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THE METRIC PROJECTION ONTO THE SOUL

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ABSTRACT. We study geometric properties of the metric projection $\pi : M \to S$ of an open manifold M with nonnegative sectional curvature onto a soul S. π is shown to be C^∞ up to codimension 3. In arbitrary codimensions, small metric balls around a soul turn out to be convex, so that the unit normal bundle of S also admits a metric of nonnegative curvature. Next we examine how the horizontal curvatures at infinity determine the geometry of M, and study the structure of Sharafutdinov lines. We conclude with regularity properties of the cut and conjugate loci of M.

The resolution of the Soul conjecture of Cheeger and Gromoll by Perelman showed that the structure of open manifolds with nonnegative sectional curvature is more rigid than expected. One of the key results in [14] is that the metric projection $\pi : M \to S$ which maps a point p in M to the point $\pi(p)$ in S that is closest to p is a Riemannian submersion. Perelman also observed that this map is at least of class C^1, and later it was shown in [9] that π is at least C^2, and C^∞ at almost every point.

The existence of such a Riemannian submersion combined with the restriction on the sign of the curvature suggests that the geometry of M must be special. The purpose of this paper is to illustrate this in several different directions. In fact, we show that most natural geometric objects classically used in the study of these spaces are intrinsically related to the structure of the map π.

The paper is essentially structured as follows: After introducing notation and recalling some basic results in section 1, we establish in section 2 the existence of maps similar to π for any submanifold of M homologous and isometric to the soul, and use this to prove the existence of convex tubular neighborhoods for them. In particular, this also proves:

Theorem 2.5. *The unit normal bundle $\nu_1(S)$ admits a metric with nonnegative sectional curvature.*

As a consequence, open manifolds with nonnegative curvature provide examples of compact manifolds with the same lower curvature bound (see also [8]). Theorem 2.5 also implies that nontrivial plane bundles over a Bieberbach manifold do not admit nonnegatively curved metrics, thereby providing a shorter proof of the main result in [13]. We are grateful to the referee for pointing out this fact to us.

Properties such as these are established via standard submersion techniques, but the fact that π is not known to be smooth everywhere prevents us from using these techniques to study the geometry of M far away from the soul. One way around

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this is to use the so-called Sharafutdinov lines: Section 3 studies the homotopy constructed by Sharafutdinov in [16] between the identity map and π. In particular, we look at the curves desribed by individual points p, and show that they generate 'singular flats' under holonomy diffeomorphisms of π. We also show that sectional curvatures of horizontal planes are nonincreasing as one travels away from the soul along one of these curves. The fact that the soul corresponds to the region of largest horizontal curvature is used to illustrate how the behavior of the horizontal curvatures at infinity influence the global type of the manifold. In section 4 we study the holonomy diffeomorphisms associated with π. They turn out to be Lipschitz maps, and their associated holonomy Jacobi fields are uniformly bounded along an infinite geodesic. This in turn is used to derive several splitting results. Section 5 starts by studying some properties of the action of the normal holonomy of the soul (Theorem 5.1), that as an application provides:

Theorem 2.10. *Let M be an open nonnegatively curved manifold with soul S of codimension 3. Then the metric projection $\pi : M \to S$ is a C^∞ Riemannian submersion. Moreover, one of the following is true:*

(1) *M splits locally isometrically over S.*
(2) $\exp : \nu(S) \to M$ *is a diffeomorphism.*
(3) *There exists a normal (global if S is simply connected) parallel vector field W along S that exponentiates to outgoing Sharafutdinov lines in the sense of section 3.*

The paper ends by showing regularity of the metric projection at regular points of the conjugate locus of the soul.

1. The basic geometry

Throughout the paper, M will denote a complete, noncompact (open, for short) manifold with nonnegative curvature K, and S a soul of M. π will be the metric projection of M onto S, which coincides with the end result of the Sharafutdinov retraction of M onto S. A and T will denote the fundamental tensors of π as defined in [12]. We write $TM = \mathcal{H} \oplus \mathcal{V}$ for the orthogonal decomposition of the tangent bundle TM of M into the horizontal and vertical distributions associated to π. We will canonically identify a fiber of the normal bundle and its tangent space at any point. Recall that horizontally lifting a geodesic $\alpha : [0, l] \to S$ to M yields a *holonomy diffeomorphism* h^α between the fibers over $\alpha(0)$ and $\alpha(l)$; cf. [7]. For generic Riemannian submersions, $dh^\alpha(u) = J(l)$, where J is the holonomy Jacobi field along the appropiate lift $\bar{\alpha}$ of α with initial conditions $J(0) = u$, $J'(0) = A_{\bar{\alpha}'}u + T_u\bar{\alpha}'$. In the case of the metric projection onto the soul, we recall from [14] and [9] that dh^α can be more explicitly described as follows (see Figure 1). Let $\gamma : [0, r] \to M$ be a radial geodesic connecting $\alpha(0)$ to $\bar{\alpha}(0)$. The vertical space at $\gamma(r)$ decomposes as a direct sum $\mathcal{V} = \mathcal{W} \oplus \mathcal{W}^\perp$, where $\mathcal{W}$ is the image of $\nu_{\alpha(0)}(S)$ under the derivative at $r\gamma'(0)$ of the normal exponential map. If P_α denotes parallel translation along α, then we have:

Proposition 1.1. (1) $dh^\alpha \circ d\exp = d\exp \circ P_\alpha$.

(2) $dh^\alpha = P_{\bar{\alpha}}$ *on* $\mathcal{W}^\perp$ [9].

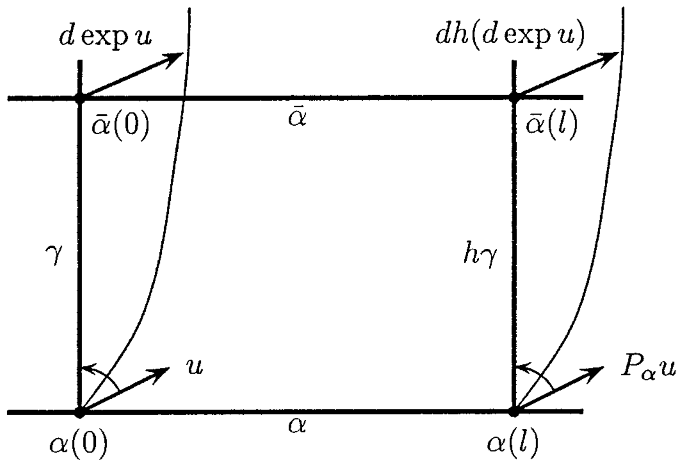

Figure 1

Finally, for X, Y basic lifts of $x, y \in TS$ along a fiber, the vector field $A_XY \circ \gamma$ along a radial geodesic γ is the Jacobi field J with initial conditions

$$J(0) = 0, \qquad J'(0) = -\frac{1}{2}R^{\nabla}(x, y)\gamma'(0), \tag{1.2}$$

where R^{∇} denotes the curvature tensor of $\nu(S)$ [18]. In particular, $A_X\gamma' = 0$. Moreover, $T_{\gamma'}X = 0$, so that $X \circ \gamma$ is a parallel Jacobi field along γ that exponentiates to a flat, totally geodesic rectangle in M.

2. Pseudosouls and convexity

The soul of an open manifold with nonnegative curvature need not be unique. The simplest example is perhaps that of Euclidean space, where any point is a soul. Nevertheless, two souls cannot significantly differ, since a result of Yim [24] (see also [16]) asserts that they must be isometric and homologous to one another. The converse is not true; i.e., there may exist submanifolds of M that are isometric and homologous to a soul, but are not souls in the Cheeger-Gromoll sense: This is, for instance, the case for any nonvertex point in a paraboloid of revolution. We recall from [24] the following:

Definition 2.1. *A submanifold B of M is called a* pseudosoul *if it is isometric and homologous to a soul S of M.*

Even though pseudosouls need not be souls, it was shown in [24] that their union $\mathcal{P}$ has an interesting structure:

Theorem 2.2. *$\mathcal{P}$ is a totally geodesic embedded submanifold which is isometric to a product manifold $S \times N$, where N is a complete manifold of nonnegative curvature diffeomorphic to a Euclidean k–space $\mathbb{R}^k$ and k is the dimension of the space of all parallel vector fields along the soul S. Furthermore, any pseudosoul in M is of the form $S \times \{p\}$ for some $p \in N$.*

A natural question to ask then is how different pseudosouls and souls really are. We will start by showing that there is a version of Perelman's theorem for each pseudosoul.

Proposition 2.3. *Let S_0 be a pseudosoul of M. Then,*

(1) *There is a C^1 Riemannian submersion $\pi_0 : M \to S_0$ which is also a retraction.*
(2) *The conclusions of Perelman's theorem hold when replacing S and π by S_0 and π_0.*
(3) *The fibers of π are totally geodesic at points of $\mathcal{P}$.*
(4) *If $\mathcal{H}_0$, $\mathcal{V}_0$ are the horizontal and vertical distributions associated to π_0, then $\mathcal{H}_0 = \mathcal{H}$ and $\mathcal{V}_0 = \mathcal{V}$.*

Proof. For S_0, there is a normal parallel vector field U along S with $S_0 = \exp(rU)$. In fact, it follows from [24], or just [14], that the map $\phi_0 : S \to S_0$ given by $\phi(p) = \exp(rU_p)$ is an isometry. The first part follows now by taking $\pi_0 = \phi_0 \circ \pi$.

For the second and third points we can use the proof of Perelman's theorem replacing π and S by π_0 and S_0, since his original argument only needed the existence of a distance nonincreasing map as the one we got in (1).

Finally, if $v \in \mathcal{V}$, $d\pi_0(v) = d\phi_0 \circ d\pi(v) = d\phi_0(0) = 0$; therefore $\mathcal{V} \subset \mathcal{V}_0$. This finishes the proof, since both spaces have the same dimension. □

Hence, pseudosouls satisfy rigidity conditions similar to the ones provided by Perelman's theorem for the soul, and are from that point of view indistinguishable from it. Furthermore, using the arguments in [9], we can show that each π_0 is also C^2. This fact will not, however, be needed here.

It is also easy to obtain from (2.1) that $\pi_0 : M \to S_0$ is C^∞ in a neighborhood of S_0 not containing any focal points. The reason is that the fiber of π_0 over a point $p \in S_0$ agrees with the image of the normal vectors at p by the normal exponential map of S_0. Recall that we use $\mathcal{H}$ and $\mathcal{V}$ for the horizontal and vertical distributions associated to this submersion.

Let $D_r = \{\, x \in M \mid d(x, S_0) \le r\}$ and $S_r = \partial D_r$. When S_0 is a point, D_r is a metric ball, and is therefore convex for small r. Our next result extends this to the case of nontrivial pseudosouls.

Proposition 2.4. *D_r is convex for small $r > 0$.*

Proof. We will show that the second fundamental form II of S_r associated to the inward-pointing unit normal vector field N is positive semidefinite. Clearly, if $\gamma : [0, r] \to M$ is a minimizing geodesic between $\bar p = \pi(p) \in S_0$ and $\gamma(r) = p$, then $\gamma'(t) \perp S_t$ for all t. Thus, $\gamma'(r) = -N(p)$. We also have a decomposition $T_pS_r = \mathcal{H}_p \oplus \widetilde{\mathcal{V}}_p$ where $\widetilde{\mathcal{V}}_p$ is the subspace of $\mathcal{V}_p$ generated by curves tangent to the metric sphere of radius r intersected with the Sharafutdinov fiber through p.

By Perelman's theorem, N is parallel along any horizontal geodesic, and therefore $II(x, *) = 0$ for any $x \in \mathcal{H}_p$.

On the other hand, observe that the set $\nu_{\bar p}(S_0, r)$ of normal vectors to S at $\bar p$ of length r satisfies

$$\exp\left(\nu_{\bar p}(S_0, r)\right) = S_r \cap \partial B_r(\bar p).$$

This set is smooth for small $r > 0$, and its tangent space at p is $\widetilde{\mathcal{V}}_p$. Since balls of small radius are strictly convex, $II_p(u, u) > 0$ for any nonzero $u \in \widetilde{\mathcal{V}}_p$, and the second form of S_r is positive semidefinite. □

Theorem 2.5. *The unit normal bundle $\nu_1(S)$ admits a metric with nonnegative sectional curvature.*

Proof. Immediate from combining the Gauss equations with the previous proposition. □

Remark 2.6. It is known that the double of D_r also admits a metric with nonnegative sectional curvature [8]. If k denotes the codimension of the soul, then it is not difficult to see that this space is the k–sphere bundle associated to the original vector bundle E which is diffeomorphic to M. Thus, a complete metric with nonnegative curvature in M provides us with two sphere bundles with nonnegative curvature metrics. It would be interesting to know whether the converse holds.

It is worth mentioning that the previous theorem may be improved as follows: By Proposition 2.4, S_r has nonnegative curvature for all $r \in (0, \varepsilon)$ when ε is small enough. For $r \to 0$, the sequence of Riemannian manifolds (S_r, g_r) has Sharafutdinov fibers shrinking to points. In particular, we have:

Corollary 2.7. *There is a sequence of nonnegatively curved metrics on the normal sphere bundle of S collapsing to S.*

How restrictive are these last observations on the vector bundle $\nu(S)$? If the base is a Bieberbach manifold, a metric of nonnegative curvature on $\nu(S)$ would force the unit normal bundle P to be nonnegatively curved by Theorem 2.5. But P is an infranilmanifold, and in this case the splitting theorem forces P to be a trivial S^1 bundle over S. Hence $\nu(S)$ must also be trivial, which gives a different proof of the main result of [13].

Another feature of Riemannian manifolds is that the distance function between two geodesics emanating from a common point is initially nondecreasing. Theorem 2.4 combined with Sharafutdinov's original construction [16] extends this to nontrivial pseudosouls.

Corollary 2.8. *Let S_0 be a pseudosoul of M. There exists an $\varepsilon > 0$ such that for any pair γ_1, γ_2 of radial geodesics starting at S_0, the function $s \mapsto d(\gamma_1(s), \gamma_2(s))$ is nondecreasing in $[0, \varepsilon]$.*

Proof. Choose ε so that D_ε is convex. By Proposition 2.4 and [3], the function $d(\cdot, \partial D_\varepsilon)$ is convex on D_ε. Therefore, the original construction of Sharafutdinov together with the fact that $d(\cdot, \partial D_\varepsilon)$ is smooth on $D_\varepsilon - S_0$ implies that the Sharafutdinov lines corresponding to $d(\cdot, \partial D_\varepsilon)$ are precisely the radial geodesics with opposite orientation. Since the Sharafutdinov retraction is distance nonincreasing, the claim follows. □

The statement in 2.8 will of course not hold in general for all values of the parameter s, since it is already false for metrics of nonnegative curvature on $\mathbb{R}^n$ which are not identically flat.

Not surprisingly, these results indicate that the existence of pseudosouls impose serious metric restrictions on the geometry of the total space. It was nevertheless already shown in [18] that even if there is a space $S \times \mathbb{R}$ of pseudosouls, the metric on M need not split accordingly as $N \times \mathbb{R}$. We initially thought that this might be due to some inherent difference between souls and pseudosouls, and that, say, the presence of more than one (real) soul should imply some local metric splitting. Our next example shows that this is not so.

Example 2.9. Let $A = \{\, (x, y, x^2) \in \mathbb{R}^3 \mid y \geq 0 \,\}$ be the set obtained by translating the parabola $P = (x, 0, x^2)$ along the positive y–axis. Now glue half a paraboloid to A and smooth out the singularity along P to get a surface with nonnegative curvature that has a soul along any point of the ray $R = \{(0, y, 0)\}$. The reason for this is that if we begin the Cheeger–Gromoll construction at a point Y, we will obtain the segment $0Y$ as the first set in the convex exhaustion of M, and the middle point of such a segment as a soul. It is also clear that this example generalizes trivially to higher dimensions. Therefore, even when there is more than one soul, M need not split off a line.

When the soul has small codimension, the metric is quite rigid. The cases of codimension 1 and 2 have been treated in [3] and [19] respectively, where it is shown that either the manifold splits locally isometrically along the soul, or the normal exponential map $\exp : \nu(S) \to M$ is a diffeomorphism. Observe that in the latter case, the Sharafutdinov vector field is the radial vector field (this is actually equivalent to the statement that the exponential is a diffeomorphism), and the soul can be constructed in one step. In codimension 3, the situation is slightly more complex.

Theorem 2.10. *Let M be an open nonnegatively curved manifold with soul S of codimension 3. Then the metric projection $\pi : M \to S$ is a C^∞ Riemannian submersion. Moreover, one of the following is true:*

(1) *M splits locally isometrically over S.*
(2) $\exp : \nu(S) \to M$ *is a diffeomorphism.*
(3) *There exists a normal (global if S is simply connected) parallel vector field W along S that exponentiates to outgoing Sharafutdinov lines in the sense of section 3.*

We will prove this theorem in section 5. Although in case (3), the Sharafutdinov lines in the parallel direction are radial geodesics, they need not be rays, as the following example shows.

Example 2.11. Let $A = \{\, (x, y, z, 0) \in \mathbb{R}^4 \mid 2x^2 + 2y^2 + z^2 \leq 1 \,\}$, and $B = \{\, (x, y, z, t) \in \mathbb{R}^4 \mid 2x^2 + 2y^2 + z^2 = 1, t \geq 0\}$. $A \cup B$ is the boundary of a convex set in $\mathbb{R}^4$, and therefore its induced singular metric has nonnegative curvature. The resulting space, which is homeomorphic to $\mathbb{R}^3$, has the origin as soul, and admits an isometric S^1-action induced by rotations in the xy plane. This metric can be smoothed into another one with nonnegative curvature, same soul and isometric S^1-action, and for which the geodesic γ from the soul which is tangent to the z-axis is not a ray. If P denotes the space thus constructed, we now use Cheeger's argument [2] to produce a nonnegatively curved metric on $M = S^n \times_{S^1} P$, where we take any free isometric action of S^1 on S^n. Since γ is normal to the orbits, it is horizontal for the quotient map $S^n \times P \to M$. Therefore γ maps to a geodesic $\widetilde{\gamma}$ in M which is not a ray because these lift to rays under Riemannian submersions. Thus the exponential map for M is not a diffeomorphism. On the other hand, the manifold does not split locally isometrically since the normal holonomy group of the soul is not trivial.

3. Sharafutdinov lines

In his paper [16], Sharafutdinov constructed the map π by deforming M onto S with the help of a Cheeger-Gromoll convex function f and its not necessarily continuous gradient vector field U. For each $p \in M$, there is a pseudointegral curve of U carrying p all the way to set C where f takes its minimum. If this set is S, we stop. Otherwise, we can continue the process using the distance function to the boundary C and its convexity to iterate the process until we wind up at S. The final point is then $\pi(p)$. Notice that U and the pseudointegral curves thus constructed are determined the moment we choose f. In this section, we will assume that such a choice has been made. Details of the Sharafutdinov construction can also be found in [23].

In this section, we will not work with the original Sharafutdinov integral curves, but rather with the ones obtained by reversing their orientation. It is clear that in this case, such curves may bifurcate as we go along the manifold. We will adopt the following terminology.

Definition 3.1. An *outgoing Sharafutdinov line (OSL)* is a curve $c : [0, \infty) \to M$ starting at the soul which is left tangent to $-U$ at every point.

Thus, each 'classical' Sharafutdinov line is contained in some finite part of an OSL, but the direction in which it is traveled is reversed. This implies that if c_1, c_2 are OSL's, then the function $t \mapsto d(c_1(t), c_2(t))$ is nondecreasing. Moreover, OSL's are completely determined by the Sharafutdinov vector field U.

In many examples, OSL's are well-behaved objects. For instance, if the normal exponential map of the soul is a diffeomorphism, then the OSL's coincide with the normal geodesics from S. This is no longer true in general: In any metric on the plane for which one of the intermediate sets of the Cheeger–Gromoll exhaustion is a rectangle with sides of different length, there are only two directions normal to the soul that correspond to OSL's. Nevertheless our next result shows that OSL's satisfy the same sort of rigidity that Perelman's theorem imposes on geodesics orthogonal to S. Part (1) of this theorem was discussed in a conversation of the first author with V. Kapovitch.

Theorem 3.2. *Let $c : [0, \infty) \to M$ be an OSL with $c(0) = p \in S$, $\alpha : [0, l] \to S$ a geodesic segment in S, and $h^s : F_{\alpha(0)} \to F_{\alpha(s)}$ the holonomy map between the fibers of π obtained by lifting α. If R denotes the rectangle $(t, s) \mapsto h^t(c(s))$, then:*

1. *$s \mapsto R(t_0, s)$ is an OSL for each $t_0 \in [0, l]$.*
2. *Any plane generated by $\partial_s^- R$ (which exists by (1)) and a horizontal vector X is flat.*
3. *The horizontal lift of a vector $x \in T_pS$ to an OSL is parallel along it.*

Proof. Recall that a horizontal geodesic β for the submersion π remains at a constant distance from the soul for all parameter values. Since the Cheeger–Gromoll convex exhaustion $\{C^b\}$ is obtained via sublevel sets of some convex function, it follows that β is contained in ∂C^b for some b. We now apply Theorem A.5 from [24] to conclude that the Sharafutdinov vector field is perpendicular to any horizontal geodesic and parallel along it. An argument similar to the one in the proof of Perelman's soul theorem as described in [9] easily implies that $dh^t(U) = P_t U$, where P_t is a parallel translation along each $t \mapsto R(t, s_0)$, as well as part (2) of the theorem. Thus, $dh^t(U) = U$, or equivalently, $\partial_s^- R(s, t) = U(R(s, t))$. (1) now follows from uniqueness of the OSL's as maximal left integral curves for the Sharafutdinov vector field.

Finally, part (3) is clear from the above paragraph for points of smoothness of $s \mapsto R(t_0, s)$. Since these points are dense in an OSL, X is parallel everywhere along it. ☐

We expect outgoing Sharafutdinov lines to be useful in general for the understanding of the geometry of M. They can be thought of as "broken" radial geodesics or even "broken" rays that may travel together part of the way but eventually bifurcate. It would be interesting to see how far this analogy can be developed. An initial step in this direction should study the topological structure of the space formed by all the OSL's. This is precisely the content of our next result.

Proposition 3.3. *Let Ω denote the collection of outgoing Sharafutdinov lines together with the topology it receives as a subspace of $C([0,\infty), M)$ in the compact–open topology. Then the map $\bar{\pi} : \Omega \to S$ induced by $\pi : M \to S$ is a locally trivial fibration.*

Proof. Given $q \in S$, let F_q denote the collection of Sharafutdinov lines from q in the compact–open topology. If p is any point in the soul, choose a convex ball B in S centered around p. Theorem 3.2 enables us to construct a natural map $\Phi : \bar{\pi}^{-1}(B) \to B \times F_p$ induced by holonomy diffeomorphisms along radial geodesics from p. The proof that Φ is a trivialization is entirely straightforward. □

Another possible approach in the study of OSL's examines how geometric invariants of M behave as we travel to infinity along one of these lines. Our next result, for example, extends Theorem 2 in [20] from rays to OSL's.

Proposition 3.4. *Let $c : [0,\infty) \to M$ denote an outgoing Sharafutdinov line starting at $p \in S$. For orthonormal x, $y \in T_pS$, and for $q \in \pi^{-1}(p)$, denote by σ_q the horizontal plane spanned by the basic lifts X_q, Y_q of x, y. Then*

(1) *The function $t \mapsto K(\sigma_{c(t)})$ is nonincreasing on $[0,\infty)$; equivalently, the norm of A_XY is nondecreasing along c.*
(2) *The maximum of the function $K(\sigma)$ (which by (1) occurs at p) is strict if and only if $R^{\nabla}(x,y) : \nu_p(S) \to \nu_p(S)$ has trivial kernel.*

Proof. The first part can be proved by combining the argument in [19, Theorem 2] with Theorem 3.2 and the fact that for two Sharafutdinov lines c_1, c_2, the function $t \mapsto d(c_1(t), c_2(t))$ is not decreasing. The statement about the A tensor is a trivial consequence of O'Neill's formula. The second part follows easily from (1.2). □

Notice that we are not claiming that if A_XY is zero at some q outside the soul, then it vanishes along geodesics joining q and $\pi(q)$; rather, there is some geodesic γ emanating from $\pi(q)$ (but not necessarily passing through q) for which $A_XY \circ \gamma = 0$. In fact, $\gamma'(0)$ is the initial tangent vector to the OSL joining $\pi(q)$ and q, which exists because $U(c(t))$ is right continuous (see [16]). It may also be worth pointing out that there is no corresponding statement for arbitrary planes: There are examples [4] where the curvature of certain vertical planes increases to infinity as one gets farther away along the manifold.

In order to present another consequence of Proposition 3.4, we define the *horizontal curvature at infinity of M* to be

$$k_\infty^H = \liminf_{r\to\infty} \{K(\pi_q) \mid \pi_q \text{ is a horizontal 2–plane at } q \text{ with } d(q,S) \geq r\}.$$

Corollary 3.5. *Suppose the sectional curvature of the soul is bounded above by $k_0 > 0$. If $k_\infty^H = k_0$, then S is a space form with constant curvature k_0 and M splits locally isometrically over S.*

Proof. For any plane σ tangent to S, denote by σ_t its horizontal lift along an OSL. The definition of k^H_∞ implies that for any $\epsilon > 0$, $K(\sigma_t) > k_0 - \epsilon$ for large t. Since this function is nonincreasing in t, $K(\sigma_t) > k_0 - \epsilon$ for all $t > 0$. Hence we obtain the following chain of inequalities:

$$k_0 \leq K(\sigma_t) \leq K(\sigma_0) \leq k_0.$$

Thus, the soul has constant curvature, and by the second part of Proposition 3.4, $R^\nabla = 0$. The main Theorem in [17] then implies that M splits locally isometrically over S. □

4. Restrictions on Jacobi fields

It was pointed out in section 1 that one can recover the A and T tensors of a Riemannian submersion π by looking at holonomy Jacobi fields along horizontal geodesics. Since these tensor fields determine the geometry of π, the behavior of Jacobi fields is crucial.

Lemma 4.1. *Let $\alpha : \mathbb{R} \to M$ denote a horizontal geodesic with $\alpha(0) = p$, u a vertical vector at p, and J the holonomy Jacobi field along α with $J(0) = u$. Then $\|J\|$ is bounded on $\mathbb{R}$.*

Proof. Let $\gamma : [0, 1] \to M$ be a minimal geodesic joining $\pi(p)$ to p. Assume first that p is not conjugate to $\pi(p)$ along γ. Then there is a geodesic variation of the form

$$V(s, t) = \exp(t(\gamma'(0) + sw)), \tag{4-2}$$

with $\partial_s V(1, 0) = u$, and w normal to the soul. Furthermore, by Proposition 1.1, the holonomy field J generated by u is given by

$$J(t) = \partial_s \exp(t(P_t\gamma'(0) + sP_t w)),$$

where P_t denotes parallel translation along the projection of α by π. Now apply the first Rauch theorem to any one of the geodesic variations obtained at each t–stage by comparing it to the corresponding variation in Euclidean space. Since the latter is independent of t, we obtain a global bound for the norm of J.

When p is conjugate to $\pi(p)$ along γ, the result follows by continuity. Alternatively, we can split $\mathcal{V}_p = \mathcal{W} \oplus \mathcal{W}^\perp$ as in Proposition 1.1. If u belongs to the first factor, the above argument yields a global bound for $\|J\|$. Since holonomy fields that belong initially to the second factor are parallel, the claim follows by linearity. □

Observe that holonomy fields are not the only Jacobi fields that are bounded: In fact, the A–tensor Jacobi fields along radial geodesics are also bounded above in norm by O'Neill's formula.

The other tensor field characterizing the submersion, namely the T-tensor given by $T_U V = \mathcal{H}\nabla_U V$, was shown to have bounded norm in [14]. This enables us to exhibit further restrictions on the holonomy diffeomorphisms between Sharafutdinov fibers.

Lemma 4.2. *Let $p, q \in S$ and $\alpha : [0, l] \to S$ be a minimal geodesic connecting p to q. If $h : F_p \to F_q$ is the corresponding holonomy diffeomorphism, then h is a Lipschitz map.*

Proof. Since $dh(u) = J(l)$ for the holonomy Jacobi field along the lift of α with $J(0) = u$, it suffices to show that $\|J(t)\|$ is bounded for any $t \in [0, l]$. But holonomy Jacobi fields satisfy $J' = A_{\alpha'}J + T_J\alpha'$, so that if $f(t) = \|J(t)\|^2$, then

$$f'(t) = 2\langle J, T_J\alpha' \rangle \leq 2Cf(t)$$

for some constant C independent of t since the second fundamental form is globally bounded. Thus, $f(l) \leq f(0)e^{2Cl}$, and the claim follows. □

The above argument also shows that the Lipschitz constant of the holonomy diffeomorphisms remains controlled when we bound the length of the geodesic connecting the basepoints of the fibers. On the other hand, a closer look at the proof of Lemma 4.1 reveals that one obtains the same conclusion assuming that we restrict ourselves to a compact set in the fiber (but no longer require a bound on the length of the geodesics inducing the holonomy diffeomorphism). This suggests the following question:

Question. *Let $\mathcal{S}$ be the set of geodesic segments of arbitrary finite length that connect two points of S. For each $\alpha \in \mathcal{S}$, let C_α be the smallest Lipschitz constant for h_α. Is $\sup_{\mathcal{S}} C_\alpha$ finite?*

As motivation for the above question, recall from [5] that if M is flat outside a compact set, then it splits locally isometrically over S. This can be generalized as follows to *vertizontal* planes, i.e., planes spanned by a vertical and a horizontal vector.

Theorem 4.3. *Let M be an open manifold with nonnegative sectional curvature and soul S.*

(1) *If outside of a compact set Ω every vertizontal plane has zero curvature then M splits locally isometrically over S.*

(2) *If $\sup_{\mathcal{S}} C_\alpha$ is finite, and if*

$$k_\infty^{VZ} = \limsup_{r\to\infty}\{K(\pi_q) \mid \pi_q \text{ is a vertizontal 2–plane at } q \text{ with } d(q, S) \geq r\}$$

is zero, then M splits locally isometrically over S.

Proof. (1) If $\alpha : \mathbb{R} \to M$ denotes a horizontal geodesic and J a holonomy Jacobi field along α, then by hypothesis,

$$J(t) = E(t) + tF(t)$$

for some parallel vector fields E, F along α. Since J has bounded norm by Lemma 4.1, it must be a parallel Jacobi field. This implies that the A tensor vanishes outside Ω. We can then use that A does not increase in norm along Sharafutdinov lines going to the soul, to conclude that A has to vanish everywhere. The main theorem of [17] finishes the proof.

(2) For any sequence of points $p_n \to \infty$ on a fixed Sharafutdinov fiber F_p, and vertical unit vectors u_n at p_n, consider the holonomy Jacobi fields J_n along lifts α_n of a fixed geodesic in S. The condition on C_α ensures that the $\|J_n\|$ will be uniformly bounded by a common constant. Combining this with the fact that the two characteristic tensors A and T have bounded norm we get that $f_n := \|J_n\|^2$, f_n', and f_n'' are three sequences of equibounded equicontinuous functions. By Ascoli–Arzela and general results on uniform convergence, we may assume that there is a subsequence of the f_n converging to some function f with their first and second derivatives converging to the first and second derivatives of f. But the condition

on the vertizontal curvatures implies that f is a bounded convex function, and therefore constant: Just observe that $f_n'' \to f''$, where

$$f_n'' = \|J_n'\|^2 - K(\alpha_n, J_n) f_n^2$$

so that $f'' \geq 0$ by the hypothesis on the curvatures. Thus $f_n' = \|A_{\alpha_n'} J_n\|^2 + \|T_{J_n}\alpha_n'\|^2 \to 0$ forcing $|A| \to 0$, and by the same argument as in (1), $A = 0$ everywhere. □

The first statement in the above theorem is related to one of the main results in [6]; the second one is a variation of the central theorem in [10]: that paper proves that the soul is flat (and consequently the local splitting of M at S) without any restrictions on $\sup_{\mathcal{S}} C_\alpha$, but assuming however that all the curvatures tend to zero at infinity.

We conclude this section with our second application of the restrictions found previously on Jacobi fields and holonomy diffeomorphisms.

Theorem 4.4. *Suppose that one of the Sharafutdinov fibers $\pi^{-1}(p)$ is flat, or more generally, suppose that every vertical Jacobi field J along a radial geodesic γ in $\pi^{-1}(p)$ vanishing at the soul has unbounded norm. Then M splits locally isometrically over S.*

Proof. By Proposition 1.1, the derivatives of holonomy diffeomorphisms map Jacobi fields along radial geodesics with $J(0) = 0$ to Jacobi fields of the same type. Since any such vector field can be obtained from those tangent to the fiber over p, Lemma 4.1 shows that every Jacobi field of this type must have unbounded norm. Thus, for any x, y tangent to S at some point, $A_X Y$ must be zero, since otherwise it would provide a vertical Jacobi field along a radial geodesic with bounded norm. Once again, the main result in [17] yields the statement. □

5. The cut and conjugate loci of the soul

The Jacobi equation along any geodesic normal to the soul splits because of Perelman's theorem. It is therefore not surprising that the structure of the conjugate and the cut loci of the soul are related to the Sharafutdinov projection in a nontrivial way. For example, in [18] it was shown that the point in the cut locus closest to the soul must be a focal point. A focal point p must in turn be a conjugate point for $\pi(p)$, and any Jacobi field that vanishes at both p and $\pi(p)$ is tangent to the fiber for all time; see [9]. Furthermore, the focal set of the soul is invariant under holonomy diffeomorphisms. It follows in fact from Proposition 1.1 that any vertical v which is orthogonal to the image of $d_u \exp$ for some $u \in \nu(S)$ remains vertical under parallel translation along horizontal geodesics, and thus generates flat, totally geodesic rectangles along them. In particular, the geodesic in direction v lies in the fiber of π. In this section, we present some further illustrations of this relation.

Denote by G the normal holonomy group of the soul. G acts in the normal space by parallel translation along closed curves contained in S. For any $u \in \nu(S)$, let G_u be the connected component of the isotropy group at u for this action, and γ_u the geodesic $t \mapsto \exp(tu)$. The following result will play a key role in the proof of Theorem 2.12.

Theorem 5.1. *If for some $u \in \nu(S)$, G_u acts without fixed points on $u^\perp$, then γ_u is an outgoing Sharafutdinov line. Furthermore, if this action is actually irreducible* *and γ_u has a conjugate point, then u extends (globally if S is simply connected) to a parallel section of $\nu(S)$, which exponentiates to Sharafutdinov lines.*

Proof. For the first part, it clearly suffices to show that the Sharafutdinov vector field is tangent to the geodesic at any nonconjugate point of γ_u. So consider such a point t_0, and let $q = \exp(t_0 u)$ be such that $d\exp_{t_0 u} : \nu_p(S) \to \mathcal{V}_q$ is an isomorphism, where $p = \gamma_u(0)$. This isomorphism induces via horizontal lifts an action of G_u on $\mathcal{V}_q$ which is clearly without fixed points on $\gamma_u'(t_0)^\perp \cap \mathcal{V}_q$. By Theorem 3.2, the Sharafutdinov vector field is invariant under this action, and must therefore be tangent to γ_u.

In the case that this action is irreducible and t_0 is a conjugate point of γ_u, it follows from Proposition 1.1 that the kernel of $d\exp_{t_0 u}$ is all of $u^\perp$. The A–tensor, which is given by radial Jacobi fields, must then be 0 at $q = \exp(t_0 u)$. Since γ_u is a Sharafutdinov line, A vanishes identically along γ_u by Proposition 3.4 (1). In particular, $R^\nabla(\cdot,\cdot)u \equiv 0$. Next, observe that the isotropy group of the parallel translate $P_\alpha u$ of u along any path α in S also acts irreducibly on the orthogonal component. Moreover, the geodesic in direction $P_\alpha u$ has t_0 as a conjugate point, so that $R^\nabla(\cdot,\cdot)P_\alpha u = 0$ for any path α in S by the above argument. But the Ambrose–Singer theorem [15] implies that the tangent space to the u–orbit of the holonomy group is generated by the collection of all $P_\alpha^{-1} \circ R^\nabla(\cdot,\cdot) \circ P_\alpha u$. If S is simply connected, this orbit is trivial. This concludes the proof. ☐

We now have the necessary tools to show that the metric projection $\pi : M \to S$ is C^∞ when S has codimension ≤ 3:

Proof of Theorem 2.10. The connected component of the normal holonomy group acts either trivially, transitively, or as an S^1-rotation of the normal unit sphere. In the first case, M splits locally isometrically over S by [17], and in the second case, the exponential from $\nu(S)$ to M is a diffeomorphism since rays are preserved under parallel translation. Moreover, in these two cases it is clear that the metric projection π is smooth. In the last case, part (3) of the theorem follows from Theorem 5.1. We now proceed to prove smoothness when M contains a totally geodesic embedded line $S \times \mathbb{R}$ of pseudosouls.

We begin by showing that if $A = 0$ at some point q outside $S \times \mathbb{R}$, then $A = 0$ on the fiber through q. To see this, recall that $A = 0$ along the Sharafutdinov line through q, so that we obtain a vector v in the normal space of some (pseudo)soul, transversal to $S \times \mathbb{R}$, with $R^\nabla(\cdot,\cdot)v = 0$. Since R^∇ is skew–symmetric and the fiber is 3–dimensional, $R^\nabla \equiv 0$ at that point. Therefore the A tensor is identically 0 along the fiber through q for the submersion onto that pseudosoul. But the fibers of the submersions onto different pseudosouls coincide, whence the claim.

Now consider a fiber F where A is not everywhere (and hence nowhere) 0. Such a fiber exists since M does not split. It suffices to establish its smoothness, since any other fiber can be realized as $h^\gamma(F)$ for some holonomy transformation h^γ; cf. [1].

We will produce for any $q \in F = \pi^{-1}(p)$ a smooth parametrization $\varphi : I \times D^2 \to F$ of the fiber with $\varphi(0,0,0) = q$, where I is an open interval around 0, and D^2 an open disk in $\mathbb{R}^2$. Let $q = \exp(u)$. If u is invariant under the holonomy, then q is not a focal point of a pseudosoul $S \times \{t\}$ for some t, and smoothness follows from Proposition 2.3. Otherwise, consider $c(t) = \exp_p(e^{it} \cdot u)$, where e^{it} denotes rotation by angle t around the axis in the normal space which is fixed by the holonomy. By the paragraph after Proposition 1.1, $c' = A_X Y \neq 0$ for some X, Y and therefore c is regular. Next, let $\mathcal{W}(t)$ be the subspace along c determined by the orthogonal decomposition $T_c M = \mathcal{H}_c \oplus \mathbb{R} \cdot c' \oplus \mathcal{W}$. We claim that $\mathcal{W}$ is smooth along c: To see this, it suffices to check smoothness of $\mathcal{H}$ along c: Just observe that Perelman's result establishes that horizontal lifts to $c(t)$ of elements of $T_p S$ are images through $d\exp$ of horizontal lifts in $\nu(S)$ with the normal connection to points of $e^{it} \cdot u$. Hence $\mathcal{H}_c$ has a smooth basis along c.

Our next observation is that $\mathcal{W}(t)$ is generated by the tangent vectors to minimal geodesics from $c(t)$ to $S \times \mathbb{R}$, since for any such geodesic γ, $A\gamma' = 0$. Thus, $\exp \mathcal{W}(t) \subset F$ by Proposition 2.3 and Perelman's Theorem. Finally, choose a smooth orthonormal basis $V_1(t), V_2(t)$ of $\mathcal{W}(t)$, and define a map $\varphi : I \times D^2 \to F$ by

$$\varphi(t, a, b) := \exp_{c(t)}(aV_1(t) + bV_2(t)).$$

This is clearly a local parametrization of F at q. □

Interestingly, when the holonomy is S^1, the image of the A-tensor generates a one-dimensional Riemannian foliation tangent to the fibers, which is singular along the line $S \times \mathbb{R}$ of pseudosouls. This follows from the proof of the above theorem by observing that the orthogonal complement of A is a totally geodesic distribution. It implies a certain degree of symmetry in the fiber, since one-dimensional Riemannian foliations can be viewed as 'weak' versions of Killing fields. Notice that we do not claim that the normal exponential map from the soul is a diffeomorphism, even in the case of nontrivial holonomy, cf. example 2.13. This contrasts significantly with the lower codimension cases.

The question of whether the metric projection π is smooth in arbitrary codimensions still remains to be answered. Perelman's Theorem implies that this is the case at points outside the focal set of S. Our next goal is to show that there is still smoothness on a large subset of focal points. In order to do this, we need to recall some terminology and results from [11] and [22].

A *ray* in $\nu(S)$ is the set of positive multiples of a fixed vector $w \in \nu(S)$. Recall from the remarks at the beginning of this section that focal points of the normal bundle to the soul are conjugate points along geodesics orthogonal to S. Suppose $\gamma_w(1)$ is conjugate to $\gamma_w(0)$ along γ_w, where γ_w is the geodesic with initial tangent vector w. w is said to be a *regular focal point* if there exists a neighborhood U of w such that for every ray r meeting U, there is at most one focal point in $r \cap U$. Otherwise, we will call w a *singular focal point*.

According to [11] and [22], the set formed by such points is open and dense in the focal set in $\nu(S)$. We will denote it by $\mathcal{R}$.

Proposition 5.2. *π is smooth at points in* $\exp(\mathcal{R})$.

Proof. Suppose p is the image by the exponential of some $w \in \mathcal{R}$. By the arguments in [22], combined with the results mentioned at the beginning of this section, it is possible to find coordinate systems $(W, \phi = (x_1, \ldots, x_s, x_{s+1}, \ldots, x_{s+l}))$ and $(W', \psi = \exp \circ \phi)$ around w and p respectively such that

(1) $(x_1, \ldots, x_s)$ is the lift of some normal coordinate chart around $\pi(p)$ with x_1 corresponding to the radial coordinate, and
(2) $(x_{s+1}, \ldots, x_{s+l})$ are invariant under holonomy diffeomorphisms for radial geodesics starting at $\bar{p}$.

By looking at the possible alternatives for the expression of exp in these coordinates described in [22], we can produce a smooth submanifold P through p which intersects each fiber of π smoothly, which is invariant under holonomy diffeomorphisms associated to short radial geodesics from $\pi(p)$, and such that directions normal to the image of $d\exp$ also vary smoothly over P. Let $\mathcal{C}$ denote the collection of such directions, and define a map $\varphi : [P \cap F_p] \times \mathcal{C} \to F_p$ by $\varphi(q, v) = \exp_q(v)$.

By the remarks at the beginning of this section, the image of φ is indeed entirely contained in F_p. Moreover, an easy dimensionality argument shows that $d\varphi$ is nonsingular at $(p, 0)$ and hence a local embedding at that point. This implies that the fiber is locally a smooth manifold, and therefore π is smooth at p. $\square$

References

1. V. Berestovskii, *Submetries of Space Forms of Negative Curvature*, Siberian Math. J. **28** (1987), 552–562.
2. J. Cheeger, *Some examples of manifolds of nonnegative curvature*, J. Differential Geometry **8** (1973), 623–628. MR **49:**6085
3. J. Cheeger and D. Gromoll, *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. **96** (1972), 413–443. MR **46:**8121
4. C. Croke and H. Karcher, *Volume of Small Balls on Open Manifolds: Lower Bounds and Examples*, Trans. Amer. Math. Soc. **309** (1988), 753–762. MR **89m:**53060
5. R.E. Greene and H. Wu, *Gap theorems for noncompact Riemannian manifolds*, Duke Math. J. **49** (1982), 731–756. MR **84m:**53050
6. ———, *Nonnegatively curved manifolds which are flat outside a compact set*, Proc. Symp. Pure Appl. Math. **54** (1993), 327–335. MR **94a:**53071
7. D. Gromoll and K. Grove, *The low-dimensional metric foliations of Euclidean spheres*, J. Differential Geometry **28** (1988), 143–156. MR **89g:**53052
8. L. Guijarro, *Improving the metric in an open manifold with nonnegative curvature*, Proc. Amer. Math. Soc. **126** (1998), 1541–1545. MR **98j:**53042
9. ———, *On the metric structure of open manifolds with nonnegative curvature*, preprint, 1997.
10. L. Guijarro and P. Petersen, *Rigidity in nonnegative curvature*, Ann. Sci. École Norm. Sup. **30** (1997), 595–603. MR **98i:**53047
11. J. J. Hebda, *The regular focal locus*, J. Differential Geometry **16** (1981), 421–429. MR **83j:**53049
12. B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966), 459–469. MR **34:**751
13. M. Ozaydin and G. Walschap, *Vector bundles with no soul*, Proc. Amer. Math. Soc. **120** (1994), 565–567. MR **94d:**53057
14. G. Perelman, *Proof of the soul conjecture of Cheeger and Gromoll*, J. Differential Geom. **40** (1994), 209–212. MR **95d:**53037
15. W. Poor, *Differential Geometric Structures*, McGraw-Hill, 1981. MR **83k:**53002
16. V. A. Sharafutdinov, *The Pogorelov–Klingenberg theorem for manifolds homeomorphic to $\mathbb{R}^n$*, Siberian Math. J. **18** (1977), 915–925. MR **58:**7488
17. M. Strake, *A splitting theorem for open nonnegatively curved manifolds*, Manuscripta Math **61** (1988), 315–325. MR **89g:**53066
18. M. Strake and G. Walschap, *Σ–flat manifolds and Riemannian submersions*, Manuscripta Math. **64** (1989), 213–226. MR **90g:**53054
19. G. Walschap, *Nonnegatively curved manifolds with souls of codimension 2*, J. Differential Geometry **27** (1988), 525–537. MR **89g:**53067
20. ———, *The soul at infinity in dimension 4*, Proc. Amer. Math. Soc. **112** (1991), 563–567. MR **92c:**53021
21. ———, *Soul-preserving submersions*, Michigan Math. J. **41** (1994), 609–617. MR **95h:**53042
22. F. W. Warner, *The conjugate locus of a Riemannian manifold*, Amer. J. Math. **87** (1965), 575–604. MR **34:**8344
23. J. W. Yim, *Distance nonincreasing retraction on a complete open manifold of nonnegative sectional curvature*, Ann. Global Anal. Geom. **6** (1988), 191–206. MR **90a:**53049

24. ———, *Space of souls in a complete open manifold of nonnegative curvature*, J. Differential Geometry **32** (1990), 429–455. MR **91j:**53023

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