setminus K_{\eta+1}$ there are an open neighborhood $U_{x,\eta}$ of x in K_η and $n \in \mathbb{N}$ such that $|F_n(y) - f(y)| \le \varepsilon/4$ for every $y \in \overline{U}_{x,\eta}$. Since K is a compact metric space we have that for every $\eta \le \theta$ there exists a countable subset $\{U_{\eta,k} : k \in \mathbb{N}\}$ of $\{U_{x,\eta} : x \in K_\eta \setminus K_{\eta+1}\}$ such that

$$\bigcup\{U_{\eta,k} : k \in \mathbb{N}\} = \bigcup\{U_{x,\eta} : x \in K_\eta \setminus K_{\eta+1}\}.$$

Let $\{U_{\eta_i,k_i} : i \in \mathbb{N}\}$ be an enumeration of $\{U_{\eta,k} : \eta \le \theta, k \in \mathbb{N}\}$. Then for every $i \in \mathbb{N}$ $\exists n_1 \in \mathbb{N}$ such that

$$(*) \qquad \|f - F_{n_i}\|_{\overline{U}_{\eta_i,k_i}} < \varepsilon/4.$$

(If M is a subspace of K we set $\|\ \|_M$ the supremum norm on $C(M)$.)

For every $i \in \mathbb{N}$ let $(f^i_m)_{m=0}^\infty \subset C(K)$ with $f^i_0 = 0$, $(f^i_m)_{m=0}^\infty$ is pointwise converging to F_{n_i} and

$$\sum_{m=0}^\infty |f^i_m(y) - f^i_m(y)| \le |F_{n_i}|_D \quad \forall y \in \overline{U}_{\eta_i,k_i}.$$

By $(*)$ and Proposition 16 we have that there exist a sequence (g^1_m) of convex blocks of (f_m) and a sequence (h^1_m) of convex blocks of (f^1_m) such that

$$(**) \qquad \|g^1_m - h^1_m\|_{\overline{U}_{\eta_1,k_1}} < \varepsilon/4 \quad \forall m \in \mathbb{N}.$$

Then for every $m_1, \ldots, m_p \in \mathbb{N}$ with $m_1 < \cdots < m_\rho$ and $y \in \overline{U}_{\eta_1,k_1}$ with $|g^1_{m_{i+1}}(y) - g^1_{m_i}(y)| \ge \varepsilon$ for all $i = 1, \ldots, \rho$ we have

$$(***) \quad \sum_{i=1}^\rho |g^1_{m_{i+1}}(y) - g^1_{m_i}(y)| \le \sum_{j=0}^\infty |f^1_{j+1}(y) - f^1_j(y)| + \frac{\varepsilon}{2}\frac{2}{\varepsilon}|F_{n_1}|_D \le 2|F_{n_1}|_D.$$

[We see this as follows: Let $p, q \in \mathbb{N}$ and $y \in \overline{U}_{\eta_1,k_1}$ with $|g^1_p(y) - g^1_q(y)| \ge \varepsilon$.

Then by $(**)$ we have

$$(1) \qquad \varepsilon \le |g^1_p(y) - g^1_q(y)| \le \frac{\varepsilon}{2} + |h^1_p(y) - h^1_q(y)| \Rightarrow |h^1_p(y) - h^1_q(y)| \ge \frac{\varepsilon}{2}.$$

Also, $\sum_{i=1}^\rho |g^1_{m_{i+1}}(y) - g^1_{m_i}(y)| \le \sum_{j=0}^\infty |f^1_{j+1}(y) - f^1_j(y)| + \frac{\varepsilon}{2}\rho \le |F_{n_1}|_D + \frac{\varepsilon}{2}\rho$. By (1) we have

$$\rho\frac{\varepsilon}{2} \le \sum_{i=1}^\rho |h^1_{m_{i+1}}(y) - h^1_{m_i}(y)| \le |F_{n_1}|_D \Rightarrow \rho \le \frac{2}{\varepsilon}|F_{n_1}|_D.$$

Hence the proof of $(***)$ is finished.]

By induction, for every $i \in \mathbb{N}$ we get a sequence (g_m^{i+1}) of convex blocks of (g_m^i) such that $\forall \rho \in \mathbb{N}$, $m_1, \dots, m_\rho \in \mathbb{N}$ with $m_1 < \cdots < m_\rho$ and $y \in \overline{U}_{\eta_i, k_i}$ with $|g_{m_{j+1}}^{i+1}(y) - g_{m_j}^{i+1}(y)| \geq \varepsilon$ for all $j = 1, \dots, \rho$ we have

$$\sum_{j=1}^{\rho} |g_{m_{j+1}}^{i+1}(y) - g_{m_j}^{i+1}(y)| \leq 2|F_{n_{i+1}}|_D.$$

We set $s_0 = 0$ and $s_n = g_n^n$ for all $n \in \mathbb{N}$. Then (s_n) is pointwise converging to f and $K^\eta((s_n), \varepsilon) \subset K_\eta$ for all $\eta \leq \theta + 1$. Hence $K^{\theta+1}((s_n), \varepsilon) = \varnothing$ and hence $\delta((s_n)) \leq \omega^{\xi - 1}$.

Sufficiency. Let $\delta((s_n)) \leq \omega^{\xi-1}$. We shall show that $\gamma((s_n)) \leq \omega^\xi$ and since $\beta(f) \leq \gamma((s_n))$ we have that $\beta(f) \leq \omega^\xi$. Hence we shall show that $P^\omega_{\varepsilon,(s_n)} \subset P'((s_n), \varepsilon/2)$ for all closed subsets P of K.

Let P be a closed subset of K and let $x \in P^\omega_{\varepsilon,(s_n)} \setminus P'((s_n), \varepsilon/2)$. Then there are a positive real number C, an open neighborhood U of x in P and $m \in \mathbb{N}$ such that $\forall p \in \mathbb{N}$, $n_1, \dots, n_p \in \mathbb{N}$ with $n_p > \cdots > n_1 \geq m$ and $y \in U$ with $|s_{n_{i+1}}(y) - s_{n_i}(y)| \geq \varepsilon/2$ for all $i = 1, \dots, p$ we have $\sum_{i=1}^p |s_{n_{i+1}}(y) - s_{n_i}(y)| \leq C$.

Then $p < \frac{C}{\varepsilon}$. Let $n \in \mathbb{N}$ with $n > \frac{C}{\varepsilon}$. Then $x \in P^n_{\varepsilon,(s_n)}$. We shall show that there are $y \in U$ and $m_1, \dots, m_{n+1} \in \mathbb{N}$ with $m_{n+1} > \cdots > m_1 \geq m$ such that $|s_{m_{j+1}}(y) - s_{m_j}(y)| > \varepsilon/2$ for all $j = 1, \dots, n$, and we shall terminate in a contradiction.

We see this as follows:

$$x \in P^n_{\varepsilon,(s_n)} \cap U \Rightarrow \exists x_1 \in P^{n-1}_{\varepsilon,(s_n)} \cap U \text{ and } m_1, m_2 \in \mathbb{N} \text{ with } m_2 > m_1 \geq m \text{ and}$$
$$|s_{m_2}(x_1) - s_{m_1}(x_1)| > \varepsilon > \varepsilon/2.$$

We set $V_1 = \{y \in U : |s_{m_2}(y) - s_{m_1}(y)| > \varepsilon/2\}$. V_1 is open and $x_1 \in V_1 \cap P^{n-1}_{\varepsilon,(s_n)}$; hence $\exists x_2 \in P^{n-2}_{\varepsilon,(s_n)} \cap V_1$ and $m_3 \in \mathbb{N}$ such that $m_3 > m_2$ and $|s_{m_3}(x_2) - s_{m_2}(x_2)| > \varepsilon/2$ (since if $|s_m(y) - s_{m_2}(y)| \leq \varepsilon/2$ for every $m \geq m_2$ and $y \in P^{n-2}_{\varepsilon,(s_n)} \cap V_1$, then $|s_m(y) - s_k(y)| \leq \varepsilon$ for all $m, k \geq m_2$ and $y \in P^{n-2}_{\varepsilon,(s_n)} \cap V_1$, that is, $x_1 \notin P^{n-1}_{\varepsilon,(s_n)}$ which is a contradiction).

We set $V_2 = \{y \in V_1 : |s_{m_3}(y) - s_{m_2}(y)| > \varepsilon/2\}$. V_2 is open in P and $x_2 \in V_2 \subset V_1$.

By induction we get $m_1, \dots, m_n \in \mathbb{N}$ with $m_n > \cdots > m_1 \geq m$, $V_1, \dots, V_{n-1}$ open subsets of P with $V_{n-1} \subset \cdots \subset V_1 \subset U$ and $x_1 \in P^{n-1}_{\varepsilon,(s_n)} \cap V_{i-1}$ for all $i = 1, \dots, n$ (where $V_0 = U$) such that $|s_{m_{i+1}}(y) - s_{m_i}(y)| > \varepsilon/2$ for all $y \in V_i$, $i = 1, \dots, n-1$. We set $V_n = \{y \in V_{n-1} : |s_{m_n}(y) - s_{m_{n-1}}(y)| > \varepsilon/2\}$. V_n is open in P and $x_{n-1} \in P'_{\varepsilon,(s_n)} \cap V_n$; hence there is $y \in V_n$ and $m_{n+1} > m_n$ such that $|s_{m_{n+1}}(y) - s_{m_n}(y)| > \varepsilon/2$.

Then $|s_{m_{j+1}}(y) - s_{m_j}(y)| > \varepsilon/2$ for all $j = 1, \dots, n$. Hence the proof of (i) is finished.

(ii) *Necessity.* By Theorem 5 we have that if $f \in \mathbb{B}_1(K)$ with $\beta(f) \leq \omega^\xi$ then there is a sequence (s_n) of convex blocks of (f_n) with $\gamma((s_n)) \leq \omega^\xi$.

Then by Remark 4 we get a conclusion.

Sufficiency. As in (i) we prove that $P^\omega_{\varepsilon,(s_n)} \subset P'((s_n), \varepsilon/2)$ for all closed subsets P of K and $\varepsilon > 0$. Then by induction we get $K^{\omega^{n+1}}_{\varepsilon,(s_n)} \subset K^{\omega^n}((s_n), \varepsilon/2)$ for all

$n \in \mathbb{N}$ and hence $K^{\omega^\omega}_{\varepsilon,(s_n)} \subset K^{\omega^\omega}((s_n), \varepsilon/2)$. Finally, by induction we get $K^{\omega^\xi}_{\varepsilon,(s_n)} \subset K^{\omega^\xi}((s_n), \varepsilon/2)$ for all $\varepsilon > 0$. □

Remark 5. If (s_n) is a sequence of continuous real-valued functions on K with $\delta((s_n)) < \omega_1$, then (s_n) converges pointwise.

[We see this is follows: As is proved in the demonstration of the sufficiency of Theorem 17 (i) we have that $P^{\omega}_{\varepsilon,(s_n)} \subset P'((s_n), \varepsilon/2)$ for all closed subsets P of K and hence

$$K^{\omega^{\xi+1}}_{\varepsilon,(s_n)} \subset K^{\omega^\xi}((s_n), \varepsilon/2) \quad \text{for all } \xi < \omega_1.$$

Assume that $\delta((s_n)) < \omega_1$. Then there is a $\xi < \omega_1$ such that $\delta((s_n)) < \omega^\xi$; hence $K^{\omega^\xi}((s_n), \varepsilon) = \varnothing$ for all $\varepsilon > 0$ and thus $K^{\omega^{\xi+1}}_{\varepsilon,(s_n)} = \varnothing$ for all $\varepsilon > 0$. Then $\gamma((s_n), \varepsilon) < \omega^{\xi+1}$ for all $\varepsilon > 0$; hence $\gamma((s_n)) \leq \omega^{\xi+1} < \omega_1$ and thus the sequence (s_n) converges pointwise (cf. [K-L]).]

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