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BOUNDARY VALUE PROBLEMS ON INFINITE INTERVALS

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ABSTRACT. We present two methods, both based on topological ideas, to the solvability of boundary value problems for differential equations and inclusions on infinite intervals. In the first one, related to the rich family of asymptotic problems, we generalize and extend some statements due to the Florence group of mathematicians Anichini, Cecchi, Conti, Furi, Marini, Pera, and Zecca. Thus, their conclusions for differential systems are as well true for inclusions; all under weaker assumptions (for example, the convexity restrictions in the Schauder linearization device can be avoided). In the second, dealing with the existence of bounded solutions on the positive ray, we follow and develop the ideas of Andres, Górniewicz, and Lewicka, who considered periodic problems. A special case of these results was previously announced by Andres. Besides that, the structure of solution sets is investigated. The case of l.s.c. right hand sides of differential inclusions and the implicit differential equations are also considered. The large list of references also includes some where different techniques (like the Conley index approach) have been applied for the same goal, allowing us to envision the full range of recent attacks on the problem stated in the title.

1. Introduction (historical remarks)

The history of boundary value problems (BVPs) on infinite intervals starts at the end of the last century with the pioneering work of A. Kneser [Kn] about monotone solutions and their derivatives on $[0,\infty)$ for second-order ordinary differential equations (ODEs). The Kneser-type results were then followed by A. Mambriani [Ma1] in 1929 and others from the beginning of the fifties until now (see e.g. [Gr], [HW], [Wo], [Sr1], [BJ], [Se3], [KS], [R1]–[R3] and the references therein).

At the beginning of the fifties the study of bounded solutions via BVPs was initiated by C. Corduneanu [Co2], [Co3], who considered second-order BVPs on the positive ray as well as on the whole real line. Since the sixties similar problems have been studied, using mostly the lower and upper solutions technique (see e.g. [Be1], [FJ], [BJ], [Av], [Sc1], [Sc2], [Sc3]).

Since the beginning of the seventies BVPs on infinite intervals have been studied systematically (see the long list of references), and we can recognize at least four very powerful techniques. The first approach (called the sequential one in §3)

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consists in investigating the limit process for the family of BVPs on infinitely increasing compact intervals. Hence, the associated function spaces for the related fixed point problems are Banach spaces. This idea has been elaborated in [Kr1], [KMP], [KMKP] for problems on the whole line, and here, in §3, we want to present a rather general method for finding bounded solutions on the positive ray. For some applications and further interesting results see e.g. [Ab], [An1]–[An6], [AMP], [Av], [Li], [PG], [P2].

If however, we work directly on the noncompact intervals, then the associated function spaces for the fixed point problems are not Banach, but Fréchet spaces, which raises some difficulties (see [Co1], [L], [Ma2] and the references therein). On the other hand, this approach can bring very strong results (see e.g. [ACZ], [CFM1], [CFM2], [CMZ1]–[CMZ4], [FP1], [FP2], [HL], [Ke1], [Ke2], [SK]). Although especially the quoted abstract results of the Italian mathematicians are very effective, they can still be generalized (which is the subject of §2).

Recently, the Conley index approach has been alternatively applied for the same goal, mainly by J. R. Ward, Jr. [MW], [Wa2]–[Wa7] and R. Srzednicki [Sr1], [Sr2], when the link with the Lefschetz index has been employed. Another remarkable recent approach consists in the application of the so-called A-mapping theory (the A-class means the approximation admissible maps); for details and some results see e.g. [Kr2], [P1].

In addition to studies of BVPs for ODEs in Euclidean spaces, there are also some contributions to the study of ODEs in function spaces (Banach spaces, Hilbert spaces, etc.); see e.g. [CP], [DR], [Ka6], [P3], [Rz], [Sz], [ZZ]. Further generalizations are related to functional problems (see e.g. [St1], [St2]) and especially those for differential inclusions (see e.g. [AZ], [CMZ1], [PG], [Se1], [ZZ]). In the present paper we will consider both differential equations and inclusions.

This paper is organized as follows: §2 deals with asymptotic BVPs as fixed point problems in Fréchet spaces. In §3, existence criteria for bounded solutions on the positive ray are obtained sequentially. In §4, the structure of solution sets for the Cauchy problem is investigated further. We consider both differential inclusions with u.s.c. and l.s.c. right hand sides. In §5 we make some remarks on the implicit differential equations on noncompact intervals. §6 consists of some concluding remarks and open problems.

2. BVPs as the fixed point problems in Fréchet spaces

2.1. **Fréchet spaces.** By a *Fréchet space* we mean a completely metrizable locally convex topological vector space. Completeness of a Fréchet space implies that, for each compact subset A, the convex closure of A ($\overline{\mathrm{conv}}A$) is compact.

Below we give two examples of Fréchet spaces which will be used in the paper.

If $J \subset \mathbb{R}$ is an arbitrary interval (not necessarily compact), then we define the following spaces: the space of all continuous functions $x : J \to \mathbb{R}^n$ with the topology of uniform convergence on compact subintervals of J (we denote it by $C(J, \mathbb{R}^n)$) and the space of all C^k real functions $u : J \to \mathbb{R}$ with the topology of uniform convergence on compact subintervals of J of all derivatives up to order k (we denote it by $C^k(J)$). A topology of the first space can be generated by the metric

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_{K_n}(x-y)}{1+p_{K_n}(x-y)},$$

where $\{K_n\}$ is a family of compact subsets of J such that $\bigcup_{n=1}^{\infty} K_n = J$, $K_n \subset K_{n+1}$ and $p_{K_n}(x) = \sup\{|x(t)| : \ t \in K_n\}$. A topology on the second space can be generated by the metric

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \frac{p^k_{K_n}(x-y)}{1+p^k_{K_n}(x-y)},$$

where $p^k_{K_n}(u) = p_{K_n}(u) + p_{K_n}(u^{(k)})$. One can check that $C(J,\mathbb{R}^n)$ and $C^k(J)$ are both Fréchet spaces. If J is compact, then these spaces are Banach.

Note that $C^k(J)$ can be embedded into a closed subspace of $C(J,\mathbb{R}^{k+1})$ via the map $u \mapsto (u^{(0)},\dots,u^{(k)})$.

Let A be a subset of $C(J,\mathbb{R}^n)$ [resp. $A \subset C^k(J)$]. One can check that A is bounded if and only if there exists a positive function $\phi : J \to \mathbb{R}$ such that $|x(t)| \le \phi(t)$ for all $t \in J, x \in A$ [resp. $|u(t)| + |u^{(k)}(t)| \le \phi(t)$ for all $t \in J, u \in A$].

Finally recall that, by Ascoli's theorem, $A \subset C(J,\mathbb{R}^n)$ [resp. $A \subset C^k(J)$] is relatively compact if and only if it is bounded and the functions [resp. the kth-order derivatives of the functions] of A are equicontinuous at each $t \in J$.

For further information concerning locally convex spaces see e.g. [RR], [Sch].

2.2. **Graph approximation theory of set-valued maps in metric spaces.** The graph approximation approach to the fixed point theory was initiated by von Neumann (see [VN]) and studied by many authors (see [Go2] and the references therein). This method is much simpler than the one based on algebraic topology and can be applied for a large class of maps.

In this subsection all spaces are metric, and by a set-valued map we always mean an upper-semicontinuous (u.s.c.) multivalued map with non-empty compact values. All single-valued maps are assumed to be continuous.

For a subset $A \subset E$ and $\varepsilon > 0$ we define the set $N_\varepsilon(A) = \{x \in E \ : \ dist(x,A) < \varepsilon\}$, i.e. $N_\varepsilon(A)$ is an open ε–neighbourhood of the set A. If $A = \{x\}$, then we put $N_\varepsilon(x) := N_\varepsilon(\{x\})$. For the Cartesian product of two spaces E, F, we define the metric

$$d_{E\times F}((x,y),(x',y')) = \max\{d_E(x,x'), d_F(y,y')\},$$

where $x, x' \in E$ and $y, y' \in F$. We denote all metrics by the same symbol d.

Let X, Y be two spaces. We say that a set-valued map $\varphi : X \rightsquigarrow Y$ is *compact*, if the set $\overline{\varphi(X)}$ is compact, where $\varphi(B) = \{y \in Y : \ \exists x \in B \ \ y \in \varphi(x)\}$ for any $B \subset X$. A set-valued map $\varphi : X \rightsquigarrow Y$ is *locally compact*, if for every $x \in X$ there exists an open neighbourhood U_x of x such that $\varphi|_{U_x}$ is compact. A set-valued map $\varphi : X \rightsquigarrow Y$ is *closed*, if $\varphi(B)$ is closed in Y for every closed subset B of X. For $A \subset Y$ we put $\varphi^{-1}(A) = \{x \in X : \ \ \varphi(x) \subset A\}$ and $\varphi_+^{-1}(A) = \{x \in X : \ \ \varphi(x) \cap A \ne \emptyset\}$. If $A = \{y\}$, then we write $\varphi^{-1}(y) := \varphi^{-1}(\{y\})$ and $\varphi_+^{-1}(y) := \varphi_+^{-1}(\{y\})$. For $\varphi : X \rightsquigarrow X$ we define $Fix(\varphi) = \{x \in X : \ \ x \in \varphi(x)\}$, the set of fixed points of φ.

By a *graph* of φ we mean the set $\Gamma_\varphi = \{(x,y) \in X \times Y : \ \ y \in \varphi(x)\}$.

Recall the following important well-known fact (see [Ga], [Go2]):

Proposition 2.1. *If X, Y are two spaces and $\varphi : X \rightsquigarrow Y$ is a set-valued map, then Γ_φ is closed in $X \times Y$.*

Let A be a subset of X, $\varepsilon > 0$, and $\varphi : X \rightsquigarrow Y$ a set-valued map. A map $f : A \to Y$ is called an *ε-approximation* (on the graph) *of φ,* if $\Gamma_f \subset N_\varepsilon(\Gamma_\varphi)$ or, equivalently,

$$f(x) \in N_\varepsilon(\varphi(N_\varepsilon(x))), \qquad x \in A.$$

If $A = X$ and f is an ε–approximation of φ, then we write $f \in a(\varphi, \varepsilon)$.

In the following result we summarize some useful properties of this notion (for the proof see [GGK], [Ga]):

Proposition 2.2. *Let X, Y be two spaces, and let $\varphi : X \multimap Y$ be a set-valued map.*

(i) Let P be a compact space and let $r : P \to X$ be a map. Then, for each $\varepsilon > 0$, there is $\delta > 0$ such that, if $f \in a(\varphi, \delta)$, then $f \circ r \in a(\varphi \circ r, \varepsilon)$.

(ii) Let C be a compact subset of X and $y \in Y$. If $\varphi_+^{-1}(y) \cap C = \emptyset$, then there exists $\varepsilon > 0$ such that, for every $f \in a(\varphi, \varepsilon)$, we have $f^{-1}(y) \cap C = \emptyset$.

(iii) Let C be a compact subset of X. Then, for every $\varepsilon > 0$, there is $\delta > 0$ such that $f|_C \in a(\varphi|_C, \varepsilon)$, whenever $f \in a(\varphi, \delta)$.

(iv) Let X be compact, and let $\chi : X \times [0,1] \multimap Y$ be a set-valued map. Then, for every $t \in [0,1]$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that $h_t \in a(\chi_t, \varepsilon)$ whenever $h \in a(\chi, \delta)$, where $\chi_t : X \multimap Y$ and $h_t : X \to Y$ are defined in the following way: $\chi_t(x) = \chi(x,t)$ and $h_t(x) = h(x,t)$, for every $x \in X$.

(v) Let Z be a space, let X, Y be compact and let $g : Y \to Z$ be a map. Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $g \circ f \in a(g \circ \varphi, \varepsilon)$ whenever $f \in a(\varphi, \delta)$.

(vi) Let Z, T be spaces and $\psi : Z \multimap T$ a set-valued map. Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, if $f \in a(\varphi, \delta)$ and $g \in a(\psi, \delta)$, then $f \times g \in a(\varphi \times \psi, \varepsilon)$, where $(f \times g)(x,z) := (f(x), g(z))$, $(\varphi \times \psi)(x,z) := \varphi(x) \times \psi(z)$.

Let us define the following classes of maps:

Definition 2.3. Let X, Y be two spaces, $C \subset X$ be a compact subset and $y \in Y$.

(i) $C(X,Y)$ is the class of all single-valued (continuous) maps from X to Y.

(ii) $A_0(X,Y)$ (resp. $A_0(X)$) is the class of all set-valued maps $\varphi : X \multimap Y$ (resp. $\varphi : X \multimap X$) such that for every $\varepsilon > 0$ there is $f \in a(\varphi, \varepsilon)$.

(iii) $A(X,Y)$ (resp. $A(X)$) is the class of all set-valued maps $\varphi : X \multimap Y$ (resp. $\varphi : X \multimap X$) such that $\varphi \in A_0(X,Y)$ (resp. $\varphi \in A_0(X)$) and, for each $\varepsilon > 0$, there is $\delta > 0$ such that, if $f, g \in a(\varphi, \delta)$, then there exists a map $h : X \times [0,1] \to Y$ (resp. $h : X \times [0,1] \to X$) such that $h_0 = f$, $h_1 = g$ and $h_t \in a(\varphi, \varepsilon)$ for every $t \in [0,1]$.

The class A_0 is adequate for obtaining many fixed point theorems, but it is not sufficient to construct the fixed point index or the topological degree. Fortunately, the class A, which is appropriate to fixed point index theory, is large enough. We shall provide some examples of set-valued maps in the class A.

First, we recall some geometric notions of subsets of metric spaces. We say that a nonempty set A is *contractible*, provided there exist $x_0 \in A$ and a homotopy $h : A \times [0,1] \to A$ such that $h(x,0) = x$ and $h(x,1) = x_0$ for every $x \in A$; A is called an *R_δ-set*, provided there exists a decreasing sequence $\{A_n\}$ of compact contractible sets such that $A = \bigcap\{A_n : n = 1, 2, \ldots\}$. Note that any R_δ-set is acyclic with respect to any continuous theory of homology (e.g. the Čech theory), so in particular, it is compact, nonempty and connected. We say that A is *R_δ-contractible* if there exists a multivalued homotopy $\chi : A \times [0,1] \multimap A$ such that

(i)	$x \in \chi(x,1)$,	for every $x \in A$,
(ii)	$\chi(x,0) = B$,	for every $x \in A$ and for some $B \subset A$,
(iii)	$\chi(x,t)$ is an R_δ-set,	for every $t \in [0,1]$ and $x \in A$,
(iv)	χ is an u.s.c. map.	

Let us remark (see [Go1]) that any R_δ-contractible set has the same homology as the one-point space $\{p\}$, so that it is acyclic and in particular connected.

A compact, nonempty subset $A \subset X$ is called ∞*–proximally connected* in X (see [Du2], [GGK]) if, for every $\varepsilon > 0$, there is $\delta > 0$ such that, for any $n = 1, 2, \dots,$ and for any map $g : \partial\Delta^n \to N_\delta(A)$, there exists an extension $g' : \Delta^n \to N_\varepsilon(A)$ ($g'|_{\partial\Delta^n} = g$); neighbourhoods are taken as subsets of X. Moreover, one can see that the above notion gives us information about embedding of A into X rather than the structure of A. In spite of this, we have the following interesting result (see [Hy]).

Proposition 2.4. *If A is an R_δ-subset of the ANR space X, then A is ∞–proximally connected.*

The following sufficient condition for R_δ-sets will be used in the sequel.

Proposition 2.5 ([BG])**.** *Let $\{A_n\}$ be a sequence of compact ARs contained in X and let A be a subset of X such that the following conditions hold:*

(i) $A \subset A_n$ for every n,
(ii) A is a set-theoretic limit of the sequence $\{A_n\}$,
(iii) for each open neighbourhood U of A in X there is a subsequence $\{A_{n_i}\}$ of $\{A_n\}$ such that $\{A_{n_i}\} \subset U$ for every n_i.

Then A is an R_δ-set.

Let us note the following simple result:

Proposition 2.6. *Let Y be a space, X be a neighbourhood retract of Y and $Z \subset X$ be a compact, ∞–proximally connected set in Y. Then Z is ∞–proximally connected in X.*

Proof. Take an arbitrary $k \in \mathbb{N}$. We will show that, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for each map $f : \partial\Delta^k \to N_\delta(Z) \cap X$, there is an extension $f' : \Delta^k \to N_\varepsilon(Z) \cap X$ of f.

Let Ω be an open subset of Y and $r : \Omega \to X$ be a retraction. Take an arbitrary $\varepsilon > 0$, put $U := r^{-1}(N_\varepsilon(Z) \cap X) \subset \Omega$ and choose $\eta > 0$ such that $N_\eta(Z) \subset U$. There is δ, $0 < \delta < \eta$, such that, for every map $f : \partial\Delta^k \to N_\delta(Z)$, we can find an extension $\bar{f} : \Delta^k \to N_\eta(Z)$. Take any $f : \partial\Delta^k \to N_\delta(Z) \cap X$. There is an $\bar{f} : \Delta^k \to N_\eta(Z)$ such that $\bar{f}|_{\partial\Delta^k} = f$. Define $f' = r \circ \bar{f} : \Delta^k \to N_\varepsilon(Z) \cap X$. The map f' is an extension of f. ☐

Let X, Y be two spaces. We say that a set-valued map $\varphi : X \rightsquigarrow Y$ is a J*–mapping* (writing $\varphi \in J(X, Y)$), provided the set $\varphi(x)$ is ∞–proximally connected for every $x \in X$. Propositions 2.4 and 2.6 imply that, if Y is a neighbourhood retract of the Fréchet space F, then $\varphi \in J(X, Y)$ if, for example, $\varphi(x)$ is an R_δ-set, for every $x \in X$. It is obvious that, if $\varphi \in J(X, Y)$ and $r : Z \to X$, then $\varphi \circ r \in J(Z, Y)$. Moreover, if $\varphi_i \in J(X_i, Y_i)$ for $i = 1, 2$, then $\varphi_1 \times \varphi_2 \in J(X_1 \times X_2, Y_1 \times Y_2)$.

Theorem 2.7 (see [GGK], Corollaries 5.10 and 5.11)**.** *Let P be a finite polyhedron, Y be a space, and $\varphi \in J(P, Y)$. Then $\varphi \in A(P, Y)$.*

One can prove (see [GGK]) that if X is a compact ANR, then also $J(X, Y) \subset A(X, Y)$ for every metric space Y. For further generalizations see [BD], [Kr3].

2.3. **Topological degree and fixed point index.** In this part, we construct a topological degree for J-mappings defined on subsets of Fréchet spaces[1] . We do this first in finite dimensional spaces, then we extend the construction to the infinite dimensional case. This will permit us to define a fixed point index on retracts of Fréchet spaces which will be used in applications.

Let E be a Fréchet space of finite dimension (we can assume that $E = \mathbb{R}^n$), and let $\Omega \subset E$ be an open subset.

We say that $\Phi : \overline{\Omega} \rightsquigarrow E$ is *decomposable* ($\Phi \in D(\overline{\Omega}, E)$) if there exist a Fréchet space F, a compact subset T of F, $\gamma \in J(\overline{\Omega}, T)$ and a map $g : T \to E$ such that $\Phi = g \circ \gamma$. We also say that a set-valued map Φ belongs to $D_{\partial\Omega}(\overline{\Omega}, E)$ if $\Phi \in D(\overline{\Omega}, E)$ and $Fix(\Phi) \cap \partial\Omega = \emptyset$. For every decomposable map we have a *decomposition*:

$$D : \overline{\Omega} \overset{\gamma}{\rightsquigarrow} T \xrightarrow{g} E.$$

We say that two decomposable maps $\Phi, \Psi \in D(\overline{\Omega}, E)$ $[\Phi, \Psi \in D_{\partial\Omega}(\overline{\Omega}, E)]$ are *homotopic* in $D(\overline{\Omega}, E)$ $[D_{\partial\Omega}(\overline{\Omega}, E)]$ if there exists a set-valued map $H = g \circ \chi : \overline{\Omega} \times [0,1] \rightsquigarrow T \to E$ such that $\chi \in J(\overline{\Omega} \times [0,1], T)$ and $\Phi = g \circ \chi(\cdot, 0)$, $\Psi = g \circ \chi(\cdot, 1)$ [and $x \notin H(x,t)$ for every $(x,t) \in \partial\Omega \times [0,1]$].

Now we prove the following simple result, which was first proven in [Be].

Theorem 2.8 ([BD, Corollary 7.3])**.** *If $\Phi \in D(E, E)$, then Φ has a fixed point.*

Proof. Let $D : E \overset{\gamma}{\rightsquigarrow} T \xrightarrow{g} E$ be a decomposition of Φ. There exists a closed cube $B \subset E$ such that $\Phi(E) \subset B$. Consider the map $\Phi' = \Phi|_B$. Since B is a finite polyhedron, we can find for every $n \in \mathbb{N}$ an approximation f_n of $\gamma|_B$ such that $g \circ f_n \in a(\Phi', 1/n)$. By the Brouwer fixed point theorem, each $g \circ f_n$ has a fixed point in B. Since Φ' is u.s.c., Proposition 2.1 implies the existence of a fixed point of Φ' that is a fixed point of Φ. □

We can define the class $FD(\overline{\Omega}, E)$ of set-valued *compact fields* associated with decomposable maps, that is, the class of all set-valued maps $\varphi = i - \Phi$, where $\Phi \in D(\overline{\Omega}, E)$. Analogously, we can define the class $FD_{\partial\Omega}(\overline{\Omega}, E)$ as a class of all set-valued maps such that $\varphi \in FD(\overline{\Omega}, E)$ and $0 \notin \varphi(\partial\Omega)$. We denote compact fields by small letters, and decomposable maps by capital ones.

Assume $\Phi \in D_{\partial\Omega}(\overline{\Omega}, E)$ $(\Phi = g \circ \gamma)$. Then $\mathcal{F} = Fix(\Phi) \subset \Omega \cap \Phi(\overline{\Omega})$ is a compact set. Thus there is an open bounded set $W \subset \Omega$ such that $\overline{W}$ is a finite polyhedron and $\mathcal{F} \subset W$. Briefly, we write $W \in N^P(\mathcal{F}, \Omega)$.

Define $\Phi_W := g \circ \gamma|_{\overline{W}}$ and $\varphi_W := i - \Phi_W$. By Proposition 2.2 we can find $\varepsilon > 0$ such that $Fix(u) \cap \partial W = \emptyset$, for every $u \in a(\Phi_W, \varepsilon)$. There exists $\varepsilon_0, 0 < \varepsilon_0 < \varepsilon$, such that $g \circ f \in a(\Phi_W, \varepsilon)$ for every $f \in a(\gamma|_{\overline{W}}, \varepsilon_0)$. We can also find $\delta, 0 < \delta < \varepsilon_0$, such that, for every $f, k \in a(\gamma|_{\overline{W}}, \delta)$, there exists a homotopy $h : \overline{W} \times [0,1] \to T$ such that $h_0 = f, h_1 = k$ and $h_t \in a(\gamma|_{\overline{W}}, \varepsilon_0)$, for each $t \in [0,1]$.

Therefore, we have a homotopy $h_W : \overline{W} \times [0,1] \to E$, $h_W(x,t) = g \circ h(x,t)$, such that $h_W(\cdot, 0) = g \circ f$, $h_W(\cdot, 1) = g \circ k$ and $h_W(\cdot, t) \in a(\Phi_W, \varepsilon)$, for every $t \in [0,1]$. It follows that $h_W(x,t) \neq x$, for every $(x,t) \in \partial W \times [0,1]$.

Take any $f \in a(\gamma|_{\overline{W}}, \delta)$. Define

$$\mathrm{Deg}\,(\varphi, D, \Omega, 0) := \mathrm{Deg}\,(\varphi_W, D|_{\overline{W}}, W, 0) := \deg\,(i - g \circ f, W, 0),$$

[1]The case of arbitrary locally convex spaces was studied by the second author in his Ph.D. thesis, 1997 (in Polish).

where $\deg(i - g\circ f, W, 0)$ denotes the Brouwer topological degree of single-valued maps and 0 denotes the origin in E. Let us note that the degree Deg may depend on a decomposition (see [Go2] for details).

By the localization property of the Brouwer degree one can show that the above definition does not depend on the choice of W. Properties of approximable maps and Theorem 2.7 imply that the definition is also independent of the choice of f.

Theorem 2.9 (Properties of Deg)**.**

(i) (Additivity) *If $\Omega, \Omega_1, \Omega_2$ are open subsets of E, $\varphi \in FD(\overline{\Omega}, E)$, $\Omega_1 \cup \Omega_2 \subset \Omega$, $\Omega_1 \cap \Omega_2 = \emptyset$ and $0 \notin \varphi(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2))$, then*

$$\mathrm{Deg}\,(\varphi, D, \Omega, 0) = \mathrm{Deg}\,(\varphi_{\Omega_1}, D|_{\overline{\Omega_1}}, \Omega_1, 0) + \mathrm{Deg}\,(\varphi_{\Omega_2}, D|_{\overline{\Omega_2}}, \Omega_2, 0),$$

where $\varphi_{\Omega_i} = \varphi|_{\overline{\Omega_i}}$.

(ii) (Existence) *If $\varphi \in FD_{\partial\Omega}(\overline{\Omega}, E)$ and $\mathrm{Deg}\,(\varphi, D, \Omega, 0) \neq 0$, then $0 \in \varphi(\Omega)$.*

(iii) (Localization) *If $\Omega' \subset \Omega$ are open subsets of E, $\varphi \in FD(\overline{\Omega}, E)$ and $0 \notin \varphi(\overline{\Omega} \setminus \Omega')$, then*

$$\mathrm{Deg}\,(\varphi, D, \Omega, 0) = \mathrm{Deg}\,(\varphi, D|_{\overline{\Omega'}}, \Omega', 0).$$

(iv) (Homotopy) *Let H be a homotopy joining φ and ψ in $FD_{\partial\Omega}(\overline{\Omega}, E)$. Then*

$$\mathrm{Deg}\,(\varphi, D_\varphi, \Omega, 0) = \mathrm{Deg}\,(\psi, D_\psi, \Omega, 0).$$

(v) (Multiplicativity) *Let Ω_1, Ω_2 be open subsets of E_1, E_2, respectively. Assume that $\varphi_i \in FD_{\partial\Omega_i}(\overline{\Omega_i}, E_i), i = 1, 2$. Then*

$$\varphi_1 \times \varphi_2 \in FD_{\partial(\Omega_1\times\Omega_2)}(\overline{\Omega_1} \times \overline{\Omega_2}, E_1 \times E_2)$$

and

$$\begin{aligned}&\mathrm{Deg}\,(\varphi_1 \times \varphi_2, D_1 \times D_2, \Omega_1 \times \Omega_2, 0)\\ &\qquad = \mathrm{Deg}\,(\varphi_1, D_1, \Omega_1, 0)\mathrm{Deg}\,(\varphi_2, D_2, \Omega_2, 0).\end{aligned}$$

The proof is an easy consequence of the definition of Deg and the analogous properties of the Brouwer degree.

Now, we give some other properties, which are needed for a construction of the degree in the infinite dimensional case.

Proposition 2.10. *If Ω is an open subset of the space E, T is a compact subset of a Fréchet space F, and $\gamma \in J(\overline{\Omega}, T)$ and $g \in C(T, E)$ are such that $\Phi = g \circ \gamma$ has no fixed points in $\partial\Omega$, then there exists $\eta > 0$ such that, for every $g' \in C(T, E)$ satisfying $||g(y) - g'(y)|| < \eta$, for each $y \in T$, we have*

(i) *Φ and $\Phi' = g' \circ \gamma$ are homotopic in $D_{\partial\Omega}(\overline{\Omega}, E)$,*

(ii) *(as a consequence of* (i)*)*

$$\mathrm{Deg}\,(\varphi, D, \Omega, 0) = \mathrm{Deg}\,(\varphi', D', \Omega, 0),$$

where $\varphi = i - \Phi, \varphi' = i - \Phi'$ and $D = g \circ \gamma$, $D' = g' \circ \gamma$ are decompositions of Φ and Φ', respectively.

Proof. By the compactness of Φ, we can find $\eta > 0$ such that $\mathrm{dist}\,(\varphi(\partial\Omega), 0) = \eta$. Let $g' \in C(T, E)$ be such that $||g(y) - g'(y)|| < \eta$ for all $y \in T$.

Define

$$k \in C(T \times [0, 1], E), \qquad k(y, t) = tg(y) + (1 - t)g'(y)$$

and

$$\gamma' \in J(\overline{\Omega} \times [0, 1], T \times [0, 1]), \qquad \gamma'(x, t) = \gamma(x) \times \{t\}.$$

Let $H : \overline{\Omega} \times [0,1] \multimap E$ be defined as $H = k \circ \gamma'$. For the proof of our assertion, it is sufficient to show that $x \notin H(x,t)$ for every $(x,t) \in \partial\Omega \times [0,1]$.

Suppose that $x \in H(x,t)$ for some $x \in \partial\Omega$ and $t \in [0,1]$. Then $x \in k \circ \gamma'(x,t) = k(\gamma(x) \times \{t\})$, which implies that there is $y \in \gamma(x)$ such that $x = k(y,t)$. However, for every $y \in T$ and $t \in [0,1]$, we have

$$||g(y) - k(y,t)|| = ||g(y) - tg(y) - (1-t)g'(y)|| = (1-t)||g(y) - g'(y)|| < \eta.$$

Thus, $||g(y) - x|| < \eta$, which contradicts the assumption $\operatorname{dist}(\varphi(\partial\Omega), 0) = \eta$.

Finally, notice that $H(\cdot,0) = k \circ \gamma'(\cdot,0) = k(\gamma(\cdot) \times \{0\}) = g' \circ \gamma = \Phi'$ and, analogously, $H(\cdot,1) = \Phi$. This completes the proof. $\square$

Proposition 2.11. *Let Ω be an open subset of the space E and $\varphi = i - \Phi \in FD_{\partial\Omega}(\overline{\Omega}, E)$. Let $D : \overline{\Omega} \stackrel{\gamma}{\multimap} T \xrightarrow{g} E$ be a decomposition of Φ. Assume that there is a subspace $G \subset E$ such that $g(T) \subset G$. Then*

$$\operatorname{Deg}(\varphi, D, \Omega, 0) = \operatorname{Deg}(\varphi^G, D^G, \Omega^G, 0),$$

where $\Omega^G = \Omega \cap G, \varphi^G = \varphi|_{\overline{\Omega^G}}$ and $D^G : \overline{\Omega^G} \stackrel{\gamma|_{\overline{\Omega^G}}}{\multimap} T \xrightarrow{g} G$.

Proof. Let us denote $\mathcal{F} = Fix(\Phi)$ and take $W \in N^P(\mathcal{F}, \Omega)$ such that $W' = W \cap G \in N^P(\mathcal{F}, \Omega^G)$. There is $\varepsilon > 0$ such that, for each $f \in a(\gamma|_{\overline{W'}}, \varepsilon)$, we have

$$\operatorname{Deg}(\varphi^G, D^G, \Omega^G, 0) = \operatorname{Deg}(\varphi^G_{W'}, D^G|_{\overline{W'}}, W', 0) = \deg(i - g \circ f, W', 0).$$

By Proposition 2.2, we can find $\delta > 0$ such that $f|_{\overline{W'}} \in a(\gamma|_{\overline{W'}}, \varepsilon)$, provided $f \in a(\gamma|_{\overline{W}}, \delta)$. Take $\beta, 0 < \beta < \delta$, such that, for every $f \in a(\gamma|_{\overline{W}}, \beta)$,

$$\operatorname{Deg}(\varphi, D, \Omega, 0) = \operatorname{Deg}(\varphi_W, D|_{\overline{W}}, W, 0) = \deg(i - g \circ f, W, 0).$$

Now, let $f \in a(\gamma|_{\overline{W}}, \beta)$. By the contraction property of the Brouwer degree, $\deg(i - g \circ f, W, 0) = \deg(i - g \circ f|_{\overline{W'}}, W', 0)$, which completes the proof. $\square$

Proposition 2.12. *Let Ω be an open subset of the space E, and let $\Phi \in D_{\partial\Omega}(\overline{\Omega}, E)$ have two decompositions*

$$D : \overline{\Omega} \stackrel{\gamma}{\multimap} T \xrightarrow{g} E \quad \text{and} \quad D' : \overline{\Omega} \stackrel{\gamma'}{\multimap} T' \xrightarrow{g'} E.$$

Assume there is $j \in C(T, T')$ such that $\gamma' = j \circ \gamma$ and $g = g' \circ j$. Then

$$\operatorname{Deg}(\varphi, D, \Omega, 0) = \operatorname{Deg}(\varphi, D', \Omega, 0).$$

Proof. Take $W \in N^P(Fix(\Phi), \Omega)$ and $\varepsilon, \varepsilon' > 0$ such that, for $f \in a(\gamma|_{\overline{W}}, \varepsilon)$ and $f' \in a(\gamma'|_{\overline{W}}, \varepsilon')$, we have

$$\operatorname{Deg}(\varphi, D, \Omega, 0) = \deg(i - g \circ f, W, 0),$$
$$\operatorname{Deg}(\varphi, D', \Omega, 0) = \deg(i - g' \circ f', W, 0).$$

Let $f \in a(\gamma|_{\overline{W}}, \varepsilon)$ be such that $j \circ f \in a(\gamma'|_{\overline{W}}, \varepsilon')$. Then

$$\begin{aligned}\operatorname{Deg}(\varphi, D', \Omega, 0) &= \deg(i - g' \circ j \circ f, W, 0) = \deg(i - g \circ f, W, 0)\\ &= \operatorname{Deg}(\varphi, D, \Omega, 0).\end{aligned}$$

Now, let E be an infinite dimensional Fréchet space. Assume that Ω is an open subset of E and $\Phi \in J(\overline{\Omega}, E)$ is compact and such that $Fix(\Phi) \cap \partial\Omega = \emptyset$. Define a compact field $\varphi = i - \Phi$. The class of such compact fields will be denoted by $F_{\partial\Omega}(\overline{\Omega}, E)$. Obviously, $0 \notin \varphi(\partial\Omega)$. One can show that φ is a closed set-valued map. Therefore, $\varphi(\partial\Omega)$ is a closed subset of $E \setminus \{0\}$, and hence $\operatorname{dist}(\varphi(\partial\Omega), 0) = \delta_0 > 0$. Let $\delta = \delta_0/2$. By the compactness of Φ and completeness of E, we can find a compact convex set $K \subset E$ such that $\Phi(\overline{\Omega}) \subset K$. Since E is locally convex, there exists a map $\pi^L : K \to L$ into a finite dimensional subspace L of E such that $d(y, \pi^L(y)) < \delta$, for every $y \in K$, and $\Omega \cap L \neq \emptyset$. Then $x - z \neq 0$, for every $x \in \partial\Omega$ and $z \in \pi^L(\Phi(x))$.

Let us denote $\Omega^L = \Omega \cap L$, $\Phi^L = \pi^L \circ \Phi|_{\overline{\Omega^L}}$, $\varphi^L = i - \Phi^L$ and $D^L : \overline{\Omega^L} \overset{\Phi|_{\overline{\Omega^L}}}{\rightsquigarrow} K \overset{\pi^L}{\to} L$.

Define

$$\mathrm{Deg}\,(\varphi, \Omega, 0) := \mathrm{Deg}\,(\varphi^L, D^L, \Omega^L, 0). \tag{1}$$

Propositions 2.10 and 2.11 imply that this definition does not depend on the choice of the space L and the map π^L. In fact, let L' and $\pi^{L'}$ be chosen like L and π^L. Define $G = L + L'$, $\Omega^G = \Omega \cap G$, $\Phi^G = \Phi|_{\overline{\Omega^G}}$, $D_L^G : \overline{\Omega^G} \overset{\Phi^G}{\rightsquigarrow} K \overset{\pi^L}{\to} G$, $D_{L'}^G : \overline{\Omega^G} \overset{\Phi^G}{\rightsquigarrow} K \overset{\pi^{L'}}{\to} G$, $\varphi_L^G = i - \pi^L \circ \Phi^G$ and $\varphi_{L'}^G = i - \pi^{L'} \circ \Phi^G$. One can see that $\mathrm{Deg}\,(\varphi_L^G, D_L^G, \Omega^G, 0)$ and $\mathrm{Deg}\,(\varphi_{L'}^G, D_{L'}^G, \Omega^G, 0)$ are well defined.

Moreover, $||\pi^L(y) - \pi^{L'}(y)|| \leq ||\pi^L(y) - y|| + ||\pi^{L'}(y) - y|| < \delta_0$. This implies (see Proposition 2.10) that

$$\mathrm{Deg}\,(\varphi_L^G, D_L^G, \Omega^G, 0) = \mathrm{Deg}\,(\varphi_{L'}^G, D_{L'}^G, \Omega^G, 0).$$

Finally, Proposition 2.11 implies that, for instance,

$$\mathrm{Deg}\,(\varphi_L^G, D_L^G, \Omega^G, 0) = \mathrm{Deg}\,(\varphi^L, D^L, \Omega^L, 0).$$

Now, we show the independence of the choice of K. It is easy to see that we need only prove it in the situation when K and K' are such that $K' \subset K$.

Let L, π^L and $L', \pi^{L'}$ be chosen for K and K', respectively. Let $G = L + L'$. Consider

$$D_L : \overline{\Omega^L} \overset{\Phi|_{\overline{\Omega^L}}}{\rightsquigarrow} K \overset{\pi^L}{\to} L, \qquad D_{L'} : \overline{\Omega^{L'}} \overset{\Phi|_{\overline{\Omega^{L'}}}}{\rightsquigarrow} K' \overset{\pi^{L'}}{\to} L',$$

$$D_L^G : \overline{\Omega^G} \overset{\Phi|_{\overline{\Omega^G}}}{\rightsquigarrow} K \overset{\pi^L}{\to} L \subset G \quad \text{and} \quad D_{L'}^G : \overline{\Omega^G} \overset{\Phi|_{\overline{\Omega^G}}}{\rightsquigarrow} K' \overset{\pi^{L'}}{\to} L' \subset G,$$

$\varphi^L = i - \pi^L \circ \Phi|_{\overline{\Omega^L}}$, $\varphi^{L'} = i - \pi^{L'} \circ \Phi|_{\overline{\Omega^{L'}}}$, $\varphi_L^G = i - \pi^L \circ \Phi|_{\overline{\Omega^G}}$ and $\varphi_{L'}^G = i - \pi^{L'} \circ \Phi|_{\overline{\Omega^G}}$.

By Proposition 2.11, we have the following equalities:

$$\mathrm{Deg}\,(\varphi^L, D^L, \Omega^L, 0) = \mathrm{Deg}\,(\varphi_L^G, D_L^G, \Omega^G, 0)$$

and

$$\mathrm{Deg}\,(\varphi^{L'}, D^{L'}, \Omega^{L'}, 0) = \mathrm{Deg}\,(\varphi_{L'}^G, D_{L'}^G, \Omega^G, 0).$$

Consider the following decomposition:

$$D_{L,K'}^G : \overline{\Omega^G} \overset{\Phi|_{\overline{\Omega^G}}}{\rightsquigarrow} K' \overset{\pi^L|_{K'}}{\longrightarrow} G$$

and put $\varphi_{L,K'}^G = i - \pi^L|_{K'} \circ \Phi|_{\overline{\Omega^G}}$.

By Proposition 2.10,

$$\mathrm{Deg}\,(\varphi_{L,K'}^G, D_{L,K'}^G, \Omega^G, 0) = \mathrm{Deg}\,(\varphi^{L'}, D^{L'}, \Omega^{L'}, 0),$$

and Proposition 2.12 implies

$$\mathrm{Deg}\,(\varphi_{L,K'}^G, D_{L,K'}^G, \Omega^G, 0) = \mathrm{Deg}\,(\varphi^L, D^L, \Omega^L, 0),$$

which completes the proof. $\square$

Remark 2.13. Note that in a normed space E, we can find for every compact set A, by a theorem of Girolo (see [Gi]), a compact neighbourhood retract K of E such that $A \subset K$. This permits us to construct the topological degree in normed spaces in an analogous way and show that Deg is independent of the choice of K.

Theorem 2.14. *The topological degree* Deg *in infinite dimensional spaces has all the standard properties (additivity, existence, localization, homotopy and multiplicativity).*

The proof is an easy consequence of Theorem 2.9 and the construction of Deg. We omit the details.

Remark 2.15. The above construction of Deg can be analogously realized in the following situation:

> "E is a Fréchet space (or a normed space), $p \in E$, Ω is an open subset of E, and $\Phi \in J(\overline{\Omega}, E)$ is a compact map such that $p \notin \varphi(x)$ for every $x \in \partial\Omega$ $(\varphi = i - \Phi)$."

We can define $\mathrm{Deg}\,(\varphi, \Omega, p)$ with the usual properties.

The following fact shows a connection between the two degrees considered above.

Proposition 2.16. (Translation) *Let* Ω *be an open subset of* E, $p \in E$, *and let* φ *be a compact field associated with* $\Phi \in J(\overline{\Omega}, E)$ *and such that* $p \notin \varphi(x)$, *for every* $x \in \partial\Omega$. *Define* $\psi : \overline{\Omega} \rightsquigarrow E$ *as follows:* $\psi(x) = \varphi(x) - p$, *for every* $x \in \overline{\Omega}$.

Then $\psi \in F_{\partial\Omega}(\overline{\Omega}, E)$ *and*

$$\mathrm{Deg}\,(\varphi, \Omega, p) = \mathrm{Deg}\,(\psi, \Omega, 0).$$

The above equality can be considered as a definition of $\mathrm{Deg}\,(\varphi, \Omega, p)$.

Finally, we show the so-called normalization property of the degree.

Proposition 2.17. (Normalization) *If* $\Phi \in J(E, E)$ *is compact and* $\varphi = i - \Phi$, *then* $\mathrm{Deg}\,(\varphi, E, 0) = 1$.

Proof. Define $H : E \times [0, 1] \rightsquigarrow E$, $H(x, t) = t\Phi(x)$. One can see that H joins Φ and the constant map $c \equiv 0$. Hence, it is sufficient to show that $\mathrm{Deg}\,(i - c, E, 0) = 1$.

Let $e \neq 0$ be an arbitrary element of E and $L = \mathrm{lin}\,\{e\}$ the one dimensional subspace of E spanned by e. Let $\pi : \{0\} \to L$ be defined by $\pi(0) = 0$. We have the constant map $c^L = \pi \circ c$ and the identity map $id : L \to L$ as a compact field associated with c^L. Denote a decomposition of c^L by D^L. Now, let $W \subset L$ be a unit ball, which implies that $W \in N^P(Fix(c^L), L)$. By the properties of the Brouwer topological degree we obtain

$$\mathrm{Deg}\,(i - c, E, 0) = \mathrm{Deg}\,(i - c^L, D^L, L, 0) = \deg\,(i, W, 0) = 1,$$

and the proof is complete. □

<u>**Fixed point index.**</u> Let E be a Fréchet space and $X \subset E$ be a retract of E. This means that there is a retraction $r : E \to X (r|_X = id)$. Let D be an open subset of X, and let $\Phi \in J(D, X)$ be compact. Assume that $\mathcal{F} = Fix(\Phi)$ is compact. Thus there is an open subset D' of X such that $\mathcal{F} \subset D' \subset \overline{D'} \subset D$. This implies that $r^{-1}(\mathcal{F}) \subset r^{-1}(D') \subset \overline{r^{-1}(D')} \subset r^{-1}(\overline{D'}) \subset r^{-1}(D)$.

Define $\Omega = r^{-1}(D')$ and $\Phi' = \Phi \circ r|_{\overline{\Omega}}$. One can show that $Fix(\Phi') \cap \partial\Omega = \emptyset$.

We define

$$\operatorname{Ind}(\Phi, X, r, D) := \operatorname{Deg}(i - \Phi', \Omega, 0) \tag{2}$$

and put $\operatorname{Ind}(\Phi, X, r, \emptyset) = 0$.

By the localization property of the degree (1) one can easily show that the above index does not depend on the choice of the set D'.

Theorem 2.18 (Properties of Ind).

(i) (Additivity) *If D_1 and D_2 are open in X, $D_1 \cup D_2 \subset D, D_1 \cap D_2 = \emptyset$, $\Phi \in J(D, X)$ is a compact map and $Fix(\Phi) \subset D_1 \cup D_2$ is compact, then*

$$\operatorname{Ind}(\Phi, X, r, D) = \operatorname{Ind}(\Phi|_{D_1}, X, r, D_1) + \operatorname{Ind}(\Phi|_{D_2}, X, r, D_2).$$

(ii) (Existence) *If* $\operatorname{Ind}(\Phi, X, r, D) \neq 0$, *then $Fix(\Phi) \neq \emptyset$.*

(iii) (Localization) *If $D' \subset D$ are open in X, $\Phi \in J(D, X)$ is a compact map and $Fix(\Phi) \subset D'$ is compact, then*

$$\operatorname{Ind}(\Phi, X, r, D) = \operatorname{Ind}(\Phi|_{D'}, X, r, D').$$

(iv) (Homotopy) *If D is open in X, $H \in J(D \times [0,1], X)$ is a compact map and the set $\{x \in D \,:\, \exists t \in [0,1] : \, x \in H(x,t)\}$ is compact, then*

$$\operatorname{Ind}(H(\cdot, 0), X, r, D) = \operatorname{Ind}(H(\cdot, 1), X, r, D).$$

(v) (Multiplicativity) *Let $X_i \subset E_i, i = 1, 2,$ be retracts of two Fréchet spaces, let D_i be open in X_i, and let $\Phi_i \in J(D_i, X_i)$ be compact maps such that $Fix(\Phi_i)$ are compact sets. Then*

$$\begin{aligned}&\operatorname{Ind}(\Phi_1 \times \Phi_2, X_1 \times X_2, r_1 \times r_2, D_1 \times D_2)\\ &\qquad = \operatorname{Ind}(\Phi_1, X_1, r_1, D_1)\operatorname{Ind}(\Phi_2, X_2, r_2, D_2).\end{aligned}$$

(vi) (Normalization) *If $D = X$, then* $\operatorname{Ind}(\Phi, X, r, D) = 1$.

The proof is immediate. It is sufficient to use the definition of Ind and apply analogous properties of Deg .

Now, let us consider a compact neighbourhood retract X of the Fréchet space E. Assume that D is an open subset of X and $\Phi \in J(D, X)$ is such that $Fix(\Phi)$ is compact.

There exist an open subset X' of E and a retraction $r : X' \to X$. Let $D' \subset X$ be an open subset such that $\mathcal{F} \subset D' \subset \overline{D'} \subset D$. Of course, $r^{-1}(D')$ is open in X' and hence, in E. But, unfortunately, $\overline{r^{-1}(D')}$ need not be a subset of X'.

By the compactness of X, we can find $\eta > 0$ such that $\overline{N_\eta(X)} \subset X'$. Define $\Omega = r^{-1}(D') \cap N_\eta(X)$. Then $\overline{\Omega} \subset X'$, and $\overline{\Omega}$ is a closed subset of E (as a closed subset of $\overline{N_\eta(X)}$). Thus we can define $\Phi' := \Phi \circ r|_{\overline{\Omega}}$ with the properties $\Phi' \in J(\overline{\Omega}, E)$ and $Fix(\Phi') \cap \partial\Omega = \emptyset$.

Define

$$\operatorname{Ind}(\Phi, X, r, D) := \operatorname{Deg}(i - \Phi', \Omega, 0). \tag{3}$$

By means of the localization property of Deg one can show the independence of the choice of η and D'.

The easy proof of the following result is similar to that of Theorem 2.18.

Theorem 2.19. *The index* Ind *defined above (on compact neighbourhood retracts of Fréchet spaces) has the usual properties the fixed point index (see Theorem 2.18).*

Remark 2.20. Remark 2.13 implies that one can construct fixed point index (by the same method) for compact J–maps defined on open subsets of retracts or for J–maps defined on open subsets of compact neighbourhood retracts of normed spaces.

Remark 2.21. Constructions of the above indices can be followed with few changes to define the fixed point index in two following cases:

(i) E is a Fréchet space (normed), X is a retract of E, $\Phi \in J(X)$, D is an open subset of X, $Fix(\Phi) \cap \partial D = \emptyset$ and Φ is compact.

(ii) E is a Fréchet space (normed), X is a compact neighbourhood retract of E, $\Phi \in J(X)$, D is an open subset of X and $Fix(\Phi) \cap \partial D = \emptyset$ (see [GGK] for another method of defining fixed point index on compact ANRs).

We often need to study fixed points for maps defined on sufficiently fine sets (possibly with empty interior), but with values outside of them. Making use of the previous results, we are in a position to make the following construction.

Assume that X is a retract of the Fréchet space E and D is an open subset of X. Let $\Phi \in J(D, E)$ be locally compact, let $Fix(\Phi)$ be compact, and let the following condition hold:

$$\text{(A)} \qquad \forall\, x \in Fix(\Phi)\ \exists U_x \ni x, U_x \text{ is open in } D: \ \ \Phi(U_x) \subset X.$$

The class of locally compact J–maps from D to E with compact fixed point set and satisfying (A) will be denoted by the symbol $J_A(D, E)$. We say that $\Phi, \Psi \in J_A(D, E)$ are homotopic in $J_A(D, E)$, if there exists a homotopy $H \in J(D \times [0, 1], E)$ such that $H(\cdot, 0) = \Phi$, $H(\cdot, 1) = \Psi$, for every $x \in D$ there is an open neighbourhood V_x of x in D such that $H|_{V_x \times [0,1]}$ is compact, and

$$(A_H) \quad \forall x \in D\ \forall t \in [0, 1]\ [x \in H(x, t) \Longrightarrow \exists U_x \ni x, U_x \text{ is open in } D: \ \ H(U_x \times [0, 1]) \subset X].$$

Note that the condition (A_H) is equivalent to the following one:

If $\{x_j\}_{j\geq 1} \subset D$ converges to $x \in H(x, t)$ for some $t \in [0, 1]$, then

$$H(\{x_j\} \times [0, 1]) \subset X \text{ for } j \text{ sufficiently large.}$$

Let $\Phi \in J_A(D, E)$. Then $Fix(\Phi) \subset \bigcup\{U_x \ : \ x \in Fix(\Phi)\} \cap V =: D' \subset D$ and $\Phi(D') \subset X$, where V is a neighbourhood of the set $Fix(\Phi)$ such that $\Phi|_V$ is compact (by the compactness of $Fix(\Phi)$ and local compactness of Φ) and U_x is a neighbourhood of x such as in (A).

Define

$$\text{(4)} \qquad \operatorname{Ind}_A(\Phi, X, r, D) = \operatorname{Ind}(\Phi|_{D'}, X, r, D').$$

The localization property of Ind defined in (2) implies that the definition is independent of the choice of D'.

In the following theorem we give some properties of Ind_A which will be used in the proof of the continuation Theorem 2.23. The simple proof is omitted.

Theorem 2.22. *(i)* (Existence) *If* $\operatorname{Ind}_A(\Phi, X, r, D) \neq 0$*, then* $Fix(\Phi) \neq \emptyset$.

(ii) (Localization) *If* $D_1 \subset D$ *are open subsets of a retract* X *of a space* E, $\Phi \in J_A(D, E)$ *is compact, and* $Fix(\Phi)$ *is a compact subset of* D_1*, then*

$$\operatorname{Ind}_A(\Phi, X, r, D) = \operatorname{Ind}_A(\Phi, X, r, D_1).$$

(iii) (HOMOTOPY) *If H is a homotopy in $J_A(D,E)$, then*

$$\operatorname{Ind}_A(H(\cdot,0),X,r,D)=\operatorname{Ind}_A(H(\cdot,1),X,r,D).$$

(iv) (NORMALIZATION) *If $\Phi\in J(X)$ is a compact map, then* $\operatorname{Ind}_A(\Phi,X,r,X)=1$.

We can now formulate the continuation principle, which is a generalization of Theorem 2.1 in [FP2] in the case of emptiness of a domain's interior.

Theorem 2.23. *Let X be a retract of the Fréchet space E, let D be an open subset of X, and let H be a homotopy in $J_A(D,E)$ such that*

(i) $H(\cdot,0)(D)\subset X$,
(ii) *There exists $H'\in J(X)$ such that $H'|_D=H(\cdot,0)$, H' is compact and $Fix(H')\cap(X\setminus D)=\emptyset$.*

Then there exists $x\in D$ such that $x\in H(x,1)$.

Proof. Applying the localization property, we obtain

$$\operatorname{Ind}_A(H(\cdot,0),X,r,D)=\operatorname{Ind}_A(H(\cdot,0),X,r,X).$$

By the normalization property, $\operatorname{Ind}_A(H(\cdot,0),X,r,X)=1$. Thus, by the homotopy property, $\operatorname{Ind}_A(H(\cdot,0),X,r,D)=\operatorname{Ind}_A(H(\cdot,1),X,r,D)=1$, which implies that $H(\cdot,1)$ has a fixed point. □

Corollary 2.24. *Let X be a retract of the Fréchet space E, and let H be a homotopy in $J_A(X,E)$ such that $H(x,0)\subset X$ for every $x\in X$ and $H(\cdot,0)$ is compact. Then $H(\cdot,1)$ has a fixed point.*

Corollary 2.25. *Let X be a retract of the Fréchet space E, D an open subset of X and H a homotopy in $J_A(D,E)$. Assume that $H(x,0)=x_0$ for every $x\in D$. Then there exists $x\in D$ such that $x\in H(x,1)$.*

Proof. It is sufficient to define $H'\in J(X)$, $H'(x)=x_0$ and to use Theorem 2.23. □

The following result generalizes the well-known Ky Fan theorem in the case of Fréchet spaces (see [Fa]).

Corollary 2.26. *Let X be a retract of the Fréchet space E, and let $\Phi\in J(X)$ be compact. Then Φ has a fixed point.*

Some problems for differential equations motivate us to consider a weaker condition on H than (A_H). Unfortunately, then we cannot use the fixed point index technique described above. However, applying fixed point Theorem 2.8 we obtain the following result generalizing Theorem 1.1 in [FP1] into the case of set-valued maps.

Theorem 2.27. *Let X be a closed convex subset of the Fréchet space E and let $H\in J(X\times[0,1],E)$ be compact. Assume that*

(i) $H(x,0)\subset X$ *for every* $x\in X$,
(ii) *for any $(x,t)\in\partial X\times[0,1)$ with $x\in H(x,t)$ there exist open neighbourhoods U_x of x in X and I_t of t in $[0,1)$ such that $H((U_x\cap\partial X)\times I_t)\subset X$.*

Then there exists a fixed point of $H(\cdot,1)$.

The idea of the proof is taken from [FP1]. We need the following fact.

Lemma 2.28. *Let X be a convex closed subset of the Fréchet space E and $K \subset E$ a compact subset such that $K \cap X \neq \emptyset$. Then, for every $\varepsilon > 0$, there exists a map $\pi_\varepsilon : K \to E$ with image contained in a finite dimensional space and such that*

(i) $\pi_\varepsilon(K \cap X) \subset X$,
(ii) $d(\pi_\varepsilon(x), x) < \varepsilon$ *for all* $x \in K$.

Proof. Let $\varepsilon > 0$ be given. By the compactness of K there exist $x_1, x_2, \dots, x_r \in K \cap X$ such that $K \cap X \subset \bigcup_{i=1}^r N_\varepsilon(x_i)$. Analogously, there are $\delta, 0 < \delta < \varepsilon$, and $x_{r+1}, x_{r+2}, \dots, x_s \in K \setminus \bigcup_{i=1}^r N_\varepsilon(x_i)$ such that

$$K \setminus \bigcup_{i=1}^{r} N_\varepsilon(x_i) \subset \bigcup_{i=r+1}^{s} N_\delta(x_i) \quad \text{and} \quad (K \cap X) \cap (\bigcup_{i=r+1}^{s} N_\delta(x_i)) = \emptyset.$$

Define $\rho_i : K \to \mathbb{R}_+$ by

$$\rho_i(x) = \begin{cases} \varepsilon - d(x, x_i), & \text{for } x \in N_\varepsilon(x_i),\ i = 1, 2, \dots, r, \\ \delta - d(x, x_i), & \text{for } x \in N_\delta(x_i),\ i = r+1, \dots, s, \\ 0 & \text{elsewhere.} \end{cases}$$

Now, let $\pi_\varepsilon : K \to E$ be defined by $\pi_\varepsilon(x) = \sum_{i=1}^s \sigma_i(x) x_i$, where

$$\sigma_i(x) = \rho_i(x)(\sum_{j=1}^{s} \rho_j(x))^{-1}.$$

It is easy to see that π_ε is continuous and $d(\pi_\varepsilon(x), x) < \varepsilon$ for every $x \in K$. By the construction, for each $x \in K \cap X$ we have $\rho_i(x) = 0$ for $i = r+1, \dots, s$, and hence $\pi_\varepsilon(x)$ belongs to the convex hull of $x_1, x_2, \dots, x_r$. By the convexity of X, $\pi_\varepsilon(K \cap X) \subset X$. The proof is complete. □

Proof of Theorem 2.27. First, let us suppose that E is finite dimensional. So, we can prove some generalization of our result. Namely, we assume that $H \in D(X \times [0,1], E)$.

Let $r : E \to X$ be a retraction which sends a point into the nearest point in X. Define

$$\mathcal{F} = \{x \in E \,:\, x \in H(r(x), \lambda) \text{ for some } \lambda \in [0,1]\},$$
$$\mathcal{F}_\lambda = \{x \in E \,:\, x \in H(r(x), \lambda)\}.$$

By Theorem 2.8, we obtain that $\mathcal{F}_\lambda \neq \emptyset$ for every $\lambda \in [0,1]$.

Notice that our assertion can be reformulated as follows: $\mathcal{F}_1 \cap X \neq \emptyset$. Suppose, for a contradiction, that $\mathcal{F}_1 \cap X = \emptyset$. Since $\mathcal{F}_1$ is compact, $\operatorname{dist}(\mathcal{F}_1, X) = 2\varepsilon > 0$ and there is an open set $V \supset X$ such that $\mathcal{F}_1 \cap \overline{V} = \emptyset$. We prove that there exists $(y, \lambda) \in \partial V \times [0,1)$ such that $H(r(y), \lambda) \ni y$. Suppose that it is not true.

By the upper semicontinuity of H, $\operatorname{dist}(\partial V, \mathcal{F} \cap \overline{V}) > 0$. Define the map $\sigma : E \to [0,1]$ as follows:

$$\sigma(x) = \max\left\{1 - \frac{\operatorname{dist}(x, \mathcal{F} \cap \overline{V})}{\operatorname{dist}(\partial V, \mathcal{F} \cap \overline{V})}, 0\right\}.$$

Obviously, σ is continuous, $\sigma(x) = 1$ in $\mathcal{F} \cap \overline{V}$, and $\sigma(x) = 0$ in $E \setminus V$.

Now, we have a decomposable map $\hat{H} : E \multimap E$, $\hat{H}(x) = H(r(x), \sigma(x))$. Thus, there exists a fixed point $y \in \hat{H}(y)$, which means that $y \in H(r(y), \sigma(y))$. Notice that $y \notin E \setminus V$, because $\mathcal{F}_0 \subset X$. Therefore, $y \in \mathcal{F} \cap \overline{V}$, which implies that $\sigma(y) = 1$ and hence $y \in \mathcal{F}_1 \cap \overline{V}$, a contradiction.

Thus, we find for V a pair $(y,\lambda) \in \partial V \times [0,1)$ such that $H(r(y),\lambda) \ni y$. Take a sequence of open neighbourhoods V_n of X defined by $V_n = \{x \in E \,:\, \mathrm{dist}\,(x,X) < \varepsilon/n\}$. Then, for every $n \in \mathbb{N}$, we can find $y_n \in \partial V_n,\ \lambda_n \in [0,1)$ and $x_n \in X$ such that $H(r(y_n),\lambda_n) \ni y_n$ and $||x_n - y_n|| < \varepsilon/n$. By the compactness of $[0,1]$ and $\overline{H(X \times [0,1])}$, we can assume that $\lambda_n \longrightarrow \lambda \in [0,1]$ and $y_n \longrightarrow y \in E$. Thus $x_n \longrightarrow y$ and $y \in X$, because X is closed. This implies that $r(y_n) \longrightarrow r(y)$ and, since $r(y) = y$ and H is u.s.c., $y \in H(y,\lambda)$. However, by the hypothesis $(\mathcal{F}_1 \cap X = \emptyset)$ we have $\lambda < 1$. By (ii) we get that there are open neighbourhoods $U_y \subset E$ and $I_\lambda \subset [0,1)$ of y and λ, respectively, such that $H((U_y \cap \partial X) \times I_\lambda) \subset X$. Notice that $r(y_n) \in U_y,\ r(y_n) \in \partial X$ (by the assumption on r), $y_n \in H(r(y_n),\lambda_n)$ and $y_n \notin X$, which is a contradiction.

Now, let E be infinite dimensional. Since H has a closed graph, it is sufficient to show that $\inf\{d(x,y) \,:\, x \in X,\ y \in H(x,1)\} = 0$.

Suppose that

$$\inf\{d(x,y) \,:\, x \in X,\ y \in H(x,1)\} > \varepsilon > 0. \tag{5}$$

It follows that $x \notin H(x,1)$ in ∂X. Thus, by (ii), we can find for every $(x,\lambda) \in \partial X \times [0,1], x \in H(x,\lambda)$, an open neighbourhood $\Omega_{(x,\lambda)}$ in $\partial X \times [0,1]$ such that $H(\Omega_{(x,\lambda)}) \subset X$. Define

$$\Omega = \bigcup \{\Omega_{(x,\lambda)} \,:\, (x,\lambda) \in \partial X \times [0,1], x \in H(x,\lambda)\}.$$

Then $H(\Omega) \subset X$. Note that, by the "closed graph" argument, we can assume ε is such that $\{(x,\lambda) \in \partial X \times [0,1] \,:\, \mathrm{dist}\,(x, H(x,\lambda)) < \varepsilon\} \subset X$.

Denote $K = \overline{H(X \times [0,1])}$. We know that K is compact and $K \cap X \neq \emptyset$, since $H(\cdot,0) \in J(X)$ and X is a retract of E. By Lemma 2.28, there exists a map $\pi_\varepsilon : K \to E$ such that $\pi_\varepsilon(K \cap X) \subset X$, $\pi_\varepsilon(K) \subset L$ $(\dim L < \infty)$ and $d(\pi_\varepsilon(x),x) < \varepsilon$, for all $x \in K$.

Let us define $H_\varepsilon := \pi_\varepsilon \circ H : (L \cap X) \times [0,1] \rightsquigarrow L$. Obviously, $H_\varepsilon \in D(X' \times [0,1], L)$, where $X' = L \cap X$. Notice that $H_\varepsilon(X' \times \{0\}) = \pi_\varepsilon(H(X' \times \{0\})) \subset \pi_\varepsilon(X \cap K) \subset X'$. We denote by $\partial_L X'$ the boundary of X' in L. Then, for $x \in \partial_L X'$ such that $x \in H_\varepsilon(x,\lambda)$ for some $\lambda \in [0,1)$, we have $x \in \partial X$ and $x = \pi_\varepsilon(y)$ for some $y \in H(x,\lambda)$. Thus $d(x,y) < \varepsilon$, and hence $(x,\lambda) \in \Omega$. But this implies that $H(\Omega'_{(x,\lambda)}) \subset X$, where $\Omega'_{(x,\lambda)}$ is an open neighbourhood of (x,λ) in $\partial_L X' \times [0,1)$, and, consequently, $H_\varepsilon(\Omega'_{(x,\lambda)}) \subset X'$.

The first part of the proof permits us to conclude that there exists a fixed point $x \in H_\varepsilon(x,1)$. By the property of π_ε, there is $y \in H(x,1)$ such that $d(x,y) < \varepsilon$. This, however, contradicts our assumption (5). □

Remark 2.29. Note that the convexity of X in Theorem 2.27 is essential only in the infinite dimensional case. For the proof we have to intersect X with a finite dimensional subspace L.

2.4. **Some applications for differential inclusions.** We are interested in existence problems for ordinary differential inclusions on noncompact intervals. Let us start with some definitions.

Let J be an interval in $\mathbb{R}$. We say that a map $x : J \to \mathbb{R}^n$ is *locally absolutely continuous* if x is absolutely continuous on every compact subset of J. The set of all locally absolutely continuous maps from J to $\mathbb{R}^n$ will be denoted by $AC_{loc}(J,\mathbb{R}^n)$.

Consider the inclusion

$$\dot{x}(t) \in F(t, x(t)), \tag{6}$$

where F is a set-valued *Carathéodory map*, i.e. it has the following properties:

(C1) The set $F(t, x)$ is nonempty, compact and convex for all $(t, x) \in J \times \mathbb{R}^n$;
(C2) The map $F(t, \cdot)$ is u.s.c. for almost all $t \in J$;
(C3) The map $F(\cdot, x)$ is measurable for all $x \in \mathbb{R}^n$.

By a solution of the inclusion (6) we mean a locally absolutely continuous map x such that (6) holds for almost all $t \in J$.

We recall two known results which are needed in the sequel.

Theorem 2.30 (cf. [AC, Theorem 0.3.4])**.** *Assume that the sequence of absolutely continuous maps $x_k : K \to \mathbb{R}^n$ (K is a compact interval) satisfies the following conditions:*

(i) *The set $\{x_k(t) |\ k \in \mathbb{N}\}$ is bounded for every $t \in K$.*

(ii) *There is an integrable function (in the sense of Lebesgue) $\alpha : K \to \mathbb{R}$ such that*

$$|\dot{x}_k(t)| \le \alpha(t) \quad \text{for a.a. } t \in K \text{ and for all } k \in \mathbb{N}.$$

Then there exists a subsequence (which we denote also by $\{x_k\}$) that converges to an absolutely continuous map $x : K \to \mathbb{R}^n$ in the following sense:

(iii) *$\{x_k\}$ converges uniformly to x;*

(iv) *$\{\dot{x}_k\}$ converges weakly in $L^1(K, \mathbb{R}^n)$ to $\dot{x}$.*

Theorem 2.31 (Mazur; cf. [Mu, Theorem 21.4])**.** *If E is a normed space and the sequence $\{x_k\} \subset E$ is weakly convergent to $x \in E$, then there exists a sequence of linear combinations $y_m = \sum_{k=1}^m a_{mk} x_k$, where $a_{mk} \ge 0$ for $k = 1, 2, \dots, m$ and $\sum_{k=1}^m a_{mk} = 1$, which is strongly convergent to x.*

The following result is crucial.

Proposition 2.32. *Let $G : J \times \mathbb{R}^n \times \mathbb{R}^m \multimap \mathbb{R}^n$ be a Carathéodory map and let S be a nonempty subset of $AC_{loc}(J, \mathbb{R}^n)$.*

Assume that:

(i) *There exists a subset Q of $C(J, \mathbb{R}^m)$ such that, for any $q \in Q$, the set $T(q)$ of all solutions of the boundary value problem*

$$\begin{cases} \dot{x} \in G(t, x(t), q(t)), & \text{for a.a. } \ t \in J, \\ x \in S, \end{cases}$$

is nonempty.

(ii) *$T(Q)$ is bounded in $C(J, \mathbb{R}^n)$.*

(iii) *There exists a locally integrable function $\alpha : J \to \mathbb{R}$ such that*

$$|G(t, x(t), q(t))| = \sup\{|y| \ : \ y \in G(t, x(t), q(t))\} \le \alpha(t), \qquad \text{a.e. in } J,$$

for any pair $(q, x) \in \Gamma_T$.

Then $T(Q)$ is a relatively compact subset of $C(J, \mathbb{R}^n)$. Moreover, under the assumptions (i) – (iii) *the multivalued operator $T : Q \multimap S$ is u.s.c. with compact values if and only if the following condition is satisfied:*

(iv) *Given a sequence $\{(q_k, x_k)\} \subset \Gamma_T$, if $\{(q_k, x_k)\}$ converges to (q, x) with $q \in Q$, then $x \in S$.*

Proof. For the relative compactness of $T(Q)$, it is sufficient to show that all elements of $T(Q)$ are equicontinuous.

By (iii), for every $x \in T(Q)$, we have $|\dot{x}(t)| \leq \alpha(t)$ for a.a. $t \in J$, and

$$|x(t_1) - x(t_2)| \leq |\int_{t_1}^{t_2} \alpha(s)ds|.$$

This implies equicontinuity of all $x \in T(Q)$.

We show that the set Γ_T is closed.

Let $\Gamma_T \supset \{(q_k, x_k)\} \longrightarrow (q, x)$. Let K be an arbitrary compact interval such that α is integrable on K. By conditions (ii) and (iii), the sequence $\{x_k\}$ satisfies the assumptions of Theorem 2.30; thus there exists a subsequence (also denoted by $\{x_k\}$) uniformly convergent to x on K (because the limit is unique) and such that $\{\dot{x}_k\}$ weakly converges to $\dot{x}$ in L^1. Therefore, $\dot{x}$ belongs to the weak closure of the set $\operatorname{conv}\{\dot{x}_m : m \geq k\}$ for every $k \geq 1$. By Theorem 2.31, $\dot{x}$ belongs also to the strong closure of this set. Hence, for every $k \geq 1$, there is $z_k \in \operatorname{conv}\{\dot{x}_m : m \geq k\}$ such that $||z_k - \dot{x}||_{L^1} \leq 1/k$. This implies that there exists a subsequence $z_{k_l} \longrightarrow \dot{x}$ a.e. in K.

Let $s \in K$ be such that

(i) $G(s, \cdot, \cdot)$ is u.s.c.;
(ii) $\lim_{l \to \infty} z_{k_l}(s) = \dot{x}(s)$;
(iii) $\dot{x}_k(s) \in G(s, x_k(s), q_k(s))$.

Let $\varepsilon > 0$. There is a $\delta > 0$ such that $G(s, z, p) \subset N_\varepsilon(G(s, x(s), q(s)))$ whenever $|x(s) - z| < \delta$ and $|q(s) - p| < \delta$. But we know that there exists $N \geq 1$ such that $|x(s) - x_m(s)| < \delta$ and $|q(s) - q_m(s)| < \delta$ for every $m \geq N$. Hence,

$$\dot{x}_k(s) \in G(s, x_k(s), q_k(s)) \subset N_\varepsilon(G(s, x(s), q(s))).$$

By the convexity of $G(s, x(s), q(s))$, for $k_l \geq N$ we have

$$z_{k_l}(s) \in N_\varepsilon(G(s, x(s), q(s))).$$

Thus $\dot{x}(s) \in \overline{N_\varepsilon(G(s, x(s), q(s)))}$, for every $\varepsilon > 0$, and so $\dot{x}(s) \in G(s, x(s), q(s))$. Since K was arbitrary, $\dot{x}(t) \in G(t, x(t), q(t))$ a.e. in J. □

We can now state one of the main results of this subsection.

Theorem 2.33. *Consider the boundary value problem*

$$\begin{cases} \dot{x} \in F(t, x(t)), & \textit{for a.a.} \quad t \in J, \\ x \in S, \end{cases} \tag{7}$$

where J is a given real interval, $F : J \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a Carathéodory map and S is a subset of $AC_{loc}(J, \mathbb{R}^n)$.

Let $G : J \times \mathbb{R}^n \times \mathbb{R}^n \times [0, 1] \rightsquigarrow \mathbb{R}^n$ be a Carathéodory map such that

$$G(t, c, c, 1) \subset F(t, c) \quad \textit{for all } (t, c) \in J \times \mathbb{R}^n.$$

Assume that the following four conditions hold:

(i) *There exist a retract Q of $C(J, \mathbb{R}^n)$ and a closed bounded subset S_1 of S such that the associated problem*

$$\begin{cases} \dot{x} \in G(t, x(t), q(t), \lambda), & \textit{for a.a.} \quad t \in J, \\ x \in S_1, \end{cases} \tag{8}$$

is solvable with R_δ-set of solutions, for each $(q, \lambda) \in Q \times [0, 1]$.

(ii) *There exists a locally integrable function* $\alpha : J \to \mathbb{R}$ *such that*

$$|G(t, x(t), q(t), \lambda)| \leq \alpha(t), \qquad \text{a.e. in } J,$$

for any $(q, \lambda, x) \in \Gamma_T$, *where* T *denotes the set-valued map which assigns to any* $(q, \lambda) \in Q \times [0, 1]$ *the set of solutions of (8).*

(iii) $T(Q \times \{0\}) \subset Q$.

(iv) *If* $Q \ni q_j \longrightarrow q \in Q, q \in T(q, \lambda)$, *then there exists* $j_0 \in \mathbb{N}$ *such that, for every* $j \geq j_0$, $\theta \in [0, 1]$ *and* $x \in T(q_j, \theta)$, *we have* $x \in Q$.

Then problem (7) has a solution.

Proof. Consider the set

$$Q' = \{y \in C(J, \mathbb{R}^{n+1}) \ : \ y(t) = (q(t), \lambda), \, q \in Q, \, \lambda \in [0, 1]\}.$$

By Proposition 2.32 we obtain that the set-valued map $T : Q \times [0, 1] \rightsquigarrow S_1$ is u.s.c., and hence it belongs to the class $J(Q \times [0, 1], C(J, \mathbb{R}^n))$. Moreover, it has a relatively compact image. Assumption (iv) implies that T is a homotopy in $J_A(Q, C(J, \mathbb{R}^n))$. Corollary 2.24 now gives the existence of a fixed point of $T(\cdot, 1)$. However, by the hypothesis, it is a solution of (7). □

Note that the conditions (iii) and (iv) in the above theorem hold if $S_1 \subset Q$. This remark permits us to obtain the generalization of Theorem 1.2 in [CFM2], where the result has been proved for a single-valued right hand side of the equation and for a convex set of parameters.

Corollary 2.34. *Consider the boundary value problem*

$$\begin{cases} \dot{x} \in F(t, x(t)), & \text{for a.a. } \ t \in J, \\ x \in S, \end{cases} \tag{9}$$

where J *is a given real interval,* $F : J \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ *is a Carathéodory map and* S *is a subset of* $AC_{loc}(J, \mathbb{R}^n)$.

Let $G : J \times \mathbb{R}^n \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ *be a Carathéodory map such that*

$$G(t, c, c) \subset F(t, c) \quad \text{for all } (t, c) \in J \times \mathbb{R}^n.$$

Assume that

(i) *there exists a retract* Q *of* $C(J, \mathbb{R}^n)$ *such that the associated problem*

$$\begin{cases} \dot{x} \in G(t, x(t), q(t)), & \text{for a.a. } \ t \in J, \\ x \in S \cap Q, \end{cases} \tag{10}$$

has an R_δ*-set of solutions for each* $q \in Q$;

(ii) *there exists a locally integrable function* $\alpha : J \to \mathbb{R}$ *such that*

$$|G(t, x(t), q(t))| \leq \alpha(t), \qquad \text{a.e. in } J,$$

for any pair $(q, x) \in \Gamma_T$;

(iii) $T(Q)$ *is bounded in* $C(J, \mathbb{R}^n)$ *and* $\overline{T(Q)} \subset S$.

Then problem (9) has a solution.

Making use of the Eilenberg-Montgomery fixed point theorem (see [EM]) and modifying the proof of Theorem 2.33, we easily obtain the generalization of Theorem 1.1 in [ACZ].

Corollary 2.35. *Consider the problem (9) and assume that all the assumptions of Corollary 2.34 hold with the convex closed set Q and acyclic sets of solutions of (10).*

Then problem (9) has a solution.

Let us remark that in applications solution sets are, in fact, R_δ sets.

Since $C^{n-1}(J)$ can be considered as a subspace of $C(J,\mathbb{R}^n)$, we can also apply the previous results for nth-order scalar differential inclusions. To solve an existence problem, one should check suitable *a priori* bounds for all the derivatives up to the order $n-1$. Our technique simplifies things. Let us describe it below.

We need the following lemma ([CFM2], Lemma 2.1) relating to the Banach space $H^{n,1}(I)$[2]:

Lemma 2.36. *Let I be a compact real interval and let $a_0, a_1, \dots, a_{n-1} : I \times \mathbb{R}^n \to \mathbb{R}$ be Carathéodory functions. Given any $q \in C^{n-1}(I)$ consider the following linear nth-order differential operator $L_q : H^{n,1}(I) \to L^1(I)$:*

$$L_q(x)(t) = x^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t, q(t), \dots, q^{(n-1)}(t))x^{(i)}(t).$$

Assume there exist a subset Q of $C^{n-1}(I)$ and an L^1 function $\beta : I \to \mathbb{R}$ such that, for any $q \in Q$ and any $i = 0, 1, \dots, n-1$ we have

$$|a_i(t, q(t), \dots, q^{(n-1)}(t))| \le \beta(t) \qquad \text{a.e. in } I.$$

Then the following two norms are equivalent in $H^{n,1}(I)$:

$$||x|| = \sum_{i=0}^{n-1} \sup_{t\in I} |x^{(i)}(t)| + \int_I |x^{(n)}(t)|dt,$$

$$||x||_Q = \sup_{t\in I} |x(t)| + \sup_{q\in Q} \int_I |L_q(x)(t)|dt.$$

Corollary 2.37. *Consider the scalar problem*

$$\text{(11)} \qquad \begin{cases} x^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t, x(t), \dots, x^{(n-1)}(t))x^{(i)}(t) \\ \qquad \in F(t, x(t), \dots, x^{(n-1)}(t)) \quad \text{for a.a. } t \in J, \\ x \in S, \end{cases}$$

where $J \subset \mathbb{R}$, $S \subset C(J)$, and a_i, F are Carathéodory maps on $J \times \mathbb{R}^n$.

Suppose that there exists a Carathéodory map $G : J \times \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \rightsquigarrow \mathbb{R}^n$ such that, for every $c \in \mathbb{R}^n$ and $\lambda \in [0,1]$, $G(t,c,c,1) \subset F(t,c)$ a.e. in J.

Then problem (11) has a solution, if the following conditions are satisfied:

(i) *There is a retract Q of the space $C^{n-1}(J)$ such that, for every $(q,\lambda) \in Q \times [0,1]$, the problem*

(12)

$$\begin{cases} x^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t, q(t), \dots, q^{(n-1)}(t))x^{(i)}(t) \\ \qquad \in G(t, x(t), \dots, x^{(n-1)}(t), q(t), \dots, q^{(n-1)}(t), \lambda) \quad \text{for a.a. } t \in J, \\ x \in S \cap Q, \end{cases}$$

has an R_δ-set of solutions.

[2]By $H^{n,1}(I)$ we denote the Banach space of all C^{n-1} functions $x : I \to \mathbb{R}$, where I is a compact interval, with absolutely continuous n-th derivative.

(ii) *There is a locally integrable function* $\alpha : J \to \mathbb{R}$ *such that, for every* $i = 0, \dots, n-1$,

$$|a_i(t, q(t), \dots, q^{(n-1)}(t))| \le \alpha(t) \quad a.e. \text{ in } J$$

and

$$|G(t, x(t), \dots, x^{(n-1)}(t), q(t), \dots, q^{(n-1)}(t), \lambda)| \le \alpha(t) \quad a.e. \text{ in } J$$

for each $(q, \lambda, x) \in Q \times [0,1] \times C^{n-1}(J)$ *satisfying (12).*

(iii) $T(Q \times \{0\}) \subset Q$, *where* T *denotes the set-valued map which assigns to any* $(q, \lambda) \in Q \times [0,1]$ *the set of solutions of (12).*

(iv) *The set* $T(Q \times [0,1])$ *is bounded in* $C(J)$ *and its closure in* $C^{n-1}(J)$ *is contained in* S *(in particular, this holds if* $S \cap C^{n-1}(J)$ *is closed in* $C^{n-1}(J)$*).*

(v) *If* $\{q_j\} \subset Q$ *converges to* $q \in Q, q \in T(q, \lambda)$ *in* $C^{n-1}(J)$, *then there exists* $j_0 \in \mathbb{N}$ *such that, for every* $j \ge j_0$, $\theta \in [0,1]$ *and* $x \in T(q_j, \theta)$, *we have* $x \in Q$.

Proof. We construct a new problem in the following way:

Define $\tilde{F} : J \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$,

$$\begin{aligned} &\tilde{F}(t, x(t), \dots, x^{(n-1)}(t)) \\ &\quad = F(t, x(t), \dots, x^{(n-1)}(t)) - \sum_{i=0}^{n-1} a_i(t, x(t), \dots, x^{(n-1)}(t)) x^{(i)}(t). \end{aligned}$$

Denote $\bar{x}(t) = (x(t), \dots, x^{(n-1)}(t)) \in \mathbb{R}^n$ and define $F' : J \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$,

$$F'(t, \bar{x}(t)) = \{(\dot{x}(t), \dots, x^{(n-1)}(t), y) : y \in \tilde{F}(t, x(t), \dots, x^{(n-1)}(t))\}.$$

So, we have a problem

$$\left\{ \begin{array}{ll} \dot{\bar{x}} \in F'(t, \bar{x}(t)), & \text{for a.a.} \quad t \in J, \\ \bar{x} \in \bar{S}, \end{array} \right. \tag{13}$$

where $\bar{S}$ is an image of $S \cap C^{n-1}(J)$ via the inclusion $i : C^{n-1}(J) \to C(J, \mathbb{R}^n)$.

Analogously, we find the associated problem

$$\left\{ \begin{array}{ll} \dot{\bar{x}} \in G'(t, \bar{x}(t), \bar{q}(t), \lambda), & \text{for a.a.} \quad t \in J, \\ \bar{x} \in \bar{S} \cap \bar{Q}. \end{array} \right. \tag{14}$$

Notice that

1. $G'(t, \bar{x}(t), \bar{q}(t), 1) \subset F'(t, \bar{x}(t))$;
2. the set $\bar{Q} = i(Q)$ is a retract of $C(J, \mathbb{R}^n)$;
3. $\bar{S} \subset AC_{loc}(J, \mathbb{R}^n)$;
4. for every $(q, \lambda) \in Q \times [0,1]$, the sets of solutions of problems (12) and (14) are the same;
5. $\overline{\bar{T}(\bar{Q} \times [0,1])} \subset \bar{S}$, where $\bar{T}$ is a suitable map corresponding to T;

and

$$\begin{aligned} |G'(t, \bar{x}(t), \bar{q}(t), \lambda)| &\le |G(t, x(t), \dots, x^{(n-1)}(t), q(t), \dots, q^{(n-1)}(t), \lambda)| \\ &\quad + \sum_{i=0}^{n-1} |a_i(t, q(t), \dots, q^{(n-1)}(t))||x^{(i)}(t)| \\ &\le \alpha(t) + \alpha(t) \sum_{i=0}^{n-1} |x^{(i)}(t)|. \end{aligned}$$

Since $T(Q \times [0,1])$ is bounded in $C(J)$, there exists a positive continuous function $m : J \to \mathbb{R}$ such that $|x(t)| \le m(t)$ for all $t \in J$ and any $x \in T(Q \times [0,1])$. We show that $T(Q \times [0,1])$ is also bounded in $C^{n-1}(J)$. It is sufficient to prove that, for any compact subinterval I in J, there is a constant $M > 0$ such that

$$p_I(x) = \sum_{i=0}^{n-1} \sup_{t \in I} |x^{(i)}(t)| \leq M,$$

for all $x \in T(Q \times [0,1])$.

Let $I \subset J$ be an arbitrary compact interval. Using the notation in Lemma 2.36 we see that $p_I(x) \leq ||x||$ and, by the equivalence of norms,

$$||x|| \leq c||x||_Q \leq c \left(\max_{t \in I} m(t) + \int_I \alpha(t)dt \right) \leq M.$$

We conclude that $T(Q \times [0,1])$ is bounded in $C^{n-1}(J)$, which implies that $\bar{T}(\bar{Q} \times [0,1])$ is bounded in $C(J,\mathbb{R}^n)$. Moreover, there exists a continuous function $\phi : J \to \mathbb{R}$ such that

$$|G'(t, \bar{x}(t), \bar{q}(t), \lambda)| \leq \alpha(t)(1 + \phi(t)).$$

Obviously, the right-hand side of the above inequality is a locally integrable function.

Finally, an easy computation shows that the condition (iv) in Theorem 2.33 holds for $\bar{Q}$ and $\bar{T}$. By Theorem 2.33 there exists a solution of (13) as well as the one of (11). □

The same argument as in Corollary 2.34 shows how to generalize the analogous result in [CFM2] for the following scalar problem:

$$(15) \qquad \begin{cases} x^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t, x(t), \dots, x^{(n-1)}(t))x^{(i)}(t) \\ \qquad \in F(t, x(t), \dots, x^{(n-1)}(t)) \quad \text{for a.a.} \quad t \in J, \\ x \in S, \end{cases}$$

where $J \subset \mathbb{R}$, $S \subset C(J)$, and a_i, F are Carathéodory maps on $J \times \mathbb{R}^n$, by means of the following linearized problem:

$$(16) \qquad \begin{cases} x^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t, q(t), \dots, q^{(n-1)}(t))x^{(i)}(t) \\ \qquad \in G(t, x(t), \dots, x^{(n-1)}(t), q(t), \dots, q^{(n-1)}(t)) \quad \text{for a.a.} \quad t \in J, \\ x \in S \cap Q, \end{cases}$$

where Q is a retract of the space $C^{n-1}(J)$.

Theorem 2.27 gives consequences similar to those of Theorem 2.23. Unfortunately, the weakness of the assumption on solutions means that we have to assume the convexity of the set Q. In spite of this, the results given below are important because of the applications.

Theorem 2.38. *Consider the boundary value problem*

$$(17) \qquad \begin{cases} \dot{x} \in F(t, x(t)), \quad \textit{for a.a.} \quad t \in J, \\ x \in S, \end{cases}$$

where J is a given real interval, $F : J \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a Carathéodory map and S is a subset of $AC_{loc}(J, \mathbb{R}^n)$.

Let $G : J \times \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \rightsquigarrow \mathbb{R}^n$ be as in Theorem 2.33.

Assume that the assumptions (i) - (iii) *of Theorem 2.33 hold, with the convexity of the set Q, and*

(iv) *If* $\partial Q \times [0,1] \supset \{(q_j,\lambda_j)\}$ *converges to* $(q,\lambda) \in \partial Q \times [0,1), q \in T(q,\lambda)$*, then there exists* $j_0 \in \mathbb{N}$ *such that, for every* $j \geq j_0$*, and* $x_j \in T(q_j,\lambda_j)$*, we have* $x_j \in Q$*.*

Then the problem (17) has a solution.

The proof can be obtained immediately by using our continuation principle presented in Theorem 2.27.

Remark 2.39. If the associated problem (8) for G is uniquely solvable for every $(q,\lambda) \in Q \times [0,1]$, then, by continuity of T, we can reformulate the condition (iv) as follows:

(iv′) *If* $\{(x_j,\lambda_j)\}$ *is a sequence in* $S_1 \times [0,1]$*, with* $\lambda_j \longrightarrow \lambda \in [0,1)$ *and* x_j *converging to a solution* $x \in Q$ *of (8) (corresponding to* (x,λ)*), then* x_j *belongs to* Q *for* j *sufficiently large.*

Thus, we have a generalization of Theorem 2.1 in [FP1].

2.5. **Nontrivial examples.** Now we will give several nontrivial examples as applications of the results from Part 2.4.

Example 2.40. Consider the equation with constant coefficients $a_j, j = 1,\dots,n$,

$$x^{(n)} + \sum_{j=1}^{n} a_j x^{(n-j)} = f(t,x,\dots,x^{(n-1)}), \tag{18}$$

where f is a continuous function. Assume the asymptotic stability for the linear part, i.e. let $\operatorname{Re}\lambda_j < 0, j = 1,\dots,n$, where λ_j are the roots of the associated characteristic polynomial $\lambda^n + \sum_{j=1}^{n} a_j \lambda^{n-j}$.

Consider now the family of equations

$$x^{(n)} + \sum_{j=1}^{n} a_j x^{(n-j)} = f(t,u(t),\dots,u^{(n-1)}(t)), \tag{19}$$

where the linear part is the same as above and

$$u(t) \in Q := \{q(t) \in C^{n-1}(\mathbb{R}) : \sup_{t\in(-\infty,\infty)} |q^{(k)}(t)| \leq D_k \text{ for } k = 0,1,\dots,n-1\}.$$

Denoting

$$F := \sup_{t\in\mathbb{R}, |x^{(k)}|\leq D_k, k=0,1,\dots,n-1} |f(t,x,\dots,x^{(n-1)})|,$$

we know (see [AT]) that, for each $u(t) \in Q$, equation (19) admits a unique entirely bounded solution

$$x(t) = e^{\lambda_1 t}\int_{-\infty}^{t} e^{(\lambda_2-\lambda_1)t}\int_{-\infty}^{t}\cdots\int_{-\infty}^{t} e^{-\lambda_n t} f(t,u(t),\dots,u^{(n-1)}(t))(dt)^n$$

such that

$$\sup_{t\in(-\infty,\infty)} |x^{(k)}(t)| \leq \frac{2^k F}{|a_n|}(1+C)^k, \quad k = 0,1,\dots,n-1, \tag{20}$$

where $C = \max(|a_1|,\dots,|a_n|)$.

So, if there exist constants D_k such that

$$\sup_{t\in\mathbb{R}, |x^{(k)}|\leq D_k, k=0,1,\dots,n-1} |f(t,x,\dots,x^{(n-1)})| \leq \frac{|a_n| D_k}{2^k(1+C)^k} \tag{21}$$

$$\text{for } k = 0,1,\dots,n-1,$$

then the bounded solution $x(t)$ belongs to a bounded closed convex subset of $C^{n-1}(\mathbb{R})$.

Therefore, our arguments apply, and consequently *equation* (18) *admits an entirely bounded solution satisfying* (20) *as well, provided there exist constants* $D_k, k = 0, 1, \ldots, n-1$, *such, that* (21) *holds.*

One can easily observe that, under the strict inequality in (21), the same is true for the equation

$$x^{(n)} + \sum_{j=1}^{n} [a_j + \alpha_j(t, x, \ldots, x^{(n-1)})] x^{(n-j)} = f(t, x, \ldots, x^{(n-1)}) \tag{22}$$

with sufficiently small continuous perturbations $\alpha_j(t, x, \ldots, x^{(n-1)})$, $j = 0, 1, \ldots, n-1$, because we can start with the following analogue of (19):

$$\begin{aligned} x^{(n)} + \sum_{j=1}^{n} a_j x^{(n-j)} \\ &= f(t, u(t), \ldots, u^{(n-1)}(t) - \sum_{j=1}^{n} \alpha_j(t, u(t), \ldots, u^{(n-1)}(t)) u^{(n-j)}(t). \end{aligned} \tag{23}$$

The analogous statement can be made for the inclusion

$$x^{(n)} + \sum_{j=1}^{n} [a_j + \alpha_j(t, x, \ldots, x^{(n-1)})] x^{(n-j)} \in f(t, x, \ldots, x^{(n-1)}) \tag{24}$$

with the Carathéodory functions $\alpha_j, j = 1, \ldots, n$, and f.

Example 2.41. Assuming, in addition to the situation in Example 2.40, that

$$\lim_{t \to \pm\infty} \alpha_j(t, x, \ldots, x^{(n-1)}) = 0 \quad \text{for } j = 1, \ldots, n-1, \tag{25}$$

$$F := \sup_{(t, x, \ldots, x^{(n-1)}) \in \mathbb{R}^{n+1}} |f(t, x, \ldots, x^{(n-1)})| < \infty, \tag{26}$$

and

$$\lim_{t \to \pm\infty} f(t, x, \ldots, x^{(n-1)}) = 0, \tag{27}$$

one can prove analogously (see [AT] and the references therein) the following.

For each $u(t) \in Q$, where again

$$Q := \{q(t) \in C^{n-1}(\mathbb{R}) : \sup_{t \in (-\infty, \infty)} |q^{(k)}(t)| \le D_k \text{ for } k = 0, 1, \ldots, n-1\},$$

equation (23) admits a unique bounded solution $x(t)$ such that

$$\sup_{t \in (-\infty, \infty)} |x^{(k)}(t)| \le \frac{2^k}{|a_n|} (1 + C)^k (F + G) \quad \text{for } k = 1, \ldots, n-1,$$

where

$$G := \sup_{t \in \mathbb{R}, |x^{(k)}| \le D_k, k = 0, 1, \ldots, n-1} \sum_{j=1}^{n} |\alpha_j(t, x, \ldots, x^{(n-1)})| D_{n-j}$$

and

$$\lim_{t \to \pm\infty} x^{(k)}(t) = 0 \quad \text{for } k = 0, 1, \ldots, n-1. \tag{28}$$

So, if there exist constants D_k such that

(29)
$$\frac{2^k}{|a_n|}(1+C)^k(F+\sup_{t\in\mathbb{R},|x^{(k)}|\le D_k,k=0,1,\dots,n-1}\sum_{j=1}^{n}|\alpha_j(t,x,\dots,x^{(n-1)})|D_{n-j})\le D_k$$
$$\text{for } k=0,1,\dots,n-1 \quad (C=\max(|a_1|,\dots,|a_n|)),$$

then the bounded solution $x(t)$, vanishing at infinities, belongs to a bounded closed convex subset Q of $C^{n-1}(\mathbb{R})$.

Therefore, *equation* (22) *admits, by the above arguments (cf. (15)), a bounded solution* $x(t)$ *satisfying* (28), *provided* (25)–(27) and (29) hold.

Obviously, condition (29) is fulfilled for sufficiently small continuous perturbations, again.

Finally, our statement can be appropriately modified for the inclusion (24).

Example 2.42. Consider the pendulum-type equation

$$\ddot{x}+a\dot{x}+b\sin x=f(t,x,\dot{x}), \tag{30}$$

where a,b are positive constants such that $a^2\ge 4b$ and f is a continuous bounded function.

Rewriting (30) into the form

$$\ddot{x}+a\dot{x}+bx=b(x-\sin x)+f(t,x,\dot{x}), \tag{31}$$

and considering the equation

$$\ddot{x}+a\dot{x}-bx=-b[x+\sin(x-\pi)]+f(t,x,\dot{x}), \tag{32}$$

we can use for both (31) and (32) the result obtained in Example 2.40.

Hence, equation (31) or (32) admits a bounded solution $x(t)$, provided that there exist constants D_0,D_1 such that (cf. (21))

$$\frac{2^k}{b}(F+B)(1+C)^k\le D_k \qquad \text{for } k=0,1, \tag{33}$$

where $C=\max(a,b)$, $F:=\sup_{(t,x,y)\in\mathbb{R}^3}|f(t,x,y)|$, and $B:=b\max_{|x|\le D_0}|x-\sin x|$ or $B:=b\max_{|x|\le D_0}|x+\sin(x-\pi)|$, respectively.

Because

$$\max_{|x|\le\frac{\pi}{2}}|x-\sin x|=\frac{\pi}{2}-1 \quad\text{or}\quad \max_{|x|\le\frac{\pi}{2}}|x+\sin(x-\pi)|=\frac{\pi}{2}-1,$$

condition (33) takes for $D_0=\frac{\pi}{2}$ an extremely simple form:

$$\sup_{(t,x,y)\in\mathbb{R}^3}|f(t,x,y)|\le b, \tag{34}$$

while the condition for $k=1$ becomes trivial.

Therefore, equation (31) or (32) has, under (34), a bounded solution $x(t)$ such that

$$\sup_{t\in(-\infty,\infty)}|x(t)|\le\frac{\pi}{2},\quad \sup_{t\in(-\infty,\infty)}|\dot{x}(t)|\le\frac{2}{b}(\frac{\pi}{2}-1+F)[1+\max(a,b)].$$

As a direct consequence, *equation* (30) *possesses, under the strict inequality in* (34), *at least two bounded solutions* $x_1(t)$ *and* $x_2(t)$ *such that*

$$\sup_{t\in(-\infty,\infty)}|x_1(t)|<\frac{\pi}{2},\quad \sup_{t\in(-\infty,\infty)}|x_2(t)-\pi|<\frac{\pi}{2},$$

and with the same as above for the derivatives. The same is certainly true for a negative coefficient a, because we can just replace t by $-t$ in (30).

Example 2.43. Consider the system

$$\begin{aligned} \dot{x}_1 &= f(t,x_1,x_2)x_1 + g(t,x_1,x_2)x_2 + e_1(t,x_1,x_2), \\ \dot{x}_2 &= -g(t,x_1,x_2)x_1 + f(t,x_1,x_2)x_2 + e_2(t,x_1,x_2), \end{aligned} \tag{35}$$

where the functions e_1, e_2, f, g are continuous on the space $\mathbb{R}_+ \times \mathbb{R}^2$, where $\mathbb{R}_+ = [0,\infty)$.

Assume, futhermore, the existence of positive constants E_1, E_2, λ, F, G such that

$$\sup_{t\in[0,\infty),|x_i|\le D, i=1,2} f(t,x_1,x_2) \le -\lambda, \tag{36}$$

$$\sup_{t\in[0,\infty),|x_i|\le D, i=1,2} |f(t,x_1,x_2)| \le F, \tag{37}$$

$$\sup_{t\in[0,\infty),|x_i|\le D, i=1,2} |g(t,x_1,x_2)| \le G, \tag{38}$$

$$\begin{cases} \displaystyle\sup_{t\in[0,\infty),|x_i|\le D, i=1,2} |e_1(t,x_1,x_2)| \le E_1, \\ \displaystyle\sup_{t\in[0,\infty),|x_i|\le D, i=1,2} |e_2(t,x_1,x_2) \le E_2, \end{cases} \tag{39}$$

where $D = \frac{1}{\lambda}(E_1 + E_2)$. Observe that, under the assumptions (37)–(39), we have

$$\sup_{t\in[0,\infty)} |\dot{x}_i(t)| \le D', \quad i = 1,2, \tag{40}$$

where $D' = (F+G)D + \max(E_1, E_2)$, so long as the solution $(x_1(t), x_2(t))$ of (35) satisfies

$$\sup_{t\in[0,\infty)} |x_i(t)| \le D, \quad i = 1,2. \tag{41}$$

Our aim is to prove, under the assumptions (36)–(39)*, the existence of the solution* $x(t) = (x_1(t), x_2(t))$ *satisfying*

$$x(0) = 0 \textit{ and } \sup_{t\in(0,\infty)} |x_i(t)| \le D \quad \textit{for } i = 1,2. \tag{42}$$

In order to apply Corollary 2.34 for this goal, define the two sets

$$Q := \{v(t) = (v_1(t), v_2(t)) \in C(\mathbb{R}^2_+) : \sup_{t\in[0,\infty)} |v_i(t)| \le D \quad \text{for } i = 1,2\},$$

$$S := \{s(t) = (s_1(t), s_2(t)) \in C(\mathbb{R}^2_+) \cap Q : |s_i(t)| \le D't \quad \text{for } i = 1,2\} \qquad \text{(observe that } s(0) = 0\text{)},$$

where Q is a closed convex subset of $C(\mathbb{R}^2_+)$ and S is a bounded closed subset of Q.

For $u(t) = (u_1(t), u_2(t)) \in Q$, consider furthermore the family of systems

$$\begin{aligned} \dot{x}_1 &= p(t)x_1 + q(t)x_2 + r_1(t), \\ \dot{x}_2 &= -q(t)x_1 + p(t)x_2 + r_2(t), \end{aligned} \tag{43}$$

where $p(t) := f(t, u(t)), q(t) := g(t, u(t)), r_1(t) = e_1(t, u(t)), r_2(t) = e_2(t, u(t))$.

To show the solvability of (35)–(42) by means of Corollary 2.34, we need to verify that, for each $u(t) \in Q$, system (43) has a (unique) solution in S.

From the theory of linear Hamiltonian systems, it is well known that the general solution $x(t,0,\xi)$, where $\xi=(\xi_1,\xi_2)\in\mathbb{R}^2$, reads as follows:

$$\begin{aligned}
x_1(t,t_0,\xi) &= [\xi_1\cos(\int_0^t q(s)ds)+\xi_2\sin(\int_0^t q(s)ds)]\exp\int_0^t p(s)ds\\
&\quad+\int_0^t[r_1(s)\exp\int_s^t p(w)dw\cos(\int_s^t q(w)dw)]ds\\
&\quad+\int_0^t[r_2(s)\exp\int_s^t p(w)dw\sin(\int_s^t q(w)dw)]ds,\\
x_2(t,t_0,\xi) &= [-\xi_1\sin(\int_0^t q(s)ds)+\xi_2\cos(\int_0^t q(s)ds)]\exp\int_0^t p(s)ds\\
&\quad-\int_0^t[r_1(s)\exp\int_s^t p(w)dw\sin(\int_s^t q(w)dw)]ds\\
&\quad+\int_0^t[r_2(s)\exp\int_s^t p(w)dw\cos(\int_s^t q(w)dw)]ds.
\end{aligned}$$

Because (see (36), (39))

$$\sup_{t\in[0,\infty)}|\int_0^t[r_i(s)\exp\int_s^t p(w)dw\cos(\int_s^t q(w)dw)]ds|$$

$$\leq E_i\sup_{t\in[0,\infty)}\int_0^t\exp[-\int_s^t|p(w)|dw]ds\leq\frac{E_i}{\lambda},$$

$$\sup_{t\in[0,\infty)}|\int_0^t[r_i(s)\exp\int_s^t p(w)dw\sin(\int_s^t q(w)dw)]ds|$$

$$\leq E_i\sup_{t\in[0,\infty)}\int_0^t\exp[-\int_s^t|p(w)|dw]ds\leq\frac{E_i}{\lambda},$$

for $i=1,2$, we have that

$$\sup_{t\in[0,\infty)}|x_i(t,0,\xi)|\leq|\xi_1|+|\xi_2|+D,\quad i=1,2,$$

and

$$x(0,0,\xi)=\xi.$$

One can readily check that the only solution of the problem (43)–(42) is $x(t,0,0)$. Moreover, in view of the indicated implication (41) $\Rightarrow$ (40), $x(t,0,0)$ belongs to S for each $u(t)\in Q$, and so Corollary 2.34 applies. Thus, *conditions* (36)–(39) *are sufficient for the solvability of the problem* (35)–(41), *indeed*.

Finally, if at least one of the inequalities (36) or (39) is sharp, then the same conclusion is true for $x(0)=0$ in (42) replaced by $x(0)=\alpha$, where α is an arbitrary constant with a sufficiently small absolute value. For bigger values of $|\alpha|$, assumptions (36) and (39) can be appropriately modified as well.

3. Sequential approach

Consider the inclusion

$$\dot{X} \in F(t, X). \tag{44}$$

In [GP] (see also [AGL]) the following is proved.

Lemma 3.1. *If F is a bounded, u.s.c. map with nonempty, compact, convex values, then the maps associated with the solutions of* (44),

$$P_k(X_0) := \{X \in C([0, kT], \mathbb{R}^n); X \text{ is a solution of } (44) \text{ with } X(0) = X_0 \in \mathbb{R}^n\},$$

are u.s.c. with R_δ-values for all $k \in \mathbb{N}$ and a positive real T.

Since in a metric space either compactness or sequential compactness imply closedness, the following Lemma 3.2 corresponds to Consequence 1 of Lemma 1, §7 in [Fi, p. 60]. Nevertheless, we will give here a simple alternative proof based only on the fact that the graphs of the maps P_k are closed.

Lemma 3.2. *Under the assumptions of Lemma 3.1, the limit of a sequence of uniformly convergent solutions of* (44) *is also a solution of* (44).

Proof. Let $X_k : [0, \infty) \to \mathbb{R}^n$ be a sequence of solutions of (44) that are uniformly convergent to $\widehat{X}$, namely $X_k \rightrightarrows \widehat{X}$. We can assume without loss of generality that $X_k|_{[0,kT]} \in P_k(X_k(0))$. It is sufficient to show that for an arbitrary k_0 we have

$$\widehat{X}|_{[0,k_0T]} \in P_{k_0}(\widehat{X}(0)).$$

Since $X_k|_{[0,k_0T]} \in P_{k_0}(u_k)$ for every $k \geq k_0$, where $u_k := X_k(0) \in \mathbb{R}^n$, we obtain that

$$u_k \to u_0 = \widehat{X}(0), \qquad X_k \rightrightarrows \widehat{X},$$

and $X_k|_{[0,k_0T]} \in P_{k_0}(u_k)$. Therefore, we get $\widehat{X} \in P_{k_0}(u_0)$, i.e. the graph P_{k_0} is closed, which completes the proof. $\square$

The following fixed-point theorem has been proved in [AGL].

Lemma 3.3. *Let E_1 and E_2 be two finite dimensional normed spaces. Assume that*

$$\varphi : [0, T] \times (E_1 \times E_2) \rightsquigarrow E_1,$$
$$\psi : [0, T] \times (E_1 \times E_2) \rightsquigarrow E_2,$$

are u.s.c. mappings with R_δ-values such that following conditions hold:

- *(i) the maps $\varphi_0 = \varphi(0, .\,)$ and $\psi_0 = \psi(0, .\,)$ are projections onto the spaces E_1 and E_2, respectively;*
- *(ii) $A \subset E_1$ and $B \subset E_2$ are open, bounded and star–shaped (with respect to the origins) subsets;*
- *(iii) $\varphi_T(\partial A \times \overline{B}) \cap \overline{A} = \emptyset$, $\psi_T(\overline{A} \times \partial B) \subset B$, where $\varphi_T = \varphi(T, .\,)$ and $\psi_T = \psi(T, .\,)$;*
- *(iv) $0 \notin \varphi([0, T] \times (\partial A \times \{0\}))$.*

Then the mapping $(\varphi_T, \psi_T) : E_1 \times E_2 \rightsquigarrow E_1 \times E_2$, i.e. $(\varphi_T, \psi_T)(x) = \varphi_T(x) \times \psi_T(x)$, has at least one fixed point in the set $A \times B$.

Now, because of these three lemmas, we are in position to give

Theorem 3.4. *Let F be a bounded, u.s.c. map with nonempty, compact, convex values. Let $\varphi_t = \varphi(t, \cdot)$ and $\psi_t = \psi(t, \cdot)$ denote the natural projections of the solutions $X(t, X_0)$ of* (44) *onto the spaces $\mathbb{R}^j$ and $\mathbb{R}^{n-j}$ $(1 \leq j \leq n-1)$, respectively. Assume we have an arbitrary positive real T and an integer $K \in \mathbb{N}$ such that for all $k \geq K, k \in \mathbb{N}$, the following conditions are satisfied:*

(i) $\varphi_{kT}(\partial A \times \overline{B}) \cap \overline{A} = \emptyset,\ \psi_{kT}(\overline{A} \times \partial B) \subset B \quad \forall k \geq K$,
(ii) $0 \notin \varphi([0, kT] \times (\partial A \times \{0\})) \quad \forall k \geq K$,

where $A \subset \mathbb{R}^j, B \subset \mathbb{R}^{n-j}$ are suitable open, bounded subsets which are star-shaped with respect to the origins.

Then there exists a sequence $\{X_k(t)\}$ of solutions $X_k(t)$ of (44), *satisfying $X_k(0) = X_k(kT)$ in the set $\mathcal{R} = A \times B$ for all $k \geq K$.*

Furthermore, if there exists some bounded neighbourhood $\mathcal{S}$ of $\mathcal{R}$ such that

$$\{X_k(t); t \in [0, kT], \forall k \geq K\} \in \mathcal{S}, \tag{iii}$$

then (44) *admits a bounded solution (on the positive ray) belonging to $\overline{\mathcal{S}}$.*

Sketch of proof. The first part of our assertion can be deduced from Lemma 3.3, applying Lemma 3.1 (for more details see [AGL]). The second part is then a direct consequence of the first, following the intuitively obvious arguments from the proof of Theorem 5, §14 in [Fi, p. 114] on the basis of Lemma 3.2. □

In the case of single-valued F, the situation simplifies as follows.

Corollary 3.5. *Let $F \in C([0, \infty) \times \mathbb{R}^n)$ and assume the global existence of solutions starting on $\overline{\mathcal{R}}$. Then the conclusion of Theorem 3.4 holds, provided (i), (ii), and (iii) hold.*

Since the uniform partial boundedness and the (uniform) partial dissipativity in the sense of Levinson (cf. [Fi], [Y]) imply the existence of a neighbourhood $\mathcal{S}_B$ of B such that the natural projections of solutions on $\mathbb{R}^{n-j}$ starting on $\overline{\mathcal{R}}$ remain entirely in $\mathcal{S}_B$ for all future times (consequently, the graph of F is compact on $[0, \infty) \times \mathcal{S}_B$), Theorem 3.4 takes in such a case the following simpler form.

Corollary 3.6. *Let F be an u.s.c. map which is bounded with respect to $t \in [0, \infty)$ and the variables from $\mathbb{R}^j$ $(1 \leq j \leq n-1)$ and which has nonempty, compact, convex values. Assume some part of the components associated with all solutions $X(t, 0)$ of* (44) *is uniformly and ultimately bounded (i.e. is uniformly partially dissipative in the sense of Levinson) and another part (related just to $\mathbb{R}^j$), starting outside some neighbourhood of the origin, which behaves as a repeller, tends uniformly (in the appropriate norm) to infinity. Then the inclusion* (44) *admits a bounded solution on the positive ray.*

Now, we can reformulate Corollary 3.6 in terms of guiding functions, which is very convenient for applications. For this purpose, let $\pi_j X$ and $\pi_{n-j} X$ denote the natural projections of the vector $X \in \mathbb{R}^n$ onto the spaces $\mathbb{R}^j$ and $\mathbb{R}^{n-j}$, respectively.

Theorem 3.7. *Assume F is an u.s.c. map which is bounded with respect to $t \in [0, \infty)$ and the variables from $\mathbb{R}^j$ $(1 \leq j \leq n-1)$ and which has nonempty, compact, convex values. Let two locally Lipschitz in X guiding functions $V(t, X)$ and $W(t, X)$*

exist such that

$$\left\{\begin{array}{l} a(\|\pi_{n-j}X\|) \le V(t,X) \le b(\|\pi_{n-j}X\|), \\ \\ \limsup\limits_{h\to 0^+} \dfrac{[V(t+h,X+hY)-V(t,X)]}{h} \le -C(\|\pi_{n-j}X\|) \\ \qquad\qquad\qquad \text{for all } Y\in F(t,X), \text{ and for } \|\pi_{n-j}X\|\ge R_2, \end{array}\right. \tag{45}$$

and

$$\left\{\begin{array}{l} A(\|\pi_j X\|) \le W(t,X) \le B(\|\pi_j X\|), \text{ for } \|\pi_{n-j}X\| \le R_3, \\ \limsup\limits_{h\to 0^+} \dfrac{[W(t+h,X+hY)-W(t,X)]}{h} \ge D(\|\pi_j X\|) \\ \qquad\qquad \text{for all } Y\in F(t,X), \text{ and for } \|\pi_j X\|\ge R_2, \|\pi_{n-j}X\|\le R_3, \end{array}\right. \tag{46}$$

where $R_1, R_2 \le R_3$ are suitable positive constants which may be large enough, the wedges $a(r), b(r), A(r), B(r)$ are continuous increasing functions such that both $a(r)\to\infty$ and $A(r)\to\infty$ as $r\to\infty$, and $C(r), D(r)$ are positive continuous functions not vanishing at infinity. Then the inclusion (44) *admits a bounded solution $X(t)$ such that*

$$\sup_{t\in[0,\infty)} \|\pi_{n-j}X(t)\| \le R_3, \quad \sup_{t\in[0,\infty)} \|\pi_j X(t)\| \le R_4, \tag{47}$$

where R_4 $(\ge R_1)$ is a sufficiently big constant.

The conclusion follows by repeating the appropriately modified arguments in [An4], where the particular form of the single-valued F was under consideration.

Remark 3.8. In the case of single-valued F, the same conclusions as those in Corollary 3.6 and Theorem 3.7 are true, assuming only that $F \in C([0,\infty)\times\mathbb{R}^n)$ and the global existence of the projections (related to $\mathbb{R}^j$) of solutions starting on $\overline{\mathcal{R}}$.

Remark 3.9. For C^1-functions $V(t,X)$ and $W(t,X)$, conditions (45) and (46) can be rewritten into the form

$$\begin{array}{cc} \lim V(t,X) = \infty, & \langle \text{grad } V(t,X), (1,Y)\rangle \le -\varepsilon_1 < 0, \\ \|\pi_{n-j}X\|\to\infty & \text{for all } Y\in F(t,X) \text{ and } \|\pi_{n-j}X\|\ge R_2, \end{array}$$

and

$$\begin{array}{cc} \lim W(t,X) = \infty, & \langle \text{grad } W(t,X), (1,Y)\rangle \ge \varepsilon_2 > 0, \\ \|\pi_j X\|\to\infty & \text{for all } Y\in F(t,X) \text{ and } \|\pi_j X\|\ge R_1, \|\pi_{n-j}X\|\le R_3, \end{array}$$

where ε_1 and ε_2 are suitable positive numbers, respectively.

Remark 3.10. In the single-valued case, we even have, for C^1-functions V and W

$$\begin{array}{rcl} \langle \text{grad } V(t,X), (1,Y)\rangle & = & V'(t,X)_{(1)}, \\ \langle \text{grad } W(t,X), (1,Y)\rangle & = & W'(t,X)_{(1)}, \end{array}$$

where $V'_{(1)}$ and $W'_{(1)}$ denote the time-derivatives along (44).

Remark 3.11. An explicit definition of sharp enough constants R_3 and R_4 in (47) might be a cumbersome problem (see e.g. [An6]).

Remark 3.12. In the single-valued case of continuous $F(t+T,X) \equiv F(t,X)$, for $n=2$, the existence of a bounded solution implies, according to the well-known Massera theorem (see e.g. [Y]), the existence of a T-periodic solution, provided all the solutions are globally extendable. Otherwise, for $n>2$, the existence of subharmonics to the inclusion (44) with the time-periodic F can only be deduced here from our statements.

Now, it is time to give some nontrivial examples.

Example 3.13. Consider (44), where $F(t,X) = AX + F_1(t,X^T)$,

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}, \quad F_1(t,X^T) = \begin{pmatrix} f(t,X^T) \\ g(t,X^T) \\ h(t,X^T) \end{pmatrix},$$

and assume the natural restrictions in the spirit of Theorem 3.7 ($j=1$); namely, take $c_1=c_2=0$ and replace c_3z, h by c_3z_*, h, where ($r \gg 0$)

$$z_* = \begin{cases} z & \text{for } |z| \le r, \\ r\, sgn\, z & \text{for } |z| \ge r, \end{cases} \qquad h_*(t,x,y,z) = \begin{cases} h(t,x,y,z) & \text{for } |z| \le r, \\ h(t,x,y,r\, sgn\, z) & \text{for } |z| \ge r. \end{cases}$$

Our aim is to show the existence of a bounded trajectory, provided the real coefficients a_i, b_i, c_i ($i=1,2,3$) and the functions f, g, h satisfy the following conditions:

$$\begin{array}{ll} & -b_2 > a_1,\ a_1b_2 > a_2b_1,\ c_3 > 0,\ |c_1| \text{ and } |c_2| \text{ are sufficiently small} \\ & (a_3 \text{ and } b_3 \text{ are arbitrary}), \end{array} \tag{48}$$

$$\begin{array}{r} \sup_{t\in[0,\infty),X\in\mathbb{R}^3}(|f|+|g|) < P(<\infty), \quad \limsup_{|z|\to\infty}\left|\frac{h(t,x,y,z)}{z}\right| < c_3 \\ \text{uniformly w.r.t. } t\ge 0, |x|+|y| \le R_3. \end{array} \tag{49}$$

Taking into account the first two lines in (44) with $c_1=c_2=0$, we can get for all solutions $X(t)$ of (44) (see [An1])

$$\limsup_{t\to\infty} |x(t)|+|y(t)| < P\left(\frac{1}{-\lambda} + \frac{2\|A_2\|}{\lambda^2}\right), \tag{50}$$

where $(0>)\ \lambda := \frac{1}{2}(a_1+b_2+\ \mathrm{Re}\sqrt{a_1^2-2a_1b_2+4a_2b_1+b_2^2})$ is the maximal real part of the eigenvalues of $A_2 := \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$, which is negative, because of the Routh–Hurwitz conditions (see (48)), and

$$\|A_2\| = \max(|a_1|+|a_2|, |b_1|+|b_2|).$$

Defining $W(z) = \frac{1}{2}z^2$, we arrive at

$$\frac{dW}{dz}z' = zz' = c_3zz_* + z[a_3x+b_3y+h_*(t,x,y,z)] \ge \varepsilon_3 > 0$$

for $t\ge 0, |z| \ge R_1, |x|+|y| \le R_3$, because of (48), (49), (50).

Hence, applying Corollary 3.6 (see also Theorem 3.7 and Remark 3.8), the desired conclusion follows. Moreover, one can readily check that the same is true for the original inclusion (44) without the additional growth restrictions on c_1, c_2, c_3 and h.

Example 3.14. Consider the same inclusion as in Example 3.13 and assume again the natural conditions in the spirit of Theorem 3.7 ($j=2$); namely, take $b_1=c_1=0$ and replace $b_2y, c_2z, b_3y, c_3z, g, h$ by $b_2y_*, c_2z_*, b_3y_*, c_3z_*, g_*, h_*$, respectively, where the "asterisk" restriction has the same meaning as in Example 3.13.

We have the same aim, this time provided

$$\begin{gathered} a_1 < 0, \quad b_2 > |b_3| + |c_2|, \quad c_3 > |b_3| + |c_2|, \\ |b_1| \text{ and } |c_1| \text{ are sufficiently small,} \\ (a_2 \text{ and } a_3 \text{ are arbitrary}) \end{gathered} \tag{51}$$

$$\begin{aligned} &\limsup_{|x|\to\infty} \left| \frac{f(t,x,y,z)}{x} \right| < -a_1, \\ &\lim_{|y|\to\infty} \left| \frac{g(t,x,y,z)}{y} \right| = 0 \quad \text{for } |x| \le R_3, \\ &\lim_{|z|\to\infty} \left| \frac{h(t,x,y,z)}{z} \right| = 0 \quad \text{for } |x| \le R_3, \\ &\text{all uniformly w.r.t. } t \ge 0 \text{ and the remaining variables.} \end{aligned} \tag{52}$$

Taking into account the first line in (44) with $b_1 = c_1 = 0$, there certainly exists a constant R such that we have for all solutions $X(t)$ of (44) (see (52))

$$\limsup_{t\to\infty} |x(t)| \le R. \tag{53}$$

This can also be seen when applying $V(x) = \frac{1}{2}x^2$, because

$$\frac{dV}{dx}x' = xx' = (a_1 + f)x \le -\varepsilon_1 < 0 \quad \text{for } |x| \ge R_2.$$

Defining $W(y,z) = \frac{1}{2}(y^2 + z^2)$, we obtain

$$\begin{aligned} \frac{\partial W}{\partial y}y' + \frac{\partial W}{\partial z}z' &= yy' + zz' \\ &= b_2 yy_* + c_3 zz_* + b_3 y_* z + c_2 yz_* + y(a_2 x + g_*) + z(a_3 x + h_*). \end{aligned}$$

In view of this and (51), there exist positive constants $\triangle, \varepsilon_2$ such that

$$\frac{\partial W}{\partial y}y' + \frac{\partial W}{\partial z}z' \ge \triangle(yy_* + zz_*) + y(a_2 x + g_*) + z(a_3 x + h_*) \ge \varepsilon_2 > 0$$
$$\text{for } t \ge 0, |y| + |z| \ge R_1, |x| \le R_3, \text{ because of (52), (53).}$$

For the same reasons as in Example 3.13, the desired conclusion follows for the original inclusion (44) without the additional growth restrictions.

Remark 3.15. The inequalities $b_2 > |b_3| + |c_2|, c_3 > |b_3| + |c_2|$ in (51) can be replaced, after small technical modifications, by $b_2 > 0$, $c_3 > 0, |b_3 + c_2| < 2\sqrt{b_2 c_3}$, which is obvious in the single-valued case.

Applying

$$W(y,z) = \frac{1}{2}(y^2 + z^2) - \frac{b_3 + c_2}{b_2 + c_3}yz$$

instead of the original (reduced) guiding function, the last inequality, $|b_3 + c_2| < 2\sqrt{b_2 c_3}$, can even be replaced by $|b_3 + c_2| < b_2 + c_3$.

The following result is well known, at least in the single-valued case (see e.g. [Ab]), but it demonstrates well the power of our method.

Example 3.16. Inclusion (44), where $X = \begin{pmatrix} x \\ y \end{pmatrix}, F(t, X^T) = \begin{pmatrix} y \\ f(t,x,y) \end{pmatrix}$, admits a bounded solution, provided that positive constants $\delta_1, \delta_2, \varepsilon_1, \varepsilon_2$ exist such that

$$f(t,x,y) \operatorname{sgn} x \ge \varepsilon_2 + \delta_1 \quad \text{for } t \ge 0, |x| \ge \varepsilon_1, |y| \le \varepsilon_2,$$

and

$$f(t,x,y)y \le -\delta_2 < 0 \qquad \text{for } t \ge 0, x \in \mathbb{R}^1, |y| \ge \varepsilon_2.$$

This can be easily verified by means of the guiding functions $V(y) = \frac{1}{2}y^2$ and $W(x,y) = \frac{1}{2}(x+y)^2$.

In the single-valued case, if moreover $f(t,x,y) \equiv f(t+T,x,y)$, the equivalent equation $x'' = f(t,x,y)$ admits a T-periodic solution (see Remark 3.12).

4. Structure of solution sets for the Cauchy problem

The essential idea of studying the structure of solution sets used below is taken from [Go2].

First, recall that, for two metric spaces X, Y and the interval J, the multivalued map $F : J \times X \rightsquigarrow Y$ is *almost upper semicontinuous* (a.u.s.c.) if for every $\varepsilon > 0$ there exists a measurable set $A_\varepsilon \subset J$ such that $m(J \setminus A_\varepsilon) < \varepsilon$ and the restriction $F|_{A_\varepsilon \times X}$ is u.s.c., where m stands for the Lebesgue measure.

It is clear that every a.u.s.c. map is Carathéodory. In general, the reverse is not true. The following Scorza-Dragoni type result describing possible regularizations of Carathéodory maps (see e.g. [JK]) will be employed.

Proposition 4.1. *Let X be a separable metric space and J be an interval. Suppose that $F : J \times X \rightsquigarrow \mathbb{R}^n$ is a nonempty compact convex valued Carathéodory map. Then there exists an a.u.s.c. map $\psi : J \times X \rightsquigarrow \mathbb{R}^n$ with nonempty compact convex values and such that:*

(i) *$\psi(t,x) \subset F(t,x)$ for every $(t,x) \in J \times X$;*
(ii) *if $\Delta \subset J$ is measurable, $u : \Delta \to \mathbb{R}^n$ and $v : \Delta \to X$ are measurable maps and $u(t) \in F(t,v(t))$ for almost all $t \in \Delta$, then $u(t) \in \psi(t,v(t))$ for almost all $t \in \Delta$.*

A single-valued map $f : J \times X \to Y$ is said to be *measurable - locally Lipschitz* if, for every $x \in X$, there exist a neighbourhood V_x of x in X and an integrable function $L_x : J \to [0,\infty)$ such that

$$||f(t,x_1) - f(t,x_2)|| \le L_x(t)||x_1 - x_2|| \text{ for every } t \in J \text{ and } x_1, x_2 \in V_x,$$

where $f(\cdot,x)$ is measurable for every $x \in X$. A map $F : J \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is said to be *integrably bounded* (resp. *locally integrably bounded*) if there exists an integrable function (resp. locally integrable function) $\mu : J \to [0,\infty)$ such that $||y|| \le \mu(t)$ for every $x \in \mathbb{R}^n,\ t \in J$ and $y \in F(t,x)$. We say that $F : J \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ has *at most linear growth* (resp. *local linear growth*) if there exist integrable functions (resp. locally integrable functions) $\mu, \nu : J \to [0,\infty)$ such that $||y|| \le \mu(t)||x|| + \nu(t)$ for every $x \in \mathbb{R}^n,\ t \in J$ and $y \in F(t,x)$.

It is obvious that F has at most linear growth if there exists an integrable function $\mu : J \to [0,\infty)$ such that $||y|| \le \mu(t)(||x|| + 1)$ for every $x \in \mathbb{R}^n,\ t \in J$ and $y \in F(t,x)$.

In the theory of differential inclusions, selectionable and σ-selectionable maps are often used for reduction of the multivalued problem to the single-valued one (see [Go2] and the references therein). Let $F : X \rightsquigarrow Y$ be a multivalued map and $f : X \to Y$ be single-valued. We say that f is a *selection* of F (writing $f \subset F$) if $f(x) \in F(x)$, for every $x \in X$.

It is convenient to consider different types of selections. Namely, we say that

(i) F is *m-selectionable* if there exists a measurable selection of F;
(ii) F is *c-selectionable* if there exists a continuous selection of F;
(iii) F is *L-selectionable* if there exists a continuous Lipschitz selection of F;
(iv) F is *LL-selectionable* if there exists a continuous locally Lipschitz selection of F;
(v) $F : J \times X \rightsquigarrow Y$ is *Ca-selectionable* if there exists a Carathéodory selection of F;
(vi) $F : J \times X \rightsquigarrow Y$ is *mLL-selectionable* if there exists a measurable - locally Lipschitz selection of F.

For examples of the above notions, see [Go2].

Adopting the proof of Theorem 4.13 in [Go2], we obtain

Theorem 4.2. *Let E, E_1 be two separable Banach spaces, J be an interval and $F : J \times E \rightsquigarrow E_1$ be an a.u.s.c. map with compact convex values. Then F is σ-Ca-selectionable i.e., it is an intersection of a decreasing sequence of Ca-selectionable mappings. The maps $F_k : J \times E \rightsquigarrow E_1$ (see the definition of σ-selectionable maps) are a.u.s.c., and $F_k(t,e) \subset \overline{\mathrm{conv}}(\bigcup_{x\in E} F(t,x))$ for all $(t,e) \in J \times E$.*

Moreover, if F is locally integrably bounded, then F is σ-mLL-selectionable.

Now, for the considerations below, fix J as the closed halfline $[0,\infty]$ and assume that $F : J \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a multivalued map. Consider the following Cauchy problem:

$$\begin{cases} \dot{x}(t) \in F(t,x(t)), \\ x(0) = x_0. \end{cases} \tag{54}$$

By $S(F,0,x_0)$ we denote the set of solutions of (54). For the characterization of the topological structure of $S(F,0,x_0)$, it will be useful to recall the following well-known uniqueness criterium (see e.g. [Fi, Theorem 1.1.2]).

Theorem 4.3. *If F is a single-valued, locally integrably bounded, measurable-locally Lipschitz map, then the set $S(F,0,x_0)$ is a singleton, for every $x_0 \in \mathbb{R}^n$.*

The following result will be employed as well (see e.g. [Go2] and the references therein).

Theorem 4.4. *If F is locally integrably bounded and mLL-selectionable, then $S(F,0,x_0)$ is contractible, for every $x_0 \in \mathbb{R}^n$.*

Proof. Let $f \subset F$ be measurable - locally Lipschitz. By Theorem 4.3, the Cauchy problem

$$\begin{cases} \dot{x}(t) = f(t,x(t)), \\ x(t_0) = u_0, \end{cases} \tag{55}$$

has exactly one solution, for every $t_0 \in J$ and $u_0 \in \mathbb{R}^n$. For the proof it is sufficient to define a homotopy $h : S(F,0,x_0) \times [0,1] \to S(F,0,x_0)$ such that

$$h(x,s) = \begin{cases} x, & \text{for } s = 1 \text{ and } x \in S(F,0,x_0), \\ \tilde{x}, & \text{for } s = 0, \end{cases}$$

where $\tilde{x} = S(f,0,x_0)$ is exactly one solution of the problem (55).

Define $\gamma : [0,1) \to [0,\infty)$, $\gamma(s) = \tan \frac{\pi s}{2}$, and put

$$h(x,s)(t) = \begin{cases} x(t), & \text{for } 0 \le t \le \gamma(s),\ s < 1, \\ S(f,\gamma(s),x(\gamma(s)))(t) & \text{for } \gamma(s) \le t < \infty,\ s < 1, \\ x(t), & \text{for } 0 \le t < \infty,\ s = 1. \end{cases}$$

Then h is a continuous homotopy, contracting $S(F,0,x_0)$ to the point $S(f,0,x_0)$. □

Analogously, we can get the following result.

Theorem 4.5. *If F is locally integrably bounded, Ca-selectionable, or in particular c-selectionable, then $S(F,0,x_0)$ is R_δ-contractible, for every $x_0 \in \mathbb{R}^n$.*

Observe that, if $F : J \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is an intersection of the decreasing sequence $F_k : J \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ i.e., $F(t,x) = \bigcap_{k=1}^{\infty} F_k(t,x)$ and $F_{k+1}(t,x) \subset F_k(t,x)$ for almost all $t \in J$ and for all $x \in \mathbb{R}^n$, then

$$S(F,0,x_0) = \bigcap_{k=1}^{\infty} S(F_k,0,x_0). \tag{56}$$

From Theorems 4.4 and 4.5 we obtain

Theorem 4.6. *Let $F : J \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ be a multivalued map.*

(i) If F is σ-mLL-selectionable, then the set $S(F,0,x_0)$ is an intersection of a decreasing sequence of contractible sets.

(ii) If F is σ-Ca-selectionable, then the set $S(F,0,x_0)$ is an intersection of a decreasing sequence of R_δ-contractible sets.

Now we can formulate the main result of this section.

Theorem 4.7. *If $F : J \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a Carathéodory map with compact convex values having at most the local linear growth, then $S(F,0,x_0)$ is an R_δ-set, for every $x_0 \in \mathbb{R}^n$.*

Sketch of proof [cf. [Go2]]. By the hypothesis, there exists a locally integrable function $\mu : J \to [0,\infty)$ such that $\sup\{||y|| \ : \ y \in F(t,x)\} \le \mu(t)(||x||+1)$, for every $(t,x) \in J \times \mathbb{R}^n$. By means of the Gronwall inequality (see [Ha]), we obtain that for every $n \ge 1$ and $t \in [0,n]$, $||x(t)|| \le (||x_0|| + \gamma_n)\exp(\gamma_n) = M_n$, where $x \in S(F,0,x_0)$ and $\gamma_n = \int_0^n \mu(s)ds$.

Define $\tilde{F} : J \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ as follows:

$$\tilde{F}(t,x) = \begin{cases} F(t,x), & \text{if } t \in [n-1,n) \text{ and } ||x|| \le M_n, \\ F(t, M_n \frac{x}{||x||}), & \text{if } t \in [n-1,n) \text{ and } ||x|| > M_n. \end{cases}$$

One can see that $\tilde{F}$ is a locally integrably bounded Carathéodory map and $S(\tilde{F},0,x_0) = S(F,0,x_0)$. By Proposition 4.1, there exists an a.u.s.c. map $G : J \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ with nonempty convex compact values such that $S(G,0,x_0) = S(\tilde{F},0,x_0)$. Applying Theorem 4.2 to the map G, we obtain the sequence of maps G_k. As in Theorem 4.6, we see that $S(G,0,x_0)$ is an intersection of the decreasing sequence $S(G_k,0,x_0)$ of contractible sets. By Ascoli's theorem and Theorem 4.3 we obtain that, for every $k \in \mathbb{N}$, the set $S(G_k,0,x_0)$ is compact and nonempty, which completes the proof. □

From the above theorem we immediately obtain the following fact, which is well-known in the case of a compact interval.

Corollary 4.8. *If $F : J \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is an u.s.c. bounded map with compact convex values, then $S(F,0,x_0)$ is an R_δ-set, for every $x_0 \in \mathbb{R}^n$.*

Using the above results and the unified approach to the u.s.c. and l.s.c. cases due to A. Bressan (cf. [Br1], [Br2]), we can obtain

Proposition 4.9. *Let $G : J \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ be a l.s.c. bounded map with closed values. Then there exists an u.s.c. map $F : J \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ with compact convex values such that for any $x_0 \in \mathbb{R}^n$ the set $S(G, 0, x_0)$ contains an R_δ-set $S(F, 0, x_0)$ as a subset.*

5. Application to implicit differential equations

The aim of this section is to use the method presented in [BiG2] to show that many types of differential equations (inclusions) on noncompact intervals whose right hand sides depend on the derivative can be reduced very easily to differential inclusions with right hand sides not depending on the derivative. We will apply this technique to ordinary differential equations of first or higher order, but other applications are possible e.g., for partial differential equations (see [BiG2], [Go2]).

Below, by X we mean the closed ball in $\mathbb{R}^n$ or the whole space $\mathbb{R}^n$. Furthermore, for a compact subset A of X, by $\dim A$ we understand the topological covering dimension.

Following [BiG2] we recall:

Proposition 5.1. *Let A be a compact subset of X such that $\dim A = 0$. Then, for every $x \in A$ and for every open neighbourhood U of x in X, there exists an open neighbourhood $V \subset U$ of x in X such that $\partial V \cap A = \emptyset$.*

In the Euclidean space $\mathbb{R}^n$ we can identify the notion of the Brouwer degree with the fixed point index (cf. [D]).

Namely, let U be an open bounded subset of $\mathbb{R}^n$ and let $g : \overline{U} \to \mathbb{R}^n$ be a continuous single-valued map such that $Fix(g) \cap \partial U = \emptyset$. We let $\widetilde{g} : \overline{U} \longrightarrow \mathbb{R}^n$,

$$\widetilde{g}(x) = x - g(x), \qquad x \in \overline{U},$$

and

$$i(g, U) = \deg(\widetilde{g}, U), \tag{57}$$

where $\deg(\widetilde{g}, U)$ denotes the Brouwer degree of $\widetilde{g}$ with respect to U; then $i(g, U)$ is called *the fixed point index* of g with respect to U.

Now all the properties of the Brouwer degree can be reformulated in terms of the fixed point index.

The proof of the following fact can be found in [Go2].

Proposition 5.2. *Let $g : X \to X$ be a compact map. Assume furthermore that the following two conditions are satisfied:*

(i) $\dim Fix(g) = 0$.

(ii) There exists an open subset $U \subset X$ such that $\partial U \cap Fix(g) = \emptyset$ and $i(g, U) \neq 0$.

Then there exists a point $z \in Fix(g)$ for which we have:

(iii) For every open neighbourhood U_z of z in X there exists an open neighbourhood V_z of z in X such that $V_z \subset U_z$, $\partial V_z \cap Fix(g) = \emptyset$ and $i(g, V_z) \neq 0$.

Now, let Y be a locally arcwise connected space and let $f : Y \times X \to X$ be a compact map. Define for every $y \in Y$ a map $f_y : X \to X$ by putting $f_y(x) = f(y, x)$ for every $x \in X$. Since X is an absolute retract, $Fix(f_y) \neq \emptyset$ for every $y \in Y$. It is easy to see that the following condition automatically holds:

$$\forall_{y \in Y} \quad \exists_{U_y} \ \ U_y \text{ is open in } X \text{ and } i(f_y, U_y) \neq 0. \tag{58}$$

Thus we can associate with a map $f : Y \times X \to X$ the following multivalued map:

$$\varphi_f : Y \rightsquigarrow X, \qquad \varphi_f(y) = Fix(f_y).$$

We immediately obtain:

Proposition 5.3. *Under all the above assumptions, the map $\varphi_f : Y \rightsquigarrow X$ is u.s.c..*

Let us remark that, in general, φ_f is not a l.s.c. map. Below we would like to formulate a sufficient condition which guarantees that φ_f has a l.s.c. selection. To this end we assume that f satisfies the following condition:

$$\forall y \in Y : \dim\, Fix(f_y) = 0. \tag{59}$$

Note that condition (59) is satisfied for several classes of maps. Namely, for some classes of maps the fixed point set $Fix(f_y)$ is a singleton for every $y \in Y$ e.g., when f_y is a k-set contraction with $0 < k < 1$ or the following assumption is satisfied (see [BiG1]):

$$\langle f(y, x_1) - f(y, x_2), x_1 - x_2 \rangle \leq k||x_1 - x_2||, \quad 0 < k < 1,\ y \in Y,\ x_1, x_2 \in X.$$

Now, in view of (58) and (59), we are able to define the map $\psi_f : Y \rightsquigarrow X$ by putting $\psi_f(y) = cl\{z \in Fix(f_y);$ for z condition (iii) from Proposition 5.2 is satisfied$\}$, for every $y \in Y$.

Theorem 5.4 (see [BiG2], [Go2]). *Under all the above assumptions we have:*
(i) *ψ_f is a selection of φ_f,*
(ii) *ψ_f is a l.s.c. map.*

For the proof see e.g. [Go2].

Observe that condition (59) is rather restrictive. Therefore it is interesting to characterize the topological structure of all mappings satisfying (59). We shall do it in the case when $Y = A$ is a closed subset of $\mathbb{R}^m$ and $X = \mathbb{R}^n$.

By $C_c(A \times \mathbb{R}^n, \mathbb{R}^n)$ we denote the Banach space of all compact (single-valued) maps from $A \times \mathbb{R}^n$ into $\mathbb{R}^n$ with the usual supremum norm. Let

$$Q = \{f \in C_c(A \times \mathbb{R}^n, \mathbb{R}^n) : f \text{ satisfies (59)}\}.$$

We have (cf. [BiG2]):

Theorem 5.5. *The set Q is dense in $C_c(A \times \mathbb{R}^n, \mathbb{R}^n)$.*

Let us remark that all the above results remain true for X an arbitrary ANR-space (see [BiG2]).

Now we shall show how to apply the above results.

We start with ordinary differential equations of the first order. According to the above consideration, we let $Y = J \times \mathbb{R}^n$, where J is a closed halfline (possibly a closed interval), $X = \mathbb{R}^n$, and we let $f : Y \times X \to X$ be a compact map. Then f satisfies condition (58) automatically, and so we need to assume only (59). Let us consider the following equation:

$$\dot{x}(t) = f(t, x(t), \dot{x}(t)), \tag{60}$$

where the solution is understood in the sense of a.e. $t \in J$.

We associate with (60) the following two differential inclusions:

$$\dot{x}(t) \in \varphi_f(t, x(t)) \tag{61}$$

and

$$\dot{x}(t) \in \psi_f(t, x(t)), \tag{62}$$

where φ_f and ψ_f are defined as before and by a solution of (61) or (62) we mean a locally absolutely continuous function which satisfies (61) (resp. (62)) a.e. in J.

Denote by $S(f), S(\varphi_f)$ and $S(\psi_f)$ the sets of all solutions of (60), (61) and (62), respectively. Then we get:

$$S(\psi_f) \subset S(f) = S(\varphi_f).$$

But the map ψ_f is a bounded, l.s.c. map with closed values, so by Corollary 4.9 we obtain that $S(\psi_f)$ contains an R_δ-set as a subset.

In particular, we have proved:

$$\emptyset \neq S(\psi_f) \subset S(\varphi_f) = S(f).$$

Observe that in (61) and (62) the right hand sides do not depend on the derivative.

In an analogous way we may consider ordinary differential equations of higher order. Let $Y = J \times \mathbb{R}^{kn}$, $X = \mathbb{R}^n$, and let $f : Y \times X \to X$ be a compact map. To study the existence problem for the following equation

$$x^{(k)}(t) = f(t, x(t), \dot{x}(t), \dots, x^{(k)}(t)),$$

we consider the following two differential inclusions:

$$x^{(k)}(t) \in \varphi_f(t, x(t), \dot{x}(t), \dots, x^{(k-1)}(t))$$

and

$$x^{(k)}(t) \in \psi_f(t, x(t), \dot{x}(t), \dots, x^{(k-1)}(t)).$$

Thus we can get the analogous conclusions.

6. Concluding remarks

As we have already pointed out, the Conley index technique represents another powerful tool for the investigation of asymptotic BVPs (see e.g. [MW], [Sr1], [Sr2], [Wa1]-[Wa7]). All such results are, however, related only to single-valued operators. On the other hand, since the Conley index can be generalized for multivalued flows (see [Mr]), the question arises how to make appropriate extensions for the differential inclusions. We want to consider this problem separately elsewhere.

In §3, the existence of bounded solutions to partially dissipative differential inclusions has been proved on the positive ray. On the other hand, the apparatus developed in [Kr1], [KMP], [KMKP] allows us to deal with entirely bounded solutions of fully dissipative systems, but only in the single-valued case. So, it seems quite natural to extend the conclusions of §3 for entirely bounded solutions of partially dissipative differential inclusions.

Since we have developed in §2 the generalized degree as well as the fixed point index for associated J–mappings defined on subsets of Fréchet spaces, we can also obtain multiplicity criteria, when using their additivity properties. This can be done quite analogously, e.g. by means of the upper and lower solutions technique, as for BVPs on compact intervals, i.e. as for maps in Banach spaces.

Although there are some uniqueness theorems for mostly second-order BVPs on infinite intervals (see e.g. [Ba], [BJ], [GGLO], [Gr], [Kn], [Ma1], [Wo] and the references therein), we feel that this problem should be elaborated systematically.

There are certainly many further related questions deserving future study, the stability and instability analysis of bounded trajectories, etc. Nevertheless, as the first step, one should look for nontrivial applications of the abstract existence theorems, especially to higher-order equations and inclusions, as well as to systems.

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