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A SCHWARZ LEMMA FOR MULTIVALUED FUNCTIONS AND DISTORTION THEOREMS FOR BLOCH FUNCTIONS WITH BRANCH POINTS

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Abstract. We give a version of the Schwarz lemma for multivalued mappings between hyperbolic plane regions. As in the original work of Nehari on this subject, the derivative must remain bounded near the branch points. Our version of the distance-decreasing principle represents a considerable strengthening of previous results. We apply it to the study of Bloch functions with branch points of specified order. We obtain upper and lower estimates for $|f'|$, an upper estimate for $|f|$, and a lower estimate for the radius of the largest schlicht disk in the image of f centered at $f(0)$. We also obtain some results requiring estimates of second order derivatives of f.

1. Introduction

Let $\mathbb{D}$ denote the unit disk in $\mathbb{C}$. In 1947 Z. Nehari proved the following theorem [N]:

Theorem. *Let $f\colon \mathbb{D} \to \mathbb{D}$ be a multivalued function, holomorphic except for a finite number of algebraic branch points, and suppose that f' is finite at each point of $\mathbb{D}$. Then $|f'(0)| \le 1$ for all of the different values that $f'(0)$ may assume, with equality for one of them if and only if $f(z) = \lambda z$, $|\lambda| = 1$.*

The restriction on the derivative is essential.

Nehari also gave a somewhat restricted version of the distance-decreasing principle (though all of his applications were applications of the theorem itself): if $f(0) = 0$ for some branch then $|f(z)| \le |z|$ for all z in the star of uniformity of this branch.

We shall give generalizations of both Nehari's theorem and the distance-decreasing principle. The latter is the main result of this paper, and is applied to the study of Bloch functions with branch points of prescribed order. We fully expect that it will have other applications.

Definition 1. Let Ω be a region in $\mathbb{C}$ and let $a \in \Omega$. Let $M_a(\Omega)$ be the class of holomorphic germs h defined at a such that h is freely continuable in $\Omega\backslash E(h)$, where $E(h)$ is a discrete subset of Ω, h has an algebraic branch point or removable singularity at each point $b \in E(h)$, and $h'(z)$ is finite at each point of Ω.

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For $h \in M_a(\Omega)$ let $h(\Omega)$ denote the set of all possible values of $h(z)$ as z ranges over Ω and all possible values of h at z are considered. If Ω is a hyperbolic region, we denote the hyperbolic density on Ω by λ_Ω. We denote the hyperbolic distance by d_Ω. (We take the hyperbolic metric to have curvature -4.)

Theorem 1. *Suppose Ω_1 and Ω_2 are hyperbolic regions in $\mathbb{C}$, $h \in M_a(\Omega_1)$, and $h(\Omega_1) \subseteq \Omega_2$. Then for $z \in \Omega_1$, $\lambda_{\Omega_2}\big(h(z)\big)|h'(z)| \le \lambda_{\Omega_1}(z)$. If equality holds at a single point for some branch of h, then h is a holomorphic covering projection of Ω_1 onto Ω_2.*

(This is a generalization of a 1983 theorem of Minda [M, Theorem 2].)

On the other hand, the distance-decreasing principle does not seem to remain valid for the full class $M_a(\Omega)$. However, there is a very useful substitute. Let $\tilde{h}(z)$ denote any of the values obtained by continuing h along any (length-minimizing) hyperbolic geodesic segment in Ω from a to z.

Theorem 2. *Suppose Ω_1 and Ω_2 are hyperbolic regions in $\mathbb{C}$, $h \in M_a(\Omega_1)$ and $h(\Omega_1) \subseteq \Omega_2$. If $b \in \Omega_1$ and $b \neq a$, then*

$$d_{\Omega_2}\big(h(a), \tilde{h}(b)\big) \le d_{\Omega_1}(a, b)\,.$$

Theorem 2 leads to a useful subordination principle for multivalued functions:

Definition 2. Let Δ be an open disk in $\mathbb{D}$ containing 0. Suppose that $k \in M_0(\mathbb{D})$, K is holomorphic (single-valued) on $\mathbb{D}$, and $k(0) = K(0)$. We say that k is subordinate to K on Δ relative to 0 (written $k \prec_0 K$), if there exists $\varphi \in M_0(\Delta)$ with $\varphi(0) = 0$, $\varphi(\Delta) \subset \Delta$, and $k = K \circ \varphi$ on Δ.

Corollary to Theorem 2 (Corollary 3 in §2)**.** *In the situation of Definition 2 let $\Delta(0, r)$ be the hyperbolic disk of center 0 and radius r relative to hyperbolic geometry on Δ. Then $\tilde{k}\big(\Delta(0, r)\big) \subseteq K\big(\Delta(0, r)\big)$.*

A holomorphic function f on $\mathbb{D}$ is called a Bloch function if the Bloch seminorm

$$\|f\|_B = \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)|$$

is finite. The class of such functions is denoted by B. We will be interested in how various results for Bloch functions (growth and distortion theorems, radius of schlichtness, etc.) depend on parameters. For $n \in \mathbb{Z}^+$ and $\alpha \in [0, 1]$ we introduce the subclasses

$$\begin{aligned} B_n(\alpha) = \big\{ f \in B \mid \|f\|_B \le 1,\ f(0) = 0,\ f'(0) = \alpha, \text{ and} \\ \text{if } f'(a) = 0 \text{ for some } a \in \mathbb{D}, \text{ then } f^{(k)}(a) = 0 \text{ for } k = 1, 2, \dots, n \big\}. \end{aligned}$$

Thus any branch points of functions in $B_n(\alpha)$ must have order at least $n + 1$. For $\alpha = 0$ we require each $f \in B_n(0)$ to have a zero of exact order $n + 1$ at the origin. We let $B_n = \bigcup_{0 \le \alpha \le 1} B_n(\alpha)$.

To state our results we need to describe certain extremal functions. Let $F_n(z) = (-1)^n C_n z^{n+1}$ where C_n is chosen so that $\|F_n\|_B = 1$. Let $F_{n,\alpha} = F_n \circ T_{m_n(\alpha)} - F_n \circ T_{m_n(\alpha)}(0)$, where $T_{m_n(\alpha)}(z) = \frac{z - m_n(\alpha)}{1 - m_n(\alpha) z}$, and $m_n(\alpha) \in \left[0, \sqrt{\frac{n}{n+2}}\right]$ is chosen so that $F'_{n,\alpha}(0) = \alpha$. Let $\Delta_n(\alpha)$ be the interior of the circle on which $(1 - |z|^2)|F'_{n,\alpha}(z)| = 1$. We note that $0 \in \Delta_n(\alpha)$ if $\alpha < 1$. Let $H_{n,\alpha}(z) = \big((1 - m_n(\alpha) z)^2 F'_{n,\alpha}(z)\big)^{1/n} = k_n(\alpha) \frac{m_n(\alpha) - z}{1 - m_n(\alpha) z}$ where $k_n(\alpha) = \big[(n + 1) C_n (1 - m_n^2(\alpha))\big]^{1/n}$.

Theorem 3. *Let $f \in B_n(\alpha)$, $\alpha \in [0,1)$. Then for any unimodular constant λ*

$$\begin{aligned}\left((1-m_n(\alpha)\bar{\lambda}z)^2 f'(z)\right)^{1/n} &\prec_0 \left((1-m_n(\alpha)\bar{\lambda}z)^2 F'_{n,\alpha}(\bar{\lambda}z)\right)^{1/n} \\ &= H_{n,\alpha}(\bar{\lambda}z) \\ &= k_n(\alpha)\frac{m_n(\alpha)-\bar{\lambda}z}{1-m_n(\alpha)\bar{\lambda}z}\end{aligned}$$

on $\lambda\Delta_n(\alpha)$.

In our applications only the case $\lambda = 1$ is needed. From Theorem 3 we deduce

Theorem 4. *Suppose $\alpha \in (0,1)$ and $f \in B_n(\alpha)$.*

(i) *For $|z| \le m_n(\alpha)$, $|f'(z)| \ge F'_{n,\alpha}(|z|)$ with equality at $z = re^{i\theta}$, $r \in \big(0, m_n(\alpha)\big)$ if and only if $f(z) = e^{i\theta}F_{n,\alpha}(e^{-i\theta}z)$.*

(ii) *For $|z| \le \frac{\sqrt{n}-\sqrt{n+2}m_n(\alpha)}{\sqrt{n+2}-\sqrt{n}m_n(\alpha)}$, $|f'(z)| \le F'_{n,\alpha}(-|z|)$ with equality at $z = -re^{i\theta}$, $r \in \left(0, \frac{\sqrt{n}-\sqrt{n+2}m_n(\alpha)}{\sqrt{n+2}-\sqrt{n}m_n(\alpha)}\right)$ if and only if $f(z) = e^{i\theta}F_{n,\alpha}(e^{-i\theta}z)$.*

(iii) *For z as in (ii), $|f(z)| \le -F_{n,\alpha}(-|z|)$ with the same conditions for equality as in (ii).*

(iv) *$r(0,f) \ge C_n m_n^{n+1}(\alpha)$ with equality if and only if $f(z) = e^{i\theta}F_{n,\alpha}(e^{-i\theta}z)$ for some $\theta \in \mathbb{R}$. ($r(0,f)$ denotes the radius of the largest schlict disk in the image of f centered at* 0.)

Bonk, Minda, and Yanagihara obtained results similar to Theorem 4 for the classes $B_1(\alpha)$ and $B_\infty(\alpha) = \bigcap_{n=1}^{\infty} B_n(\alpha)$, using ordinary subordination [BMY1, BMY2]. It is also of interest to compare our results with those of Liu and Minda [LM] for the class $B_n(1)$. The results of that paper can be recovered by letting $\alpha \nearrow 1$ in ours, but cannot be obtained directly from the subordination principle for multivalued functions. Instead, a version of Julia's lemma for multivalued functions is applied [LM]. Thus our approach simultaneously gives a unified approach to known distortion theorems for Bloch functions and yields new results for Bloch functions with branch points of specified order.

The plan of the paper is as follows: In Section 2 we prove the multivalued Schwarz lemma results and some corollaries. Section 3 is devoted to properties of the extremal functions in the class $B_n(\alpha)$. Section 4 applies the multivalued subordination principle to the solution of extremal problems for Bloch functions (Theorems 3 and 4). In Section 5 we consider some additional extremal problems involving second order derivatives and determine the radius of convexity for the class $B_n(\alpha)$.

2. Proof of the multivalued Schwarz lemma and related results

In this section we prove Theorems 1 and 2 and give some corollaries.

We begin with a standard example of a multi-valued function in $M_a(\Omega)$ (Definition 1); in fact, this is the only type we consider in this paper.

Example. Suppose g is holomorphic in Ω and each zero of g has multiplicity at least n, where n is a fixed positive integer. If $g(a) \ne 0$, then $h = g^{1/n}$ (select any fixed branch at a) belongs to $M_a(\Omega)$ with $E(h) = \{z \in \Omega \mid g(z) = 0\}$. In this special case $|h|$ and $|h'|$ are single-valued on Ω.

Proof of Theorem 1. For $\Omega_1 = \mathbb{D}$ this result is contained in [M]. The more general situation is easily reduced to this special case. Let $\pi\colon \mathbb{D} \to \Omega_1$ be a holomorphic universal covering projection with $\pi(0) = a$. Then $h\circ\pi \in \mathcal{M}_a(\mathbb{D})$ and $h\circ\pi(\mathbb{D}) \subset \Omega_2$. Therefore, by [M, Theorem 2]

$$\lambda_{\Omega_2}(h(\pi(z)))|h'(\pi(z))||\pi'(z)| \le \lambda_{\mathbb{D}}(z) = \lambda_{\Omega_1}(\pi(z))|\pi'(z)|\,,$$

so that

$$\lambda_{\Omega_2}(h(\pi(z)))|h'(\pi(z))| \le \lambda_{\Omega_1}(\pi(z))\,.$$

Since π is surjective, this establishes the inequality in the theorem. If equality holds, then $h \circ \pi\colon \mathbb{D} \to \Omega_2$ is a covering projection. Since $\pi\colon \mathbb{D} \to \Omega_1$ is a covering projection, this forces $h\colon \Omega_1 \to \Omega_2$ to be a covering projection also.

Corollary 1. *Suppose Ω is a simply connected region, $\Omega \ne \mathbb{C}$, $\varphi \in \mathcal{M}_a(\Omega)$, $\varphi(a) = a$ and $\varphi(\Omega) \subset \Omega$. Then $|\varphi'(a)| \le 1$ with equality if and only if φ is a conformal automorphism of Ω fixing a. In particular, $\varphi'(a) = 1$ if and only if φ is the identity function.*

Proof. For $z = a$ the theorem gives

$$\lambda_\Omega(a)|\varphi'(a)| = \lambda_\Omega(\varphi(a))|\varphi'(a)| \le \lambda_\Omega(a)\,,$$

or $|\varphi'(a)| \le 1$. Equality implies φ is a covering of Ω onto itself with $\varphi(a) = a$. Since Ω is simply connected, a covering is a conformal self-mapping.

Next, we explain the notation $\tilde{h}$ in Theorem 2 in a little more detail. Consider any $h \in M_a(\Omega)$. For $z \in \Omega$ we limit the possible values for the continuation of h to z as follows: Suppose γ is any (length-minimizing) hyperbolic geodesic segment in Ω from a to z. Then h can be analytically continued along γ. We let $\tilde{h}(z)$ denote a value at z determined by continuing h along a hyperbolic geodesic from a to z. If γ does not meet $E(h)$, then $\tilde{h}(z)$ is uniquely determined by γ. If γ meets $E(h)$, then $\tilde{h}(z)$ has finitely many possible values since there are finitely many ways to continue h through each algebraic branch point in $\gamma \cap E(h)$. Different hyperbolic geodesics in Ω from a to z (if they exist) can result in additional values for $\tilde{h}(z)$. Thus, $\tilde{h}(z)$ denotes any of the values obtained by continuing h along any hyperbolic geodesic in Ω from a to z.

Proof of Theorem 2. Suppose $\tilde{h}(b)$ is obtained by continuing h along the hyperbolic geodesic γ from a to b. Then

$$\begin{aligned} d_{\Omega_1}(a,b) &= \int_\gamma \lambda_{\Omega_1}(z)|dz| \\ &\ge \int_\gamma \lambda_{\Omega_2}(h(z))|h'(z)||dz| \\ &= \int_{h\circ\gamma} \lambda_{\Omega_2}(w)|dw| \\ &\ge d_{\Omega_2}(h(a), \tilde{h}(b)) \end{aligned}$$

since $h \circ \gamma$ is a path in Ω_2 connecting $h(a)$ to $\tilde{h}(b)$. If equality holds, then $\lambda_{\Omega_1}(z) = \lambda_{\Omega_2}(h(z))|h'(z)|$ along γ. This implies $h\colon \Omega_1 \to \Omega_2$ is a covering.

Corollary 2. *Suppose Ω is a simply connected region, $\Omega \ne \mathbb{C}$, $\varphi \in \mathcal{M}_a(\Omega)$, $\varphi(a) = a$ and $\varphi(\Omega) \subset \Omega$. If $b \in \Omega$, $b \ne a$ and $\tilde{\varphi}(b) = b$, then φ is the identity function.*

Proof. From the theorem

$$d_\Omega(a,b) \geq d_\Omega(\varphi(a), \tilde{\varphi}(b)) = d_\Omega(a,b)\,.$$

Since equality holds, φ: $\Omega \to \Omega$ is a covering. Because Ω is simply connected, φ must be a conformal automorphism of Ω. It is well-known that a conformal automorphism of a simply-connected region $\Omega \subsetneq \mathbb{C}$ which fixes two distinct points must be the identity.

Proof of Corollary 3. This follows if we observe that φ satisfies the hypotheses of Theorem 2, so that $\tilde{\varphi}(\Delta(0,r)) \subseteq \Delta(0,r)$.

3. Extremal functions in the class $B_n(\alpha)$

We investigate certain n-sheeted branched coverings of $\mathbb{D}$ onto other disks. These elementary functions turn out to be extremal for a number of problems for the class $B_n(\alpha)$.

Let $d_{\mathbb{D}}$ denote the hyperbolic distance on $\mathbb{D}$. We denote by $D_{\mathbb{D}}(a,r)$ the open hyperbolic disk of center a and radius $r > 0$, and by $C_{\mathbb{D}}(a,r)$ the hyperbolic circle of the same center and radius. Let $D_e(a,r)$ denote the Euclidean disk of center a and radius r.

Set

$$C_n = \frac{n+2}{2(n+1)}\Big(\frac{n+2}{n}\Big)^{n/2}$$

and

$$F_n(z) = (-1)^n C_n z^{n+1}\,.$$

Then F_n is an $(n+1)$-sheeted branched covering of $\mathbb{D}$ onto $D_e(0, C_n)$ with $F_n(0) = F_n'(0) = \ldots = F_n^{(n)}(0) = 0$, $F_n^{(n+1)}(0) \neq 0$ and $\|F_n\|_B = 1$. In fact we have

$$(1-|z|^2)|F_n'(z)| = M_n(|z|)\,,$$

where

$$M_n(t) = (n+1)C_n t^n(1-t^2)$$

is increasing on $\Big[0, \sqrt{\frac{n}{n+2}}\Big]$, decreasing on $\Big[\sqrt{\frac{n}{n+2}}, 1\Big]$ and $M_n\Big(\sqrt{\frac{n}{n+2}}\Big) = 1$. In particular, $(1-|z|^2)|F_n'(z)| = 1$ if and only if $|z| = \sqrt{\frac{n}{n+2}}$. We let m_n: $[0,1] \to \Big[0, \sqrt{\frac{n}{n+2}}\Big]$ be the inverse function of the restriction of M_n to the interval $\Big[0, \sqrt{\frac{n}{n+2}}\Big]$. The function m_n is increasing with $m_n(0) = 0$ and $m_n(1) = \sqrt{\frac{n}{n+2}}$.

By precomposing F_n with certain conformal automorphisms of $\mathbb{D}$ and normalizing at the origin, we obtain extremal functions for the classes $B_n(\alpha)$. For $a \in (-1,1)$ the function

$$T_a(z) = \frac{z-a}{1-az}$$

is a conformal automorphism of $\mathbb{D}$, so $F_n \circ T_a$ has Bloch seminorm 1. Computation yields $(F_n \circ T_a)'(0) = M_n(a)$. For each $\alpha \in [0,1]$ there is a unique $a \in \Big[0, \sqrt{\frac{n}{n+2}}\Big]$ with $M_n(a) = \alpha$, or $a = m_n(\alpha)$. We define

$$F_{n,\alpha}(z) = F_n \circ T_{m_n(\alpha)}(z) - F_n \circ T_{m_n(\alpha)}(0)\,.$$

This function belongs to $B_n(\alpha)$ and is an $(n+1)$-sheeted branched covering of $\mathbb{D}$ onto $D_e\big(C_n m_n(\alpha)^{n+1}, C_n\big)$. Its derivative has a zero of order n at $m_n(\alpha)$. Also, $F_{n,\alpha}$ is increasing on $[0, m_n(\alpha)]$ and maps this interval onto $[0, C_n m_n(\alpha)^{n+1}]$. Next, we note that $(1-|z|^2)|F'_{n,\alpha}(z)| \le 1$ with equality if and only if $|T_{m_n(\alpha)}(z)| = \sqrt{\frac{n}{n+2}}$; that is, $d_{\mathbb{D}}\big(m_n(\alpha), z\big) = \operatorname{artanh}\Big(\sqrt{\frac{n}{n+2}}\Big)$. Moreover, $(1-|z|^2)|F'_{n,\alpha}(z)|$ is constant on hyperbolic circles centered at $m_n(\alpha)$. We set $\Delta_n(\alpha) = D_{\mathbb{D}}\Big(m_n(\alpha), \operatorname{artanh}\Big(\sqrt{\frac{n}{n+2}}\Big)\Big)$. This hyperbolic disk $\Delta_n(\alpha)$ is the same as the Euclidean disk

$$D_e\Big(\frac{2m_n(\alpha)}{(n+2) - nm_n^2(\alpha)}, \frac{\sqrt{n(n+2)}(1-m_n^2(\alpha))}{(n+2) - nm_n^2(\alpha)}\Big).$$

The boundary of $\Delta_n(\alpha)$ meets the real axis in the points

$$\frac{\sqrt{n+2}m_n(\alpha) + \sqrt{n}}{\sqrt{n}m_n(\alpha) + \sqrt{n+2}} \in \left[\sqrt{\frac{n}{n+2}}, \frac{\sqrt{n(n+2)}}{n+1}\right]$$

and

$$-\frac{\sqrt{n} - \sqrt{n+2}m_n(\alpha)}{\sqrt{n+2} - \sqrt{n}m_n(\alpha)} \in \left[-\sqrt{\frac{n}{n+2}}, 0\right].$$

The left-hand endpoints of the intervals correspond to $\alpha = 0$ and the right-hand endpoints to $\alpha = 1$.

There are important auxiliary functions associated with each $F_{n,\alpha}$. For $\alpha \in [0,1]$ we set

$$\begin{aligned} G_{n,\alpha}(z) &= \big(1 - m_n(\alpha)z\big)^2 F'_{n,\alpha}(z) \\ &= (-1)^n (n+1) C_n \big(1 - m_n^2(\alpha)\big)\big(T_{m_n(\alpha)}(z)\big)^n \end{aligned}$$

and

$$H_{n,\alpha}(z) = G_{n,\alpha}^{1/n}(z) = -k_n(\alpha) T_{m_n(\alpha)}(z)$$

where $k_n(\alpha) = \big[(n+1)C_n\big(1 - m_n^2(\alpha)\big)\big]^{1/n}$. The function $H_{n,\alpha}$ is a Möbius transformation. For $d_{\mathbb{D}}\big(m_n(\alpha), z\big) = r$, $|H_{n,\alpha}(z)| = k_n(\alpha)\tanh(r)$, so that $H_{n,\alpha}$ is a conformal mapping of $D_{\mathbb{D}}\big(m_n(\alpha), r\big)$ onto $D_e\big(0, k_n(\alpha)\tanh(r)\big)$. The function $H_{n,\alpha}$ is decreasing on $\Delta_n(\alpha) \cap \mathbb{R}$ with

$$H_{n,\alpha}\left(\frac{\sqrt{n+2}m_n(\alpha) + \sqrt{n}}{\sqrt{n}m_n(\alpha) + \sqrt{n+2}}\right) = -\frac{\sqrt{n}}{\sqrt{n+2}}\big((n+1)C_n\big)^{1/n}\big(1 - m_n^2(\alpha)\big)^{1/n},$$

$$H_{n,\alpha}\left(-\frac{\sqrt{n} - \sqrt{n+2}m_n(\alpha)}{\sqrt{n+2} - \sqrt{n}m_n(\alpha)}\right) = \frac{\sqrt{n}}{\sqrt{n+2}}\big((n+1)C_n\big)^{1/n}\big(1 - m_n^2(\alpha)\big)^{1/n}.$$

4. Distortion theorems via subordination

We first prove the theorem which sets up the application of the subordination principle to the study of the class $B_n(\alpha)$.

Proof of Theorem 3. It suffices to establish this result in the special case $\lambda = 1$: the general case then follows by considering $\bar{\lambda} f(\lambda z)$. Let $g(z) = \big(1 - m_n(\alpha)z\big)^2 f'(z)$

and $h(z) = g(z)^{1/n}$. The function h is multi-valued on $\mathbb{D}$: the branch at the origin is selected so that $h(0) = \alpha^{1/n} \geq 0$. For $d_{\mathbb{D}}\big(m_n(\alpha), z\big) = r$,

$$\begin{aligned} |g(z)| &= \frac{|1 - m_n(\alpha)z|^2}{1 - |z|^2}(1 - |z|^2)|f'(z)| \\ &\leq \frac{|1 - m_n(\alpha)z|^2}{1 - |z|^2} = \frac{1 - m_n^2(\alpha)}{1 - \tanh^2(r)}\,, \end{aligned}$$

so h maps $D_{\mathbb{D}}\big(m_n(\alpha), r\big)$ into $D_e\Big(0, \Big(\frac{1-m_n^2(\alpha)}{1-\tanh^2(r)}\Big)^{1/n}\Big)$ for $r > 0$. Since $H_{n,\alpha}$ is a conformal map of $D_h\big(m_n(\alpha), r\big)$ onto $D_e\big(0, k_n(\alpha)\tanh(r)\big)$, $h \prec_0 H_{n,\alpha}$ on $D_{\mathbb{D}}\big(m_n(\alpha), r\big)$ if

$$\left(\frac{1 - m_n^2(\alpha)}{1 - \tanh^2(r)}\right)^{1/n} \leq k_n(\alpha)\tanh(r)\,,$$

or

$$\begin{aligned} 1 &\leq (n+1)C_n \tanh^n(r)\big(1 - \tanh^2(r)\big) \\ &= M_n\big(\tanh(r)\big)\,. \end{aligned}$$

This happens only for $\tanh(r) = \sqrt{\frac{n}{n+2}}$, or $r = \operatorname{artanh}\Big(\sqrt{\frac{n}{n+2}}\Big)$. Note that r is independent of α. For this choice of r the function $H_{n,\alpha}$ maps $\Delta_n(\alpha)$ conformally onto $D_e\Big(0, \sqrt{\frac{n}{n+2}}, k_n(\alpha)\Big)$.

Corollary 4. *Suppose $f \in B_n(\alpha)$, $\alpha \in (0,1)$. If $a \in \mathbb{D}$ and $f'(a) = 0$, then $|a| \geq m_n(\alpha)$. Also, $a = m_n(\alpha)e^{i\theta}$ if and only if $f(z) = e^{i\theta}F_{n,\alpha}(e^{-i\theta}z)$.*

Proof. Without loss of generality we may assume $a > 0$; if not, replace f by an appropriate rotation of f. We want to prove that $a \geq m_n(\alpha)$ and equality implies $f = F_{n,\alpha}$. For simplicity we write $\Delta = \Delta_n(\alpha)$ and $\Omega = D_e\Big(0, \sqrt{\frac{n}{n+2}}k_n(\alpha)\Big)$. Then $h \in \mathcal{M}_0(\Delta)$ and $f'(a) = 0$ implies $\tilde{h}(a) = 0$, where $\tilde{h}(a)$ is obtained by analytically continuing h along a hyperbolic geodesic. Hence

$$d_\Delta(0, a) \geq d_\Omega\big(h(0), \tilde{h}(a)\big) = d_\Omega(\alpha^{1/n}, 0)\,.$$

Now, $H_{n,\alpha}\colon \Delta \to \Omega$ is a conformal mapping, so $H_{n,\alpha}(0) = \alpha^{1/n}$ and $H_{n,\alpha}\big(m_n(\alpha)\big) = 0$ imply

$$d_\Omega(\alpha^{1/n}, 0) = d_\Delta\big(0, m_n(\alpha)\big)\,.$$

Since $\Delta_n(\alpha)$ is symmetric about $\mathbb{R}$, $\Delta_n(\alpha) \cap \mathbb{R}$ is a hyperbolic geodesic. Therefore, $0 < a, m_n(\alpha)$ and $d_\Delta\big(0, m_n(\alpha)\big) \leq d_\Delta(0, a)$ imply $m_n(\alpha) \leq a$. Suppose equality holds. Then $H_{n,\alpha}^{-1} \circ h \in \mathcal{M}_0(\Delta)$ maps Δ into itself and fixes both 0 and $m_n(\alpha)$, so is the identity, or $h = H_{n,\alpha}$, which gives $f = F_{n,\alpha}$.

Remark. Since $f'(z) \neq 0$ for $|z| < m_n(\alpha)$ when $f \in B_n(\alpha)$, we conclude that $h(z) = \big((1 - m_n(\alpha)z)^2 f'(z)\big)^{1/n}$ is holomorphic and single-valued in the disk $D_e\big(0, m_n(\alpha)\big)$.

Corollary 5. *Suppose $f \in B_n(\alpha)$, $\alpha \in (0,1)$. If $\tilde{h}(z) = H_{n,\alpha}(z)$ for some $z \in \Delta_n(\alpha)$, $z \neq 0$, then $h = H_{n,\alpha}$ and hence $f = F_{n,\alpha}$.*

Proof. Set $\varphi = H_{n,\alpha}^{-1} \circ h$. Then $\varphi \in \mathcal{M}_0(\Delta)$, $\varphi(0) = 0$ and $\varphi(\Delta) \subset \Delta$. Also $\tilde{\varphi}(z) = z$, so φ is the identity function, or $h = H_{n,\alpha}$.

We can now prove the main results about functions in the class $B_n(\alpha)$.

Proof of Theorem 4. Set $g(z) = \big(1 - m_n(\alpha)z\big)^2 f'(z)$ and $h(z) = g(z)^{1/n}$.

(i) Since $B_n(\alpha)$ is rotationally invariant, it is enough to establish (i) when $z = x \in \big(0, m_n(\alpha)\big)$ and show that equality implies $f = F_{n,\alpha}$. Actually, it suffices to show $|\tilde{h}(x)| \ge H_{n,\alpha}(x)$ with equality if and only if $h = H_{n,\alpha}$.

This inequality follows from $h \prec_0 H_{n,\alpha}$ on $\Delta_n(\alpha)$. Let δ_x be the hyperbolic circle (relative to hyperbolic geometry on $\Delta_n(\alpha)$) with center 0 which passes through x. Then $\tilde{h}(\delta_x)$ is contained in the closed disk bounded by the circle $H_{n,\alpha}(\delta_x)$. Now, $H_{n,\alpha}(\delta_x)$ is a hyperbolic circle relative to hyperbolic geometry on $D_e\Big(0, \sqrt{\frac{n}{n+2}}k_n(\alpha)\Big)$ with hyperbolic center $k_n(\alpha)m_n(\alpha) = \alpha^{1/n}$ and is symmetric about $\mathbb{R}$. Because $H_{n,\alpha}$ is decreasing on $\Delta_n(\alpha) \cap \mathbb{R}$, the point of $H_{n,\alpha}(\delta_x)$ with smallest real part is $H_{n,\alpha}(x)$. Consequently,

$$|\tilde{h}(x)| \ge \operatorname{Re}\, \tilde{h}(x) \ge H_{n,\alpha}(x)\,.$$

If equality holds, then we must have $\tilde{h}(x) = H_{n,\alpha}(x)$ which implies $h = H_{n,\alpha}$.

(ii) It is sufficient to prove the inequality for $z = -x$, $x \in \Big(0, \frac{\sqrt{n}-\sqrt{n+2}m_n(\alpha)}{\sqrt{n+2}-\sqrt{n}m_n(\alpha)}\Big)$ and equality implies $f = F_{n,\alpha}$. This is a consequence of $|\tilde{h}(-x)| \le H_{n,\alpha}(-x)$ for $x \in \Big(0, \frac{\sqrt{n}-\sqrt{n+2}m_n(\alpha)}{\sqrt{n+2}-\sqrt{n}m_n(\alpha)}\Big)$ and equality implies $h = H_{n,\alpha}$. Note that $-x \in \Delta_n(\alpha)$. Let δ_{-x} be the hyperbolic circle (relative to hyperbolic geometry on $\Delta_n(\alpha)$) with center 0 which passes through $-x$. Then $\tilde{h}(\delta_{-x})$ is contained in the closed disk bounded by the circle $H_{n,\alpha}(\delta_{-x})$ which is a hyperbolic circle relative to hyperbolic geometry on $D_e\Big(0, k_n(\alpha)\sqrt{\frac{n}{n+2}}\Big)$ with hyperbolic center $k_n(\alpha)m_n(\alpha) = \alpha^{1/n}$. Since $D_e\Big(0, k_n(\alpha)\sqrt{\frac{n}{n+2}}\Big)$ is centered at the origin and the hyperbolic center of $H_{n,\alpha}(\delta_{-x})$ is nonnegative, it follows that the euclidean center of $H_{n,\alpha}(\delta_{-x})$ is also nonnegative. Because $H_{n,\alpha}$ is decreasing on $\Delta_n(\alpha) \cap \mathbb{R}$ and $H_{n,\alpha}(\delta_{-x})$ is symmetric about $\mathbb{R}$, we deduce that for all w in the closed disk bounded by $H_{n,\alpha}(\delta_{-x})$, $|w| \le H_{n,\alpha}(-x)$ with equality if and only if $w = H_{n,\alpha}(-x)$. Thus, $|\tilde{h}(-x)| \le H_{n,\alpha}(-x)$ and equality forces $\tilde{h}(-x) = H_{n,\alpha}(-x)$, or $h = H_{n,\alpha}$.

(iii) Since we have the freedom to replace f by a rotation of f, it suffices to consider $z = -x \in \Big(\frac{\sqrt{n}-\sqrt{n+2}m_n(\alpha)}{\sqrt{n+2}-\sqrt{n}m_n(\alpha)}\Big)$ and show equality implies $f = F_{n,\alpha}$. In this case the result follows immediately from part (ii) of the theorem:

$$\begin{aligned} |f(-x)| &= \Big| \int_0^x f'(-t)dt \Big| \le \int_0^x |f'(-t)|dt \\ &\le \int_0^x F'_{n,\alpha}(-t)dt = -F_{n,\alpha}(-x)\,. \end{aligned}$$

Equality forces $|f'(-t)| = F'_{n,\alpha}(-t)$ for $0 < t < x$ and so $f = F_{n,\alpha}$.

(iv) From the definition of $r(0, f)$, there is a simply connected region $\Omega \subset \mathbb{D}$ containing 0 such that f maps Ω conformally onto a disk centered at $0 = f(0)$ with radius $r(0, f)$. This disk must either touch the boundary of the Riemann surface $f(\mathbb{D})$ or a branch point lies on the boundary. Hence, there is a radial segment Γ in this disk joining $f(0)$ to either a boundary point of $f(\mathbb{D})$ or a branch point. Let γ

be the inverse image of Γ under $f \mid \Omega$. Then either γ goes from 0 to $\partial\mathbb{D}$ or from 0 to a point $c \in \mathbb{D}$ with $f'(c) = 0$. Since $f'(z) \neq 0$ for $|z| < m_n(\alpha)$, γ must eventually meet the circle $|z| = m_n(\alpha)$ in either case. Thus,

$$\begin{aligned} r(0, f) &= \int_\Gamma |dw| = \int_\gamma |f'(z)||dz| \\ &\geq \int_\gamma |f'(z)|d|z| \\ &\geq \int_0^{m_n(\alpha)} F'_{n,\alpha}(|z|)d|z| \\ &= F_{n,\alpha}\big(m_n(\alpha)\big) = C_n m_n^{n+1}(\alpha)\,. \end{aligned}$$

Equality implies $|f'(z)| = F'_{n,\alpha}(|z|)$ for $|z| < m_n(\alpha)$ and so f must be a rotation of $F_{n,\alpha}$.

In terms of the operator $D_1 f(z) = (1 - |z|^2)f'(z)$ (see §5), parts (i) and (ii) of the theorem can be expressed as follows:

Corollary 6. *Suppose $f \in B_n(\alpha)$, $\alpha \in (0, 1)$.*

(i) *For $|z| \leq m_n(\alpha)$, $|D_1 f(z)| \geq D_1 F_{n,\alpha}(|z|)$ with equality at $z = re^{i\theta}$, $r \in \big(0, m_n(\alpha)\big)$, if and only if $f(z) = e^{i\theta} F_{n,\alpha}(e^{-i\theta} z)$.*

(ii) *For $|z| \leq \frac{\sqrt{n}-\sqrt{n+2}m_n(\alpha)}{\sqrt{n+2}-\sqrt{n}m_n(\alpha)}$, $|D_1 f(z)| \leq D_1 F_{n,\alpha}(-|z|)$ with equality at $z = -re^{i\theta}$, $r \in \Big(0, \frac{\sqrt{n}-\sqrt{n+2}m_n(\alpha)}{\sqrt{n+2}-\sqrt{n}m_n(\alpha)}\Big)$, if and only if $f(z) = e^{i\theta} F_{n,\alpha}(e^{-i\theta} z)$.*

5. Estimates for second order derivatives

The subordination principle for multivalued functions can also be applied to extremal problems involving second order derivatives. This allows us to determine the radius of convexity for $B_n(\alpha)$. Some of the steps are analogous to results of [BMY2] for the class $B_1(\alpha)$, so we omit the proofs.

First we introduce two differential operators which are suitably covariant under disk automorphisms. Let f be holomorphic on $\mathbb{D}$. Then we define $D_j f$ $(j = 1, 2)$ by

$$\begin{aligned} D_1 f(z) &= (1 - |z|^2)f'(z), \\ D_2 f(z) &= (1 - |z|^2)^2 f''(z) - 2\bar{z}(1 - |z|^2)f'(z)\,. \end{aligned}$$

For any $a \in \mathbb{D}$, the map $T(z) = (z + a)/(1 + \bar{a}z)$ is a conformal automorphism of $\mathbb{D}$ which takes 0 to a and $D_j f(a) = (f \circ T)^{(j)}(0)$, $j = 1, 2$. Thus $D_j f(0)$ is just the ordinary jth derivative of f at 0. These differential operators are invariant in the sense that

$$|D_j(S \circ f \circ T)| = |D_j f| \circ T, \qquad j = 1, 2,$$

where S is an oriented Euclidean motion of $\mathbb{C}$ and T is a disk automorphism.

Lemma 1. *For $\alpha \in [0, 1]$ and $z \in \overline{\Delta_n(\alpha)}$*

$$\begin{aligned} |D_2 F_{n,\alpha}(z)| = &(n+1) C_n m_n^{n-1}(|D_1 F_{n,\alpha}(z)|)\big(1 - m_n^2(|D_1 F_{n,\alpha}(z)|)\big) \\ &\times \big(n - (n+2)m_n^2(|D_1 F_{n,\alpha}(z)|)\big). \end{aligned}$$

The application of the subordination principle occurs in the following theorem:

Theorem 5. *Suppose $\alpha \in (0,1)$ and $f \in B_n(\alpha)$. Then*

$$\begin{aligned}|f''(0)| &\le -F''_{n,\alpha}(0)\\ &= (n+1)C_n m_n^{n-1}(\alpha)\big(1-m_n^2(\alpha)\big)\big(n-(n+2)m_n^2(\alpha)\big)\,.\end{aligned}$$

Equality holds if and only if $f(z) = \bar{\lambda} F_{n,\alpha}(\lambda z)$ for some unimodular constant λ.

Proof. There is nothing to prove if $f''(0) = 0$. Therefore, without loss of generality we may assume $f''(0) < 0$ and then we must prove $-f''(0) \le -F''_{n,\alpha}(0)$ with equality implying $f = F_{n,\alpha}$. As usual, set $g(z) = \big(1 - m_n(\alpha)z\big)^2 f'(z)$ and $h(z) = g(z)^{1/n}$. Then $h'(0) = \frac{1}{n}\alpha^{\frac{1}{n}-1}[f''(0) - 2\alpha m_n(\alpha)]$. Since $h \prec_0 H_{n,\alpha}$ on $\Delta_n(\alpha)$, there is $\varphi \in \mathcal{M}_0\big(\Delta_n(\alpha)\big)$ with $\varphi(0) = 0$, $\varphi\big(\Delta_n(\alpha)\big) \subset \Delta_n(\alpha)$ and $h = H_{n,\alpha} \circ \varphi$. Now,

$$\begin{aligned}\frac{1}{n}\alpha^{\frac{1}{n}-1}[f''(0) - 2\alpha m_n(\alpha)] &= h'(0) = H'_{n,\alpha}(0)\varphi'(0)\\ &= \frac{1}{n}\alpha^{\frac{1}{n}-1}[F''_{n,\alpha}(0) - 2\alpha m_n(\alpha)]\varphi'(0)\,.\end{aligned}$$

The two expressions in brackets are negative, so $\varphi'(0) > 0$. But $|\varphi'(0)| \le 1$, so $0 < \varphi'(0) \le 1$ and

$$\frac{1}{n}\alpha^{\frac{1}{n}-1}[f''(0) - 2\alpha m_n(\alpha)] \ge \frac{1}{n}\alpha^{\frac{1}{n}-1}[F''_{n,\alpha}(0) - 2\alpha m_n(\alpha)]\,,$$

or

$$-f''(0) \le -F''_{n,\alpha}(0)\,.$$

If equality holds, then $\varphi'(0) = 1$, or φ is the identity function which implies $h = H_{n,\alpha}$ and so $f = F_{n,\alpha}$.

Corollary 7. *If $f \in B_n$, then*

$$\begin{aligned}|D_2 f(z)| \le (n+1)C_n m_n^{n-1}&\big(|D_1 f(z)|\big)\big(1 - m_n^2(|D_1 f(z)|)\big)\\ &\times \big(n - (n+2)m_n^2(|D_1 f(z)|)\big)\,.\end{aligned}$$

Equality holds at a point $z_0 \in \mathbb{D}$ where $D_2 f(z_0) \neq 0$ if and only if

$$f(z) = \lambda F_{n,\alpha}\Big(\mu \frac{z - z_0}{1 - \bar{z}_0 z}\Big) + C$$

for some $\alpha \in (0,1)$, unimodular constants λ and μ, and $C \in \mathbb{C}$. Also,

$$\begin{aligned}|D_2 f(z)| \le (n+1)C_n &\left[\frac{(n+1)^2 - \sqrt{5n^2+10n+1}}{(n+2)(n+3)}\right]^{\frac{n-1}{2}}\\ &\cdot \frac{3n+5+\sqrt{5n^2+10n+1}}{(n+2)(n+3)}\,\frac{n-1+\sqrt{5n^2+10n+1}}{n+3}\end{aligned}$$

with equality at a point $z_0 \in \mathbb{D}$ if and only if

$$f(z) = \lambda F_{n,\alpha}\Big(\mu \frac{z - z_0}{1 - \bar{z}_0 z}\Big) + C$$

for unimodular constants λ and μ and $C \in \mathbb{C}$, where

$$\alpha = (n+1)C_n \frac{3n+5+\sqrt{5n^2+10n+1}}{2(n+3)} \left[\frac{(n+1)^2 - \sqrt{5n^2+10n+1}}{n(n+3)}\right]^{n/2}.$$

Theorem 6. *Suppose $f \in B_n(\alpha)$, $\alpha \in (0,1)$. Then for $|z| < m_n(\alpha)$*

$$\left|\frac{D_2 f(z)}{D_1 f(z)}\right| \leq \frac{-D_2 F_{n,\alpha}(|z|)}{D_1 F_{n,\alpha}(|z|)},$$

or

$$\left|(1-|z|^2)\frac{f''(z)}{f'(z)} - 2\bar{z}\right| \leq -\frac{(n+2)m_n^2(\alpha) - n - 4m_n(\alpha)|z| + \big((n+2) - nm_n^2(\alpha)\big)|z|^2}{\big(m_n(\alpha)-|z|\big)\big(1-m_n(\alpha)|z|\big)}$$

Equality holds at $z = re^{i\theta}$, $r \in \big(0, m_n(\alpha)\big)$, if and only if $f(z) = e^{i\theta}F_{n,\alpha}(e^{-i\theta}z)$.

Corollary 8. *Suppose $f \in B_n(\alpha)$, $\alpha \in (0,1)$. Then for $|z| < m_n(\alpha)$*

$$\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} \geq \frac{|z|F''_{n,\alpha}(|z|)}{F'_{n,\alpha}(|z|)} = -\frac{|z|\big(n-(n+2)m_n^2(\alpha)+2m_n(\alpha)|z|\big)}{\big(m_n(\alpha)-|z|\big)\big(1-m_n(\alpha)|z|\big)}.$$

Equality holds at $z = re^{i\theta}$, $r \in \big(0, m_n(\alpha)\big)$, if and only if $f(z) = e^{i\theta}F_{n,\alpha}(e^{-i\theta}z)$. In particular, the radius of convexity for $B_n(\alpha)$ is

$$R_c(n,\alpha) = \frac{2m_n(\alpha)}{(n+1)\big(1-m_n^2(\alpha)\big) + \sqrt{(n+1)^2\big(1-m_n^2(\alpha)\big)^2 + 4m_n^2(\alpha)}}.$$

Proof. From the theorem we have

$$\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)} - \frac{2|z|^2}{1-|z|^2}\right\} = \operatorname{Re}\left\{\frac{zD_2f(z)}{(1-|z|^2)D_1f(z)}\right\} \geq \frac{|z|F''_{n,\alpha}(|z|)}{F'_{n,\alpha}(|z|)} - \frac{2|z|^2}{1-|z|^2}$$

which is the desired inequality. The equality statement follows from the theorem. Next,

$$1 + \operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} \geq 1 + \frac{|z|F''_{n,\alpha}(|z|)}{F'_{n,\alpha}(|z|)}.$$

Since the right-hand side is positive for $|z| < R_c(n,\alpha)$ and vanishes for $|z| = R_c(n,\alpha)$, this establishes the radius of convexity result.

References

[BMY1] M. Bonk, D. Minda and H. Yanagihara, Distortion theorems for locally univalent Bloch functions, *J. Analyse Math.* **69** (1996), 73–95. MR **98g:**30058

[BMY2] M. Bonk, D. Minda and H. Yanagihara, Distortion theorems for Bloch functions, *Pacific J. Math.* **179** (1997), 241–262. MR **98g:**30059

[LM] X. Liu and D. Minda, Distortion theorems for Bloch functions, *Trans. Amer. Math. Soc.* **333** (1992), 325–338. MR **92k:**30041

[M] D. Minda, Lower bounds for the hyperbolic metric in convex regions, *Rocky Mountain J. Math.* **13** (1983), 61–69. MR **84:**30039

[N] Z. Nehari, A generalization of Schwarz' Lemma, *Duke Math. J.* **14** (1947), 1035–1049. MR **9:**340i

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