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WHEN ALMOST MULTIPLICATIVE MORPHISMS ARE CLOSE TO HOMOMORPHISMS

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Abstract. It is shown that approximately multiplicative contractive positive morphisms from $C(X)$ (with dim $X \le 2$) into a simple C^*-algebra A of real rank zero and of stable rank one are close to homomorphisms, provided that certain K-theoretical obstacles vanish. As a corollary we show that a homomorphism $h : C(X) \to A$ is approximated by homomorphisms with finite dimensional range, if h gives no K-theoretical obstacle.

0. Introduction

Let X be a compact metric space and let A be a C^*-algebra. A contractive positive linear map $\psi : C(X) \to A$ is said to be δ-$\mathcal{F}$-multiplicative if

$$\|\psi(fg) - \psi(f)\psi(g)\| < \delta$$

for all $f \in \mathcal{F}$. A homomorphism is certainly δ-$\mathcal{F}$-multiplicative. The problem when an almost multiplicative contractive positive linear morphism is close to a homomorphism has been studied for a long time and recently there has been some important progress. A classical problem is whether for any $\varepsilon > 0$ there is $\delta > 0$ such that for any n and any pair of selfadjoint matrices $x, y \in M_n(\mathbf{C})$ such that $\|x\|, \|y\| \le 1$ and $\|xy - yx\| < \delta$, there exists a commuting pair $x', y' \in M_n(\mathbf{C})$ of selfadjoint matrices with $\|x' - x\| + \|y - y'\| < \varepsilon$. It was an old open problem for decades in linear algebra and operator theory which was solved affirmatively recently (see [Ln7]). This result is equivalent to the following: For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \in C(D)$, where D is the unit disk, there is $\delta > 0$ and a finite subset $\mathcal{G} \in C(D)$ such that for any finite-dimensional C^*-algebra A and any δ-$\mathcal{G}$-multiplicative contractive positive linear morphism $\psi : C(D) \to A$, there is a homomorphism $h : C(D) \to A$ (with finite-dimensional range) such that

$$\|\psi(f) - h(f)\| < \varepsilon \quad \text{for all } f \in \mathcal{F}.$$

The problem mentioned above appears in many different areas of mathematics, e.g., linear algebras, operator theory, as well as in the study of C^*-algebras.

In general, a δ-$\mathcal{G}$-multiplicative contractive positive linear morphism is not close to a homomorphism no matter how small δ is and how large $\mathcal{G}$ is. This was first discovered by D. Voiculescu (see [V]). K-theoretical obstacle was later explained by T. Loring (see [Lr2]). Therefore, what we are hoping for is that a δ-$\mathcal{G}$-multiplicative contractive positive linear morphism is close to a homomorphism, provided that

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δ is sufficiently small and $\mathcal{G}$ is sufficiently large and that a K-theoretical obstacle vanishes. Since every compact metric space X is a subspace of a contractible space Ω, a contractive positive linear morphism $\psi : C(X) \to A$ can always be viewed as a contractive positive linear morphism from $C(\Omega)$ into A. Therefore, some injectivity condition has to be imposed so that we know which obstacle has to vanish.

With the restriction that A is a unital simple C^*-algebra of real rank zero, stable rank one and unique normalized quasi-trace, it is shown in [GL2] that, if $dim(X) \le 2$, an approximately injective and sufficiently multiplicative contractive positive linear morphism $L : C(X) \to A$ is close to a homomorphism $h : C(X) \to A$ with finite dimensional range, provided that a K-theoretical obstacle vanishes. It is also shown (in [GL2]) that if X is a finite CW complex with dimension greater than 2, an approximately injective and δ-$\mathcal{G}$-multiplicative contractive positive linear morphism may not be close to any homomorphism even if all reasonable K-theoretical obstructions disappear, no matter how small δ is and how large $\mathcal{G}$ is. However, for higher dimensional spaces, with an additional injectivity condition, an affirmative solution to the problem can be found in [GL1]. On the other hand, we showed in [Ln11] that, for a given $\varepsilon > 0$ and a given finite subset $\mathcal{F} \subset C(X)$, there exist $\delta > 0$ and a finite subset $\mathcal{G} \subset C(X)$ such that, for any unital purely infinite simple C^*-algebra A and any δ-$\mathcal{G}$-multiplicative contractive positive linear morphism $L : C(X) \to A$, there is a homomorphism $h : C(X) \to A$ such that

$$\|L(f) - h(f)\| < \varepsilon$$

for all $f \in C(X)$. It is the assumption that A is finite that creates many obstacles as well as technical problems. The above mentioned results about almost multiplicative morphisms have many applications including those to classification of C^*-algebras of real rank zero and to C^*-algebra extension theory (see [EGLP], [Ln4], [Ln5], [Ln7], [LP1], [GL1], [Ln13]).

In this paper, we consider a class of simple C^*-algebras of real rank zero (denoted by $\mathbf{B}$) which consists of all purely infinite simple C^*-algebras as well as all simple C^*-algebras of real rank zero and stable rank one. It should be noted that there is no known example of a simple C^*-algebra of real rank zero which is neither purely infinite nor stable rank one. We will study almost multiplicative contractive positive linear morphisms from $C(X)$ into C^*-algebras in $\mathbf{B}$, with $dim(X) \le 2$. Roughly, the main result of this paper is that, with the above X, a δ-$\mathcal{G}$-multiplicative contractive positive linear morphism from $C(X)$ into any $A \in \mathbf{B}$ is close to a homomorphism, if a K-theoretical obstacle vanishes (see, for example, 1.12 and 2.5).

Here are some conventions which are needed in the rest of this paper.

Definition 0.1. Let $\psi : C(X) \to C$ be a homomorphism, where C is a C^*-algebra. Let Ω be the compact subset such that

$$ker(\psi) = \{f \in C(X) : f(\xi) = 0 \text{ for all } \xi \in \Omega\}.$$

We will denote Ω by $sp(\psi)$.

Definition 0.2 (cf. 1.2 of [LP1])**.** Let ψ be a contractive positive linear map from $C(X)$ to a C^*-algebra A, where X is a compact metric space. Fix a finite subset $\mathcal{F}$ contained in the unit ball of $C(X)$. For $\varepsilon > 0$, we denote by $\Sigma_\varepsilon(\psi, \mathcal{F})$ (or simply $\Sigma_\varepsilon(\psi)$) the closure of the set of those points $\lambda \in X$ for which there is a nonzero hereditary C^*-subalgebra B of A satisfying

$$\|(f(\lambda) - \psi(f))b\| < \varepsilon \quad \text{and} \quad \|b(f(\lambda) - \psi(f))\| < \varepsilon$$

for $f \in \mathcal{F}$ and $b \in B$ with $\|b\| \le 1$. Note that if $\varepsilon < \sigma$, then $\Sigma_\varepsilon(\psi) \subset \Sigma_\sigma(\psi)$.

For a δ-$\mathcal{G}$-multiplicative contractive positive linear morphism ψ and a subset $\mathcal{F} \subset \mathcal{G}$, we say ψ is σ-injective with respect to δ and $\mathcal{F}$, or σ-$\mathcal{F}$-injective, if $\Sigma_\delta(\psi, \mathcal{F})$ is σ-dense in X. If ψ is a homomorphism, then ψ is σ-injective, if $sp(\psi)$ is σ-dense in X. It follows from 1.12 in [Ln11] that, for any $\varepsilon > 0$ and $\mathcal{F}$, when δ is sufficiently small and $\mathcal{G}$ is sufficiently large, $\Sigma_\varepsilon(\psi, \mathcal{F})$ is not empty.

Definition 0.3. Let B be a C^*-algebra and X be a compact metric space. A homomorphism $\psi : C(X) \to B$ has *finite dimensional range* if (and only if) there exist a finite subset $\{\xi\}_{i=1}^l \subset X$ and a finite subset of mutually orthogonal projections $\{p_i\}_{i=1}^l \subset B$ such that

$$\psi(f) = \sum_{i=1}^{l} f(\xi_i) p_i \quad \text{for all } f \in C(X).$$

Definition 0.4. Let X be a finite CW complex and let A be a unital C^*-algebra. Suppose that $\phi : C(X) \to A \otimes \mathcal{K}$ is a homomorphism and $\xi_1, \xi_2, ..., \xi_m \in X$ are points in each connected component of X. Let $Y = X \setminus \{\xi_1, ..., \xi_m\}$. Homomorphism ϕ gives a homomorphism $\phi_0 : C_0(Y) \to A \otimes \mathcal{K}$. Let $[\phi]$ be the element in $KK(C(X), A)$ and let $[\phi_0]$ be an element in $KK(C_0(Y), A)$. We denote by $\mathcal{N}'(X, A)$ (or just $\mathcal{N}'$ if X and A are understood) the set of those elements in $KK(C(X), A)$ which are represented by those ϕ such that $[\phi_0] = 0$. Given m mutually orthogonal projections $p_1, p_2, ..., p_m \in A \otimes \mathcal{K}$, define $\phi'(f) = \sum_{i=1}^m f(\xi_i) p_i$ for $f \in C(X)$. Then $[\phi'] \in \mathcal{N}'$. Conversely, if $[\phi] \in \mathcal{N}'$, let $f_1, f_2, ..., f_m$ be projections in $C(X)$ corresponding to each component of X, and let $\phi(f_i) = p_i$, $i = 1, 2, ..., m$. Then $[\phi] - [\phi'] = 0$ in $KK(C(X), A)$. In fact, from the six-term exact sequence in KK-theory, the map from $KK(C(X), A)$ to $KK(C_0(Y), A)$ maps both $[\phi]$ and $[\phi']$ into zero. So they both are in the image of the map from $KK(C(X)/C_0(Y), A)$ (to $KK(C(X), A)$). Note that $C(X)/C_0(Y)$ is m copies of $\mathbf{C}$ corresponding to the m components. From the choice of ϕ', they both induce the same element in $KK(C(X), A)$.

Now let X be any compact metric space. Then $C(X) = lim_{n\to\infty} C(X_n)$, where X_n is a finite CW complex. There is a surjective map $s : KK(C(X), A) \to lim_{n\to\infty} KK(C(X_n), A)$. We denote by $\mathcal{N}'$ the set of those elements x in $KK(C(X), A)$ such that $s(x) \in lim_{n\to\infty} \mathcal{N}'(X_n, A)$ for any sequence of finite CW complexes $\{X_n\}$.

Recall that $KL(C(X), A)$ is the quotient $KK(C(X), A)$ by the subgroup of pure extensions in $Ext(K_*(C(X)), K_{*-1}(A))$ (see [Rr2]).

We denote by $\mathcal{N}$ the image of $\mathcal{N}'$ in $KL(C(X), A)$. If $\phi : C(X) \to A$ is a homomorphism, we write $\Gamma(\phi) \in \mathcal{N}$, if $[\phi] \in \mathcal{N}$.

Definition 0.5. The standard definition of mod-p K theory for C^*-algebras as given by Schochet in [Sch] is

$$K_i(A; \mathbf{Z}/n) = K_i(A \otimes C_0(X_n)),$$

where $X_n = C_n \setminus \{\xi\}$, ξ is a base point of C_n and C_n is the 2-dimensional CW-complex obtained by attaching a 2-cell to S^1 via the degree n map from S^1 to S^1, (notice that $K_0(C_0(X_n)) = \mathbf{Z}/n\mathbf{Z}$ and $K_1(C_0(X_n)) = \{0\}$). Let A be a C^*-algebra. Following [DL3], we write

$$\underline{K}(A) = \bigoplus_{i=0,1,n\ge 0} K_i(A; \mathbf{Z}/n\mathbf{Z}).$$

By [DL3], there is an isomorphism from $KL(C(X), A)$ onto $Hom_{\Lambda}(\underline{K}(C(X)), \underline{K}(A))$. Note that

$$K_0(A \otimes C_0(C_m \times S^1)) \cong K_0(A) \oplus K_1(A) \oplus K_0(A; \mathbf{Z}/m) \oplus K_1(A; \mathbf{Z}/m).$$

There is an obvious surjective from $\bigcup_{m>0} K_0(A \otimes C(C_m \times S^1))$ onto $\underline{K}(A)$. (Please see [DL3] for more information.)

The reason that we introduce mod-p K-theory is to deal with the case when $K_i(C(X))$ has torsion, $i = 0, 1$. If the reader is only interested in the case when $K_i(C(X))$ is torsion free, then it will be enough to consider $K_0(C(X)) \oplus K_0(C(X) \otimes C(S^1))$. Please see Remark 1.13.

0.6. Let A be a C^*-algebra. Denote by $\mathbf{P}(A)$ the set of projections in

$$\bigcup_{m \geq 0} M_\infty(A \otimes C(C_m \times S^1)).$$

Let $\mathcal{P}$ be a finite subset in $\mathbf{P}(A)$. There are a finite subset $\mathcal{G}(\mathcal{P}) \subset A$ and $\delta(\mathcal{P}) > 0$ such that if B is any C^*-algebra and $\phi : A \to B$ is a $*$-preserving linear map which is $\delta(\mathcal{P})$-$\mathcal{G}(\mathcal{P})$-multiplicative, then

$$\|((\phi \otimes id)(p))^2 - (\phi \otimes id)(p)\| < 1/4$$

for all $p \in \mathcal{P}$. Hence, for each $p \in \mathcal{P}$, there is a projection $q \in \mathbf{P}(B)$ such that

$$\|(\phi \otimes id)(p) - q\| < 1/2.$$

Furthermore, if q' is another projection satisfying the same condition, then $\|q - q'\| < 1$, hence q is unitarily equivalent to q'. Let $\bar{\mathcal{P}}$ be the image of $\mathcal{P}$ in $\underline{K}(A)$. For each $p \in \mathcal{P}$, we set $\phi_*([p]) = [q]$. This defines a map $\phi_* : \bar{\mathcal{P}} \to \underline{K}(B)$.

Let $\alpha : \bar{\mathcal{P}} \to \underline{K}(B)$. Suppose that there is a homomorphism $\psi : C(X) \to M_k(B)$ for some integer k with finite dimensional range such that $\psi_* = \alpha : \bar{\mathcal{P}} \to \underline{K}(B)$. Then we write $\alpha(\bar{\mathcal{P}}) \in \mathcal{N}$.

1. Approximation by homomorphisms with finite dimensional range

1.1. Recall (3.1 in [FR]) that a simple C^*-algebra A has property (IR) if the set of invertible elements is dense in the set of those elements in A which are either invertible or not one-sided invertible. We have the following fact:

Proposition 1.2. *For a simple C^*-algebra A, the following are equivalent:*
(1) A has the property (IR)*;*
(2) A is extremally rich;
(3) A is either of stable rank one or A is purely infinite.

Proof. Suppose that A is a simple C^*-algebra which has (IR). Let $x \in A$ which is not quasi-invertible (see p. 118 in [BP2]). Then, since A is simple, x is neither invertible nor one-sided invertible. But A has (IR). So x is approximated by invertible elements. Therefore the set of quasi-invertible elements is dense in A. So A is extremally rich (section 3 in [BP2]).

Now suppose that A is a simple C^*-algebra which is extremally rich. It follows from 10.5 in [BP3] that A is either purely infinite simple or has stable rank one.

Suppose that A is either purely infinite simple or has stable rank one. Then, by 3.1 in [FR], A has property (IR). □

1.3. Let $\mathbf{B}$ denote the family of those unital simple C^*-algebras with real rank zero which satisfy one of the conditions in Proposition 1.2.

Theorem 1.4 ([Ln12]). *Let $A \in \mathbf{B}$. Then, for any $\varepsilon > 0$, if x is a normal element in A and*

$$\lambda - x \in Inv_0(A)$$

for all $\lambda \notin \{t : dist(\lambda, sp(x)) \geq r\}$ for some $0 \leq r < \varepsilon$, then there is a normal element $y \in A$ with finite spectrum such that

$$\|x - y\| < \varepsilon.$$

We note that the condition that A is simple is necessary (see [Ln12]).

Theorem 1.5 ([FR]). *Let D be the unit disk in the plane. For any $\varepsilon > 0$ and any finite subset of the plane $\mathcal{F} \subset C(D)$, there exist $\delta > 0$ and a finite subset $\mathcal{G} \subset C(D)$ satisfying the following: if $\Lambda : C(D) \to A$ is a contractive positive linear morphism which is δ-$\mathcal{G}$-multiplicative, then there exists a homomorphism $h : C(D) \to A$ such that*

$$\|\Lambda(f) - h(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$, whenever A is a C^*-algebra with (IR).

Suppose that u is a unitary in a unital C^*-algebra A and $\Lambda : A \to B$ is a δ-$\mathcal{G}$-multiplicative contractive positive linear morphism, where B is also a unital C^*-algebra. It is easy to see that $\Lambda(u)$ is close to a unitary in B, if δ is small enough and $\mathcal{G}$ is large enough. Since two unitaries with distance less than 1 are connected to each other by a continuous path, $\Lambda(u)$ defines an element in $U(A)$, the unitary group of A. We will denote it by $\Lambda_*([u])$. Let $\mathcal{U}$ be a finite subset of $U(A)$. There is a finite subset $\mathcal{G}(\mathcal{U}) \subset A$ and $\delta(\mathcal{U}) > 0$ such that if $\Lambda : A \to B$ is a $\delta(\mathcal{U})$-$\mathcal{G}(\mathcal{U})$-multiplicative contractive positive linear morphism, then $\Lambda_*([u])$ is well defined for all $u \in \mathcal{U}$.

Theorem 1.6. *Let X be a compact subset of the plane. For any $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset C(X)$, there are a finite subset $\mathcal{U}$ of the generators of $U(C(X))$, the unitary group of $C(X)$, a positive number δ $(< \delta(\mathcal{U}))$, a finite subset $\mathcal{G} \subset C(X)$ containing $\mathcal{F}$ and $1 > \sigma > 0$ satisfying the following: if $A \in \mathbf{B}$ and $\Lambda : C(X) \to A$ is a contractive positive linear morphism which is δ-$\mathcal{G}$-multiplicative and σ-$\mathcal{F}$-injective with*

$$\Lambda_*([u]) = 0 \quad \text{for all } u \in \mathcal{U},$$

then there exists a homomorphism $h : C(X) \to A$ with finite dimensional range such that

$$\|\Lambda(f) - h(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$.

Proof. Without loss of generality, we may assume that $X \subset D$. Let $s : C(D) \to C(X)$ be the surjective map. Assume given a contractive positive linear morphism $\Lambda : C(X) \to A$. Let $\psi = \Lambda \circ s$. By applying Theorem 1.5, for any $\varepsilon_1 > 0$ with sufficiently small δ and sufficiently large $\mathcal{G}$, there is a homomorphism $h_1 : C(D) \to A$ such that

$$\|\psi(f) - h_1(f)\| < \varepsilon_1$$

for all $f \in \mathcal{F}_1$, where $\mathcal{F}_1$ is a finite subset of $C(D)$ such that $s(\mathcal{F}_1) = \mathcal{F}$, provided that Λ is δ-$\mathcal{G}$-multiplicative. Let $\sigma > 0$. Suppose that Λ is also σ-injective. With small ε_1 and a large $\mathcal{G}$, we may further assume that

$$sup\{dist(\xi, sp(h_1)), dist(X, \zeta) : \xi \in X, \zeta \in sp(h_1)\} < 2\sigma.$$

Let $\Omega = \{\xi : dist(\xi, X) \leq 2\sigma\}$. Note that Ω is homeomorphic to a finite CW complex on the plane. Let x be the generator of $h_1(C(D))$ with $sp(x) = sp(h_1)$. It is easy to compute that, with a sufficiently large $\mathcal{U}$, the condition that $\Lambda_*([u]) = 0$ for all $u \in \mathcal{U}$ implies that $\lambda - x \in Inv_0(A)$ for all $\lambda \in \Omega$. So, by Theorem 1.4, with sufficiently small σ and ε_1 and with sufficiently large $\mathcal{G}$, there is a homomorphism $h : C(X) \to A$ with finite dimensional range such that

$$\|h_1(f) - h \circ s(f)\| < \varepsilon/2$$

for all $f \in \mathcal{F}_1$. From this, we conclude that

$$\|\Lambda(f) - h(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$, if δ is sufficiently small and $\mathcal{G}$ is sufficiently large. □

Remark 1.7. In Theorem 1.6, if we assume that $X \subset D$, σ depends on ε and $\mathcal{F}$ only. Any $\sigma > 0$ that satisfies

$$|f(t) - f(t')| < \varepsilon/2,$$

whenever $dist(t, t') < 2\sigma$ and $f \in \mathcal{F}$, will do.

For future use, given an $\varepsilon > 0$, a compact metric space X and a finite subset $\mathcal{F} \subset C(X)$, let $\sigma = \sigma_{X,\mathcal{F},\varepsilon}$ be the largest positive number such that

$$|f(\xi) - f(\xi')| < \varepsilon,$$

whenever $dist(\xi, \xi') < \sigma$ and $f \in \mathcal{F}$. Note that

$$\sigma_{F,s(\mathcal{F}),\varepsilon} \geq \sigma_{X,\mathcal{F},\varepsilon}$$

if $F \subset X$ is a compact subset and $s : C(X) \to C(F)$ is the surjective map. We also note that, in 1.6, the assumption that A is simple is necessary since it is necessary in Theorem 1.4.

Lemma 1.8. *Let X be a compact metric space, $F \subset X$ be a compact subset of X which is homeomorphic to a contractible compact subset of the plane and F_0 be a compact subset of F with $dist(F_0, X \setminus F) > 0$. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset C(X)$, there exist $\delta > 0$, $\sigma > 0$, $\eta > 0$ and a finite subset $\mathcal{G} \subset C(X)$ (containing $\mathcal{F}$) satisfying the following: for any C^*-algebra $A \in \mathbf{B}$ and two mutually orthogonal projections $p, q \in A$, any contractive positive linear morphism $\Lambda : C(X) \to pAp$ which is δ-$\mathcal{G}$-multiplicative and σ-injective, and any homomorphism $h_0 : C(F_0) \to qAq$, if there is a homomorphism $h_1 : C(X) \to (p+q)A(p+q)$ with finite dimensional range such that*

$$\|\Lambda(f) \oplus h_0 \circ s(f) - h_1(f)\| < \eta$$

for all $f \in \mathcal{G}$, where $s : C(X) \to C(F_0)$ is the surjective map, then there is a homomorphism $h : C(X) \to pAp$ with finite dimensional range such that

$$\|\Lambda(f) - h(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$.

Proof. We may assume that $p+q=1$ and $\mathcal{F}$ is a subset of the unit ball of $C(X)$.

It follows from 1.6 that, for any $\varepsilon_1>0$ and a finite subset $\mathcal{F}_1\in C(F)$, there are $\delta_1>0$, $\sigma_1>0$ and a finite subset $\mathcal{G}_1\in C(F)$ satisfying the following: if $\lambda: C(F)\to B$ is a δ_1-$\mathcal{G}_1$-multiplicative and σ_1-injective contractive positive linear morphism, then there exists a homomorphism $H_1: C(F)\to B$ with finite dimensional range such that

$$\|\lambda(f)-H_1(f)\|<\varepsilon_1$$

for all $f\in\mathcal{F}_1$, whenever $B\in\mathbf{B}$.

Let $s_1: C(X)\to C(F)$ be the surjective map. Let $f_1,f_2\in C(X)$ such that $0\le f_1\le 1$, $f_1(x)=0$ if $x\in F_0$, $f_1(x)=1$ if $x\in X\setminus F$, and $0\le f_1\le 1$, $f_2(x)=1$ if $x\in F_0$, $f_2(x)=0$ if $x\in X\setminus F$ and $f_1f_2=0$.

Let $\mathcal{G}_2\subset C(X)$ be a finite subset containing f_1,f_2 and containing a finite subset of $C(X)$ such that its image under s_1 contains $\mathcal{G}_1$. Let $\delta>0$, $\eta>0$ and $\sigma>0$. Suppose that Λ, h_0 and h_1 are given as stated in the lemma for above δ, σ and $\mathcal{G}$. Note that we may assume that $\mathcal{G}$ is a subset of the unit ball of $C(X)$.

Write $h_1=\sum_{i=1}^{l} f(\xi_i)p_i$ for all $f\in C(X)$, where $\{x_i\}$ are fixed points in X and $\{p_i\}$ are finitely many mutually orthogonal projections in A. Set $\phi_1(f)=\sum_{\xi\in X\setminus F} f(\xi_i)p_i$, $\phi_2(f)=\sum_{\xi\in F} f(\xi_i)p_i$ for all $f\in C(X)$ and let $r=\sum_{\xi_i\in X\setminus F}p_i$. Since r commutes with h_1 and q commutes with $\Lambda\oplus h_0\circ s$, we have

$$h_1(f_1)r=rh_1(f_1)=r$$

and

$$(\Lambda\oplus h_0\circ s)(f_2)q=q((\Lambda\oplus h_0\circ s)(f_2))=q.$$

Therefore

$$\begin{aligned}\|rq\|&=\|rh_1(f_1)(\Lambda\oplus h_0\circ s)(f_2)q\|\\&\le\|r(h_1(f_1)-(\Lambda(f_1)\oplus h_0\circ s(f_1)))\|+\|r(\Lambda\oplus h_0\circ s)(f_1)(\Lambda\oplus h_0\circ s)(f_2)q\|\\&<\eta+\delta.\end{aligned}$$

Consequently,

$$\|r-prp\|<2(\eta+\delta).$$

If $\delta+\eta<1/4$, by [Eff, A8], there is a projection $r'\le p$ such that

$$\|r'-r\|<2(\eta+\delta)$$

and there is a unitary $u_1\in(p+q)A(p+q)$ such that

$$\|u_1-(p+q)\|<4(\eta+\delta)\quad\text{and}\quad u_1^*ru_1=r'.$$

Thus

$$\|u_1^*\phi_i(f)u_1-\phi_i(f)\|<8(\eta+\delta)\|f\|,\qquad i=1,2,$$

for all $f\in C(X)$. Then

$$\begin{aligned}\|p\phi_2(f)-\phi_2(f)p\|&\le\|p\phi_1(f)-\phi_1(f)p\|+\|ph_1(f)-h_1(f)p\|\\&\le\|pr\phi_1(f)-\phi_1(f)rp\|+2(\eta+\delta)<4(\eta+\delta)\end{aligned}$$

for all $f\in\mathcal{G}$. Since

$$\|u_1^*\phi_2(f)u_1-\phi_2(f)\|<8(\eta+\delta)\|f\|$$

for all $f \in C(X)$, we obtain

$$\|p(u_1^*\phi_2(f)u_1) - (u_1^*\phi_2(f)u_1)p\| \le 4(\eta+\delta) + 16(\eta+\delta) = 20(\eta+\delta)\|f\|$$

for all $f \in \mathcal{G}$. Put $p' = p - r'$. Since

$$r'u_1^*\phi_2(f)u_1 = u_1^*\phi_2(f)u_1 r' = 0$$

for all $f \in C(X)$,

$$\|p'(u_1^*\phi_2(f)u_1) - (u_1^*\phi_2(f)u_1)p'\| < 22(\eta+\delta)$$

for all $f \in \mathcal{G}$. Define $L(f) = p'(u_1^*\phi_2 \circ s_1(f)u_1)p'$ for all $f \in C(F)$. We have

$$\|ph(f)p - u_1^*\phi_1(f)u_1 \oplus L(f)\| < 8(\eta+\delta)$$

for all $f \in \mathcal{G}$.

It is clear that if σ, δ and η are small enough and $\mathcal{G}$ is large enough, $\{\xi_i : \xi_i \in F\}$ is σ_1-dense in F. So, when σ, δ and η are small enough and $\mathcal{G}$ is large enough, L is a δ_1-$\mathcal{G}_1$-multiplicative and σ_1-injective contractive positive linear morphism from $C(F)$ into $p'Ap'$. By applying Lemma 1.4, there is a homomorphism $H_2 : C(F) \to p'Ap'$ with finite dimensional range such that

$$\|L(f) - H_2(f)\| < \varepsilon_1$$

for all $f \in s(\mathcal{F})$.

Since $\Lambda(f) = p(\Lambda \oplus h_0 \circ s)p$, we have

$$\|\Lambda(f) - u_1^*\phi_1(f)u_1 \oplus H_2 \circ s(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$, provided that ε_1, δ and η are small enough. □

Lemma 1.9. *Let X be a compact metric space and let $F_0 \subset X$ be a compact subset of X which is homeomorphic to a contractible compact subset of the plane. Suppose that there is $d > 0$ such that there is a retraction $r : X_d \to F_0$, where $X_d = \{\xi \in X : dist(x, F_0) \le d\}$. For any $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset C(X)$, there exist $\delta > 0$, $\eta > 0$ and a finite subset $\mathcal{G} \subset C(X)$ (containing $\mathcal{F}$) satisfying the following: For any C^*-algebra $A \in \mathbf{B}$ and two mutually orthogonal projections $p, q \in A$, any contractive positive linear morphism $\Lambda : C(X) \to pAp$ which is δ-$\mathcal{G}$-multiplicative and $\sigma_{X,\mathcal{F},\varepsilon}$-injective, and any homomorphism $h_0 : C(F_0) \to qAq$, if there is a homomorphism $h_1 : C(X) \to (p+q)A(p+q)$ with finite dimensional range such that*

$$\|\Lambda(f) \oplus h_0 \circ s(f) - h_1(f)\| < \eta$$

for all $f \in \mathcal{G}$, where $s : C(X) \to C(F_0)$ is the canonical surjective map, then there is a homomorphism $h : C(X) \to pAp$ with finite dimensional range such that

$$\|\Lambda(f) - h(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$.

Proof. The proof is a minor modification of that of Lemma 1.8. Since F_0 is compact, there is $0 < a < d$ such that

$$\|f \circ r' - f\| < \varepsilon/2$$

for all $f \in s(\mathcal{F})$, where $r' = r|_{X_a}$, $X_a = \{\xi \in X_d : dist(\xi, F_0) \le a\}$ and $s : C(X) \to C(X_d)$.

In the proof of Lemma 1.8, we let $F = X_a$. Let $j : C(F_0) \to C(F)$ by defining $j(f) = f \circ r'$ for $f \in C(F_0)$ ($F = X_a$) and $s' : C(F) \to C(F_0)$ be the surjective map. Then

$$\|j \circ s'(f) - f\| < \varepsilon/2$$

for all $f \in s(\mathcal{F})$. Define $L' = L \circ j$, where L is as defined in the proof of Lemma 1.8. Now $L : C(F_0) \to p'Ap'$ is a homomorphism and F_0 is homeomorphic to a contractive subset of the plane. We then apply Lemma 1.4 to L as in the proof of Lemma 1.8 to obtain a homomorphism $H_2 : C(F_0) \to p'Ap'$ such that

$$\|L'(f) - H_2(f)\| < \varepsilon_1$$

for all $f \in s' \circ s(\mathcal{F})$. We note that

$$\|L(f) - L' \circ s'(f)\| < \varepsilon/2$$

for all $f \in s(\mathcal{F})$. Thus, as in the proof of Lemma 1.8, we will have

$$\|\Lambda(f) - u_1^* \phi_1(f) U_1 \oplus H_2 \circ s' \circ s(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$, provided that ε_1, δ and η are small enough. $\square$

Corollary 1.10. *Let X be a compact metric space and let $F \subset X$ be a compact subset of X which is a finite CW complex of dimension no more than 1. Suppose that there is $d > 0$ such that there is a retraction $r : X_d \to F$, where $X_d = \{\xi \in X : dist(x, F) \leq d\}$. For any $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset C(X)$, there exist $\delta > 0$, $\sigma > 0$, $\eta > 0$ and a finite subset $\mathcal{G} \subset C(X)$ (containing $\mathcal{F}$) satisfying the following: for any C^*-algebra $A \in \mathbf{B}$ and two mutually orthogonal projections $p, q \in A$, any contractive positive linear morphism $\Lambda : C(X) \to pAp$ which is δ-$\mathcal{G}$-multiplicative and σ-injective, and any homomorphism $h_0 : C(F) \to qAq$ with finite dimensional range, if there is a homomorphism $h_1 : C(X) \to (p+q)A(p+q)$ with finite dimensional range such that*

$$\|\Lambda(f) \oplus h_0 \circ s(f) - h_1(f)\| < \eta$$

for all $f \in \mathcal{G}$, where $s : C(X) \to C(F)$ is the surjective map, then there is a homomorphism $h : C(X) \to pAp$ with finite dimensional range such that

$$\|\Lambda(f) - h(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$.

Proof. We note that F is homeomorphic to $F' = \bigcup_{j=1}^m X_i$, where each X_i is a point or a closed line segment, and these line segments only intersect at the end points and do not intersect with anything else. Note that each line segment is a compact contractive subset of the plane. Let Y_i be the compact subset of F which is homeomorphic to X_i. It is clear that, for each i, there is a retraction $r_i : X_d^{(i)} \to Y_i$, where $X_d^{(i)} = \{x \in X : dist(x, Y_i) \leq d\}$, since there is a retraction $r : X_d \to F$. Thus the corollary follows by a repeated (m times) application of Lemma 1.8. $\square$

Lemma 1.11. *Let X be a compact metric space with dimension no more than 2 and let $\mathcal{F}$ be a finite subset of (the unit ball of) $C(X)$. For any $\varepsilon > 0$, there exist a finite subset $\mathcal{P}$ of projections in $\mathbf{P}(C(X))$, positive numbers $\delta > 0$ ($\delta < \delta(\mathcal{P})$ in 0.6) and $\sigma > 0$ and a finite subset $\mathcal{G}$ of (the unit ball of) $C(X)$ such that whenever $A \in \mathbf{B}$ and whenever $\psi : C(X) \to A$ is a contractive unital positive linear map* *which is δ-$\mathcal{G}$-multiplicative, σ-$\mathcal{G}$-injective and $\psi_*(\bar{\mathcal{P}}) \in \mathcal{N}$, then there exists a unital homomorphism $\varphi : C(X) \to A$ with finite dimensional range such that*

$$\|\psi(f) - \varphi(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$.

Proof. First, we will reduce the general case to the case when X is a polyhedron. We can write $C(X) = lim_{n\to\infty}(C(X_n), f_{n,n+1})$, where $f_{n,n+1}$ is the homomorphism from $C(X_n)$ to $C(X_{n+1})$ and by X_n is a polyhedron of dimension no more than 2. We also denote by f_n the homomorphism from $C(X_n)$ into $C(X)$ and by $f_{n,m} : C(X_n) \to C(X_m)$ $(m > n)$ the holomorphism induced by the direct limit system. It is easy to see that we may assume that all f_n are surjective. In fact we may assume that $X \subset X_{n+1} \subset X_n$ for all n and $X = \bigcap_{n=1} X_n$. We may also assume that there exists a finite subset $\mathcal{F}_1 \subset C(X_1)$ such that $f_1(\mathcal{F}_1) = \mathcal{F}$. Since $X = \bigcap_{n=1} X_n$ and $X_{n+1} \subset X_n$, there is, for any $\varepsilon > 0$, an integer N such that for any $\xi \in X_N$, there exists a point $\zeta \in X$ such that

$$|f(\xi) - f(\zeta)| < \varepsilon$$

for all $f \in \mathcal{F}_1$. It follows from Lemma 1.1 in [M] that, for any $r > 0$, there exist a polyhedron Q of dimension no more than 2, an injective homomorphism $\tau_1 : C(Q) \to C(X)$ and a homomorphism $\tau_2 : C(X_N) \to C(Q)$ such that

$$\|f_N(g) - \tau_1 \circ \tau_2(g)\| < r$$

for all $g \in \mathcal{F}_1$. Set $\mathcal{F}_2 = \tau_2(\mathcal{F}_1)$. Note that τ_1 gives a homomorphism from $KL(C(X), A)$ into $KL(C(Q), A)$ and that homomorphism maps $\mathcal{N}$ in $KL(C(X), A)$ to $\mathcal{N}$ in $KL(C(Q), A)$. In particular, if a finite subset $\mathcal{P}_1 \subset \mathbf{P}(C(Q))$ is given, then there is a finite subset $\mathcal{P} \subset \mathbf{P}(C(X))$ such that

$$(\psi \circ \tau_1)_*(\bar{\mathcal{P}}_1) \in \mathcal{N} \qquad \text{(see §0.6 for the notation)}$$

whenever

$$\psi_*(\bar{\mathcal{P}}) \in \mathcal{N}.$$

We also note that $\psi \circ \tau_1$ is σ-$\mathcal{F}_2$-injective. Furthermore, we note that a homomorphism $\psi' : C(X_N) \to A$ with finite dimensional range is close (within ε on $f_{1,N}(\mathcal{F})$) to a homomorphism $\psi : C(X) \to A$ with finite dimensional range on $\mathcal{F} = f_1(\mathcal{F}_1)$. Thus, without loss of generality, we may assume that X is a polyhedron of dimension no more than 2.

We decompose X and write $X = F \cup Y$, where F is finite CW complex with dimension 1 and Y is finitely (m) many disjoint subsets each of which is homeomorphic to the open unit disk. Let $d > 0$ and $F_d = \{\xi \in X : dist(\xi, F) \leq d\}$. Choose a small $d > 0$ so that the closure Y_d of $X \setminus F_d$ is a disjoint union of finitely many compact subsets $\{Y_j\}_{j=1}^m$, where each Y_j is homeomorphic to the (closed) unit disk on the plane.

Since $\psi_*(\mathcal{P}) \in \mathcal{N}$ and A is either purely infinite or has stable rank one, it follows from Theorem 1.6 in [Ln11] that, for any $\eta > 0$ and any finite subset $\mathcal{G}_1 \subset C(X)$, when δ is sufficiently small, $\mathcal{P}$ and $\mathcal{G}$ are sufficiently large,

$$\|\psi(f) \oplus h_0(f) - h_1(f)\| < \eta$$

for all $f \in \mathcal{G}$, where $h_0 : C(X) \to M_L(A)$ and $h_1 : C(X) \to M_{L+1}(A)$ are homomorphisms with finite dimensional range. Write $h_0(f) = \sum_{i=1}^l f(\xi_i)p_i$, where $\{\xi_i\}$ are finitely many points in X and $\{p_i\}$ are finitely many mutually orthogonal projections in $M_L(A)$. Without loss of generality, we may assume that $\{\xi_i\}$ are in the union of the interior of F_d and the interior of $\bigcup_{j=1}^m Y_j$. Let $s_j : C(X) \to C(Y_j)$ and $s_d : C(X) \to C(F_d)$ be the surjective maps. We may write

$$h_0 = \bigoplus_{j=1}^m h_0^{(j)} \circ s_j \oplus (h_d \circ s_d)$$

where $h_0^{(j)} : C(Y_j) \to q_j M_L(A) q_j$ and $h_d : C(F_d) \to q_d M_L(A) q_d$ are homomorphisms with finite dimensional range and $q_d, q_1, q_2, ..., q_m$ are mutually orthogonal projections in $M_L(A)$. For any $\varepsilon_1 > 0$ and any finite subset $\mathcal{G}_1 \in C(X)$, with sufficiently small δ, σ and η, and sufficiently large $\mathcal{G}$, by repeatedly applying Lemma 1.8 (as in Corollary 1.10), there is a homomorphism $h_2 : C(X) \to (1+q_d)M_{1+L}(A)(1+q_d)$ with finite dimensional range such that

$$\|\Lambda(f) \oplus h_d \circ s_d(f) - h_2(f)\| < \varepsilon_1/2$$

for all $f \in \mathcal{G}_1$.

There is a retraction $r : F_d \to F$. Let $j : C(F) \to C(F_d)$ be the injection induced by r and $s_0 : C(X_d) \to C(F)$ be the surjective map.

We now choose $\mathcal{G}_1$ and ε_1 so that we can apply Corollary 1.10 for the one (or zero) dimensional finite CW complex F (and for $\varepsilon > 0$ and $\mathcal{F}$) as below. With this ε_1 and $\mathcal{G}_1$, we can choose small $d > 0$ at the beginning so that

$$\|j \circ s_0(s_d(f)) - s_d(f)\| < \varepsilon_1/2$$

for all $f \in \mathcal{G}_1$. Denote $h_d \circ j : C(F) \to q_d M_L(A) q_d$ by H. Then we have

$$\|\Lambda(f) \oplus H \circ (s_0 \circ s_d)(f) - h_2(f)\| < \varepsilon_1$$

for all $f \in \mathcal{G}_1$.

Then, by applying Corollary 1.10, we obtain a homomorphism $h : C(X) \to A$ with finite dimensional range such that

$$\|\Lambda(f) - h(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$, provided that ε_1 is small enough and $\mathcal{G}_1$ is large enough.

This proves the theorem for the case when X is a two-dimensional polyhedron. The above certainly also work if we replace disk by line segments. So the theorem holds for the case when X is one-dimensional polyhedron. (It also follows from [Lr3] and [Ln3].) The case when X is zero dimensional is trivial, since A has real rank zero. □

Theorem 1.12. *Let X be a compact metric space with dimension no more than 2 and let $\mathcal{F}$ be a finite subset of (the unit ball of) $C(X)$. For any $\varepsilon > 0$, there exist a finite subset $\mathcal{P}$ of projections in $\mathbf{P}(C(X))$, a positive number $\delta > 0$ ($\delta < \delta(\mathcal{P})$) and a finite subset $\mathcal{G}$ of (the unit ball of) $C(X)$ such that whenever $A \in \mathbf{B}$ and whenever $\psi : C(X) \to A$ is a contractive unital positive linear map which is δ-$\mathcal{G}$-multiplicative and $(1/2)\sigma_{X,\mathcal{F},\varepsilon/4}$-$\mathcal{F}$-injective, and $\psi_*(\bar{\mathcal{P}}) \in \mathcal{N}$, then there exists a unital homomorphism $\varphi : C(X) \to A$ with finite dimensional range such that*

$$\|\psi(f) - \varphi(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$.

Proof. The only difference between Lemma 1.11 and Theorem 1.12 is the requirement about σ. It follows from [GL2] that we may assume that A is not elementary. Let $\sigma = 1/2\sigma_{X,\mathcal{F},\varepsilon/4}$. Suppose that $\psi : C(X) \to A$ is a contractive positive linear morphism satisfying the conditions in the theorem. □

Given any $\varepsilon_1 > 0$ and $\mathcal{G}_1$, with sufficiently small δ and sufficiently large $\mathcal{G}$, by Lemma 1.5 in [LP1], without loss of generality, we may write that

$$\psi(f) = \sum_{i=1}^{m} f(\zeta_i)p_i \oplus \psi_1(f)$$

for all $f \in C(X)$, where $\{\zeta_i\}$ is 2σ-dense in X and ψ_1 is a δ_1-$\mathcal{G}_1$-multiplicative contractive positive linear morphism. Since we now assume that A is not elementary and simple, there exists a nonzero projection $e \in A$ such that $e \preceq p_i$ for each i. Again, for any $\sigma_1 > 0$, since eAe is nonelementary and simple, there exists a homomorphism $h_0 : C(X) \to eAe$ which is σ_1-$\mathcal{G}_1$-injective. Now we apply Lemma 1.11 to the map $\psi_1 \oplus h_0$ (with sufficiently small δ_1 and sufficiently large $\mathcal{G}_1$). We obtain a homomorphism $h_1 : C(X) \to QM_2(A)Q$ (with $Q = diag(1 - \sum_{i=1}^m p_i, e)$) with finite dimensional range such that

$$\|\psi_1(f) \oplus h_0(f) - h_1(f)\| < \varepsilon/3$$

for all $f \in \mathcal{F}$. Since $\{\zeta_i\}$ is σ-dense in X, by changing h_0 slightly, we obtain a homomorphism $h_2(f) = \sum_{i=1}^m f(\zeta_i)q_i$, where $\{q_i\}$ are mutually orthogonal projections in eAe such that

$$\|\psi_1(f) \oplus h_2(f) - h_1(f)\| < \varepsilon/3 + \varepsilon/4$$

for all $f \in \mathcal{F}$. Note that $q_i \preceq p_i$ for each i. There is a unitary $U \in M_2(A)$ such that

$$U^*q_iU \le p_i \quad \text{and} \quad U^*(1 - \sum_{i=1}^{m} p_i) = (1 - \sum_{i=1}^{m} p_i)U = (1 - \sum_{i=1}^{m} p_i).$$

Thus

$$\|\psi(f) - [\sum_{i=1}^{m} f(\xi_i)(p_i - U^*q_iU) \oplus U^*h_2(f)U]\| < \varepsilon$$

for all $f \in \mathcal{F}$.

Remark 1.13. Theorem 1.12 is in the best form in the following sense: the condition that $dim(X) \le 2$ is necessary by [GL2]. Note that a homomorphism ϕ can be approximated (pointwise) by homomorphisms with finite dimensional range only if $[\phi] \in \mathcal{N}$ (see 5.4 in [Rr1]). So the condition that $\psi_*(\bar{\mathcal{P}}) \in \mathcal{N}$ is also necessary. The conditions that A is simple and has real rank zero are necessary (note that the condition that A is simple is necessary in Theorem 1.4). The "injective" condition cannot be more relaxed than $\sigma_{X,\mathcal{F},\varepsilon}$-$\mathcal{F}$-injective otherwise as stated earlier in the introduction, we would not know which KK-theoretical obstacle needs to vanish. We also note that we do not know if there is any simple C^*-algebra of real rank zero which is not in $\mathbf{B}$. If $K_i(C(X))$ is torsion free, then $KL(C(X), A) = Hom(K_*(C(X)), K_*(A))$. So, from the proof of Theorem 1.6 in [Ln11], one sees that the condition that $\psi_*(\bar{\mathcal{P}}) \in \mathcal{N}$ in Lemma 1.11 and in Theorem 1.12 can be replaced by the condition that $\psi_*(\bar{\mathcal{Q}}) = h_*(\bar{\mathcal{Q}})$ for some homomorphism $h : C(X) \to A \otimes \mathcal{K}$ with finite dimensional range and some (large) finite subset $\mathcal{Q}$ of projections in $\bigcup_m M_m(C(X)) \oplus M_m(C(X) \otimes C(S^1))$.

However, if $K_0(C(X))$ has torsion, then we have to use the mod-p K-theory since $KL(C(X), A) \neq Hom(K_*(C(X)), K_*(A))$. The following is an easy example. Let $D_n = C(C_n)$, where C_n is as in 0.5 and $n > 2$. Note that $dim(C_n) = 2$, $K_0(D_n) = \mathbf{Z} \oplus \mathbf{Z}/n\mathbf{Z}$ and $K_1(D_n) = \{0\}$. Let A be a separable unital (nuclear) purely infinite simple C^*-algebra with $K_0(A) = 0$ and $K_1(A) = \mathbf{Z}$. Such a C^*-algebra is given in [Rr1]. From the Universal Coefficient Theorem, since $Hom(K_*(C(C_n)), K_*(A)) = 0$, $KK(D_n, A) = ext^1_{\mathbf{Z}}(\mathbf{Z}/n\mathbf{Z}, \mathbf{Z})$. Choose a nontrivial element $\alpha \in ext^1_{\mathbf{Z}}(\mathbf{Z}/n\mathbf{Z}, \mathbf{Z})$. Then, by Theorem 1.17 in [Ln11], there is a unital homomorphism $h : D_n \to A$ such that $[h] = \alpha$. Let $F : D_n \to A$ be a point-evaluation. Then $[F] = 0$ in $KK(D_n, A)$. We see that h cannot be approximated by homomorphisms with finite dimensional range, since $[h] \neq [F]$ in $KK(D_n, A)$ (5.4 in [Rr1]). However,

$$h_*([p]) = F_*([p]) = 0$$

for any projection in $\bigcup_m M_m(D_n) \oplus M_m(D_n \otimes C(S^1))$. So to reveal the hidden obstacle in $ext^1_{\mathbf{Z}}(K_0(D_n), K_1(A))$, we use mod-$p$ K-theory.

Corollary 1.14. *Let X be a compact metric space of dimension no more than two. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset C(X)$, there is $\sigma > 0$ such that, for any unital C^*-algebra $A \in \mathbf{B}$ and σ-injective homomorphism $\phi : C(X) \to A$ with*

$$\Gamma(\phi) \in \mathcal{N}$$

there is a homomorphism $h : C(X) \to A$ with finite dimensional range such that

$$\|\phi(f) - h(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$.

2. Non-injective maps

While we have seen that the "injective" condition is necessary for Theorem 1.12, in some cases, however, this condition can be removed, if we do not require the homomorphism ϕ in Theorem 1.12 to have finite dimensional range. Furthermore, a homomorphism $\phi : C(X) \to A$ may have the property that $\Gamma(\phi) \notin \mathcal{N}$. So if we just want to have homomorphisms to approximate a δ-$\mathcal{G}$-multiplicative contractive positive linear morphism Λ, it is not necessary to have $\Lambda_*(\mathcal{P}) \in \mathcal{N}$. However, in general, some K-theoretical condition is still needed.

2.1. Let X be a compact metric space and A be a C^*-algebra. There is a short exact sequence

$$kerd_X \to K_0(C(X)) \to C(X, \mathbf{Z}) \to 0,$$

where the map $d : K_0(C(X)) \to C(X, \mathbf{Z})$ is the dimension map. Suppose that there is a homomorphism $h : C(Y) \to C(X)$. Then the induced map h_* maps $kerd_Y$ into $kerd_X$.

Let $h : C(X) \to A$ be a homomorphism and A be a C^*-algebra with normalized quasitraces τ_a. Note that each τ_a gives a state on $K_0(A)$. For each quasitrace of A, $\tau_a \circ h$ gives a trace of $C(X)$. It is then clear that, if $b \in kerd_X$, then $t(b) = 0$, where t is the state on $K_0(C(X))$ defined by $\tau_a \circ h$. So, in general, not every element $\alpha \in KL(C(X), A)$ is given by a homomorphism from $C(X)$ into A.

We now introduce the following definition. Denote by $\mathcal{N}k$ the set of those elements α in $KL(C(X), A)$ such that $\gamma(\alpha)|_{kerd_X} = 0$, where $\gamma : KL(C(X), A) \to Hom(K_*(C(X)), K_*(A))$ is the usual surjective map.

Lemma 2.2. *Let X be a locally compact mertric space and $G \subset X$ be an open subset. For any $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset C_0(X)$, there exist $\delta > 0$, a finite subset $\mathcal{G}_1 \subset I$, where $I = \{f \in C_0(X) : f(\xi) = 0, \xi \in X \subset G\}$, such that, if A is a C^*-algebra and $\phi : C_0(X) \to A$ is a positive linear map with*

$$\|\phi(f)\| < \delta$$

for all $f \in \mathcal{G}_1$, then there is a positive linear map $\Lambda : C_0(X \setminus G) \to A$, such that

$$\|\phi(f) - \Lambda(s \circ f)\| < \varepsilon$$

for all $f \in \mathcal{F}$, where $s : C_0(X) \to C_0(X \setminus G)$ is the surjective map.

Proof. Suppose that the lemma is false. Let $\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_n, ...$ be a sequence of finite subsets of the unit ball of I such that $\mathcal{G}_n \subset \mathcal{G}_{n+1}$ and the union $\bigcup_{n=1}^{\infty} \mathcal{G}_n$ is dense in the unit ball of $C(X)$. Then there are a positive number $\varepsilon > 0$, a finite subset $\mathcal{F} \subset C(X)$, a sequence $\{\delta_n\}$ with $\delta_n \to 0$, unital C^*-algebras B_n and contractive positive linear morphisms $\psi_n : C_0(X) \to B_n$ with $\|\psi_n(f)\| < \delta_n$ for $f \in \mathcal{G}_n$ such that

$$inf\{sup_{f\in\mathcal{F}}\{\|\psi_n(f) - \Lambda_n(s \circ f)\|\}\} \geq \varepsilon.$$

Here the infimum is taken over all contractive positive linear morphisms

$$\Lambda_n : C_0(X \setminus G) \to A.$$

Let $\Psi = \{\psi_n\}$. Then $\Psi : C(X) \to \prod_n B_n$ is a contractive positive linear morphism. Let $\pi : \prod_n B_n \to \prod_n B_n / \bigoplus_n B_n$ be the quotient map. Then $\pi \circ \Psi : C(X) \to \prod_n B_n / \bigoplus_n B_n$ is a linear map with its kernel containing I. Thus there is a contractive positive linear morphism $L : C(F) \to \prod_n B_n / \bigoplus_n B_n$ such that $\pi \circ \Psi = L \circ s$. It follows from [CE] that there is a contractive positive linear morphism $L_1 : C(F) \to \prod_n B_n$ such that $\pi \circ L_1 = L$. Write $L_1 = \{\Lambda_n\}$, where $\Lambda_n : C(F) \to B_n$ are contractive positive linear morphisms. Thus, for any $f \in C(F)$,

$$\|\psi_n(f) - \Lambda_n(s \circ f)\| \to 0 \quad \text{as} \quad n \to \infty.$$

This ends the proof. □

Lemma 2.3 (cf. Lemma 2.1 in [Ln7] and Lemma 2.12 in [GL2])**.** *Let X be a locally compact metric space, $G \subset X$ be an open subset,*

$$I = \{f \in C_0(X) : f(x) = 0 \ \ if \ \ x \notin G\}.$$

For any $\varepsilon > 0$, $\eta > 0$ and a finite subset $\mathcal{F} \in C_0(X)$, there exist $\delta > 0$, $\gamma > 0$, $a > 0$ and a finite subset $\mathcal{G} \subset C(X)$ satisfying the following: if A is a C^*-algebra of real rank zero, $\phi : C_0(X) \to A$ is a contractive positive linear map which is γ-$\mathcal{G}$-multiplicative and if

$$\|\phi(g_{a/8}f) - \sum_{k=1}^{m} g_{a/8}f(\xi_k)p_k\| < \delta$$

for all $f \in \mathcal{G}$, where $\xi_k \in G$, $\{p_k\}$ are mutually orthogonal projections in A and where $g_\beta \in C_0(X)$, $0 \leq g_\beta \leq 1$, $g_\beta(t) = 0$ if $dist(t, X \setminus G) < \beta/2$ and $g_\beta(t) = 1$ if $dist(t, X \setminus G) \geq \beta$, if $\beta > 0$, then there exists a projection $p \in A$, a contractive positive linear morphism $\Lambda : C_0(X \setminus G) \to pAp$ which is η-$s(\mathcal{F})$-multiplicative, finitely many points $\{\zeta_k\} \subset G$, and finitely many mutually orthogonal projections

$\{p_k\} \subset pAp$, where $F = \{\xi \in X : dist(\xi, G) \geq \sigma\}$ and $s : C_0(X) \to C_0(X \setminus G)$ is the surjective map, such that

$$\|\phi(f) - \Lambda(s \circ f) \oplus \sum_k f(\zeta_k)p_k\| < \varepsilon$$

for all $f \in \mathcal{F}$.

Proof. The proof is a modification of that of Lemma 2.11 in [GL2]. We may assume that $\mathcal{F}$ is a subset of the unit ball of $C(X)$.

Given $\eta > 0$ and a finite subset $\mathcal{G}_1$ which is in the unit ball of I, there exists $a > 0$ such that

$$\|g_a f - f\| < \eta$$

for all $f \in \mathcal{G}_1 \cup \mathcal{F} \cap I$.

Set $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{F} \cup \{g_{a/4}\}$. Suppose that

$$\|\phi(g_{a/8}f) - \sum_{k=1}^{m} g_{a/8}f(\xi_k)p_k\| < \delta$$

for all $f \in \mathcal{G}$, where $\delta > 0$ will be chosen later.

Let $p = \sum_{\xi_j \in \Omega_{a/2}} p_j$, where $\Omega_{a/2} = \{\xi \in G : dist(\xi, X \setminus G) \geq a/2\}$. A direct computation shows, with sufficiently small γ as well as sufficiently small δ, as in the proof of Lemma 2.11 in [GL2],

$$\|p\phi(f)p - \sum_{\xi_j \in \Omega_{a/2}} f(\xi_j)p_j\| < \varepsilon/4$$

for all $f \in \mathcal{F}$ and

$$\|(1-p)\phi(g_a)\| < \sigma$$

for any given $\sigma > 0$. We also have, with sufficiently small δ and η,

$$\|(1-p)\phi(f)(1-p)\| < \sigma$$

for all $f \in \mathcal{G}_1 \cup \mathcal{F} \cap I$. Thus, by applying Lemma 2.2, with sufficiently small σ and sufficiently large $\mathcal{G}_1$, we obtain a contractive positive linear morphism $\Lambda : C(F) \to (1-p)A(1-p)$ such that

$$\|(1-p)\phi(f)(1-p) - \Lambda(s \circ f)\| < \varepsilon/4$$

for all $f \in \mathcal{F}$. Therefore

$$\|\phi(f) - [\Lambda(s \circ f) \oplus \sum_{\xi_j \in \Omega_{a/2}} f(\xi_j)p_j]\| < \varepsilon$$

for all $f \in \mathcal{F}$. □

Lemma 2.4. *Let X be a finite CW complex of dimension no more than two. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \in C(X)$, there exist a finite subset $\mathcal{P} \in \mathbf{P}(C(X))$, $\delta > 0$ ($\delta < \delta(\mathcal{P})$) and a finite subset $\mathcal{G} \in C(X)$ satisfying the following: if $A \in \mathbf{B}$ and $\Lambda : C(X) \to A$ is a contractive positive linear morphism which is δ-$\mathcal{G}$-multiplicative and $1/2\sigma_{X,\mathcal{F},\varepsilon/4}$-$\mathcal{F}$-injective with*

$$\Lambda_*(\bar{\mathcal{P}}) = \alpha(\bar{\mathcal{P}})$$

for some $\alpha \in \mathcal{N}k$, then there exists a homomorphism $h : C(X) \to A$ such that

$$\|\Lambda(f) - h(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$.

Proof. We assume that X is a polyhedron. If X is a polyhedron of dimension zero, then, the lemma follows easily. The case when X is a polyhedron of dimension 1 follows from Theorem 5.1 in [Lr4].

We now assume that X is a finite CW complex with dimension 2. Suppose that $F \subset X$ is a one-dimensional finite CW complex, $\{Y_i\}_{i=1}^m$ are m disks (of the plane) and f_i are continuous functions from the boundaries of Y_i into F such that X is the result of gluing Y_i on F by maps f_i.

Let $I = \{f \in C(X) : f(x) = 0 \text{ for } x \in F\}$ be the ideal of $C(X)$. Since now we assume that X is a finite CW complex, $K_0(C(X))$ is finitely generated. Let $\Lambda : C(X) \to A$ be a contractive positive linear morphism which is δ-$\mathcal{G}$-multiplicative and σ-injective, where δ, $\mathcal{G}$ and σ will be determined. We also assume that

$$\Lambda_*(\bar{\mathcal{P}}) = \alpha(\bar{\mathcal{P}})$$

for some $\alpha \in \mathcal{N}k$ and for some finite subset $\mathcal{P} \subset \mathbf{P}(C(X))$. Let $\lambda : I \to A$ be the restriction of Λ on I and $L : C(\Omega) \to A$ be the contractive positive linear morphism induced by λ, where $C(\Omega) = \tilde{I}$. Note that Ω is the one-point compactification of the union of interior Y_i's. Note that Ω is the space of finitely many 2-spheres glued at a common point. Since $K_i(C(\Omega))$ is finitely generated and torsion free, one computes that $KL(C(\Omega), A) = Hom(K_*(C(\Omega)), K_*(A))$. Therefore, for space Ω, $\mathcal{N}_k = \mathcal{N}$. From K-theory, the natural injective map from $C(\Omega)$ into $C(X)$ induces an injective map from $Hom(K_0(C(\Omega)), K_0(A))$ into $Hom(K_0(C(X)), K_0(A))$. Let $\mathcal{Q} \subset \mathbf{P}(C(\Omega))$ be a finite subset. Then the condition that

$$\Lambda_*(\bar{\mathcal{P}}) = \alpha(\bar{\mathcal{P}})$$

for some $\alpha \in \mathcal{N}k$, implies that

$$L_*(\bar{\mathcal{Q}}) \in \mathcal{N},$$

provided that $\mathcal{P}$ is sufficiently large. By Theorem 1.12, for any $\varepsilon_1 >$ and any finite subset $\mathcal{G}_1 \in C(\Omega)$, there is a homomorphism $h_0 : C(\Omega) \to A$ with finite dimensional range such that

$$\|L(f) - h_0(f)\| < \varepsilon_1$$

for all $f \in \mathcal{G}_1$, provided that σ and δ are sufficiently small and $\mathcal{G}$ is sufficiently large.

Denote by s_F the canonical surjective map from $C(X)$ onto $C(F)$. For $\delta_1 > 0$ a finite subset $\mathcal{G}_2 \in C(X)$, by applying Lemma 2.3, we obtain a projection $p \in A$, a contractive positive linear morphism $L_0 : C(F) \to pAp$ which is δ_1-$s_F(\mathcal{G})$-multiplicative and σ-injective, a finite set of mutually orthogonal projections $\{p_i\} \subset (1-p)A(1-p)$ and a finite set $\{\zeta_i\}$ in the interior of $\bigcup_{j=1}^m Y_j$ such that

$$\|\Lambda(f) - (L_0 \circ s_F(f) \oplus \sum_i f(\zeta_i)p_i)\| < \varepsilon_1$$

for all $f \in \mathcal{G}_2$, provided that δ is sufficiently small and $\mathcal{G}$ is sufficiently large. Since F is one-dimensional, by Theorem 5.1 in [Lr4], there is a homomorphism $h_1 : C(F) \to pAp$ such that

$$\|L_0(f) - h_1(f)\| < \varepsilon/2$$

for all $f \in s_F(\mathcal{F})$, provided that δ_1 is sufficiently small and $\mathcal{G}$ is sufficiently large. We then define $h(f) = h_1 \circ s_F \oplus \sum_i f(\zeta_i)p_i$ for $f \in C(X)$. Thus

$$\|\Lambda(f) - h(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$, provided that ε_1 is sufficiently small (which requires that δ and σ be small enough) and $\mathcal{G}_2$ is sufficiently large (which requires that $\mathcal{G}$ be large enough). $\square$

Theorem 2.5. *Let X be a finite CW complex of dimension no more than two. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \in C(X)$, there exist a finite subset $\mathcal{P} \in \mathbf{P}(C(X))$, a positive number $\delta > 0$ ($\delta < \delta(\mathcal{P})$) and a finite subset $\mathcal{G} \in C(X)$ satisfying the following: If $A \in \mathbf{B}$ and $\Lambda : C(X) \to A$ is a contractive positive linear morphism which is δ-$\mathcal{G}$-multiplicative with*

$$\Lambda_*(\bar{\mathcal{P}}) = \alpha(\bar{\mathcal{P}})$$

for some $\alpha \in \mathcal{N}k$, then there exists an homomorphism $h : C(X) \to A$ such that

$$\|\Lambda(f) - h(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$.

Proof. Let $\sigma = 1/4\sigma_{X,\mathcal{F},\varepsilon/4}$, where $\sigma_{X,\mathcal{F},\varepsilon/4}$ is as defined in Remark 1.7. Since X is compact, there are compact subsets $F_1, F_2, ..., F_l \subset X$ which are themselves finite CW complex such that for any compact subset $F \subset X$, there is an F_j with $F \subset F_j$ and

$$sup\{dist(x, F_j) + dist(F, y) : x \in F, y \in F_j\} < \sigma.$$

Note that $\sigma_{F_j,s_j(\mathcal{F}),\varepsilon/4} \geq \sigma_{X,\mathcal{F},\varepsilon/4}$, where $s_j : C(X) \to C(F_j)$ is the surjective map. Let $\delta_j > 0$, $\mathcal{P}_j \subset \mathbf{P}(C(F_j))$ and finite subset $\mathcal{G}_j \subset C(F_j)$ be as required in Lemma 2.4, for ε_1, $s_j(\mathcal{F})$ and space F_j, $j = 1, 2, ..., l$. Let $\delta' = min\{\delta_i : i = 1, 2, ..., l\}$, $\mathcal{G}' = \bigcup_i H_i$, where H_i is a finite subset of $C(X)$ such that $s_i(H_i) = \mathcal{G}_i$. It follows from Lemma 1.17 in [Ln11] that with sufficiently large $\mathcal{G}$ and sufficiently small δ, there is a compact subset $F \subset X$ and a contractive positive linear morphism $L' : C(F) \to A$ which is $\delta'/2$-$\mathcal{G}'$-multiplicative and σ-$s(\mathcal{G}')$-injective, where $s : C(X) \to C(F)$ is the surjective map, such that

$$\|\psi(f) - L'(f)\| < \varepsilon/2$$

for all $f \in \mathcal{G}'$. Choose F_j above so that $F \subset F_j$ and

$$sup\{dist(x, F_j) + dist(F, y) : x \in F, y \in F_j\} < \sigma/2.$$

Let $s_0 : C(F_j) \to C(F)$ be the quotient map and $L = L' \circ s_0 : C(F_j) \to A$. Then L is $\delta/2$-$s_j(\mathcal{G}')$-multiplicative and σ-$s_j(\mathcal{G}')$-injective. It follows from Lemma 2.2 in [GL2] that, since $dim(X) \leq 2$, there is a surjective map from $kerd_X$ onto $kerd_{F_j}$. Thus, with sufficiently large $\mathcal{P} \subset \mathbf{P}(C(X))$, the condition that $\Lambda_*(\bar{\mathcal{P}}) \in \mathcal{N}_k$ implies that

$$L_*(\bar{\mathcal{P}}_j) \in \mathcal{N}k.$$

By Lemma 2.4, there is a homomorphism $h_1 : C(F_j) \to A$ such that

$$\|L(f) - h_1(f)\| < \varepsilon/2$$

for all $f \in s_j(\mathcal{F})$. Note that $L' \circ s_0 \circ s_j = L \circ s'$. We have

$$\|\phi(f) - h_1 \circ s_j(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$. Take $h = h_1 \circ s_j$. $\square$

Definition 2.6. Let X be a compact metric space and A be a unital C^*-algebra. Denote by $\mathcal{H}$ the subset of elements in $KL(C(X), A)$ which is represented by a homomorphism $h : C(X) \to A \otimes \mathcal{K}$. Note that if $\alpha \in \mathcal{H}$ and γ is the map from $KL(C(X), A)$ onto $Hom(K_*(C(X)), K_*(A))$, then $\gamma(\alpha)$ preserves the order on $K_0(C(X))$. Fix X. We denote by $\mathbf{B}_X$ the set of those simple C^*-algebra $A \in \mathbf{B}$ satisfying the property that, for any nonzero projection $p \in A$ and $\alpha \in \mathcal{H}$, there exists a homomorphism $h : C(X) \to eAe$ such that

$$(h)_*|_{kerd_X} = \alpha|_{kerd_X}$$

for every compact subset of X. It follows from [Ln11] that every purely infinite simple C^*-algebra is in $\mathbf{B}_X$.

Theorem 2.7. *Let X be a compact metric space of dimension no more than two. For any $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset C(X)$, there exist a finite subset $\mathcal{G} \subset C(X)$, a positive number $\delta > 0$ ($\delta < \delta(\mathcal{P})$) and a finite subset $\mathcal{P} \subset \mathbf{P}(C(X))$ satisfying the following: For any unital C^*-algebra $A \in \mathbf{B}_X$, if $\Lambda : C(X) \to A$ is a contractive positive linear morphism which is δ-$\mathcal{G}$-multiplicative, $1/4\sigma_{X,\mathcal{F}\varepsilon/4}$-$\mathcal{F}$-injective and*

$$\Lambda_* = \alpha \text{ on } \bar{\mathcal{P}}$$

for some $\alpha \in \mathcal{H}$, then there exists a homomorphism $h : C(X) \to A$ such that

$$\|\Lambda(f) - h(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$.

Proof. We first would like to point out that, in the case when A is an elementary C^*-algebra, Theorem 2.7 follows from Theorem 1.12 directly. This is because $K_1(A) = 0$ and $K_0(A)$ has no infinitesimal element and is free. Thus $\mathcal{H} = \mathcal{N}$.

So now we assume that A is a nonelementary C^*-algebra among other conditions. Let Λ be as in the theorem, with δ, $\mathcal{G}$ and $\mathcal{P}$ to be chosen.

By Lemma 1.5 in [LP1], without loss of generality, we may write that

$$\Lambda(f) = \sum_{i=1}^{m} f(\zeta_i)p_i \oplus \psi_1(f)$$

for all $f \in C(X)$, where $\{\zeta_i\}$ is 2σ-dense in X and $\psi_1(f)$ is δ-$\mathcal{G}$-multiplicative (by letting Λ be δ'-$\mathcal{G}'$-multiplicative with sufficiently small δ' and sufficiently large $\mathcal{G}'$). Since A is a nonelementary simple C^*-algebra of real rank zero, there is a projection $e \in A$ such that

$$e \oplus e \oplus e \oplus e \oplus e \preceq p_i$$

for each i. By the assumption, for any given $\mathcal{P}$, with small enough δ and large enough $\mathcal{G}$, there is a homomorphism $\phi : C(X) \to eAe$, such that

$$(\psi_1)_{*\bar{\mathcal{P}}} = \phi_{*\bar{\mathcal{P}}}.$$

By [EG] (note that $dim(X) \leq 2$), there is a homomorphism $\bar{\phi} : C(X) \to M_4(eAe)$ such that

$$(\phi \oplus \bar{\phi})_* \in \mathcal{N}.$$

Note that, since A is a nonelementary simple C^*-algebra, ϕ (and $\bar{\phi}$) can always be chosen so that it is σ-injective. Now we have

$$(\psi_1 \oplus \bar{\psi})_{*\bar{\mathcal{P}}} = (\phi \oplus \bar{\phi})_{*\bar{\mathcal{P}}}.$$

Applying Theorem 1.12, with sufficiently small δ and sufficiently large $\mathcal{G}$, there are homomorphisms $h_1 : C(X) \to QM_5(A)Q$ (with $Q = diag(1 - \sum_i p_i, e, e, e, e)$) and $h_2 : C(X) \to M_5(eAe)$ both with finite dimensional range such that

$$\|\psi_1(f) \oplus \bar{\phi}(f) - h_1(f)\| < \varepsilon/4 \quad \text{and} \quad \|\phi(f) \oplus \bar{\phi}(f) - h_2(f)\| < \varepsilon/4$$

for all $f \in \mathcal{F}$. Without loss of generality (with sufficiently small σ), we may write $h_2(f) = \sum_{i=1}^n f(\zeta_i)d_i$, where $\{d_i\}$ are mutually orthogonal projections in $M_6(eAe)$. There is a unitary $U \in M_7(A)$ such that

$$U^*d_iU \le p_i \quad \text{and} \quad U(\sum_{i=1}^n p_i) = (\sum_{i=1}^n p_i)U = (\sum_{i=1}^n p_i)$$

for $i = 1, 2, ..., n$. We estimate that

$$\begin{aligned}
&\|\Lambda(f) - [\sum_{i=1}^n f(\zeta_i)(p_i - U^*d_iU) \oplus U^*(h_1(f) \oplus \phi(f))U]\| \\
&\quad \le \|\psi(f) - [\sum_{i=1}^n f(\zeta_i)(p_i - U^*d_iU) \oplus \phi_1(f) \oplus U^*h_2(f)U]\| \\
&\qquad + \|[\sum_{i=1}^n f(\zeta_i)(p_i - U^*d_iU) \oplus \phi_1(f) \oplus U^*h_2(f)U - U^*(h_1 \oplus \phi(f))U\| \\
&\quad < \varepsilon/4 + \varepsilon/4 < \varepsilon
\end{aligned}$$

for all $f \in \mathcal{F}$. □

Remark 2.8. Note as in Remark 1.13 that if $K_i(C(X))$ is torsion free ($i = 0, 1$), then we only need to consider projections in $\bigcup_m M_m(C(X)) \oplus M_m(C(X) \otimes C(S^1))$. Suppose that $A \in \mathbf{B}$ such that $K_0(A)$ is a dimension group. Suppose that B is a (simple) AF-algebra with $K_0(A) = K_0(B)$ and $\alpha \in \mathcal{H}$. A result in [Li] says that, for any nonzero projection $p \in B$, there is a homomorphism $h : C(X) \to pBp$ with $h_*|_{kerd_X} = \alpha|_{kerd_X}$. It follows from 2.9 in [Ln1] that there is a unital inclusion $j : B \to A$ such that j induces an isomorphism from $K_0(B)$ onto $K_0(A)$. This implies that $A \in \mathbf{B}_X$. Now let X be a compact metric space of dimension no more than 2. Then, in 2.7, the condition that $A \in \mathbf{B}_X$ can be replaced by $K_0(A)$ is a dimension group. More significantly, the injective condition can be removed. This is because if $F \subset X$, the surjection maps $kerd_X$ onto $kerd_F$. Thus, one can combine the proof of 2.5 with the proof of 2.7 to get the following corollary. Note that the condition that $\Lambda_* = \alpha$ on $\bar{\mathcal{P}}$ for some $\alpha \in \mathcal{H}$ is necessary.

Corollary 2.9. *Let X be a compact metric space of dimension no more than two. For any $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset C(X)$, there exist a finite subset $\mathcal{G} \subset C(X)$, a positive number $\delta > 0$ ($\delta < \delta(\mathcal{P})$) and a finite subset $\mathcal{P} \subset \mathbf{P}(C(X))$ satisfying the following: For any unital C^*-algebra $A \in \mathbf{B}$ with $K_0(A)$ being a dimension group, if $\Lambda : C(X) \to A$ is a contractive positive linear contractive positive linear morphism which is δ-$\mathcal{G}$-multiplicative and*

$$\Lambda_* = \alpha \ \text{ on } \ \bar{\mathcal{P}}$$

for some $\alpha \in \mathcal{H}$, then there exists a homomorphism $h : C(X) \to A$ such that

$$\|\Lambda(f) - h(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$.

2.10. Now we give a very special example. Let

$$f(e^{2\pi it}) = \begin{cases} 1-2t, & \text{if } 0 \le t \le 1/2, \\ -1+2t, & \text{if } 1/2 \le t \le 1, \end{cases}$$

$$g(e^{2\pi it}) = \begin{cases} (f(t)-f(t)^2)^{1/2}, & \text{if } 0 \le t \le 1/2, \\ 0, & \text{if } 1/2 \le t \le 1, \end{cases}$$

$$h(e^{2\pi it}) = \begin{cases} 0, & \text{if } 0 \le t \le 1/2, \\ (f(t)-f(t)^2)^{1/2}, & \text{if } 1/2 \le t \le 1. \end{cases}$$

Let A be a (unital) C^*-algebra. For any pair of unitaries u and v in A, define

$$e(u,v) = \begin{pmatrix} f(v) & g(v)+h(v)u^* \\ g(v)+uh(v) & 1-f(v) \end{pmatrix},$$

where f, g and h are as above. If u commutes with v, then $e(u,v)$ is a projection. In the general case, $e(u,v)$ is always selfadjoint. If $\|uv-vu\|$ is small, then

$$\|e(u,v)^2 - e(u,v)\|$$

is small (see [Lr1, 3.5 and 3.6]). There is a positive number δ_0 (which is independent of A, u and v) such that whenever

$$\|uv-vu\| < \delta_0,$$

then

$$sp(e(u,v)) \subset [-1/4, 1/4] \cup [3/4, 1+1/4]$$

(see [Lr1, 3.5 and 3.6]). Let χ be the characteristic function for the subset $[1/2, 3/2]$. Then $\chi(e(u,v))$ is a projection in the C^*-subalgebra of A generated by $e(u,v)$ and

$$\|\chi(e(u,v)) - e(u,v)\| \le 1/4.$$

The Exel-Loring index (cf. [EL2]) $\kappa(u,v)$ is defined by

$$\kappa(u,v) = [\chi(e(u,v))] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] \quad \text{in } K_0(A).$$

Let A be a C^*-algebra in $\mathbf{B}$ which is not elementary. It is known and (easy to show) that for any $\alpha_1 \in Hom(K_1(C(\mathbf{T}^2)), K_1(A))$, there is a homomorphism $h_1 : C(\mathbf{T}^2) \to eAe$ for any nonzero projection $e \in A$ such that h_1 induces α_1. Thus if $\alpha \in Hom(K_*(C(\mathbf{T}^2)), K_*(A))$ with $\alpha|_{kerd_{\mathbf{T}^2}} = 0$ and $\alpha([1_{C(\mathbf{T}^2)}]) = [1_A]$, then there is a unital homomorphism $h : C(\mathbf{T}^2) \to A$ such that $[h] = \alpha$. If $K_0(A)$ is a dimension group, then, by a result in [Li], $\alpha \in \mathcal{H}$ if and only if $\alpha|_{kerd_{\mathbf{T}^2}}$ are infinitesimal elements in $K_0(A)$, or equivalently, $t(\alpha(x)) = 0$ for each $x \in kerd_{\mathbf{T}^2}$ and all normalized quasitraces of A.

Corollary 2.11 (cf. [Lr5]). *For any $\varepsilon > 0$, there is $\delta > 0$ so that, whenever $A \in \mathbf{B}$, if u and v are two unitaries in A,*

$$\|uv-vu\| < \delta \quad \text{and} \quad \kappa(u,v) = 0,$$

then there exist commuting unitaries u_1, $v_1 \in A$ such that

$$\|u-u_1\| < \varepsilon \quad \text{and} \quad \|v-v_1\| < \varepsilon.$$

Furthermore,

(a) if $K_1(A) = 0$, u_1 and v_1 can be required to have finite spectrum;

(b) if A is a purely infinite simple C^*-algebra, the condition that $\kappa(u,v)=0$ is not needed;

(c) if $K_0(A)$ is a dimension group, the condition that $\kappa(u,v)=0$ can be replaced by $\tau(\kappa(u,v))=0$ for all normalized quasitraces τ of A.

Proof. (See the proof of 3.21 in [GL2].) Define two homomorphisms $h_1, h_2 : C(S^1) \to A$ by the unitaries u and v in A. It follows from Lemma 2.1 in [LP1] that, for any $\varepsilon_1 > 0$ and any finite subset $\mathcal{F}_1 \in C(\mathbf{T}^2)$ there exists a contractive positive linear map $L : C(\mathbf{T}^2) \to A$ which is ε_1-$\mathcal{F}_1$-multiplicative such that

$$\|L(z_1) - u\| < \varepsilon \quad \text{and} \quad \|L(z_2) - v\| < \varepsilon,$$

where z_1 and z_2 are standard unitary generators of $C(\mathbf{T}^2)$. So, without loss of generality, we may assume that $L(z_1) = u$ and $L(z_2) = v$. Therefore the first part of the corollary follows from the last part of Corollary 2.11 and Theorem 2.5 (and Remark 2.8).

Part (a) is clear. Part (b) follows from [Ln5]. Part (c) follows from 2.10 and the last part of Corollary 2.11. □

We note that, in the case when $K_0(A)$ is a dimension group, the condition $\tau(\kappa(u,v))=0$ is also necessary.

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