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OPTIMAL INDIVIDUAL STABILITY ESTIMATES FOR C_0-SEMIGROUPS IN BANACH SPACES

VOLKER WROBEL

Abstract. In a previous paper we proved that the asymptotic behavior of a C_0-semigroup is completely determined by growth properties of the resolvent of its generator and geometric properties of the underlying Banach space as described by its Fourier type. The given estimates turned out to be optimal. The method of proof uses complex interpolation theory and reflects the full semigroup structure. In the present paper we show that these uniform estimates have to be replaced by weaker ones, if individual initial value problems and local resolvents are considered because the full semigroup structure is lacking. In a different approach this problem has also been studied by Huang and van Neerven, and a part of our straightforward estimates can be inferred from their results. We mainly stress upon the surprising fact that these estimates turn out to be optimal. Therefore it is not possible to obtain the optimal uniform estimates mentioned above from individual ones. Concerning Hardy-abscissas, individual orbits and their local resolvents behave as badly as general vector valued functions and their Laplace-transforms. This is in strict contrast to the uniform situation of a C_0-semigroup itself and the resolvent of its generator where a simple dichotomy holds true.

1. Introduction

Given a generator $A : X \supseteq D(A) \to X$ of a C_0-semigroup U_A on a complex Banach space X, let $\sigma(A;X)$ and $\rho(A;X)$ denote the spectrum and the resolvent set of A, respectively, and let $R(.,A) : \rho(A;X) \to L(X)\,,\ z \mapsto (z-A)^{-1}$ denote the resolvent function. Given $x \in X$ we shall be concerned with analytic extensions, so-called local resolvents, $\tilde{R}(.,A)x$ of $R(.,A)x$. For fixed $\mu \in \rho(A;X)$ one has

$$R(\mu,A)\,\tilde{R}(.,A)x = \tilde{R}(.,A)\,R(\mu,A)x$$

by the resolvent equation and the uniqueness of analytic extensions.

For $1 \le p < \infty$ and $\omega \in \mathbb{R}$ let

$$H^p(\omega;X) := \left\{ f : \{z \in \mathbb{C} : Re\, z > \omega\} \to X \,\middle|\, f \text{ analytic, and } \sup_{\beta>\omega} \int_{\mathbb{R}} \|\, f(\beta+is)\,\|^p \; ds < \infty \right\}$$

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and

$$H^\infty(\omega;X)=\left\{f:\{z\in\mathbb{C}:Re\,z>\omega\}\to X\,\middle|\, f \text{ analytic, and } \sup_{Re\,z>\omega}\| f(z)\|<\infty\right\}.$$

For $x\in X$ and $1\le p\le\infty$, let

$$s^p(x):=\inf\{\omega\in\mathbb{R}:\tilde{R}(.,A)x\in H^p(\omega;X)\}\ . \tag{1.1}$$

Moreover, let

$$s^p(A):=\sup\{s^p(x):x\in X\}\ . \tag*{$(1.1)_U$}$$

By a Baire argument, one can show that $s^p(x)=s^p(A)$ for x in a residual subset of X. On the other hand [HP, Thm 6.4.2] says that

$$H^p(\omega;X)\subseteq H^\infty(\omega+\varepsilon;X)\ (\varepsilon>0)\ ,$$

and since $L_p(\mathbb{R};X)\cap L_\infty(\mathbb{R};X)\subseteq L_q(\mathbb{R};X)$ for $q\ge p$, this yields

$$s^q(x)\le s^p(x)\ (1\le p\le q\le\infty\,,\,x\in X)\ , \tag{1.2}$$

and thus

$$s^q(A)\le s^p(A)\ (1\le p\le q\le\infty)\ . \tag*{$(1.2)_U$}$$

But if $s^p(A)<\infty$, then the resolvent equation and $(1.2)_U$ imply $s^\infty(A)=s^p(A)$ (cf. [vNSW, Prop. 3.1]). In the last section, we shall present an example showing that this dichotomy $s^p(A)=\infty$ or $s^p(A)<\infty$ and $s^\infty(A)=s^p(A)$ is a typical uniform result. Indeed, for each $1<p\le 2$ we present an example of a Banach space of Fourier type p and a C_0-semigroup U_A such that for given $0<r<1$ and all $n\in\mathbb{N}$ there exists $x\in X$ such that

$$s^\infty(x)=r^n\le r^{n-\frac{1}{q}}=s^q(x)<r^{n-\frac{1}{p}}=s^p(x) \tag*{$(1.2)_S$}$$

for $\frac{p}{p-1}<q\le\infty$.

This may be interesting for this constitutes a very simple, natural function that has exactly those prescribed Hardy-abscissas. Applying the Hahn-Banach theorem and Baire's theorem one obtains $\mathbb{C}$-valued functions with these properties.

Aside from these Hardy-abscissas we study the following:

For $x\in X$ let

$$\omega(x):=\inf\{\omega\in\mathbb{R}:\sup_{t\ge 0}\| e^{-\omega t}U_A(t)x\|<\infty\}\ , \tag{1.3}$$

and

$$\omega_\alpha(A):=\sup\{\omega(x):x\in D((\mu-A)^\alpha)\}\ , \tag*{$(1.3)_U$}$$

where $\mu>\omega_0(A)$, and $(\mu-A)^\alpha$ denotes the fractional power of order α in the sense of Komatsu [Ko]. Finally, given $x\in X$ and $\beta\ge 0$, let

$$s_\beta(x):=\inf\{\omega\in\mathbb{R}:\ \|\tilde{R}(a+ib,A)x\|\ =O(|b|^\beta),\ \text{as } |b|\to\infty,\ \text{and } a\ge\omega\} \tag{1.4}$$

and

$$s_\beta(A)=\sup\{s_\beta(x):x\in X\}\ . \tag*{$(1.4)_U$}$$

Observe that $s_0(x)=s^\infty(x)$.

Recall that a Banach space X has Fourier type $1 \le p \le 2$, provided the vector valued Fourier transform defined on $\mathcal{S}(\mathbb{R}; X)$ has a continuous extension

$$\mathcal{F} : L_p(\mathbb{R}; X) \longrightarrow L_q(\mathbb{R}; X) \; (\frac{1}{p} + \frac{1}{q} = 1) .$$

In [WW] the following optimal estimates for $\omega_\alpha(A)$ in terms of $s_\beta(A)$ have been shown.

Theorem 1.1. *Suppose $A : X \supseteq D(A) \to X$ generates a C_0–semigroup U_A on a complex Banach space X. If X has Fourier type $1 \le p \le 2$, then with $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\omega_{\alpha + \frac{1}{p} - \frac{1}{q}}(A) \le s_{Re\,\alpha}(A) \; (0 < Re\,\alpha, \; or \; \alpha = 0) \tag*{$(1.5)_U$}$$

and these estimates are best possible.

Consequently,

$$\omega_{\alpha+1}(A) \le s_{Re\,\alpha}(A) \; (0 < Re\,\alpha, \; or \; \alpha = 0) . \tag*{$(1.6)_U$}$$

The method of proof is complex interpolation that makes use of the full semigroup structure, and so [WW] gives no indication whether individual analogues of $(1.5)_U$ and/or $(1.6)_U$ hold true. Meanwhile van Neerven [vN] proved the following individual result that implies $(1.6)_U$:

$$\omega(R(\mu, A)x) \le s_0(x) \; (\mu \in \rho(A; X) \, , \, x \in X). \tag{1.6}$$

Keeping this in mind it seems to be a natural question whether the following individual version of $(1.5)_U$ holds true

$$\omega(R(\mu, A)^{\frac{1}{p} - \frac{1}{q}} x) \le s_0(x) \; (\mu \in \rho(A; X) \, , \, x \in X) \, , \tag{1.5}$$

provided X has Fourier type p.

By a Baire argument one can prove that

$$\omega_{\frac{1}{p} - \frac{1}{q}}(A) = \omega(R(\mu, A)^{\frac{1}{p} - \frac{1}{q}} x) \, , \text{ and } s_0(A) = s_0(x)$$

for x in a residual subset of X, and thus (1.5) holds true for these x.

On the other hand, if X has Fourier type $1 \le p \le 2$, then we shall prove that

$$\omega(x) \le s^p(x) \; (x \in X) \tag{1.7}$$

and

$$\omega(R(\mu, A)^{\frac{1}{p} + \varepsilon} x) \le s^p(R(\mu, A)^{\frac{1}{p} + \varepsilon} x) \le s_0(x) \, (x \in X \, , \, \varepsilon > 0) \, . \tag{1.8}$$

Surprisingly these straightforward estimates turn out to be optimal, which especially means that Theorem 1.1 $(1.5)_U$ cannot be inferred from individual estimates. This is the contents of

Theorem 1.2. *Suppose $A : X \supseteq D(A) \to X$ generates a C_0-semigroup U_A on a complex Banach space X. If X has Fourier type $1 \le p \le 2$, then*

(1) (*van Neerven* [vN])

$$\omega(R(\mu, A)\, x) \le s_0(x) \; (x \in X \, , \, \mu \in \rho(A; X)) \, ;$$

(2) *For all $\varepsilon > 0$ and $1 < p \le 2$ we have*

$$\omega(R(\mu, A)^{\frac{1}{p} + \varepsilon} x) \le s^p(R(\mu, A)^{\frac{1}{p} + \varepsilon} x) \le s_0(x) \; (x \in X) \, , \tag{1.8}$$

and $\frac{1}{p}$ is optimal for (1.8) to be true.

We mention that the inequality

$$\omega(R(\mu, A)^{\frac{1}{p}+\varepsilon}) \leq s_0(x) \ (x \in X\,,\, \varepsilon > 0)$$

is Theorem 0.1 of [HvN].

The failure of (1.5) is of course due to the fact that we cannot exploit the full semigroup structure as done in [WW]. Perhaps the following estimate is known to experts, but since we did not find any reference for it, we state and prove this useful fact here. One should compare it with inequality (1.8).

Proposition 1.3. *Suppose $A : X \supseteq D(A) \to X$ generates a C_0-semigroup. Then*

$$s_\beta(x) \leq \omega(R(\mu, A)^\beta x) \ (x \in X\,,\, \beta \geq 0) \tag{1.9}$$

and consequently,

$$s_\beta(A) \leq \omega_\beta(A) \ (\beta \geq 0)\,. \tag*{$(1.9)_U$}$$

Proof. Fix $\delta > 0\,,\, \mu > \omega_0(A) + 2\delta$. For $\beta = 0$, inequality (1.9) follows from the properties of the Laplace-transform. So let $\beta > 0$. Then the analytic functional calculus gives

$$R(\mu, A)^{\beta+1} = \frac{1}{2\pi} \int_{\mathbb{R}} (\mu - \omega_o(A) - \delta - it)^{-\beta-1}\, R(\omega_0(A) + \delta + it, A)dt\,.$$

For $\xi < \omega_0(A) + \delta$ the resolvent equation gives

$$\begin{aligned}
&\tilde{R}(\xi + i\eta(A)\, R(\mu, A)^{\beta+1}x \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{R(\omega_0(A) + \delta + it, A)\, x\, dt}{(\mu - \omega_0(A) - \delta - it)^{\beta+1}\, (\xi + i\eta - \omega_0(A) - \delta - it)} \\
&+ \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\tilde{R}(\xi + i\eta, A)\, x\, dt}{(\mu - \omega_0(A) - \delta - it)^{\beta+1}\, (it - (\xi + i\eta - \omega_0(A) - \delta)}.
\end{aligned}$$

Since $(\mu - A)\, \tilde{R}\, (\lambda, A)x = (\mu - \lambda)\, \tilde{R}\, (\lambda, A)\, x - x$ and since the integrand remain Bochner-integrable when $(\mu - A)$ is applied, we obtain by Cauchy's theorem

$$\begin{aligned}
&\tilde{R}(\xi + i\eta, A)\, R\, (\mu, A)^\beta\, x \\
&\quad = (\mu - A)\, \tilde{R}\, (\xi + i\eta, A)\, R\, (\mu, A)^{\beta+1}x \\
&\quad = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{R(\omega_0(A) + \delta + it, A)\, x\, dt}{(\mu - \omega_0(A) - \delta - it)^{\beta}\, (\xi + i\eta - \omega_o(A) - \delta - it)} \\
&\qquad + \frac{1}{2\pi} \int_{\mathbb{R}} \frac{(\mu - (\xi + i\eta))\, \tilde{R}\, (\xi + i\eta, A)\, x\, dt}{(\mu - \omega_0(A) - \delta - it)^{\beta+1}\, (it - (\xi + i\eta - \omega_0(A) - \delta))}\,.
\end{aligned}$$

The first integral is uniformly bounded with respect to η for each ξ by a standard estimate using Young's inequality, whereas the second integral yields $(\mu - (\xi + i\eta))^{-\beta}\, \tilde{R}\, (\xi + i\eta, A)x\,.$

As $\eta \mapsto \tilde{R}(\xi{+}i\eta, A)\, R\, (\mu, A)^\beta x$ is bounded for $\xi > \omega(R(\mu, A)^\beta x)$ by the Riemann-Lebesgue Lemma, we obtain that $\eta \mapsto (\mu - (\xi{+}\ i\eta))^{-\beta}\, \tilde{R}\, (\xi + i\eta, A)x$ is bounded for $\xi > \omega(R(\mu, A)^\beta x)$, too, and thus $s_\beta(x) \leq \omega(R(\mu, A)^\beta x)$.

2. The proof of Theorem 1.2

We start with a simple consequence of Lusin's theorem.

Lemma 2.1. *If $f, g : \mathbb{R} \to X$ are Lebesgue measurable functions such that*

$$N_\varphi := \{t \in \mathbb{R} : \varphi \circ g(t) \neq \varphi \circ f(t)\}$$

is a Lebesgue null set for every $\varphi \in X'$, then

$$A := \{t \in \mathbb{R} : g(t) \neq f(t)\}$$

is a Lebesgue null set.

Proof. Given $\delta > 0$ there exists a Lebesgue measurable subset $M_\delta \subseteq \mathbb{R}$ such that $\lambda(M_\delta) < \delta$ and $f - g\big|_{\mathbb{R}\setminus M_\delta}$ is continuous by Lusin's theorem.

Hahn-Banach theorem and the continuity of $f - g\big|_{\mathbb{R}\setminus M_\delta}$ yield:

$$\forall\ t_0 \in A\ \exists\ U(t_0) \text{ open } \exists\ \varphi \in X'\ \forall\ t \in U(t_0) \cap (\mathbb{R}\setminus M_\delta) : \varphi(f(t) - g(t)) \neq 0\,.$$

Consequently,

$$\forall\ t_0 \in A\ \exists\ U(t_0) \text{ open } \exists\ \varphi \in X' : U(t_0) \cap (\mathbb{R}\setminus M_\delta) \subseteq N_\varphi\,.$$

Since A contains a countable dense subset, $A \cap (\mathbb{R}\setminus M_\delta)$ is a Lebesgue null set, whereas $\lambda(A \cap M_\delta) < \delta$. Consequently, $\lambda(A) < \delta$ for every $\delta > 0$.

The next result is mathematical folklore, too.

Lemma 2.2. *Suppose $A : X \supseteq D(A) \to X$ generates a C_0-semigroup U_A. Let $\xi_0 \in \mathbb{R}$, $x \in X$, $\varphi \in X'$ and assume that $R(.,A)x$ has an analytic extension $\tilde{R}(.,A)x$ upon the half-plane $Re(z) > \xi_0$ such that*

$$\langle \varphi, \tilde{R}(.,A)x\rangle \in H^2(\xi_0;\mathbb{C})\,. \tag{2.1}$$

Then

$$t \mapsto \langle \varphi, U_{A-\xi_0}(t)x\rangle \in L_2([0,\infty[;\mathbb{C})$$

and

$$\langle \varphi, \tilde{R}(\xi + i\eta, A)x\rangle = \int_0^\infty e^{-(\xi+i\eta)t}\langle \varphi, U_A(t)x\rangle dt$$

for $\xi > \xi_0$, $\eta \in \mathbb{R}$.

Proof. Since $R(z, A-\xi_0) = R(z+\xi_0, A)$, $R(., A-\xi_0)x$ has an analytic extension $\tilde{R}(., A-\xi_0)x$ upon the half-plane $Re\, z > 0$ and $\langle \varphi, \tilde{R}(., A-\xi_0)x\rangle \in H^2(0;\mathbb{C})$.

By Paley-Wiener's theorem there exists a unique $f_\varphi \in L_2([0,\infty[;\mathbb{C})$ such that

$$\langle \varphi, \tilde{R}(z, A-\xi_0)x\rangle = \int_0^\infty e^{-zt}\, f_\varphi(t)dt$$

for $Re(z) > 0$.

On the other hand, if $Re(z) > \omega_0(A-\xi_0) = \omega_0(A) - \xi_0$, then

$$\langle \varphi, R(z, A-\xi_0)x\rangle = \int_0^\infty e^{-zt}\, \langle \varphi, U_{A-\xi_0}(t)x\rangle dt\,.$$

By the uniqueness property of the Laplace transformation, there exists a Lebesgue null set N_φ such that

$$f_\varphi(t) = \langle \varphi, U_{A-\xi_0}(t)x\rangle\ (t \in \mathbb{R}\setminus N_\varphi)\,.$$

Finally, if $\xi > \xi_0$, and $\eta \in \mathbb{R}$, then

$$\begin{aligned} \langle \varphi, \tilde{R}(\xi + i\eta, A)x \rangle &= \langle \varphi, \tilde{R}(\xi - \xi_0 + i\eta, A - \xi_0)x \rangle \\ &= \int_0^\infty e^{-(\xi - \xi_0 + i\eta)t} \langle \varphi, U_{A-\xi_0}(t)x \rangle dt \\ &= \int_0^\infty e^{-(\xi + i\eta)t} \langle \varphi, U_A(t)x \rangle \, dt \,. \end{aligned}$$

If X has Fourier type $1 \le p \le 2$, then the "inverse" Fourier transform $f \mapsto (\eta \mapsto \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\eta t} f(t) dt)$ admits a continuous extension $\mathcal{F} : L_p(\mathbb{R}; X) \to L_q(\mathbb{R}; X)$ ($\frac{1}{p} + \frac{1}{q} = 1$), as well.

Corollary 2.3. *Suppose X has Fourier type $1 \le p \le 2$. Let $\xi_0 \in \mathbb{R}$, $x \in X$ and suppose that $R(.,A)x$ has an analytic extension $\tilde{R}(.,A)x$ upon the half-plane $Re\, z > \xi_0$ and that $\eta \mapsto \tilde{R}(\xi + i\eta, A)x \in L_p(\mathbb{R}; X)$ for $\xi > \xi_0$. If $\langle \varphi\, ,\, \tilde{R}(.,A)x \rangle \in H^2(\xi_0, \mathbb{C})$ for all $\varphi \in X'$, then the inverse Fourier-transform of $\eta \mapsto \tilde{R}(\xi + i\eta, A)x$ equals $\frac{1}{2\pi} U_{A-\xi}(.)x \in L_q([0, \infty[\,; X)$, and consequently $\omega(x) \le \xi$.*

Proof. Since X has Fourier type p, let $f \in L_q(\mathbb{R}; X)$ denote the inverse Fourier transform of $\eta \mapsto \tilde{R}(\xi + i\eta, A)x$.

By Lemma 2.2 for $\varphi \in X'$ we have

$$\langle \varphi, \tilde{R}(\xi + i\eta, A)x \rangle = \int_0^\infty e^{-i\eta t} \langle \varphi, U_{A-\xi}(t)x \rangle dt$$

and consequently there exists a Lebesgue null set N_φ such that

$$\frac{1}{2\pi} \langle \varphi, H(t)\, U_{A-\xi}(t)x \rangle \;=\; \langle \varphi, f(t) \rangle \, (t \in \mathbb{R} \backslash N_\varphi) \,,$$

H denoting the Heaviside function. An application of Lemma 2.1 to f and $g = \frac{1}{2\pi} H U_{A-\xi}(.)x$ gives $\frac{1}{2\pi} U_{A-\xi}(.)x = f$ almost everywhere, and thus $U_{A-\xi}(.)x \in L_q([0, \infty[; X)$. By the Datko-Pazy Lemma (cf. [WW] for a version needed here), $\omega(x) \le \xi$.

The next two results comprise the major steps in the proof of Theorem 1.2.

Corollary 2.4. *Suppose X has Fourier type $1 \le p \le 2$. Then with $\frac{1}{p} + \frac{1}{q} = 1$ we have*

$$s^\infty(x) \le s^q(x) \le \omega(x) \le s^p(x) \;\; (x \in X) \,.$$

Proof. Since $U_{A-\omega(x)-\varepsilon}(.)x \in L_p([0, \infty[\,; X)$ and since X has Fourier type p, we obtain $s^q(x) \le \omega(x)$.

Since $s^2(x) \le s^p(x)$ by (1.2), (2.1) is fulfilled for all $\varphi \in X'$. So if $\xi > s^p(x)$, Corollary 2.3 gives $\omega(x) \le \xi$ and thus $\omega(x) \le s^p(x)$.

Lemma 2.5. *Suppose $A : X \supseteq D(A) \to X$ generates a C_0-semigroup. Fix $\mu, \delta, \beta \in \mathbb{R}_{>0}$ such that $\beta \le 1$, $\mu > \omega_0(A) + \delta$ and suppose that*

$$\eta \mapsto (\mu - \omega_0(A) - \delta - i\eta)^{-\beta}\, R(\omega_0(A) + \delta + i\eta, A)x \in L_1(\mathbb{R}; X)$$

for all $x \in X$. Let $u \in \,]\, \frac{1+\beta\varepsilon}{1+\beta(\varepsilon-1)}\, , \infty\,]$.

Then $R(.,A)\, R(\mu, A)^\beta x$ has an analytic extension $\tilde{R}(.,A)\, R(\mu, A)^\beta x$ upon the half-plane $Re(z) > s^u(x)$ and moreover $\tilde{R}(.,A)\, R(\mu, A)^\beta \in H^r(\alpha, X)$, where $\alpha >$

$s^u(x)$ *and*

$$r = \begin{cases} \frac{1}{\beta} + \varepsilon \ , & \textit{if } u = \infty \\ & \qquad (\varepsilon > 0) \\ \frac{u(\frac{1}{\beta}+\varepsilon)}{u+\frac{1}{\beta}+\varepsilon}, & \textit{if } u \neq \infty \,. \end{cases}$$

Proof. Let $z \mapsto z^{-\beta}$ denote the principal branch of the complex power on $\mathbb{C}\backslash\,]-\infty, 0]$.

By means of the analytic functional calculus we obtain

$$\begin{aligned} R(\mu, A)^\beta x &= \frac{1}{2\pi i}\int_{\omega_0(A)+\delta+i\mathbb{R}} (\mu - z)^{-\beta} R(z, A)x\, dz \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} (\mu - \omega_0(A) - \delta - it)^{-\beta} R(\omega_0(A) + \delta + it, A)x\, dt\,. \end{aligned}$$

If $\xi < \omega_0(A) + \delta$ the resolvent equation gives

$$\begin{aligned} &\tilde{R}(\xi + i\eta, A)\, R(\mu, A)^\beta x \\ &\quad = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{R(\omega_0(A) + \delta + it, A)x\, dt}{(\mu - \omega_0(A) - \delta - it)^\beta\, (\xi + i\eta - \omega_0(A) - \delta - it)} \\ &\qquad + \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\tilde{R}(\xi + i\eta, A)x\, dt}{(\mu - \omega_0(A) - \delta - it)^\beta\, (it - (\xi + i\eta - \omega_0(A) - \delta))}. \end{aligned}$$

The second integral equals $(\mu - (\xi + i\eta))^{-\beta}\, \tilde{R}(\xi + i\eta, A)x$ by Cauchy's theorem and thus

$$\eta \mapsto (\mu - (\xi + i\eta))^{-\beta}\, \tilde{R}(\xi + i\eta, A)x \in L_r(\mathbb{R}; X)$$

for $s^u(x) < \xi < \omega_0(A) + \delta$.

We are going to estimate the first integral for $s^u(x) < \xi < \omega_0(A) + \delta$ by means of the continuous Minkowski inequality, i.e.,

$$\begin{aligned} &\left(\int_{\mathbb{R}} \left\| \int_{\mathbb{R}} \frac{R(\omega_0(A) + \delta + it, A)x\, dt}{(\mu - \omega_0(A) - \delta - it)^\beta\, (\omega_0(A) + \delta + it - (\xi + i\eta))} \right\|^r d\eta \right)^{1/r} \\ \leq\ & \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left\| \frac{R(\omega_0(A) + \delta + it, A)x}{(\mu - \omega_0(A) - \delta - it)^\beta\, (\omega_0(A) + \delta + it - (\xi + i\eta))} \right\|^r d\eta \right)^{1/r} dt \\ =\ & \int_{\mathbb{R}} \frac{\|R(\omega_0(A) + \delta + it, A)x\|}{|\mu - \omega_0(A) - \delta - it|^\beta} \left(\int_{\mathbb{R}} \frac{d\eta}{|\omega_0(A) + \delta + it - (\xi + i\eta)|^r} \right)^{1/r} dt < \infty. \end{aligned}$$

Consequently,

$$\eta \mapsto \tilde{R}(\xi + i\eta, A)\, R(\mu, A)^\beta x \in L_r(\mathbb{R}; X)$$

for $s^u(x) < \xi < \omega_0(A) + \delta$.

Finally, if $\omega_0(A) + \frac{\delta}{2} < \xi_1$, and $\omega_0(A) < \xi_0 < \omega_0(A) + \frac{\delta}{2}$, then by the resolvent equation

$$\begin{aligned} &R(\xi_1 + i\eta, A)\, R(\mu, A)^\beta x \\ &\quad = R(\xi_0 + i\eta, A)\, R(\mu, A)^\beta x + (\xi_0 - \xi_1)\, R(\xi_1 + i\eta, A)\, R(\xi_0 + i\eta, A)\, R(\mu, A)^\beta x \end{aligned}$$

and since $\sup_{\eta\in\mathbb{R}} \|R(\xi_1 + i\eta, A)\| < \infty$, we obtain

$$\eta \mapsto R(\xi_1 + i\eta, A)\, R(\mu, A)^\beta x \in L_r(\mathbb{R}; X)\,,$$

since $\eta \mapsto R(\xi_0 + i\eta, A)\, R(\mu, A)^\beta x \in L_r(\mathbb{R}; X)$ by the first step. Consequently,

$$\tilde{R}(., A)\, R(\mu, A)^\beta x \in H^r(\alpha; X) \quad \text{for} \quad \alpha > s^u(x)\,.$$

Now we complete

2.6 Proof of Theorem 1.2. Let $1 < p \le 2$ denote the Fourier type of X, $\frac{1}{p} + \frac{1}{q} = 1$, fix $x \in X$ and $\varepsilon > 0$. By the Datko-Pazy Lemma it is enough to prove

$$U_{A-\xi}(.)R(\mu, A)^{\frac{1}{p}+\varepsilon} x \in L_r([0, \infty[\,; X)$$

for some $r \in [1, \infty[$, where $\xi = s_0(x) + \delta$, $\delta > 0$. So fix $\delta > 0$, and let $\mu > \omega_0(A) + \delta$. Then $U_{A-\omega_0(A)-\delta}(.)x \in L_p([0, \infty[\,; X)$, and so its Fourier transform $\eta \mapsto \int_0^\infty e^{-i\eta t} U_{A-\omega_0(A)-\delta}(t)x\, dt = R(\omega_0(A) + \delta + i\eta, A)x$ belongs to $L_q(\mathbb{R}; X)$. Let $u = \infty$ and $\beta := \frac{1}{p} + \varepsilon$. Then the assumptions of Lemma 2.5 are fulfilled and thus $R(., A)\, R(\mu, A)^{\frac{1}{p}+\varepsilon} x$ has an analytic extension $\tilde{R}(., A)\, R(\mu, A)^{\frac{1}{p}+\varepsilon} x$ upon the half-plane $Re(z) > s_0(x)$, such that $\tilde{R}(., A) R(\mu, A)^{\frac{1}{p}+\varepsilon} x \in H^p(\alpha, X)$ $(\alpha > s_0(x))$, i.e., $s^p(R(\mu, A)^{\frac{1}{p}+\varepsilon} x) \le s_0(x)$, and thus $\omega(R(\mu, A)^{\frac{1}{p}+\varepsilon} x) \le s^p(R(\mu, A)^{\frac{1}{p}+\varepsilon} x) \le s_0(x)$ by Corollary 2.4, i.e., (1.8) holds true.

The optimality as well as the strictness,

$$s_0(x) < s^q(x) < s^p(x) < \infty \;\; (1 < p < q < \infty) \tag{$(1.2)_S$}$$

will be demonstrated in the next section.

3. Optimality of the estimates (1.8) and (1.9)

We shall modify an example due to Zabczyk [Za] as displayed in [Wr].

Example 3.1. Fix $0 < r < 1$, $1 < p \le 2$ and let

$$m(k) := -\left\lfloor \frac{\ln k}{\ln r} \right\rfloor \;\; (k \in \mathbb{N}_{\ge 2})\,.$$

For $\ell \in \mathbb{N}_{\ge 2}$ define

$$N_\ell : \ell_p^\ell \to \ell_p^\ell\,,\; (\xi_j)_{j=1}^\ell \mapsto (\xi_2, \xi_3, ..., \xi_\ell, 0)\,,$$

and let

$$\begin{aligned} X_p &:= \ell_p - \bigoplus_{k=2}^\infty \ell_p^{m(k)}, \\ A_p &:= (ik + N_{m(k)})_{k=2}^\infty : X_p \supseteq D(A_p) \to X_p\,. \end{aligned}$$

Observe that X_p has Fourier type p and that A_p generates a C_0-group U_{A_p} on X_p given by

$$U_{A_p}(t)(x_k)_{k=2}^\infty := (e^{itk}\, e^{tN_{m(k)}}\, x_k)_{k=2}^\infty\,.$$

Finally, $\sigma(A_p; X_p) = \{ik : k \in \mathbb{N}_{\ge 2}\}$.

As has been done in [Wr] for $p = 2$, the following can be proved for general $p \in [1, \infty]$:

$$\omega_n(A_p) = s_n(A_p) = r^n \;\; (n \in \mathbb{N} \cup \{0\})\,. \tag{3.1}$$

Next we are going to establish a sharpened version of $(1.2)_S$. We shall do these calculations for all powers r^n of r simultaneously for two reasons. The first one is didactical, for the appearance of $n - \frac{1}{q}$ for $n = 1$ might lead the reader to wrongly think of conjugate indices. The second one is that it might be useful to have these estimates in between abscissas where the growth of $R(.,A)y$ is generically of a high polynomial order.

So fix n and let $x_{m(k)}$ denote the $m(k)$-th unit vector of $\ell_p^{m(k)}$.

With $x = (k^{-n}x_{m(k)})_{k=2}^{\infty}$ throughout we shall prove

$$\begin{aligned}(1.2)'_S \qquad s_0(x) = r^n \;&=\; \omega(R(\mu,A)^{\frac{1}{p}}x) < r^{n-\frac{1}{q}}\\ &= \omega(R(\mu,A)^{\frac{1}{p}-\frac{1}{q}} \;\leq\; s^q(x) \leq r^{m-\frac{1}{p}}\\ &= s^p(x) \;=\; \omega(x)\,.\end{aligned}$$

For $q > \frac{p}{p-1}$ we show $s^q(x) = r^{m-\frac{1}{q}} < s^p(x)$, and thus $(1.2)_S$ is contained in $(1.2)'_S$.

On the other hand $(1.2)'_S$ also establishes the optimality of (1.8).

Indeed, for $0 < z < 1$ we have

$$\|(z - N_{m(k)})^{-1}\, k^{-n}\, x_{m(k)}\|^p \geq k^{-np}\, z^p\, k^{p\frac{\ln z}{\ln r}} - k^{-np}\,,$$

and thus $s_0((k^{-n}\,x_{m(k)})_{k=2}^{\infty}) \geq r^n$. But an elementary calculation also yields

$$\begin{aligned}&\left\| \sum_{j=0}^{m(k)-1} (z+it)^{-j-1}\, N_{m(k)}^j\, k^{-n}\, x_{m(k)} \right\|^p\\ \leq\;& k^{-np} \sum_{j=0}^{m(k)-1} z^{-p(j+1)} = k^{-np}\, \frac{1 - z^{-m(k)p}}{z^p - 1}\\ \leq\;& \frac{1}{1-z^p}\, k^{-np}\, k^{p\frac{\ln z}{\ln r}}\,,\end{aligned}$$

and so $s_0((k^{-n}\,x_{m(k)})_{k=2}^{\infty}) \leq r^n$.

Fix $0 < \delta$ small. Then with $q \geq p$ we have $(0 < \xi < \sqrt{1-\delta^2})$

$$\begin{aligned}&\int_{\mathbb{R}} \left\| R\big(\xi + i\eta, A_p\big)\, x \right\|_p^q d\eta\\ =\;& \int_{\mathbb{R}} \left(\sum_{k=2}^{\infty} \left\| (\xi + i(\eta-k) - N_{m(k)})^{-1}\, k^{-n}\, x_{m(k)} \right\|^p \right)^{q/p} d\eta\\ \geq\;& \sum_{k=2}^{\infty} \int_{k-\delta}^{k+\delta} \left\| \Big(\xi + i(\eta-k) - N_{m(k)}\Big)^{-1} k^{-n}\, x_{m(k)} \right\|^q d\eta\\ \geq\;& 2\delta \sum_{k=2}^{\infty} k^{-nq} \left(\frac{\left(\frac{1}{|\xi+i\delta|}\right)^{pm(k)} - 1}{1-\xi^p} \right)^{q/p}\\ \geq\;& 2\delta \sum_{k=2}^{\infty} k^{-nq} \left(\left(\frac{1}{|\xi+i\delta|}\right)^{pm(k)} - 1 \right)^{q/p}\\ \geq\;& 2\delta \sum_{k=2}^{\infty} k^{-nq} \left(k^{p\frac{\ln|\xi+i\delta|}{\ln r}} - 1 \right)^{q/p}.\end{aligned}$$

Since $|\xi + i\delta| < 1$ by assumption, this means that

$$\sum_{k=2}^{\infty} k^{-nq} \, |\xi + i\delta|^q \, k^{q \frac{\ln |\xi + i\delta|}{\ln r}} < \infty \, ,$$

and so

$$-nq + q \, \frac{\ln |\xi + i\delta|}{\ln r} < -1$$

which means

$$|\xi + i\delta| > r^{n - \frac{1}{q}},$$

and since this has to be true for all sufficiently small $\delta > 0$, we obtain

$$s^q \, (x) \geq r^{n - \frac{1}{q}} \, . \tag{3.2}$$

On the other hand, using Minkowski's inequality, we obtain

$$\begin{aligned}
&\left(\int_{\mathbb{R}} \left\| R\Big(\xi + i\eta, A_p \Big) \; x \right\|_p^q d\eta \right)^{p/q} \\
= \quad & \left(\int_{\mathbb{R}} \left(\sum_{k=2}^{\infty} \left\| \Big(\xi + i(\eta - k) - N_{m(k)} \Big)^{-1} k^{-n} \, x_{m(k)} \right\|_p^p \right)^{q/p} d\eta \right)^{p/q} \\
\leq \quad & \sum_{k=2}^{\infty} \left(\int_{\mathbb{R}} \left\| \Big(\xi + i(\eta - k) - N_{m(k)} \Big)^{-1} k^{-n} \, x_{m(k)} \right\|_p^q d\eta \right)^{p/q} \\
\leq \quad & \sum_{k=2}^{\infty} \left(\int_{k-1}^{k+1} \left\| \Big(\xi + i(\eta - k) - N_{m(k)} \Big)^{-1} k^{-n} \, x_{m(k)} \right\|_p^q d\eta \right. \\
& \left. + \int_{\mathbb{R} \setminus [k-1, k+1]} \left\| \Big(\xi + i(\eta - k) - N_{m(k)} \Big)^{-1} k^{-n} \, x_{m(k)} \right\|_p^q d\eta \right)^{p/q} \\
\leq \quad & \sum_{k=2}^{\infty} \left(2 \, k^{-nq} \cdot \frac{|\xi|^{-qm(k)}}{(1 - |\xi|^p)^{q/p}} + k^{-nq} \int_{\mathbb{R} \setminus [k-1, k+1]} \frac{d\eta}{(|\xi + i(\eta - k)|^p - 1)^{q/p}} \right)^{p/q} \\
\leq \quad & c \sum_{k=2}^{\infty} k^{-np} \, |\xi|^{-pm(k)} \leq c \sum_{k=2}^{\infty} k^{-np} \, k^{p \frac{\ln \xi}{\ln r}} \, ,
\end{aligned}$$

with a suitable constant $c > 0$.

The right hand side is finite iff $-np + p \frac{\ln \xi}{\ln r} < -1$, i.e., $\xi > r^{n - \frac{1}{p}}$ and thus

$$s^p(x) = r^{n - \frac{1}{p}} \, , \tag{3.3}$$

whereas for $q > p$ we have

$$r^{n - \frac{1}{q}} \leq s^q(x) \leq r^{n - \frac{1}{p}} \, . \tag{3.4}$$

In order to obtain $(1.2)_S$ we have to work a little bit harder.

$$\begin{aligned}
&\int_{\mathbb{R}} \left\| R(\xi + i\eta, A_p) \, x_{k=2}^{\infty} \right\|_p^q d\eta \\
&\leq \sum_{j \in \mathbb{Z}} \int_{j-1}^{j+1} \left(\sum_{k=2}^{\infty} \left\| \Big(\xi + i(\eta - k) - N_{m(k)} \Big)^{-1} k^{-n} \, x_{m(k)} \right\|_p^p \right)^{q/p} d\eta.
\end{aligned} \tag{3.5}$$

We are now going to estimate each of the integrals separately. If $j \leq 0$, then

$$\begin{aligned}
(3.6)\qquad & \int_{j-1}^{j+1} \left(\sum_{k=2}^{\infty} \left\| \left(\xi + i(\eta - k) - N_{m(k)} \right)^{-1} k^{-n}\, x_{m(k)} \right\|_p^p \right)^{q/p} d\eta \\
& \leq \int_{j-1}^{j+1} \left(\sum_{k=2}^{\infty} k^{-np} \, \frac{1}{|\xi + i(\eta - k)|^p - 1} \right)^{q/p} d\eta \\
& \leq \int_{j-1}^{j+1} \left(\sum_{k=2}^{\infty} k^{-np} \, \frac{1}{|\xi + i(j - 1)|^p - 1} \right)^{q/p} d\eta
\end{aligned}$$

and consequently

$$\int_{-\infty}^{1} \left\| R(\xi + i\eta, A_p)\, x \right\|_p^q d\eta < \infty$$

for $0 \leq \xi \leq 1$.

Next let $j \geq 4$. Then

$$\begin{aligned}
(3.7)\qquad & \int_{j-1}^{j+1} \left(\sum_{k=2}^{\infty} k^{-np} \left\| \left(\xi + i(\eta - k) - N_{m(k)} \right)^{-1} x_{m(k)} \right\|_p^p \right)^{q/p} d\eta \\
& \leq \int_{j-1}^{j+1} \Bigg((j-1)^{-np} \, \frac{|\xi|^{-pm(j-1)}}{1 - |\xi|^p} + j^{-mp} \, \frac{|\xi|^{-pm(j)}}{1 - |\xi|^p} \\
& \qquad + (j+1)^{-np} \, \frac{|\xi|^{-pm(j+1)}}{1 - |\xi|^p} \\
& \qquad + \sum_{\substack{k=2 \\ k \notin \{j-1, j, j+1\}}}^{\infty} k^{-np} \, \frac{1}{|\xi + i(\eta - k)|^p - 1} \Bigg)^{q/p} d\eta \,.
\end{aligned}$$

Moreover,

$$\begin{aligned}
(3.8)\qquad & \sum_{\substack{k=2 \\ k \notin \{j-1, j, j+1\}}}^{\infty} k^{-np} \, \frac{1}{|\xi + i(\eta - k)|^p - 1} \\
& \leq \sum_{k=2}^{j-2} k^{-np} \, \frac{1}{|\xi + i(j - 1 - k)|^p - 1} + \sum_{k=j+2}^{\infty} k^{-np} \, \frac{1}{|\xi + i(j + 1 - k)|^p - 1} \\
& \leq (j-4) \cdot \frac{2^{-np}}{(j-3)^p - 1} + (j+2)^{-np} \sum_{k=1}^{\infty} \frac{1}{|\xi + ik|^p - 1} \,.
\end{aligned}$$

Observing that

$$\left(\sum_{\ell=1}^{k} \alpha_\ell \right)^r \leq k^r \sum_{\ell=1}^{k} \alpha_\ell^r \quad (\alpha_j \geq 0)\,,$$

by the above estimates (3.7) and (3.8), we obtain

$$\begin{aligned}
(3.9)\qquad & \int_{j-1}^{j+1}\left(\sum_{k=2}^{\infty} k^{-np}\left\|\Big(\xi+i(\eta-k)-N_{m(k)}\Big)^{-1}x_{m(k)}\right\|_p^p\right)^{q/p} d\eta \\
& \leq 2\cdot 5^{q/p}\Bigg((j-1)^{-nq}\ \frac{|\xi|^{-qm(j-1)}}{(1-|\xi|^p)^{q/p}}+j^{-nq}\ \frac{|\xi|^{-qm(j)}}{(1-|\xi|^p)^{q/p}} \\
& +\ (j+1)^{-nq}\ \frac{|\xi|^{-qm(j+1)}}{(1-|\xi|^p)^{q/p}}+(j-4)^{q/p}\ \frac{2^{-nq}}{((j-3)^p-1)^{q/p}} \\
& +(j+2)^{-nq}\left(\sum_{k=1}^{\infty}\ \frac{1}{|\xi+ik|^p-1}\right)^{q/p}\Bigg)\ .
\end{aligned}$$

If $q > \dfrac{p}{p-1}$, then $\displaystyle\sum_{j=4}^{\infty}\frac{2^{-nq}(j-4)^{q/p}}{((j-3)^p-1)^{q/p}} < \infty\,.$

If $\xi = (1+\varepsilon)\,r^{n-\frac{1}{q}}$, then

$$\begin{aligned}
|\xi|^{-qm(j)} &\leq j^{q\cdot\left(\frac{(n-\frac{1}{q})\cdot\ln r+\ln(1+\varepsilon)}{\ln r}\right)} \\
&= j^{qn-1+\frac{\ln(1+\varepsilon)}{\ln r}}
\end{aligned}$$

and so the first three positively indexed terms on the right hand side of (3.5) are summable with respect to j, too.

Consequently, by (3.7)–(3.9),

$$\int_{\mathbb{R}}\left\|R(\xi+i\eta,A_p)\,x\right\|_p^q d\eta<\infty$$

for $\xi > r^{n-\frac{1}{q}}$, i.e., $s^q((k^{-n}\,x_{m(k)})_{k=2}^{\infty} \leq r^{n-\frac{1}{q}}$.

Together with the estimates (3.3) and (3.4) this establishes $(1.2)_S$.

It remains to prove that $r^{n-\frac{1}{q}} = \omega(R(\mu,A)^{\frac{1}{p}-\frac{1}{q}}\,x)$ for $q\in[p,\infty]$ and x as above. Take $s > r^n$ and $\beta > 0$ such that $\omega(R(\mu,A_p)^{\beta}\,x) < s$.

Then for all $1 \leq u \leq \infty$ one has

$$U_{A_p-s}\,(.)\,R(\mu,A_p)^{\beta}\,x \in L_u([0,\infty[\,;X)\,. \tag{3.10}$$

By means of the minimal equation this will turn out to be equivalent to

$$U_{A_p-s}\,(.)\,((\mu-ik)^{-\beta}\,k^{-n}\,x_{m(k)})_{k=2}^{\infty} \in L_u([0,\infty[\,;X)\,. \tag{3.11}$$

For purposes of calculation (3.11) is much more convenient, and we shall try to distill the mutual relations between β and s from (3.11).

Take the equivalence of (3.10) and (3.11) for granted for the moment. Observing that

$$\int_0^{\infty} e^{-\alpha t}\,\frac{t^j}{j!}\,dt = \alpha^{-(j+1)} \text{ and}$$

$$\|(y_j)_{j=1}^n\|_1 \leq n^{1-\frac{1}{p}}\,\|(y_j)_{j=1}^n\|_p \quad\text{for}\quad (y_j)_{j=1}^{\infty}\in\mathbb{K}^n\,,$$

we obtain by means of Minkowski's inequality

$$
\begin{aligned}
(3.12)\qquad & \left(\sum_{k=2}^{\infty}\left||\mu-ik|^{-\beta}\,k^{-n}\,m(k)^{\frac{1}{p}-1}\sum_{j=0}^{m(k)-1}\left(\frac{1}{s}\right)^{j+1}\right|^{p}\right)^{1/p}\\
&=\left(\sum_{k=2}^{\infty}\left||\mu-ik|^{-\beta}\,k^{-n}\,m(k)^{\frac{1}{p}-1}\sum_{j=0}^{m(k)-1}\int_0^\infty e^{-st}\,\frac{t^j}{j!}\,dt\right|^{p}\right)^{1/p}\\
&\le\left(\sum_{k=2}^{\infty}\left||\mu-ik|^{-\beta}\,k^{-n}\int_0^\infty e^{-st}\left(\sum_{j=0}^{m(k)-1}\left(\frac{t^j}{j!}\right)^{p}\right)^{1/p}dt\right|^{p}\right)^{1/p}\\
&\le\int_0^\infty\left(\sum_{k=2}^{\infty}|\mu-ik|^{-\beta p}\,k^{-np}\,e^{-pst}\sum_{j=0}^{m(k)-1}\left(\frac{t^j}{j!}\right)^{p}\right)^{1/p}dt\\
&=\left\|U_{A_p-s}\,(.)\,((\mu-ik)^{-\beta}\,k^{-n}\,x_{m(k)})_{k=2}^{\infty}\right\|_{L_1([0,\infty[;X_p)}
\end{aligned}
$$

By the assumption upon β, the right hand side of (3.12) is finite. But the left hand side of (3.12) yields the following condition upon β:

$$\sum_{k=2}^{\infty}|\mu-ik|^{-\beta p}\,k^{-np}\,m(k)^{1-p}\,s^{-pm(k)}<\infty\ :$$

Since $m(k)^{1-p}$ is a logarithmic term in k we must have

$$-\beta p-np+p\,\frac{\ln s}{\ln r}<-1\,,\quad\text{i.e.,}\quad\beta>\frac{1}{p}-n+\frac{\ln s}{\ln r}\ .$$

Thus given s we obtain a condition on β and vice versa

$$\ln s>(\beta+n-\frac{1}{p})\ \ln r\ .$$

First let $s=(r+\delta)^{n-\frac{1}{q}}$. Then each $\beta>\frac{1}{p}-\frac{1}{q}+(n+\frac{1}{q})\,(\frac{\ln(r+\delta)}{\ln r}-1)$ yields

$$\omega(R(\mu,A)^{\beta}x)<(r+\delta)^{n-\frac{1}{q}}\ (\delta>0)$$

and consequently

$$\omega(R(\mu,A)^{\frac{1}{p}-\frac{1}{q}}x)\le r^{n-\frac{1}{q}}\ .$$

On the other hand given $\beta=\frac{1}{p}-\frac{1}{q}$, for $\omega(R(\mu,A)^{\frac{1}{p}-\frac{1}{q}}x)<s$ to hold we must have $\ln s>(n-\frac{1}{q})\ln r$ by the above and therefore $r^{n-\frac{1}{q}}\le\omega(R(\mu,A)^{\frac{1}{p}-\frac{1}{q}}x)$.

So $(1.2)'_S$ and thus the optimality of estimate (1.8) is established.

So the equivalence of (3.10) and (3.11) has to be shown.

Without loss of generality assume $\mu > 3$. By the minimal equation [DS, VII.3.22] we have

$$\begin{aligned} & e^{tN_{m(k)}}\, R(\mu - ik, N_{m(k)})^{\beta} \\ = \quad & e^{tN_{m(k)}} \sum_{j=0}^{m(k)-1} \binom{\beta + j - 1}{j} (\mu - ik)^{-\beta - j}\, N_{m(k)}^{j} \\ = \quad & e^{tN_{m(k)}} (\mu - ik)^{-\beta} \left(\sum_{j=0}^{m(k)-1} \binom{\beta + j - 1}{j} (\mu - ik)^{-j}\, N_{m(k)}^{j} \right) \end{aligned}$$

and since

$$\left\| \sum_{j=1}^{m(k)-1} \binom{\beta + j - 1}{j} (\mu - ik)^{-j}\, N_{m(k)}^{j} \right\| \le \frac{1}{|\mu - ik| - 1} < \frac{1}{2}\,,$$

we obtain with $B_k := \displaystyle\sum_{j=0}^{m(k)-1} \binom{\beta + j - 1}{j} (\mu - ik)^{-j}\, N_{m(k)}^{j}$ and $(y_k)_{k=2}^{\infty} \in X_p$, since $\sup \|B_k^{-1}\| \le 2$ and $\sup \|B_k\| < \frac{3}{2}$,

$$\begin{aligned} & \|\, U_{A_p}(t)\, ((\mu - ik)^{-\beta}\, y_k)_{k=2}^{\infty}\, \| \\ \le \quad & 2\, \|\, U_{A_p}(t)\, ((\mu - ik)^{-\beta}\, B_k\, y_k)_{k=2}^{\infty}\, \| \\ = \quad & 2\, \|\, U_{A_p}(t)\, R(\mu, A_p)^{\beta}\, (y_k)_{k=2}^{\infty}\, \| \\ \le \quad & 3\, \|\, U_{A_p}(t)\, ((\mu - ik)^{-\beta}\, y_k)_{k=2}^{\infty}\, \| \end{aligned}$$

which especially establishes the equivalence of (3.10) and (3.11).

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Mathematisches Seminar, Universität Kiel, D-24098 Kiel, Germany