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RARIFIED SUMS OF THE THUE-MORSE SEQUENCE

MICHAEL DRMOTA AND MARIUSZ SKAŁBA

ABSTRACT. Let q be an odd number and $S_{q,0}(n)$ the difference between the number of $k < n$, $k \equiv 0 \bmod q$, with an even binary digit sum and the corresponding number of $k < n$, $k \equiv 0 \bmod q$, with an odd binary digit sum. A remarkable theorem of Newman says that $S_{3,0}(n) > 0$ for all n. In this paper it is proved that the same assertion holds if q is divisible by 3 or $q = 4^N + 1$. On the other hand, it is shown that the number of primes $q \le x$ with this property is $o(x/\log x)$. Finally, analoga for "higher parities" are provided.

1. INTRODUCTION

The Thue-Morse sequence [9], [5] is defined by

$$t_n = (-1)^{s(n)}, \tag{1}$$

where $s(n)$ denotes the number of ones in the binary representation of n. For any positive integer q and $i \in \mathbf{Z}$ we denote

$$S_{q,i}(n) = \sum_{0 \le j < n, j \equiv i (\mathrm{mod}\ q)} t_j. \tag{2}$$

In 1969 Newman [10] proved a remarkable conjecture of L. Moser saying that for any $n \ge 1$

$$S_{3,0}(n) > 0.$$

More precisely, he proved that

$$\frac{3^\alpha}{20} < \frac{S_{3,0}(n)}{n^\alpha} < 5 \cdot 3^\alpha \qquad \text{with } \alpha = \frac{\log 3}{\log 4}.$$

In 1983 Coquet [1] provided an explicit precise formula for $S_{3,0}(n)$ by the use of a continuous function $\psi_3(x)$ with period 1 which is nowhere differentiable ($\eta_3(n) \in \{-1, 0, 1\}$):

$$S_{3,0}(n) = n^{\frac{\log 3}{\log 4}} \cdot \psi_3\left(\frac{\log n}{\log 4}\right) - \frac{\eta_3(n)}{3}. \tag{3}$$

Furthermore, he was able to identify $\min \psi([0,1]) > 0$ and $\max \psi([0,1])$.

In general, (asymptotic) representations similar to (3) exist for any $S_{q,i}(n)$ (see [5] and section 2). But it is a non-trivial problem to decide whether the continuous function $\psi_{q,i}(x)$ has a zero or not. The only known examples where $\psi_q(x) = \psi_{q,0}(x)$ has no zero are $q = 3^k 5^l$ ([6]) and $q = 17$ ([7]). (Note that the assertion that $\psi_{q,i}(x)$

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has no zero is more or less equivalent to $S_{q,i}(n) > 0$ for almost all n or to $S_{q,i}(n) < 0$ for almost all n; see section 2.)

Our first result provides infinitely many new examples where $\psi_q(x)$ has no zero.

Theorem 1. *Suppose that q is divisible by 3 or $q = 4^N + 1$. Then $S_{q,0}(n) > 0$ for almost all n.*[1]

However, if q is prime then we can prove that there are only a few exceptions (e.g. Fermat primes). Let $\mathbf{P}_t$, $t \ge 1$, denote the set of those primes p where the order $\mathrm{ord}_p(2)$ of 2 in the multiplicative group $(\mathbf{Z}/p\mathbf{Z})^*$ equals $\mathrm{ord}_p(2) = (p-1)/t$.

Theorem 2. *There exists a constant $C > 0$ such that for any $t \ge 1$ the primes $p \in \mathbf{P}_t$ satisfying $S_{p,0}(n) > 0$ for almost all n are bounded by*

$$p \le Ct^2 \log^2 t.$$

Furthermore, the total number of primes $p \le x$ with $S_{p,0}(n) > 0$ for almost all n is $o(x/\log x)$ as $x \to \infty$.

The first part of Theorem 2 generalizes a result by the authors [2], where it is shown that 3 and 5 are the exceptional primes of $\mathbf{P}_1$ and 17 and possibly 41 those of $\mathbf{P}_2$. (In fact, $p = 41$ is not exceptional, see section 3.)

It is surely a very difficult problem to decide whether there are infinitely many primes p satisfying $S_{p,0}(n) > 0$ for almost all n or not. Unfortunately our methods are not strong enough to settle this problem. But it should be noted that if there were only finitely many primes with this property, Theorem 1 would imply that there were only finitely many Fermat primes.

However, the methods to be developed are essentially sufficient to decide this problem for any concrete value q. For example, we can prove the following theorem.

Theorem 3. *The only primes $p \le 1000$ satisfying $S_{p,0}(n) > 0$ for almost all n are $p = 3, 5, 17, 43, 257, 683$.*

Note that $p = 43 \in \mathbf{P}_3$ and $p = 683 \in \mathbf{P}_{31}$ are not Fermat primes.[2]

We will prove Theorems 1 and 2 in sections 4 and 5. The negative part of Theorem 3 is proved at the end of section 3 and the positive part at the end of section 4. Section 6 is devoted to the case of higher parities where similar phenomena appear. In section 2 we collect some basic facts on the fractal structure of $S_{q,i}(n)$, and in section 3 we discuss two different kinds of positivity phenomena.

2. Basic Facts

For any fixed positive integer q and $i \in \mathbf{Z}$, set

$$S_{q,i}(y,n) = \sum_{j<n,\, j\equiv i \bmod q} y^{s(j)}, \tag{4}$$

[1] The phrase "*almost all*" means "*all but finitely many*", i.e. there might be finitely many exceptions.

[2] Note that both 43 and 684 are of the form $(2^{2N+1}+1)/3$. Recently, by extending the methods of section 4, Leinfellner [8] showed that q of the form $(2^{2N+1}+1)/3$ have the property that $S_{q,0}(n) > 0$ for almost all n.

in which $n \geq 0$ and y is a (complex) parameter. With help of these expressions we can determine the numbers

$$A_{q,i;r,m}(n) = |\{j < n : \ j \equiv i \bmod q, \ s(j) \equiv m \bmod r\}| \tag{5}$$

$$= \frac{1}{r} \sum_{l=0}^{r-1} \zeta_r^{-ml} S_{q,i}(\zeta_r^l, n), \tag{6}$$

where r is a positive integer (which will be called a *parity*), $m \in \mathbf{Z}$, and ζ_r denotes the r-th primitive root of unity, $\zeta_r = \exp\left(\frac{2\pi i}{r}\right)$.

Note that $S_{q,i}(y,n)$, $0 \leq i < q$, satisfies a simple generating relation if n is a power of 2:

$$\sum_{i=0}^{q-1} S_{q,i}(y, 2^k)\zeta_q^{li} = \prod_{j=0}^{k-1} \left(1 + y\zeta_q^{l2^j}\right), \tag{7}$$

in which $\zeta_q = \exp\left(\frac{2\pi i}{q}\right)$ denotes the q-th primitive root of unity and $l \in \mathbf{Z}$. Hence we directly obtain

$$S_{q,i}(y, 2^k) = \frac{1}{q} \sum_{l=0}^{q-1} \zeta_q^{-li} \prod_{j=0}^{k-1} \left(1 + y\zeta_q^{l2^j}\right). \tag{8}$$

Moreover, the obvious relation

$$S_{q,i}(y, 2^k + n') = S_{q,i}(y, 2^k) + yS_{q,i-2^k}(y, n') \quad (n' < 2^k) \tag{9}$$

can be used to calculate $S_{q,i}(n)$ inductively for any integer $n \geq 0$.

We will further need

$$S(y,n) = \sum_{j<n} y^{s(j)} = \sum_{i=0}^{q-1} S_{q,i}(y,n) \tag{10}$$

and the numbers

$$A_{r,m}(n) = |\{j < n : \ s(j) \equiv m \bmod r\}| \tag{11}$$

$$= \frac{1}{r} \sum_{l=0}^{r-1} \zeta_r^{-ml} S(\zeta_r^l, n). \tag{12}$$

$S(y, 2^k)$ is given by

$$S(y, 2^k) = (1+y)^k \tag{13}$$

and satisfies

$$S(y, 2^k + n') = S(y, 2^k) + yS(y, n') \quad (n' < 2^k).$$

Our first aim is to describe the asymptotic behaviour of $A_{q,i;r,m}(n)$. The natural leading term is $\frac{1}{q}A_{r,m}(n)$:

$$A_{q,i;r,m}(n) = \frac{1}{q}A_{r,m}(n) + R_{q,i;r,m}(n). \tag{14}$$

From (6), (8), (12), and (13) we obtain the representations

$$A_{q,i;r,m}(2^k) = \frac{1}{rq} \sum_{l_1=0}^{r-1} \zeta_r^{-l_1 m} \sum_{l_2=0}^{q-1} \zeta_q^{-l_2 i} \prod_{j=0}^{k-1} \left(1 + \zeta_r^{l_1} \zeta_q^{l_2 2^j}\right) \tag{15}$$

and

$$A_{r,m}(2^k) = \frac{1}{r}\sum_{l_1=0}^{r-1} \zeta_r^{-l_1 m}\left(1+\zeta_r^{l_1}\right)^k, \tag{16}$$

so that

$$R_{q,i;r,m}(2^k) = \frac{1}{rq}\sum_{l_1=0}^{r-1} \zeta_r^{-l_1 m}\sum_{l_2=1}^{q-1} \zeta_q^{-l_2 i}\prod_{j=0}^{k-1}\left(1+\zeta_r^{l_1}\zeta_q^{l_2 2^j}\right). \tag{17}$$

These Fourier expansions will be frequently used in the proofs of our main results.

From now on let q be an odd positive integer and let $s = \mathrm{ord}_q(2)$ be the order of the multiplicative subgroup $\langle 2\rangle$ of $(\mathbf{Z}/q\mathbf{Z})^*$. (Since we are mainly interested in $A_{q,0,r,m}(n)$, it is no real restriction to assume that q is odd.) Furthermore, let

$$\mathbf{S}_q(y,n) = (S_{q,0}(y,n),\ldots,S_{q,q-1}(y,n))^t$$

denote the vector of $S_{q,i}(y,n)$. Let $\mathbf{e}_0,\ldots,\mathbf{e}_{q-1}$ denote the canonical basis of the q-dimensional vector space $\mathbf{C}^q$ and let $\mathbf{T}$ denote the matrix defined by $\mathbf{T}\mathbf{e}_i = \mathbf{e}_{i+1}$ ($\mathbf{e}_q = \mathbf{e}_0$). The identity matrix is denoted by $\mathbf{I}$.

The following observations are more or less direct generalizations of [5].

Proposition 1. *Let* $\mathbf{M}(y)$ *be defined by*

$$\mathbf{M}(y) = \prod_{m=0}^{s-1}\left(\mathbf{I} + y\mathbf{T}^{2^m}\right). \tag{18}$$

Then

$$\mathbf{S}_q(y,2^s n) = \mathbf{M}(y)\mathbf{S}_q(y,n). \tag{19}$$

Proof. By using the relations $s(2j)=s(j)$ and $s(2j+1)=s(j)+1$ we obtain

$$\begin{aligned} S_{q,i}(y,2n) &= \sum_{j<2n,\, j\equiv i \bmod q} y^{s(j)} \\ &= \sum_{2j<2n,\, 2j\equiv i \bmod q} y^{s(2j)} + \sum_{2j+1<2n,\, 2j+1\equiv i \bmod q} y^{s(2j+1)} \\ &= \sum_{j<n,\, j\equiv 2^{-1} i \bmod q} y^{s(j)} + y\sum_{j<n,\, j\equiv 2^{-1}(i-1) \bmod q} y^{s(j)} \\ &= S_{q,2^{-1}i}(y,n) + yS_{q,2^{-1}(i-1)}(y,n). \end{aligned}$$

Hence, denoting by $\mathbf{U}$ the matrix defined by $\mathbf{U}\mathbf{e}_i = \mathbf{e}_{2i}$, we have

$$\mathbf{S}_q(y,2n) = (\mathbf{U} + y\mathbf{U}\mathbf{T})\mathbf{S}_q(y,n).$$

By using the property $\mathbf{U}\mathbf{T} = \mathbf{T}^2\mathbf{U}$ it follows by induction that

$$(\mathbf{U}+y\mathbf{U}\mathbf{T})^i = \left(\prod_{m=1}^{i}\left(\mathbf{I}+y\mathbf{T}^{2^m}\right)\right)\mathbf{U}^i.$$

Since $\mathbf{T}^q = \mathbf{U}^s = \mathbf{I}$, we directly obtain (19) by setting $i=s$. □

The eigenvalues of $\mathbf{T}$ are exactly the q-th roots of unity ζ_q^l, $0 \le l < q$, with corresponding eigenvectors $\mathbf{v}_l = \sum_{i=0}^{q-1} \zeta_q^{-il}\mathbf{e}_i$ which are orthogonal. Since $\mathbf{M}(y)$ is a polynomial in $\mathbf{T}$, the eigenvalues of $\mathbf{M}(y)$ are given by

$$\lambda_l(y) = \prod_{m=0}^{s-1}\left(1 + y\zeta_q^{l2^m}\right) \tag{20}$$

It is clear that $\lambda_l(y) = \lambda_{l'}(y)$ if and only if $l'\langle 2\rangle = l\langle 2\rangle$. (Observe that $\mathbf{l} = l\langle 2\rangle$ contains $\mathrm{ord}_{(q/(q,l))}(2)$ elements, where (q,l) denotes the greatest common divisor of q and l.) Appropriately we will write $\lambda_{\mathbf{l}}(y)$ instead of $\lambda_l(y)$ if $l \in \mathbf{l}$. Let L denote the system of equivalence classes $\mathbf{l} = l\langle 2\rangle$. Then a basis of the eigenspace $V_{\mathbf{l}}$ corresponding to $\lambda_{\mathbf{l}}(y)$, $\mathbf{l} \in L$, is given by $\mathbf{v}_l$, $l \in \mathbf{l}$. All these eigenspaces are orthogonal. $\mathbf{P}_{\mathbf{l}}$, $\mathbf{l} \in L$, will denote the orthogonal projection on $V_{\mathbf{l}}$. Furthermore, let $V^{(0)}$ denote the eigenspace corresponding to the eigenvalue 0 (if 0 is an eigenvalue), $V^{(s)}$ the subspace corresponding to eigenvalues of modulus < 1, $V^{(1)}$ the subspace corresponding to those of modulus 1, $V^{(u)}$ corresponding to those with modulus > 1, and $V^{(m)}$ that corresponding to those eigenvalues with maximal modulus. Furthermore, let $\mathbf{P}^{(0)}$, $\mathbf{P}^{(s)}$, $\mathbf{P}^{(1)}$, $\mathbf{P}^{(u)}$, and $\mathbf{P}^{(m)}$ denote the orthogonal projections on $V^{(0)}$, $V^{(s)}$, $V^{(1)}$, $V^{(u)}$, and $V^{(m)}$, respectively.

Using these notations and the same methods as in [5], we immediately obtain a fractal representation for $\mathbf{S}_q(y,n)$.

Proposition 2. *There exists a contiuous function* $\mathbf{F}(y,\cdot) : \mathbf{R}^+ \to V^{(u)}$ *satisfying*

$$\mathbf{F}(y, 2^s x) = \mathbf{M}(y)\mathbf{F}(y,x) \quad (x > 0)$$

and $\mathbf{P}_u\mathbf{S}_q(y,n) = \mathbf{F}(y,n)$. *Consequently*

$$\mathbf{S}_q(y,n) = \mathbf{F}(y,n) + \begin{cases} \mathcal{O}(1) & \text{if } V^{(1)} = \{0\}, \\ \mathcal{O}(\log n) & \text{if } V^{(1)} \ne \{0\}, \end{cases}$$

Let $|\lambda_{\mathbf{l}}(y)| > 1$. Then $\mathbf{G}_{\mathbf{l}}(y,t) = \lambda_{\mathbf{l}}(y)^{-t}\mathbf{P}_{\mathbf{l}}\mathbf{F}(y, 2^{st})$ is a continuous function $\mathbf{G}_{\mathbf{l}}(y,\cdot) : \mathbf{R} \to V_{\mathbf{l}}$ which satisfies $\mathbf{G}_{\mathbf{l}}(y,t+1) = \mathbf{G}_{\mathbf{l}}(y,t)$. With $\alpha_{\mathbf{l}}(y) = (\log\lambda_{\mathbf{l}}(y))/(s\log 2)$ we finally obtain a fractal representation for $\mathbf{S}_q(y,n)$:

$$\mathbf{S}_q(y,n) = \sum_{|\lambda_{\mathbf{l}}(y)|>1} n^{\alpha_{\mathbf{l}}(y)}\mathbf{G}_{\mathbf{l}}\left(y, \frac{\log n}{s\log 2}\right) + \mathcal{O}(\log n). \tag{21}$$

We want to mention also that it is quite easy to evaluate $\mathbf{G}_{\mathbf{l}}(y,t)$ for special values of t by using the representation (8):

$$\begin{aligned} S_{q,i}(y, 2^k) &= \frac{1}{q}\sum_{l=0}^{q-1}\zeta_q^{-li}\lambda_l(y)^a\prod_{j=0}^{b-1}\left(1 + y\zeta_q^{l2^j}\right) \\ &= \frac{1}{q}\sum_{\mathbf{l}\in L}(2^k)^{\alpha_{\mathbf{l}}(y)}\lambda_{\mathbf{l}}^{-b/s}\sum_{l\in\mathbf{l}}\zeta_q^{-il}\prod_{j=0}^{b-1}\left(1 + y\zeta_q^{l2^j}\right), \end{aligned} \tag{22}$$

where $k = as + b$, $0 \le b < s$. In particular, the first component of $\mathbf{G}_{\mathbf{l}}(y,0)$ is non-zero.

Sometimes it would be more convenient to operate with real exponents instead of in general complex exponents $\alpha_{\mathbf{l}}(y)$. For example, if $\lambda_{\mathbf{l}}(y)^{r'}$ is real and positive for some positive integer r', then we can use $\tilde{\mathbf{G}}_{\mathbf{l}}(y,t) = \lambda_{\mathbf{l}}(y)^{-r't}\mathbf{P}_{\mathbf{l}}\mathbf{F}(y, 2^{r'st})$ instead of $\mathbf{G}_{\mathbf{l}}(y,t)$ and $\tilde{\alpha}_{\mathbf{l}}(y) = \Re(\alpha_{\mathbf{l}}(y))$ instead of $\alpha_{\mathbf{l}}(y)$. (Compare with [5].)

For the evaluation of $A_{q,i;r,m}(n)$ we will need $\mathbf{S}_q(\zeta_r^m, n)$, $0 \le m < r$. It is an easy exercise to show that $\arg(\lambda_{\mathbf{l}}(\zeta_r^m)) = sm\pi/r + m'\pi$ for some $m' \in \mathbf{Z}$. Thus $\lambda_{\mathbf{l}}(\zeta_r^m)^r$ is real and $\lambda_{\mathbf{l}}(\zeta_r^m)^{2r} > 0$. Hence it is always possible to operate with positive exponents.

Finally, observe that $S(y,n)$ can be treated in a similar fashion as above but much more easily. Using the relation $S(y,2n) = (1+y)S(y,n)$, it follows that there is a continuous function $F(y,x)$ satisfying $F(y,2x) = (1+y)F(y,x)$ in the case $|1+y| > 1$ such that

$$S(y,n) = F(y,n) = n^{\alpha} G\left(y, \frac{\log n}{\log 2}\right),$$

where $\alpha(y) = \log(1+y)/\log 2$ and $G(y,t) = (1+y)^{-t}F(2^t)$. Furthermore, $S(y,n) = \mathcal{O}(1)$ if $|1+y| < 1$ and $S(y,n) = \mathcal{O}(\log n)$ if $|1+y| = 1$.

Now the fractal representations for $A_{r,m}(n)$ and $R_{q,i;r,m}(n)$ follow immediately.

Theorem 4. *Let q, r be positive integers such that q is odd and $r \ge 2$. Set*

$$\begin{aligned}\alpha_r &= \frac{\log\left(2\cos\frac{\pi}{r}\right)}{\log 2} \quad (r > 2),\\ \alpha_{q,r} &= \max_{0<m<r,\,0<l<q} \frac{\log|\lambda_l(\zeta_r^m)|}{s\log 2}.\end{aligned}$$

Furthermore, let r' be the least positive integer such that $\lambda_l(\zeta_r^m)^{r'} > 0$ for those $\lambda_l(\zeta_r^m)$, $0 < l < q$, $0 < m < r$, with largest modulus.

Then there exist real valued periodic continuous functions $\psi_{r,m}(x)$, $\psi_{q,i;r,m}(x)$, $0 \le m < r$, $0 \le i < q$, with period 1 such that

$$\begin{aligned}A_{r,m}(n) &= \frac{n}{r} + \begin{cases} (-1)^m\eta_n/2 & (\textit{if } r = 2),\\ n^{\alpha_r}\cdot\psi_{r,m}\left(\frac{\log n}{2r\log 2}\right) + \mathcal{O}(n^{\beta_r}) & (\textit{if } r > 2),\end{cases}\\ R_{q,i;r,m}(n) &= n^{\alpha_{q,r}}\cdot\psi_{q,i;r,m}\left(\frac{\log n}{2r's\log 2}\right) + \mathcal{O}(n^{\beta_{q,r}}),\end{aligned}$$

where $\beta_r < \alpha_r$, $\beta_{q,r} < \alpha_{q,r}$, and $\eta_n = 0$ if $n \equiv 0 \bmod 2$ and $\eta_n = t_n$ if $n \equiv 1 \bmod 2$.

Proof. Since $A_{r,m}(n)$ is given by (12) and $A_{q,i;r,m}$ by (6) (compare also with (16) and (17)), it follows that the asymptotic leading term of $A_{r,m}(n) - n/r$ depends on the largest eigenvalue $\lambda_0(\zeta_r^m) = (1+\zeta_r^m)^s$, $0 < m < r$, and the asymptotic leading term of $R_{q,i;r,m}(n)$ on the largest eigenvalue $\lambda_l(\zeta_r^m)$, $0 < l < q$, $0 \le m < r$.

Since $|1+\zeta_r^m| = 2|\cos(\frac{m\pi}{r})|$ is maximal for $m = 1$, we immediately obtain the asymptotic expansion for $A_{r,m}(n)$. (Note that $\beta_r = \log(2\cos\frac{2\pi}{r})/\log 2$.)

Furthermore, since $\lambda_l(1) = 1 + \zeta_q^l + \zeta_q^{2l} + \cdots + \zeta_q^{(2^s-1)l} = 0$ for $0 < l < q$, it is clear that $\alpha_{q,r}$ is the correct exponent in the asymptotic leading term of $R_{q,i;r,m}(n)$.

Finally, $A_{2,m}(n)$ can be directly evaluated. □

Remark. In this paper we will only discuss binary digits. But the above concept easily applies for arbitrary b-ary digit expansions. Let $s(j)$ be a sequence satisfying $s(bn+c) = s(n) + s(c)$ for $n \ge 0$ and $0 \le c < b$. Let $\mathbf{S}_q(y,n)$ be defined as above and assume that b and q are relatively prime. Then

$$\mathbf{S}_q(y,bn) = \mathbf{U}_b \left(\sum_{c=0}^{b-1} y^{s(c)} \mathbf{T}^c \right) \mathbf{S}_q(y,n),$$

where $\mathbf{U}_b \mathbf{e}_i = \mathbf{e}_{bi}$, $0 \le i < q$, and $s = \operatorname{ord}_q(b)$. Hence $\mathbf{S}_q(y, b^s n) = \mathbf{M}_b(y)\mathbf{S}_q(y,n)$, where

$$\mathbf{M}_b(y) = \prod_{m=0}^{b-1} \left(\sum_{c=0}^{b-1} y^{s(c)} \mathbf{T}^{cb^m} \right),$$

and we are in the same position as above. All eigenvalues and eigenvectors of $\mathbf{M}_b(y)$ are known, and we immediately obtain a fractal representation for $\mathbf{S}_q(y,n)$. (In [5] only the case $b = r$ is mentioned.)

3. Newman-like Phenomena

We want to discuss two kinds of positivity pheonmena:

(**N1**) $\quad A_{q,0;r,0}(n) > \max_{0<m<r} A_{q,0;r,m}(n)$ for almost all $n \ge 0$,

(**N2**) $\quad R_{q,0;r,0}(n) > 0$ for almost all n ≥ 0.

Newman's theorem $S_{3,0}(n) > 0$ $(n \ge 0)$ is precisely the same as

$$A_{3,0;2,0}(n) > A_{3,0;2,1}(n).$$

Therefore (**N1**) is a natural generalization of this property. Recall that $R_{q,0;r,m}(n)$ is the remainder term of $A_{q,0;r,m}(n)$ if $\frac{1}{q}A_{r,m}(n)$ is considered as the "natural" leading term of $A_{q,0;r,m}(n)$ (see section 2). Hence, (**N2**) means that the remainder term $R_{q,0;r,0}(n)$ is positive (for almost all n). We will now show that (**N1**) implies (**N2**) if $\alpha_r \ne \alpha_{q,r}$.

The following lemma provides a necessary condition for (**N1**).

Lemma 1. *If* (**N1**) *holds then* $\alpha_r \le \alpha_{q,r}$.

Proof. Suppose that $\alpha_r > \alpha_{q,r}$. In this case (see Theorem 4) the asymptotic behaviour of $A_{q,0;r,m}(n)$ is determined by $A_{r,m}(n)$. However, we will show that $A_{r,0}(2^{(2a+1)r}) < A_{r,m}(2^{(2a+1)r})$ for all $m \not\equiv 0 \bmod r$ and sufficiently large a. Therefore (**N1**) cannot occur.

Combining (13) and Theorem 4, we obtain

$$A_{r,m}(2^k) - \frac{2^k}{r} \sim 2\Re\left(\zeta_r^{-m}(1+\zeta_r)^k\right).$$

Since $(1+\zeta_r)^r$ is real and negative, everything follows. □

Hence, if $\alpha_r \ne \alpha_{q,r}$ then (**N1**) implies

$$R_{q,0;r,0}(n) > \max_{0<m<r} R_{q,0;r,m}(n) \quad \text{for almost all } n \ge 0. \tag{23}$$

Finally, (23) always implies (**N2**). This follows from the following property.

Lemma 2.

$$\sum_{m=0}^{r-1} R_{q,i;r,m}(n) = \mathcal{O}(\log n) \tag{24}$$

for all $i = 0, \ldots, q-1$.

Proof. From (17) we get

$$\sum_{m=0}^{r-1} R_{q,i;r,m}(2^k) = \frac{1}{q}\sum_{l=1}^{q-1} \zeta_q^{-li} \prod_{j=0}^{k-1}\left(1+\zeta_q^{l2^j}\right).$$

This means that the asymptotic behaviour of this sum is determined by the eigenvalues $\lambda_l(1)$, which are given by

$$\lambda_l(1) = \prod_{j=0}^{s-1}\left(1+\zeta_q^{l2^j}\right) = 1.$$

Hence (24) follows. □

Note that there are situations where (**N2**) holds although (**N1**) fails; see Theorem 8. However, in the "classical" case $r = 2$ it is easy to verify that (**N1**) and (**N2**) are equivalent to $S_{q,0}(-1, n) > 0$ (for almost all n).

Before we prove further necessary conditions for (**N1**) and (**N2**), we want to mention that "converse" phenomena of the form $A_{q,0;r,0}(n) < \min_{0<m<r} A_{q,0;r,m}(n)$ or $R_{q,0;r,0}(n) < 0$ for almost all $n \geq 0$ do not exist.

Lemma 3. *There exist infinitely many $n \geq 0$ such that*

$$A_{q,0;r,0}(n) > \max_{0<m<r} A_{q,0;r,m}(n) \tag{25}$$

and

$$R_{q,0;r,0}(n) > 0. \tag{26}$$

Proof. Let $s = \mathrm{ord}_q(2)$ and let $n = 2^{2rs\cdot a}$ for some $a \geq 0$. Then $\lambda_{\mathbf{l}}(\zeta_r^l)^{2ra} > 0$ for all $\mathbf{l} \in L$ and $l = 0, \ldots, q-1$. Hence (25) and (26) follow from

$$\begin{aligned} A_{q,0;r,m}(n) &= \frac{1}{q}A_{r,m}(n) + R_{q,0;r,m}(n) \\ &= \frac{1}{rq}\sum_{l=0}^{r-1}\cos\left(2\pi\frac{lm}{r}\right)\lambda_0(\zeta_r^l)^{2ra} + \frac{1}{rq}\sum_{l_1=0}^{r-1}\cos\left(2\pi\frac{l_1 m}{r}\right)\sum_{\mathbf{0}\neq\mathbf{l}\in L}|\mathbf{l}|\lambda_{\mathbf{l}}(\zeta_r^{l_1})^{2ra}. \end{aligned}$$ □

Theorem 5. *Let q, r be positive integers such that q is odd and $r \geq 2$. If $s = \mathrm{ord}_q(2)$ and r are coprime or if there exists an integer $r' > 0$ such that $\lambda_l(\zeta_r^m)^{r'} < 0$ for those $\lambda_l(\zeta_r^m)$, $0 < l < q$, $0 < m < r$, with maximal modulus, then* (**N1**) *and* (**N2**) *fail.*

Proof. We only prove that (**N2**) fails. Since $\lambda_0(\zeta_r)^r < 0$, the following proof can be extended to contradict (**N1**).

Let L_m denote the set of pairs $(\mathbf{l}, m)$, $\mathbf{l} \in L$, $0 < m < r$, such that the eigenvalues $\lambda_{\mathbf{l}}(\zeta_r^m)$ have maximal modulus ρ. Then the asymptotic leading term of $R_{q,0;m,0}(n)$ only depends on these eigenvalues. In particular, we have

$$R_{q,0;m,0}(2^{ks}) \sim \frac{1}{rq}\sum_{(\mathbf{l},m)\in L_m}|\mathbf{l}|\lambda_{\mathbf{l}}(\zeta_r^m)^k.$$

If there exists an integer $r' > 0$ such that $\lambda_{\mathbf{l}}(\zeta_r^m)^{r'} < 0$ for $(\mathbf{l}, m) \in L_m$, then $R_{q,0;m,0}(2^{a2rs+r's}) < 0$ for all $a \geq 0$.

Now suppose that r and s are coprime. Since $\arg(\lambda_{\mathbf{l}}(\zeta_r^m)) = ms\pi/r + \eta\pi$, where $\eta \in \{0,1\}$, any eigenvalue $\lambda_{\mathbf{l}}(\zeta_r^m)$ is not real. Set $\eta_{\mathbf{l},m} = \lambda_{\mathbf{l}}(\zeta_r^m)/\rho$ for $(\mathbf{l},m) \in L_m$. Then $\eta_{\mathbf{l},m}$ are non-real $(2r)$-th roots of unity. Thus

$$\sum_{b=0}^{2r-1} \sum_{(\mathbf{l},m)\in L_m} |\mathbf{l}|\eta_{\mathbf{l},m}^b = 0,$$

and consequently there exists b_0, $0 < b_0 < 2r$, such that

$$\sum_{(\mathbf{l},m)\in L_m} |\mathbf{l}|\lambda_{\mathbf{l}}(\zeta_r^m)^{b_0} = \rho^{b_0} \sum_{(\mathbf{l},m)\in L_m} |\mathbf{l}|\Re(\eta_{\mathbf{l},m}^b) < 0.$$

Hence $R_{q,0;m,0}(2^{a2rs+b_0 s}) < 0$ for sufficiently large a. □

With the help of Theorem 5 we will prove the negative part of Theorem 3 saying that primes $p \leq 1000$, $p \neq 3, 5, 17, 43, 257, 683$, do not satisfy $S_{p,0}(-1,n) > 0$ for almost all n. First, we only have to consider $p \in \mathbf{P}_t$ with $t > 2$. In [2] it is shown that $p = 3$ and $p = 5$ are the only exceptional primes in $\mathbf{P}_1$, and $p = 17$ and possibly $p = 41$ those of $\mathbf{P}_2$. (We will treat the case $p = 41$ in a moment.) Next, it follows from Theorem 5 that we only have to pay attention to those primes $p \in \mathbf{P}_t$, $t > 2$, with even $s = \mathrm{ord}_p(2)$, e.g. for $p = 109 \in \mathbf{P}_3$ we have $s = 36$. Finally, if there is $k < s$ with

$$S_{p,0}^{(m)}(-1, 2^k) = \frac{1}{p} \sum_{j=0}^{s-1} \prod_{i=0}^{k-1} \left(1 - \zeta_p^{l_m 2^{i+j}}\right) < 0,$$

in which $\lambda_m = \lambda_{l_m}(-1)$ is the largest eigenvalue, then $S_{p,0}(-1, 2^{as+k}) < 0$ for sufficiently large a. For example, for $p = 109$ we have $l_m = 9$ and $S_{109,0}^{(m)}(-1, 2^6) < 0$. Hence, for $p = 109$ there is no phenomenon of type (**N1**). Similarly it follows that $S_{41,0}^{(m)}(-1, 2^8) < 0$, and we really have to consider just primes $p \in \mathbf{P}_t$ with $t > 2$.

Table 1 gives a list of all primes $p \leq 1000$, $p \in \mathbf{P}_t$, $t > 2$, such that s is even. Furthermore the largest eigenvalue $\lambda_m = \lambda_{l_m}(-1)$ is represented by l_m, and if there is $k < s$ such that $S_{p,0}^{(m)}(-1, 2^k) < 0$ then k is listed.

The only primes for which this method provides no answer are $p = 43, 257, 683$. At the end of section 4 it will be shown that for these primes $S_{p,0}(-1,n) > 0$ for almost all n. This completes the proof of the negative part of Theorem 3.

Remark. It is also an interesting problem to consider $A_{q,i;r,m}(n)$ and $R_{q,i;r,m}(n)$ $(0 \leq m < r)$ for some fixed $i \not\equiv 0 \mod q$. For example, it is known that $A_{3,1;2,0}(n) < A_{3,1;2,1}(n)$ for almost all $n \geq 0$ (see [3]). Most of our methods can be applied in these cases too. However, for the sake of shortness we restrict ourselves to the case $i = 0$. □

Table 1

p	s	t	l_m	k	p	s	t	l_m	k
43	14	3	7	–	499	166	3	11	12
109	36	3	9	6	571	114	5	25	13
113	28	4	5	13	577	144	4	13	15
157	52	3	9	9	593	148	4	9	14
229	76	3	3	8	617	154	4	17	13
241	24	10	35	6	641	64	10	43	10
251	50	5	17	8	643	214	3	11	6
257	16	16	43	–	673	48	14	51	23
277	92	3	3	19	683	22	31	113	–
281	70	4	15	16	691	230	3	3	18
283	94	3	3	9	733	244	3	9	11
307	102	3	7	14	739	246	3	9	12
331	30	11	25	13	811	270	3	5	38
353	88	4	7	18	953	68	14	51	11
397	44	9	23	17	971	194	5	25	10
433	72	6	21	13	997	332	3	17	6
457	76	6	31	20					

4. Proof of Theorem 1

In the case of the usual parity $r = 2$ we just have to discuss $S_{q,i}(-1, n)$ to obtain all informations needed. For short we will write $S_{q,i}(n)$, λ_l, and $\mathbf{M}$ instead of $S_{q,i}(-1, n)$, $\lambda_l(-1)$, and $\mathbf{M}(-1)$.

From an heuristic point of view integers of the form $q = 4^N + 1$ or $q = 4^N - 1$ are 'good candidates' for a phenomenon of type (**N1**). In both cases we have $s(j) \equiv 0 \bmod 2$ for $j \equiv 0 \bmod q$, $j < q4^N + 1$, i.e. $S_{q,0}(n)$ is as positive as possible. (The first case is trivial. For the second case see Proposition 4.) In fact, Theorem 1 says that $S_{q,0}(n) > 0$ (for almost all n) for these q. However, an heuristic argument of this kind does not work in all cases. Suppose that $q = 2^{2N+1} - 1$. Then $s(j) \equiv 1 \bmod 2$ for $j \equiv 0 \bmod q$, $j < q2^{2N+1} + 1$, i.e. $S_{q,0}(n)$ is as negative as possible. Furthermore, $s = \mathrm{ord}_q(2) = 2N + 1$ is odd. Hence, by Theorem 5 $S_{q,0}(n) < 0$ for infinitely many n. But we know from Lemma 3 that we also have $S_{q,0}(n) > 0$ for infinitely many n.

Let $\mathbf{S}_q^{(m)}(n) = (S_{q,0}^{(m)}(n), \ldots, S_{q,q-1}^{(m)}(n))^t = \mathbf{P}^{(m)}\mathbf{S}_q(n)$. According to the above considerations it is sufficient to show that

$$S_{q,0}^{(m)}(n) \gg n^{(\log \lambda_m)/(s \log 2)},$$

where λ_m denotes the maximal eigenvalue, resp. $\min \psi_{q,0;m,0} > 0$.

First we will discuss the case $3|q$, where it is rather easy to identify λ_m.

Lemma 4. *Suppose that q is a positive odd integer. Then any eigenvalue*

$$\lambda_l = \prod_{m=0}^{s-1} \left(1 - \zeta_q^{l2^m}\right)$$

of $\mathbf{M}$ is bounded by $|\lambda_l| < 3^{s/2}$ or $\lambda_l = 3^{s/2}$.

The case $\lambda_l = 3^{s/2}$ *appears if and only if* $q \equiv 0 \bmod 3$ *and* $l \equiv q/3 \bmod q$ *or* $l \equiv 2q/3 \bmod q$.

Proof. It is an elementary exercise to show that

$$|1 - z^2| < \sqrt{3} \quad \text{and} \quad \left|(1-z)(1-z^2)\right| < 3$$

if $|z| = 1$ and $|1 - z| > \sqrt{3}$. Furthermore $|1 - z^2| = \sqrt{3}$ if $|z| = 1$ and $|1 - z| = \sqrt{3}$.

Now let $\lambda_l = \prod_{m=0}^{s-1} \left(1 - \zeta_q^{l2^m}\right)$ be an eigenvalue of $\mathbf{M}$. Let us consider a partition M_0, M_1 ,M_2, M_3 of the set $\{0, 1, \ldots, s-1\}$, where M_0 consists of those m with $|1 - \zeta_q^{l2^m}| = \sqrt{3}$, M_1 of those with $|1 - \zeta_q^{l2^m}| > \sqrt{3}$, and $M_2 = M_1 + 1$. It is clear that either $M_0 = \emptyset$ or $M_0 = \{0, 1, \ldots, s-1\}$. Furthermore M_1, M_2, M_3 are pairwise disjoint. If $M_0 = \emptyset$ then

$$|\lambda_l| = \prod_{m \in M_1} |(1 - \zeta_q^{l2^m})(1 - \zeta_q^{l2^{m+1}})| \prod_{m \in M_3} |1 - \zeta_q^{l2^m}| < 3^{|M_1|} 3^{|M_3|/2} = 3^{s/2}.$$

On the other hand, if $M_0 = \{0, 1, \ldots, s-1\}$, then s is even and $\lambda_l = 3^{s/2}$. Furthermore, the case $M_0 = \{0, 1, \ldots, s-1\}$ occurs only if $q \equiv 0 \bmod 3$ and $l \equiv q/3 \bmod q$ or $l \equiv 2q/3 \bmod q$. □

Lemma 5. *Suppose that* q *is an odd multiple of* 3. *Then*

$$\left|S_{q,i}^{(m)}(2^k)\right| \leq \frac{2}{q}\, 3^{k/2} \quad (0 \leq i < q), \tag{27}$$

$$S_{q,-2^j}^{(m)}(2^k) \leq 0 \quad (0 \leq j < s), \tag{28}$$

$$S_{q,0}^{(m)}(2^k) \geq \frac{\sqrt{3}}{q}\, 3^{k/2}. \tag{29}$$

Proof. Set $\omega = \zeta_3$. By (8) we have

$$S_{q,i}^{(m)}(2^k) = \frac{1}{q}\left(\omega^{-i} \prod_{j=0}^{k-1}\left(1 - \omega^{2^j}\right) + \omega^i \prod_{j=0}^{k-1}\left(1 - \omega^{-2^j}\right)\right).$$

Since $\omega^{2^j} = \omega^{(-1)^j}$ and $|1 - \omega^{\pm 1}| = \sqrt{3}$, we immediately obtain the estimate (27). Furthermore,

$$\prod_{j=0}^{k-1}\left(1 - \omega^{2^j}\right) = \begin{cases} 3^{k/2} & \text{if } k \text{ is even,} \\ 3^{(k-1)/2}(1-\omega) & \text{if } k \text{ is odd.} \end{cases}$$

Hence

$$S_{q,-2^i}^{(m)}(2^k) = \begin{cases} -q^{-1}3^{k/2} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd and } i \text{ is even,} \\ -q^{-1}3^{(k+1)/2} & \text{if } k \text{ and } i \text{ are odd,} \end{cases}$$

and

$$S_{q,0}^{(m)}(2^k) = \begin{cases} 2q^{-1}3^{k/2} & \text{if } k \text{ is even,} \\ q^{-1}3^{(k+1)/2} & \text{if } k \text{ is odd,} \end{cases}$$

which prove (28) and (29). □

Now suppose that $n = 2^k + \delta 2^{k-1} + r$, where $\delta \in \{0,1\}$ and $r < 2^{k-1}$. Then by using (9), (27), (28), and (29) we immediatly obtain

$$\begin{aligned} S_{q,0}^{(m)}(n) &= S_{q,0}^{(m)}(2^k) - \delta S_{q,-2^k}^{(m)}(2^{k-1}) + \sum_{j=0}^{k-2} \eta_j S_{q,i_j}^{(m)}(2^j) \\ &\geq \left(\frac{\sqrt{3}}{2} - \frac{(1-3^{-1/2})^{-1/2}}{3} \right) \frac{2}{q}\, 3^{k/2} \\ &> 0.077 \cdot \frac{2}{q}\, 3^{k/2} \gg n^{(\log \lambda_m)/(s \log 2)}. \end{aligned}$$

This proves Theorem 1 in the case $3|q$.

The case $q = 4^N + 1$ is a little bit more involved. The first step is to identify the largest eigenvalue λ_m. Note that $s = 4N$.

Lemma 6. *If $q = 4^N + 1$ then λ_m is given by*

$$\lambda_m = \prod_{j=0}^{4N-1} \left(1 - \zeta_q^{l_m 2^j}\right) = c3^{2N} \left(1 + \mathcal{O}(2^{-2N})\right),$$

where $l_m = (q+1)/3$ and $c = 0.363247\dots > 0$. Moreover, if $l \notin \mathbf{l}_m = l_m\langle 2\rangle$ then $|\lambda_l| < \lambda_m$.

Proof. First observe that for $0 \le i < N$

$$\arg \zeta_q^{l_m 2^{2i}} \in I_1 = \left(\frac{2\pi}{3}, \frac{5\pi}{6}\right), \qquad \arg \zeta_q^{l_m 2^{2i+1}} \in I_2 = \left(-\frac{2\pi}{3}, -\frac{\pi}{3}\right),$$

$$\arg \zeta_q^{l_m 2^{2N+2i}} \in I_3 = \left(-\frac{5\pi}{6}, -\frac{2\pi}{3}\right), \qquad \arg \zeta_q^{l_m 2^{2N+2i+1}} \in I_4 = \left(\frac{\pi}{3}, \frac{2\pi}{3}\right).$$

This means that there are exactly N elments $\zeta_q^{l_m 2^i}$, $0 \le i < 4N$, satisfying $\arg \zeta_q^{l_m 2^i} \in I_1$. Furthermore, the eigenvalue λ_m is calculated by

$$\begin{aligned} \lambda_m &= \prod_{i=0}^{N-1} \left|1 - \zeta_q^{l4^i}\right|^2 \left|1 - \zeta_q^{2l4^i}\right|^2 \\ &= \prod_{i=0}^{N-1} 16 \sin^2\left(\frac{\pi}{3} + \frac{\pi 4^i}{3q}\right) \sin^2\left(\frac{2\pi}{3} + \frac{2\pi 4^i}{3q}\right) \\ &= 3^{2N} \prod_{j=1}^{N} \frac{16}{9} \sin^2\left(\frac{\pi}{3} + \frac{\pi}{3\, 4^j}\right) \sin^2\left(\frac{2\pi}{3} + \frac{2\pi}{3\, 4^j}\right) \left(1 + \mathcal{O}(2^{-2N-j})\right) \\ &= 3^{2N} \left(\prod_{j=1}^{\infty} \frac{16}{9} \sin^2\left(\frac{\pi}{3} + \frac{\pi}{3 \cdot 4^j}\right) \sin^2\left(\frac{2\pi}{3} + \frac{2\pi}{3 \cdot 4^j}\right) \right) \left(1 + \mathcal{O}(2^{-2N})\right) \\ &= c3^{2N} \left(1 + \mathcal{O}(2^{-2N})\right). \end{aligned}$$

If $\arg \zeta_q^l \in I_1$ for some $l \not\equiv 0 \bmod q$, then $\arg \zeta_q^{2l} \in I_2$, $\arg \zeta_q^{2^{2N} l} \in I_3$, and $\arg \zeta_q^{2^{2N+1} l} \in I_4$. Hence, the number N_0 of elements $\zeta_q^{l2^i}$, $0 \le i < 4N$, satisfying $\arg \zeta_q^{l2^i} \in I_1$ is always bounded by $N_0 \le N$.

The most interesting case appears if $N_0 = N$. It is clear that this occurs if and only if $\arg\zeta_q^{l2^i} \notin \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$ for all $i \geq 0$. Let us classify those $x \in (0,1)$ such that $z = e^{2\pi(1+x)i/3}$ satisfies $\arg z^{2^i} \notin \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$ for all $i \geq 0$.

Since $z \notin \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$ it follows that $z \notin \left[\frac{5\pi}{6}, \frac{7\pi}{6}\right]$, and consequently $z \notin \left[\frac{5\pi}{12}, \frac{7\pi}{12}\right] \cup \left[-\frac{7\pi}{12}, -\frac{5\pi}{12}\right]$ etc. By induction it follows that $\arg z$ must be contained in a zero set quite similar to the Cantor set. More precisely, the only possible values $x \in (0,1)$ are given by

$$x = \sum_{n\geq 1} a_n 4^{-n},$$

where $a_n \in \{0,3\}$ and there exist $n_1, n_2 \geq 1$ with $a_{n_1} = 0$ and $a_{n_2} = 3$. If z is in addition a q-th root of unity then x must be of the form $x = k/q$, where $k \equiv 1 \bmod 3$ and $1 \leq k \leq 4^N$. Since

$$\frac{1}{q} = \frac{4^N - 1}{4^{2N} - 1} = \sum_{p\geq 0} \sum_{n=2pN+N+1}^{2(p+1)N} 3 \cdot 4^{-n},$$

we immediately obtain

$$\frac{k}{q} = k \sum_{p\geq 1} (4^N - 1) 4^{-2pN} = \sum_{p\geq 1} \left((k-1)4^N + \left((4^N - 1) - (k-1)\right)\right) 4^{-2pN},$$

and observe that the 4-adic digits a_n of the digit expansion of k/q, $1 \leq k \leq 4^N$, satisfy $a_n \in \{0,3\}$ for all $n \geq 1$ if and only if the 4-adic digit expansion of $k-1$ has the same property. (Evidently $k \equiv 1 \bmod 3$ in these cases.) This means that if we choose digits $b_n \in \{0,3\}$, $1 \leq n \leq N$, and set

$$k = 1 + \sum_{n=1}^{N} b_n 4^{N-n},$$

then

$$\frac{k}{q} = \sum_{p\geq 0} \left(\sum_{n=1}^{N} b_n 4^{-2Np-n} + \sum_{n=1}^{N} (3 - b_n) 4^{-2Np-N-n} \right).$$

In this way we get all q-th roots of unity $z = \zeta_q^l$ with $\arg\zeta_q^l \in I_1 \cup I_3$ such that $N_0 = N$. Furthermore, the digits b_n, $1 \leq n \leq N$, encode the distribution of $\zeta_q^{l4^i}$. If $\zeta_q^l = e^{2\pi(1+x_0)/3}$ with $x_0 = \sum_{n\geq 1} c_n 4^{-n}$ ($c_{2Np+n} = b_n$, $c_{2Np+N+n} = 3 - b_n$, $1 \leq n \leq N$, $p \geq 0$), then $\zeta_q^{l4^i} = e^{2\pi(1+x_i)/3}$, where $x_i = \sum_{n\geq 1} c_{n+i} 4^{-n}$. The periodicity $\zeta_q^{l4^{2N+i}} = \zeta_q^{l4^i}$ is reflected by the periodic digit expansion of x_0. In particular, $\zeta_q^{l_m}$ corresponds to the digits $b_n = 0$, $1 \leq n \leq N$. This means that $\zeta_q^{l_m 4^i} = e^{2\pi(1+x_{im})/3}$ are the only q-th roots of unity (with $N_0 = N$), where one period of the digits of x_{im} contains just one subblock of the form 03. In other words, there is exactly one element $\zeta_q^{l_m 4^i}$, $0 \leq i < N$, satisfying $\arg\zeta_q^{l_m 4^i} \in [19\pi/24, 5\pi/6]$, namely $\zeta_q^{l_m 4^{N-1}}$. For any other ζ_q^l, $l \notin \mathbf{l}_m$ (with $N_0 = N$), there are at least two subblocks of the form 03 in any period of the digit expansion of x_0. Thus there exist $0 \leq i_1 < i_2 < N$ with $\arg\zeta_q^{l4^{i_1}}, \arg\zeta_q^{l4^{i_2}} \in [19\pi/24, 5\pi/6]$. Consequently

$$\lambda_l < 3^{2N} \frac{16^2}{9^2} \sin^4\left(\frac{19\pi}{24}\right) \sin^4\left(\frac{19\pi}{48}\right) = 0.34899 \cdots 3^{2N} < \lambda_m.$$

The case $N_0 < N$ is much easier. Let J_1 denote the set of j, $0 \le j < 4N$, such that $\arg \zeta_q^{l2^j} \in I_1$. We assume that the elements j_i, $0 \le i < N_0$, of $J_1 = \{j_0, j_1, \ldots, j_{N_0-1}\}$ are 'ordered' in such a way that $\arg \zeta_q^{l2^{j_i}} \le \arg \zeta_q^{l2^{j_{i+1}}}$, $0 \le i < N_0 - 1$. (Recall that $|J_1| = N_0$.) Our first aim is to show that for any i, $0 \le i < N_0$, we have

$$\arg \zeta_q^{l_m 4^i} < \arg \zeta_q^{l2^{j_i}}. \tag{30}$$

Let b_i, $1 \le i < N$, denote the number of $j \in J_1$ satisfying $\arg \zeta_q^{l2^j} \in I^{(i)} = \left(\arg \zeta_q^{l_m 4^{i-1}}, \arg \zeta_q^{l_m 4^i}\right)$. Furthermore set $c_i = \sum_{1 \le j \le i} b_j$. Observe that

$$c_i \le i, \quad 1 \le i < N, \tag{31}$$

immediately implies (30). Since $\arg \zeta_q^{l2^j} \in I^{(i)}$, $1 \le i < N-1$, implies $\arg \zeta_q^{l2^{j+2}} \in I^{(i+1)}$, we always have $b_{i+1} \ge b_i$. Set $a_1 = b_1$ and $a_i = b_i - b_{i-1}$, $2 \le i < N$. Then $a_i \ge 0$, $b_i = \sum_{1 \le j \le i} a_j$, and $c_i = \sum_{1 \le j \le i} (i - j + 1) a_j$.

Since $c_{N-1} = N_0 \le N - 1$, condition (31) is satisfied for $i = N-1$. Now we show that $c_i \le i$ implies $c_{i-1} \le i - 1$. Suppose that $c_{i-1} \ge i$; then we obtain $a_1 + \cdots + a_i = c_i - c_{i-1} \le 0$. Thus $a_j = 0$, $1 \le j \le i$, which implies $c_{i-1} = 0$ and contradicts $c_{i-1} \ge i$. This completes the proof of (31) and consequently that of (30).

Let J_2 denote the set of j, $0 \le j < 4N$, such that $\arg \zeta_q^{l2^j} \in (5\pi/6, \pi)$, and J_3 the set of those j, $0 \le j < 4N$, such that $\arg \zeta_q^{l2^j} \in (0, \pi/3)$. Clearly $N_0 + |J_2| + |J_3| = N$ and

$$|1 - \zeta_q^{l2^j}| \cdot |1 - \zeta_q^{l2^{j+1}}| < |1 - \zeta_q^{l_m 2^{N-1}}| \cdot |1 - \zeta_q^{l_m 2^N}|$$

for $j \in J_2 \cup J_3$. Therefore we can estimate λ_l by

$$\begin{aligned}
\lambda_l &= \prod_{j \in J_1 \cup J_2 \cup J_3} \left(|1 - \zeta_q^{l2^j}|^2 |1 - \zeta_q^{l2^{j+1}}|^2 \right) \\
&= \prod_{i=0}^{N_0-1} \left(16 \sin^2 \left(\frac{\arg \zeta_q^{l2^{j_i}}}{2} \right) \sin^2 \left(\arg \zeta_q^{l2^{j_i}} \right) \right) \\
&\quad \cdot \prod_{j \in J_2 \cup J_3} \left(16 \sin^2 \left(\frac{\arg \zeta_q^{l2^j}}{2} \right) \sin^2 \left(\arg \zeta_q^{l2^j} \right) \right) \\
&< \prod_{i=0}^{N_0-1} \left(16 \sin^2 \left(\frac{\arg \zeta_q^{l_m 2^i}}{2} \right) \sin^2 \left(\arg \zeta_q^{l_m 2^i} \right) \right) \\
&\quad \cdot \left(16 \sin^2 \left(\frac{\arg \zeta_q^{l_m 2^{N-1}}}{2} \right) \sin^2 \left(\arg \zeta_q^{l_m 2^{N-1}} \right) \right)^{|J_2| + |J_3|} \\
&< \lambda_m,
\end{aligned}$$

which finishes the proof of Lemma 6. □

In order to complete the proof of Theorem 1 we need an analogon to Lemma 5. However, the situation is much more delicate. For the following estimates we use the notation

$$c_j = \prod_{i>j} \left(\frac{2}{\sqrt{3}} \sin\left(\frac{\pi}{3} + (-1)^i \frac{\pi}{3} 2^{-i} \right) \right) = 1 + \mathcal{O}(2^{-j}). \tag{32}$$

The proof is completely elmentary and just uses the Fourier expansion (8) of $S_{q,i}(2^k)$, or its dominant term $S_{q,i}^{(m)}(2^k)$ corresponding to $\zeta_q^{l_m}$.

Lemma 7. *Suppose that $q = 4^N + 1$ and $0 \le k \le 2N$. Furthermore, let $i = 0$ or $i = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_l}$, in which $k < k_1 < k_2 < \cdots < k_l \le 2N$, and set*

$$w_1 = \sum_{l'=1}^{l} (-1)^{k_{l'}-k}, \qquad w_2 = \sum_{l'=1}^{l} 2^{k_{l'}-k},$$
$$w_3 = \sum_{l'=1}^{l} (-1)^{k_{l'}-2N}, \qquad w_4 = \sum_{l'=1}^{l} 2^{k_{l'}-2N}.$$

If $k \equiv 0 \bmod 2$, then

$$\begin{aligned} S_{q,-i}^{(m)}(2^k) &= \frac{3^{k/2}}{q} \Bigg(2 \sum_{j=1}^{2N-k} c_j \cos\left(\frac{\pi}{3} 2^{-j} + (-1)^j \frac{2\pi}{3} w_1 + \frac{2\pi}{3} 2^{-j} w_2 \right) \\ &\qquad + 2c_0 \sum_{j=1}^{k} \frac{c_{j+2N-k}}{c_j} \sin\Big((-1)^j \frac{\pi}{6} + \frac{\pi}{3} 2^{-j} + \frac{\pi}{3} 2^{-j-2N+k} \\ &\qquad + (-1)^j \frac{2\pi}{3} w_3 + \frac{2\pi}{3} 2^{-j} w_4 \Big) + \mathcal{O}(2^{-k}) \Bigg) \\ &= \frac{3^{k/2}}{q} \Bigg(2(2N-k) \cos\left(\frac{2\pi}{3} w_1 \right) + C_1(k; k_1, \dots, k_l) \\ &\qquad + C_2(k; k_1, \dots, k_l) + \mathcal{O}(2^{-k}) + \mathcal{O}(2^{k-2N}) \Bigg), \end{aligned}$$

where the constants $C_1(k; k_1, \dots, k_l)$, $C_2(k; k_1, \dots, k_l)$ are given by

$$\begin{aligned} &C_1(k; k_1, \dots, k_l) \\ &\quad = 2 \sum_{j \ge 1} \left(c_j \cos\left(\frac{\pi}{3} 2^{-j} + (-1)^j \frac{2\pi}{3} w_1 + \frac{2\pi}{3} 2^{-j} w_2 \right) - \cos\left(\frac{2\pi}{3} w_1 \right) \right) \end{aligned}$$

$$\begin{aligned} &C_2(k; k_1, \dots, k_l) \\ &\quad = 2c_0 \sum_{j \ge 1} \Bigg(c_j^{-1} \sin\left((-1)^j \frac{\pi}{6} + \frac{\pi}{3} 2^{-j} + (-1)^j \frac{2\pi}{3} w_3 + \frac{2\pi}{3} 2^{-j} w_4 \right) \\ &\qquad\qquad - \sin\left((-1)^j \frac{\pi}{6} + (-1)^j \frac{2\pi}{3} w_3 \right) \Bigg), \end{aligned}$$

and $C_2(k; k_1, \dots, k_l)$ is uniformly bounded by $|C_2(k; k_1, \dots, k_l)| \le 3.64$.

If $k \equiv 1 \bmod 2$, then

$$\begin{aligned} S_{q,-i}^{(m)}(2^k) &= \frac{3^{k/2}}{q}\Bigg(2\sum_{j=1}^{2N-k} c_j \cos\left((-1)^j\frac{\pi}{6}+\frac{\pi}{3}2^{-j}+(-1)^j\frac{2\pi}{3}w_1+\frac{2\pi}{3}2^{-j}w_2\right) \\ &\quad + 2c_0\sum_{j=1}^{k}\frac{c_{j+2N-k}}{c_j}\sin\left(\frac{\pi}{3}2^{-j}+\frac{\pi}{3}2^{-j-2N+k}+(-1)^j\frac{2\pi}{3}w_3+\frac{2\pi}{3}2^{-j}w_4\right) \\ &\quad + \mathcal{O}(2^{-k})\Bigg) \\ &= \frac{3^{k/2}}{q}\Bigg(2(2N-k)\cos\left(\frac{\pi}{6}+\frac{2\pi}{3}w_3\right)+D_1(k;k_1,\dots,k_l) \\ &\quad + D_2(k;k_1,\dots,k_l)+\mathcal{O}(2^{-k})+\mathcal{O}(2^{k-2N})\Bigg), \end{aligned}$$

where the constants $D_1(k;k_1,\dots,k_l)$, $D_2(k;k_1,\dots,k_l)$ are given by

$$D_1(k;k_1,\dots,k_l) = 2\sum_{j\ge 0}\left(c_j\cos\left((-1)^j\frac{\pi}{6}+\frac{\pi}{3}2^{-j}+(-1)^j\frac{2\pi}{3}w_1+\frac{2\pi}{3}2^{-j}w_2\right) - \cos\left(\frac{\pi}{6}+\frac{2\pi}{3}w_3\right)\right),$$

$$D_2(k;k_1,\dots,k_l) = 2c_0\sum_{j\ge 1}\left(c_j^{-1}\sin\left(\frac{\pi}{3}2^{-j}+(-1)^j\frac{2\pi}{3}w_3+\frac{2\pi}{3}2^{-j}w_4\right) - \sin\left((-1)^j\frac{2\pi}{3}w_3\right)\right)$$

and $D_2(k;k_1,\dots,k_l)$ is uniformly bounded by $|D_2(k;k_1,\dots,k_l)|\le 2.22$.

Corollary 1. *Suppose that $q=4^N+1$ and $0\le k\le 2N$. Then*

$$\left|S_{q,-i}^{(m)}(2^k)\right| \le \frac{3^{k/2}}{q}\left(2(2N-k)+3.65\right),\quad (2^{k+1}\le i<4^N+1), \tag{33}$$

$$-S_{q,-2^{k+1}}^{(m)}(2^k) \ge \begin{cases} q^{-1}3^{k/2}\left((2N-k)-2.674\right) & (k\equiv 0 \bmod 2),\\ q^{-1}3^{k/2}\cdot 1.453 & (k\equiv 1 \bmod 2),\end{cases} \tag{34}$$

$$-S_{q,-2^{k+2}}^{(m)}(2^k) \ge \begin{cases} q^{-1}3^{k/2}\left((2N-k)-0.669\right) & (k\equiv 0 \bmod 2),\\ q^{-1}3^{k/2}\left(\sqrt{3}(2N-k)-5.12\right) & (k\equiv 1 \bmod 2),\end{cases} \tag{35}$$

$$-S_{q,-2^{k+3}}^{(m)}(2^k) \ge \begin{cases} q^{-1}3^{k/2}\left((2N-k)-2.358\right) & (k\equiv 0 \bmod 2),\\ q^{-1}3^{k/2}\cdot 4.791 & (k\equiv 1 \bmod 2),\end{cases} \tag{36}$$

$$S_{q,-2^{k+1}-2^{k+2}}^{(m)}(2^k) \ge \begin{cases} q^{-1}3^{k/2}\left(2(2N-k)-5.984\right) & (k\equiv 0 \bmod 2),\\ q^{-1}3^{k/2}\left(\sqrt{3}(2N-k)-3.699\right) & (k\equiv 1 \bmod 2),\end{cases} \tag{37}$$

$$S_{q,0}^{(m)}(2^k) \ge \begin{cases} q^{-1}3^{k/2}\left(2(2N-k)+0.831\right) & (k\equiv 0 \bmod 2),\\ q^{-1}3^{k/2}\left(\sqrt{3}(2N-k)+1.262\right) & (k\equiv 1 \bmod 2),\end{cases} \tag{38}$$

where all error terms $\mathcal{O}(2^{-2N})$ are neglected.

Proof. (33) follows from Lemma 7 and the fact that

$$\sum_{i=1}^{n} c_i \le n + 0.05 \quad (n \ge 1).$$

The constants in (34)–(38) are easy to calculate. □

Now, let $2^{4Na} \le n \le 2^{4Na+2N}$ for some $a \ge 0$. Then the binary digit expansion of n is given by

$$n = d_0 d_1 \cdots d_{4Na+k} = \sum_{j=0}^{2na+k} d_j 2^{4Na+k-j},$$

in which $d_0 = 1$ and $0 \le k \le 2N$. Furthermore, let d_{j_i}, $0 \le i < s(n)$, denote exactly those digits with $d_{j_i} = 1$. Then

$$\begin{aligned} S_{q,0}^{(m)}(n) &= \sum_{i=0}^{s(n)-1} (-1)^i S_{q,-n_i}^{(m)}(2^{4Na+k-j_i}) \\ &= S_{q,0}^{(m)}(2^{4Na+k}) - S_{q,-2^k}^{(m)}(2^{4Na+k-j_1}) + S_{q,-2^k-2^{k-j_1}}^{(m)}(2^{4Na+k-j_2}) \mp \cdots, \end{aligned}$$

where

$$n_i = \sum_{j<j_i} d_j 2^{4Na+k-j}.$$

Since $S_{q,i}^{(m)}(2^{4Na+k}) = \lambda_m^a S_{q,i}^{(m)}(2^k)$, we can use Corollary 1 in order to estimate $S_{q,0}^{(m)}(n)$ and $S_{q,0}(n)$.

First, suppose that $k \equiv 0 \bmod 2$. In the case $d_0 = 1$, $d_1 = d_2 = d_3 = 0$ we have $j_1 \ge 4$, and consequently

$$\begin{aligned} S_{q,0}^{(m)}(n) &= S_{q,0}^{(m)}(2^{4Na+k}) + \sum_{i\ge 1}(-1)^i S_{q,-n_{j_i}}^{(m)}(2^{4Na+k-j_i}) \\ &\ge \frac{\lambda_m^a 3^{k/2}}{q}\left(2(2N-k) + 0.831 - \sum_{i\ge 4}(2(2N-k)+2i+3.65)3^{-i/2}\right) \\ &\ge \frac{\lambda_m^a 3^{k/2}}{q}\left(1.474(2N-k) - 2.951\right). \end{aligned}$$

Hence, if $k \le 2N-3$ and $k \equiv 0 \bmod 2$ (i.e. $k \le 2N-4$), then $S_{q,0}^{(m)}(n) > 0$. If $d_0 = 1$, $d_1 = d_2 = 0$, $d_3 = 1$, then we obtain in the same way

$$\begin{aligned} S_{q,0}^{(m)}(n) &= S_{q,0}^{(m)}(2^{4Na+k}) - S_{q,-2^k}^{(m)}(2^{4Na+k-3}) + \sum_{i\ge 2}(-1)^i S_{q,-n_{j_i}}^{(m)}(2^{4Na+k-j_i}) \\ &\ge \frac{\lambda_m^a 3^{k/2}}{q}\Bigg(2(2N-k) + 0.831 + 3^{-3/2}4.791 \\ &\qquad - \sum_{i\ge 4}(2(2N-k)+2i+3.65)3^{-i/2}\Bigg) \\ &\ge \frac{\lambda_m^a 3^{k/2}}{q}\left(1.474(2N-k) - 2.029\right). \end{aligned}$$

Thus, $S_{q,0}^{(m)}(n) > 0$ if $k \le 2N-2$. Next, let $d_0 = 1$, $d_1 = 0$, $d_2 = 1$. Here we can verify that

$$\begin{aligned} 2(2N-k) + 0.831 + 3^{-1}((2N-k+2) - 0.669) \\ - \sum_{i\ge 3}(2(2N-k)+2i+3.65)3^{-i/2} \\ = 1.422(2N-k) - 4.363 > 0 \end{aligned}$$

for $k \le 2N-4$. In the case $d_0 = d_1 = 1$, $d_2 = 0$ we have

$$\begin{aligned} 2(2N-k) + 0.831 + 3^{-1/2}\cdot 1.453 - 2(2N-k)\cdot 0.456 - 5.638 \\ = 1.088(2N-k) - 3.979 > 0 \end{aligned}$$

if $k \le 2N-4$. Finally, if $d_0 = d_1 = d_2 = 1$ we can check that

$$\begin{aligned} 2(2N-k) + 0.831 + 3^{-1/2}\cdot 1.453 + 3^{-1}(2(2N-k+2) - 5.984) \\ - 2(2N-k)\cdot 0.456 - 5.638 = 1.754(2N-k) - 4.578 > 0 \end{aligned}$$

for $k \le 2N-3$.

Next, suppose that $k \equiv 1 \bmod 2$. If $d_0 = 1$, $d_1 = d_2 = d_3 = 0$, then

$$\begin{aligned} \sqrt{3}(2N-k) + 1.262 - \sum_{i\ge 4}\Big(2(2N-k)+2i+3.65)3^{-i/2}\Big) \\ = 1.206(2N-k) - 2.52 > 0 \end{aligned}$$

for $k \le 2N-3$. If $d_0 = 1$, $d_1 = d_2 = 0$, $d_3 = 1$, then

$$\begin{aligned} \sqrt{3}(2N-k) + 1.262 + 3^{-3/2}((2N-k+3) - 2.358) \\ - 2(2N-k)\cdot 0.263 - 3.782 \\ = 1.786(2N-k) - 2.397 > 0 \end{aligned}$$

for $k \le 2N-2$. If $d_0 = 1$, $d_1 = 0$, $d_1 = 1$, then

$$\begin{aligned} \sqrt{3}(2N-k) + 1.262 + 3^{-1}(\sqrt{3}(2N-k+2) - 5.12) \\ - 2(2N-k)\cdot 0.456 - 5.638 \\ = 1.397(2N-k) - 4.928 > 0 \end{aligned}$$

for $k \le 2N-4$. If $d_0 = d_1 = 1$, $d_2 = 0$, then

$$\begin{aligned} \sqrt{3}(2N-k) + 1.262 + 3^{-1/2}((2N-k+1) - 2.674) \\ - 2(2N-k)\cdot 0.456 - 5.638 \\ = 1.397(2N-k) - 5.343 > 0 \end{aligned}$$

for $k \le 2N-4$. Finally, if $d_0 = d_1 = d_2 = 1$, then

$$\begin{aligned} \sqrt{3}(2N-k) + 1.262 + 3^{-1/2}((2N-k+1) - 5.12) \\ + 3^{-1}(\sqrt{3}(2N-k+2) - 3.699) - 2(2N-k)\cdot 0.456 - 5.638 \\ = 1.974(2N-k) - 5.007 > 0 \end{aligned}$$

for $k \le 2N-3$.

This implies $S_{q,0}(2^{4Na+k} + \cdots) > 0$ if $k \le 2N-4$. The remaining cases $k = 2N$, $k = 2N-1$, $k = 2N-2$, and $k = 2N-3$ must be treated separately.

First let $k = 2N$. By Lemma 7 it is easy to calculate $S_{q,0}^{(m)}(2^k)$, $S_{q,-2^k}^{(m)}(2^{k-1})$, etc. up to an error term $\mathcal{O}(2^{-k}) = \mathcal{O}(2^{-2N})$. Let us consider a first example: $d_0 = 1$, $d_1 = 0$, $d_2 = 1$, $d_3 = 0$, $d_4 = 1$. We have

$$S_{q,0}^{(m)}(2^{4aN+2N}) = \lambda_m^a 3^N \left(2.20605\cdots + \mathcal{O}(2^{-k})\right),$$
$$S_{q,-2^{2N}}^{(m)}(2^{4aN+2N-2}) = \lambda_m^a 3^{N-1} \left(-4.4423\cdots + \mathcal{O}(2^{-k})\right),$$
$$S_{q,-2^{2N}-2^{2N-2}}^{(m)}(2^{4aN+2N-4}) = \lambda_m^a 3^{N-2} \left(-0.1559\cdots + \mathcal{O}(2^{-k})\right),$$

and

$$\begin{aligned} \sum_{i=3}^{s(n)-1} (-1)^i S_{q,-n_i}(2^{4Na+2N-j_i}) &\leq \lambda_m^a 3^N \sum_{i\geq 5} (2i+3.56)3^{-i/2} \\ &\leq 2.4865 \lambda_m^a 3^N. \end{aligned}$$

Hence

$$\begin{aligned} S_{q,0}^{(m)}(n) &\geq 3^{2N}\left(2.20605 + 3^{-1}4.4423 - 3^{-2}0.1559 - 2.4865 + \mathcal{O}(2^{-2N})\right) \\ &> (3.6695 - 2.4865)3^{2N}, \end{aligned}$$

which gives $S_{q,0}(2^{4aN}(2^{2N} + 2^{2N-2} + 2^{2N-4} + \cdots)) > 0$ for sufficiently large a.

All other cases can be treated in the same fashion. For completeness all relevant values are provided in Tables 2–5. The first column corresponds to the leading digits $d_0 d_1 d_2 \cdots d_j$ of $n = 2^{4aN}(d_0 2^k + d_1 2^{k-1} + \cdots + d_i 2^{k-j} + \cdots))$, the second one to the (approximate) value of the constant c in

$$S_{q,0}^{(m)}(2^{4aN}(d_0 2^k + d_1 2^{k-1} + \cdots + d_j 2^{k-j})) = \lambda_m^a 3^{k/2}(c + \mathcal{O}(2^{-k}))$$

and the third one to the error estimate

$$d = \sum_{i\geq j+1} (2(2N-k) + 2i + 3.65)3^{-i/2}.$$

For example, if $k = 2N$ and $d_0 \cdots d_j = 10101$, then $j = 4$, $c = 3.669508$ and $d = 2.4865$.

Since $c > d$, in any case we have proved that $S_{q,0}(n) > 0$ for $2^{4aN} \leq n \leq 2^{4aN+2N}$ if a and N are sufficiently large. The remaining cases $2^{4aN+2N} < n < 2^{4(a+1)N}$ can be tackled in the same fashion. We just need to find an analoge to Lemma 7 and to consider several cases. Thus we have proved the second part of Theorem 1 for sufficiently large N. The above proof has neglected the error terms $\mathcal{O}(2^{-2N})$. It is an easy but messy job to take these errors into account. In fact, it turns out that the above proof gives the second part of Theorem 1 for $N \geq 5$. Therefore we just have to check the two cases $N = 3$ and $N = 4$. We omit the details, but it is clear how to proceed in these cases in order to prove that $S_{4^N+1,0}(n) > 0$ for almost all n.

In the same fashion it is possible to prove $S_{43,0}(n) > 0$ and $S_{683,0}(n) > 0$ for almost all n. (Of course, a simple computer program assists us.) This completes the proof of Theorem 3. □

Table 2. $k = N$

$d_0 \cdots d_j$	c	d
100000	2.206052	1.611
100001	2.820907	1.611
10001	2.928609	2.4865
10010	3.377196	2.4865
10011	3.609011	2.4865
10100	3.686833	2.4865
10101	3.669508	2.4865
10110	3.456902	2.4865
10111	3.956975	2.4865
11000	3.691269	2.4865
11001	4.500393	2.4865
1101	4.628219	3.781
1110	4.908283	3.781
1111	4.78918	3.781

Table 3. $k = N - 1$

$d_0 \cdots d_j$	c	d
10000	3.82099	2.79
10001	3.96445	2.79
10010	4.13854	2.79
10011	4.36108	2.79
10100	4.16599	2.79
10101	3.72492	2.79
10110	3.51240	2.79
10111	3.75527	2.79
11000	3.45378	2.79
11001	4.26506	2.79
11010	4.27048	2.79
11011	4.79882	2.79
11100	4.88619	2.79
11101	4.67129	2.79
11110	4.07916	2.79
11111	4.55637	2.79

Table 4. $k = N - 2$

$d_0 \cdots d_j$	c	d
1000	5.09167	4.832
1001	6.11387	4.832
1010	6.48795	4.832
1011	6.12713	4.832
1100	6.26171	4.832
1101	7.11082	4.832
1110	7.30221	4.832
1111	7.18199	4.832

Table 5. $k = N - 3$

$d_0 \cdots d_j$	c	d
1000	6.65934	5.3581
1001	7.36031	5.3581
1010	7.89254	5.3581
1011	7.13277	5.3581
1100	7.42538	5.3581
1101	8.63985	5.3581
1110	9.29522	5.3581
1111	8.79174	5.3581

5. Proof of Theorem 2

The crucial step of the proof of Theorem 2 is contained in the following lemma.

Lemma 8. *Let p be an odd prime number and $s = \mathrm{ord}_p(2)$. Then*

$$S_{p,0}(2^{4ks-2}) = \frac{1}{p} \sum_{\mathbf{l} \in L} \lambda_{\mathbf{l}}^{4k} \left(\frac{s}{2} - \frac{1}{4} \sum_{l \in \mathbf{l}} \frac{1}{1 - \Re \zeta_p^l} \right). \tag{39}$$

Proof. Since $\lambda_{\mathbf{l}}^4$ is real for all eigenvalues $\lambda_{\mathbf{l}} = \prod_{l \in \mathbf{l}} (1 - \zeta_p^l)$ and since

$$S_{p,0}(2^{4ks-2}) = \frac{1}{p} \sum_{\mathbf{l} \in L} \lambda_{\mathbf{l}}^{4k} \sum_{l \in \mathbf{l}} \frac{1}{(1 - \zeta_p^l)(1 - \zeta_p^{2l})},$$

(39) follows from

$$\Re \left(\frac{1}{(1 - z)(1 - z^2)} \right) = \frac{1}{2} - \frac{1}{4(1 - \Re z)}, \tag{40}$$

in which $z \in \mathbf{C}$ has modulus $|z| = 1$. □

The next lemma ensures that

$$\frac{s}{2} < \frac{1}{4}\sum_{l\in\mathbf{l}}\frac{1}{1-\Re\zeta_p^l}$$

for all $\mathbf{l}\in L$ if $p\in\mathbf{P}_t$ is sufficiently large. Hence $S_{p,0}(2^{4ks-2})<0$ for all $k\geq 1$.

Lemma 9. *Suppose that $p\in\mathbf{P}_t$ and that $p\geq(2t\log p)^2$. Then*

$$\sum_{l\in\mathbf{l}}\frac{1}{1-\Re\zeta_p^l} > \frac{1}{8\pi^2}\frac{p^{3/2}}{t^2\log p}.$$

Proof. By assumption $p\geq 2tp^{1/2}\log p$. Hence by the Polya-Vinogradov inequality [12, p. 86, Aufgabe 12 b]

$$|\{k\in\mathbf{l}:\ 0<k\leq 2tp^{1/2}\log p\}| > p^{1/2}\log p$$

for all $\mathbf{l}\in L\setminus\{0\}$. Consequently

$$\begin{aligned}\sum_{l\in\mathbf{l}}\frac{1}{1-\Re\zeta_p^l} &= \sum_{l\in\mathbf{l}}\frac{1}{2\sin^2\left(\frac{l\pi}{p}\right)} > \frac{p^2}{2\pi^2}\sum_{l\in\mathbf{l}}\frac{1}{l^2}\\ &> \frac{p^2}{2\pi^2}\frac{p^{1/2}\log p}{(2tp^{1/2}\log p)^2} = \frac{1}{8\pi^2}\frac{p^{3/2}}{t^2\log p}.\end{aligned}$$ □

Now the first part of Theorem 2 follows from the next proposition.

Proposition 3. *Suppose that $p\in\mathbf{P}_t$ satisfies $S_{p,0}(n)>0$ for almost all n. Then*

$$p^{1/2}\leq 16\pi^2 t\log p, \tag{41}$$

i.e., if $S_{p,0}(n)>0$ for almost all n, then $s=\mathrm{ord}_p(2)\leq 16\pi^2p^{1/2}\log p$.

Proof. It is clear that we just have to consider primes p with $p^{1/2}\geq 2t\log p$. If $p^{1/2}>16\pi^2t\log p$, then Lemma 9 would imply

$$\frac{s}{2}-\frac{1}{4}\sum_{l\in\mathbf{l}}\frac{1}{1-\Re\zeta_p^l} < \frac{p}{2t}-\frac{1}{32\pi^2}\frac{p^{3/2}}{t^2\log p} < 0,$$

and by using Lemma 8 we would obtain $S_{p,0}(2^{4ks-2})<0$ for all $k\geq 1$. □

In order to finish the proof of Theorem 2 we just have to mention a result by Erdős [4] saying that for any sequence $\varepsilon_p\to 0$ (as $p\to\infty$)

$$\left|\{p\leq x:\ s=\mathrm{ord}_p(2)<p^{1/2+\varepsilon_p}\}\right| = o\left(\frac{x}{\log x}\right). \tag{42}$$

□

Remark. Theorem 2 also says that the number A_t of primes $p\in\mathbf{P}_t$ satisfying $S_{p,0}(n)>0$ for almost all n is bounded by $A_t\leq Cp^2\log^2 p$. However, this bound can be essentially sharpened. A theorem of Titchmarsh [11, p. 147] says that for all a, $0<a<1$, there exists a constant $C=C(a)$ such that

$$\pi(x;k,l) < C\frac{x}{\varphi(k)\log x}$$

for all $1\leq k\leq x^a$ and $0\leq l<k$ with $\gcd(l,k)=1$. Since $p\in\mathbf{P}_t$ satisfies $p\equiv 1 \bmod t$, we get

$$A_t=\mathcal{O}(t^2(\log t)/\varphi(t)).$$

Furthermore, $\varphi(t) > ct/(\log\log t)$ for some constant $c > 0$ (see [11, p. 24]). Hence

$$A_t = \mathcal{O}(t \log t \log\log t).$$

Comparing the above properties with Theorem 4, we find that the fractal function $\psi_p(x) = \psi_{p,0}(x)$ has a zero near $x = 1$. It is also an interesting problem to determine other zeroes and sign changes of $\psi_p(x)$. In [2] it is shown that for almost all primes $p \in \mathbf{P}_1$ the fractal function $\psi_p(x)$ has a zero near $x = 1/2$. Furthermore, a similar result may be expected for $\mathbf{P}_2$. If $|\Im(L(2,\chi))| > 40\pi^2 p^{-3/2}$, where χ denotes the biquadratic character $\bmod\, p \in \mathbf{P}_2$, then $\psi_p(x)$ has a zero near $x = 1/2$. Hence there is a connection between zeroes of $\psi_p(x)$ and properties of Dirichlet L-series. In what follows we will extend this connection to arbitrary t. However, we are unable to prove the properties of L-series. Nevertheless by numerical evidence (see [2]) the zeroes of ψ_p seem to be very well dispersed. Therefore we conjecture that the L-series in question satisfy the proposed properties (43) and (44).

Let $p \in \mathbf{P}_t$, and denote by λ_m the eigenvalue of largest modulus. If $s = \mathrm{ord}_p(2)$ is odd, then all eigenvalues λ_l are imaginary and $r' = 4$, which means that $\psi_p(\frac{1}{2}) < 0$ corresponds to $S_{p,0}^{(m)}(2^{(4a+2)s}) < 0$. Hence the same arguments as in the proof of Theorem 2 give

$$S_{p,0}^{(m)}(2^{(4a+2)s-2}) > 0,$$

providing a sign change of $\psi_p(x)$ near $x = \frac{1}{2}$ for sufficiently large p. If $s = \mathrm{ord}_p(2)$ is even, then $2^{s/2} \equiv -1 \bmod p$, and consequently all eigenvalues λ_l are real and positive. Hence $\lambda_m > 0$ and $r' = 1$. Let $\lambda_m = \prod_{i=0}^{s-1}(1 - \zeta_p^{l_m 2^i})$ and set

$$a_j = \prod_{i=0}^{s/2-1} (1 - \zeta_p^{l_m 2^{j+i}}).$$

Then

$$\begin{aligned}
S_{p,0}^{(m)}(2^{as+s/2}) &= \frac{\lambda_m^a}{p} \sum_{j=0}^{s-1} a_j, \\
S_{p,0}^{(m)}(2^{as+s/2-1}) &= \frac{\lambda_m^a}{p} \sum_{j=0}^{s-1} \frac{a_j}{1 - \zeta_p^{l_m 2^j}}, \\
S_{p,0}^{(m)}(2^{as+s/2-2}) &= \frac{\lambda_m^a}{p} \sum_{j=0}^{s-1} \frac{a_j}{(1 - \zeta_p^{l_m 2^j})(1 - \zeta_p^{l_m 2^{j+1}})}.
\end{aligned}$$

Since $2^{s/2} \equiv -1 \bmod p$ it follows that $\zeta_p^{l_m 2^{s/2}} = \zeta_p^{-l_m}$. Hence $a_{j+1} = -a_j \zeta_p^{-l_m 2^j}$ and

$$\begin{aligned}
\sum_{j=0}^{s-1} a_j &= a_0 \zeta_p^{l_m} \sum_{j=0}^{s-1} (-1)^j \zeta_p^{-l_m 2^j}, \\
\sum_{j=0}^{s-1} \frac{a_j}{1 - \zeta_p^{l_m 2^j}} &= a_0 \zeta_p^{l_m} \sum_{j=0}^{s-1} (-1)^j \frac{\zeta_p^{-l_m 2^j}}{1 - \zeta_p^{l_m 2^j}}, \\
\sum_{j=0}^{s-1} \frac{a_j}{(1 - \zeta_p^{l_m 2^j})(1 - \zeta_p^{l_m 2^{j+1}})} &= a_0 \zeta_p^{l_m} \sum_{j=0}^{s-1} (-1)^j \frac{\zeta_p^{-l_m 2^j}}{(1 - \zeta_p^{l_m 2^j})(1 - \zeta_p^{l_m 2^{j+1}})}.
\end{aligned}$$

First, suppose that $s \equiv 2 \bmod 4$, i.e. $s/2$ is odd. Then $\overline{a_0} = a_{s/2} = (-1)^{s/2} a_0 \zeta_p^{2l_m}$ implies that $a_0 \zeta_p^{l_m}$ is imaginary. Since

$$\frac{1}{z(1-z)} = \frac{1}{z} + \frac{1}{1-z}$$

and

$$\Im\left(\frac{1}{1-z}\right) = \frac{\Im z}{2(1-\Re z)}$$

for $|z| = 1$, we directly get

$$\sum_{j=0}^{s-1} (-1)^j \frac{\zeta_p^{-l_m 2^j}}{1-\zeta_p^{l_m 2^j}} = \sum_{j=0}^{s-1} (-1)^j \zeta_p^{-l_m 2^j} + \frac{1}{2} \sum_{j=0}^{s-1} (-1)^j \frac{i \Im \zeta_p^{l_m 2^j}}{1 - \Re \zeta_p^{l_m 2^j}}.$$

Let b be a generator of $G = (\mathbf{Z}/p\mathbf{Z})^*/\langle 4 \rangle$, i.e. all residue classes $\bmod\ p$ are parametrized by $b^i 4^j$, $0 \le i \le 2t-1$, $0 \le j \le s/2-1$, and χ_k, $1 \le k \le 2t$, Dirichlet characters defined by $\chi_k(b^i 4^j) = \zeta_{2t}^{ik}$. (Obviously the χ_k, $1 \le k \le 2t$, constitute the character group of G.) If

$$g_{\chi_k} = \sum_{n=0}^{p-1} \chi_k(n) \zeta_p^n,$$

denote the corresponding Gauss sums

$$S_1 = \sum_{j=0}^{s-1} (-1)^j \zeta_p^{-l_m 2^j} = \frac{1}{2t} \sum_{k=1}^{2t} \zeta_{2t}^{k i_m} (1 - (-1)^k) g_{\chi_k},$$

in which $b^{i_m} \equiv l_m \bmod p$. Furthermore, its absolute value can be estimated by $|S_1| \le \sqrt{p}$. Now set

$$h_{\chi_k} = \sum_{n=0}^{p-1} \chi_k(n) \frac{\Im \zeta_p^n}{1 - \Re \zeta_p^n} = \sum_{n=0}^{p-1} \chi_k(n) \cot \frac{n\pi}{p} = \frac{p}{\pi} (1 - (-1)^k) L(1, \chi_k).$$

Then

$$S_2 = \frac{1}{2} \sum_{j=0}^{s-1} (-1)^j \frac{i \Im \zeta_p^{l_m 2^j}}{1 - \Re \zeta_p^{l_m 2^j}} = i \frac{p}{2\pi t} \sum_{k=1}^{2t} \zeta_{2t}^{k i_m} (1 - (-1)^k)^2 L(1, \chi_k).$$

Note that S_1 and S_2 are imaginary. This representation is interesting if $|S_2| > \sqrt{p}$. If $\operatorname{sgn}(iS_1) \ne \operatorname{sgn}(iS_2)$, then it is clear that there is a sign change of $\psi_p(x)$ near $x = \frac{1}{2}$. If $\operatorname{sgn}(iS_1) = \operatorname{sgn}(iS_2)$, then it is an easy exercise to show that $S_{p,0}^{(m)}(2^{as+s/2})$ and $S_{p,0}^{(m)}(2^{as}(2^{s/2} + 2^{s/2-1})$ have different signs. Therefore, if $p \in \mathbf{P}_t$, $s \equiv 2 \bmod 4$, and

$$\left| \frac{\sqrt{p}}{4\pi t} \sum_{k=1}^{2t} \zeta_{2t}^{k i_m} (1 - (-1)^k)^2 L(1, \chi_k) \right| > 1, \tag{43}$$

then there is a sign change of $\psi_p(x)$ near $x = \frac{1}{2}$. For example, if $p \in \mathbf{P}_1$ and $p > 163$, then Dirichlet's class number formula and the fact that the class number h of the corresponding quadratic field satisfies $h > 1$ show that this case appears (see [2]).

Finally, suppose that $p \in \mathbf{P}_t$ and that $s/2$ is even, i.e. $s \equiv 0 \bmod 4$. Here $a_0\zeta_p^{l_m}$ is real and consequently S_1 is real, too. Furthermore, $\Re(1/(1-z)) = \frac{1}{2}$ for $|z| = 1$. Hence

$$\sum_{j=0}^{s-1}(-1)^j \frac{\zeta_p^{-l_m 2^j}}{1-\zeta_p^{l_m 2^j}} = \sum_{j=0}^{s-1}(-1)^j \zeta_p^{-l_m 2^j}$$

and $S_{p,0}^{(m)}(2^{as+s/2}) = S_{p,0}^{(m)}(2^{as+s/2-1})$. Since

$$\Re\left(\frac{1}{z(1-z)(1-z^2)}\right) = \Re z + \frac{1}{4} - \frac{1}{4(1-\Re z)},$$

for $|z| = 1$ we obtain as above

$$\begin{aligned}
\sum_{j=0}^{s-1}(-1)^j \frac{\zeta_p^{-l_m 2^j}}{(1-\zeta_p^{l_m 2^j})(1-\zeta_p^{l_m 2^{j+1}})} &= S_1 - \frac{1}{4}\sum_{j=0}^{s-1}(-1)^j \frac{1}{1-\Re\zeta_p^{l_m 2^j}} \\
&= S_1 - \frac{p^2}{8\pi^2 t}\sum_{k=1}^{2t} \zeta_{2t}^{k i_m}(1-(-1)^k)L(2,\chi_k) \\
&= S_1 - S_3.
\end{aligned}$$

Again, if

$$\left|\frac{p^{3/2}}{8\pi^2 t}\sum_{k=1}^{2t} \zeta_{2t}^{i_m}(1-(-1)^k)L(2,\chi_k)\right| > 1 \tag{44}$$

the above representation yields a sign change of $\psi_p(x)$ near $x = \frac{1}{2}$ if $|S_3| > \sqrt{p}$. (If $\operatorname{sgn}(S_1) \ne \operatorname{sgn}(S_3)$, then consider $S_{p,0}^{(m)}(2^{as}(2^{s/2}+2^{s/2-2}))$.) If $p \in \mathbf{P}_1$ and $p \ge 17$, this concept can be used to prove a sign change of $\psi_p(x)$ near $x = \frac{1}{2}$ (see [2]).

However, if $t > 1$ we do not know a general concept to decide whether (43) or (44) are satisfied or not. Nevertheless, it seems to be an interesting problem to consider linear combinations of values of Dirichlet L-series (with coefficients in a proper number field) and to quantify lower bounds in terms of p and not only in terms of the heights of coefficients. We conjecture that (43) and (44) are true for sufficiently large $p \ge c(t)$.

6. Higher Parities

The purpose of this section is to show that Newman's phenomenon $S_{q,0}(-1,n) > 0$ (which is the same as $A_{q,0;2,0}(n) > A_{q,0;2,1}(n)$) has generalizations for higher parities $r > 2$. However, the situation is more difficult than in the case $r = 2$. We show that direct analoga of Newman's theorem appear just for $r \le 6$ (Theorem 6). For $r > 6$ we do not know whether a phenomonen of type (**N1**) occurs or not. But Theorem 2 has a direct analogon (Theorem 10).

Our first observation suggest that $q = 2^r - 1$ is a good choice for a phenomenon of type (**N1**) for a parity r.

Proposition 4. *Let $q = 2^r - 1$, $r \ge 2$. Then $s(kq) = r$ for $k \le 2^r$, i.e. $A_{q,0;r,m}(n) = 0$ for $n < 2^{2r}$ and $m \not\equiv 0 \bmod r$.*

Proof. Since $k(2^r-1) = (k-1)2^r + ((2^r-1)-(k-1))$ it is clear that $s(k(2^r-1)) = r$ if $k-1 < 2^r$. ∎

However, we will prove the following theorem, showing that (**N1**) holds just for $r \le 6$.

Theorem 6. *The equality*

$$A_{2^r-1,0;r,0}(n) > \max_{0<m<r} A_{2^r-1,0;r,m}(n) \quad \text{for almost all } n \ge 0 \tag{45}$$

holds exactly for $2 \le r \le 6$.

If $r > 6$ it is very easy to disprove (45).

Proposition 5. *Suppose that* $r > 6$. *Then (45) fails.*

Proof. We show that $\alpha_r > \alpha_{q,r}$. By Lemma 1 this contradicts (45).

The largest eigenvalue $\lambda_0(\zeta_r^m)$, $0 < m < r$, corresponding to α_r is given by

$$\lambda_0(\zeta_r) = \left(2\cos\frac{\pi}{m}\right)^r = -2^r\left(1 - \frac{\pi^2}{2r} + \mathcal{O}(r^{-2})\right).$$

Now consider any q-th root of unity $\zeta_q^l = e^{2\pi x_0 i}$, $0 < l < q$ $(q = 2^r - 1)$. Then

$$x_0 = \frac{l}{q} = \sum_{j\ge 1} l2^{-jr} = \sum_{k\ge 1} c_k 2^{-k}$$

has a periodic digit expansion $c_{k+r} = c_k$, and for $\zeta_q^{l2^m} = e^{2\pi i x_m}$ we have

$$x_m = \sum_{k\ge 1} c_{k+m} 2^{-k}.$$

Furthermore there exists a k_0 with $c_{k_0} = 1$ and $c_{k_0+1} = 0$. Hence $1/2 \le x_{k_0} \le 3/4$, and consequently $|x_{k_0} - x_{k_0+1}| \ge 1/4$. Thus, for any m

$$\min\left(|1 + \zeta_r^m \zeta_q^{l2^{k_0}}|| 1 + \zeta_r^m \zeta_q^{l2^{k_0+1}}|\right) \le 2\cos\frac{\pi}{8},$$

which implies

$$|\lambda_l(\zeta_r^m)| \le 2^r \cos\frac{\pi}{8}.$$

Hence there are only finitely many $r \ge 2$ such that $\alpha_r \le \alpha_{q,r}$. It is an easy task to verify that this occurs exactly for $r \le 6$. ☐

First, consider the case $r = 3$ and set $\omega = \zeta_3 = e^{2\pi i/3}$. Since

$$S_{7,0}(\omega, n) = \sum_{m=0}^{2} A_{7,0;3,m}(n)\omega^m$$

(45) is equivalent to the following proposition.

Proposition 6. *We have*

$$\arg\left(S_{7,0}(\omega, n)\right) \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right) \quad \text{for almost all } n \ge 0. \tag{46}$$

Proof. First, let us determine the corresponding eigenvalues $\lambda_1 = \lambda_{\{1,2,4\}}(\omega)$, $\lambda_2 = \lambda_{\{3,5,6\}}(\omega)$, and $\lambda_3 = \lambda_{\{0\}}(\omega)$. Set $R = \zeta_7 + \zeta_7^2 + \zeta_7^4$ and $N = \zeta_7^3 + \zeta_7^5 + \zeta_7^6$. Since $R + N = -1$ and

$$R - N = \sum_{i=1}^{6} \left(\frac{i}{7}\right) \zeta_7^i = i\sqrt{7},$$

Table 6

j	c'_{j0}	d'_{j0}	c'_{j1}	d'_{j1}	c'_{j2}	d'_{j2}
0	18	0	$18+2\sqrt{21}$	$-3+\sqrt{21}$	$21+5\sqrt{21}$	$-3+\sqrt{21}$
1	$-3+\sqrt{21}$	$2\sqrt{21}$	$-3+\sqrt{21}$	$18+2\sqrt{21}$	$2\sqrt{21}$	$18+4\sqrt{21}$
2	$-3+\sqrt{21}$	$2\sqrt{21}$	$-3-\sqrt{21}$	$-3+\sqrt{21}$	$-2\sqrt{21}$	$18+2\sqrt{21}$
3	$-3-\sqrt{21}$	$-2\sqrt{21}$	$-3-3\sqrt{21}$	$-3-3\sqrt{21}$	$-21-5\sqrt{21}$	$-24-4\sqrt{21}$
4	$-3+\sqrt{21}$	$2\sqrt{21}$	$-3+3\sqrt{21}$	$-3+3\sqrt{21}$	$2\sqrt{21}$	$-3+\sqrt{21}$
5	$-3-\sqrt{21}$	$-2\sqrt{21}$	$-3-3\sqrt{21}$	$-3-3\sqrt{21}$	0	$-3-3\sqrt{21}$
6	$-3-\sqrt{21}$	$-2\sqrt{21}$	$-3+\sqrt{21}$	$-3-\sqrt{21}$	$-2\sqrt{21}$	$-3-\sqrt{21}$

we have $R=(-1+i\sqrt{7})/2$ and $N=(-1-i\sqrt{7})/2$. Hence

$$\begin{aligned}\lambda_1 &= (1+\omega\zeta_7)(1+\omega\zeta_7^2)(1+\omega\zeta_7^4) = 2+\omega R+\omega^2 N\\ &= \frac{5-\sqrt{21}}{2} = 0.20871\cdots\end{aligned}$$

Similarly we obtain $\lambda_2=(1+\omega\zeta_7^3)(1+\omega\zeta_7^5)(1+\omega\zeta_7^5)=(5+\sqrt{21})/2=4.79128\cdots$ and $\lambda_3=(1+\omega)^3=-1$. Thus, $\lambda_m=\lambda_2$ is the largest eigenvalue.

Next we will estimate $S_{7,0}^{(m)}(\omega,n)=c_{n0}+\omega d_{n0}$. Clearly it is sufficient to prove that $c_{n0}>|d_{n0}|$ for almost all $n\geq 0$. For this purpose we define c'_{jk} and d'_{jk} by

$$S_{7,j}^{(m)}(2^k)=\frac{\lambda_2^{[\frac{k}{3}]}}{42}(c'_{jk}+\omega d'_{jk}).$$

Observe that c'_{jk} and d'_{jk} are periodic in k with period 3. We use (8) in order to calculate their values. First we have

$$S_{7,0}^{(m)}(2^{3l})=\frac{3}{7}\lambda_2^l.$$

Next we obtain

$$\begin{aligned}S_{7,0}^{(m)}(2^{3l+1}) &= \frac{\lambda_2^l}{7}\left((1+\omega\zeta_7^3)+(1+\omega\zeta_7^5)+(1+\omega\zeta_7^6)\right)=\frac{\lambda_2^l}{7}(3+\omega N)\\ &= \frac{\lambda_2^l}{42}\left((18+2\sqrt{21})+(-3+\sqrt{21})\omega\right),\end{aligned}$$

Here and in what follows we use the representations

$$R=\frac{(-3+\sqrt{21})+2\sqrt{21}\omega}{6},\quad L=\frac{(-3-\sqrt{21})-2\sqrt{21}\omega}{6}.$$

Similarly,

$$\begin{aligned}&S_{7,0}^{(m)}(2^{3l+2})\\ &\quad=\frac{\lambda_2^l}{7}\left((1+\omega\zeta_7^3)(1+\omega\zeta_7^6)+(1+\omega\zeta_7^5)(1+\omega\zeta_7^3)+(1+\omega\zeta_7^6)(1+\omega\zeta_7^5)\right)\\ &\quad=\frac{\lambda_2^l}{7}\left((3-R)+(2N-R)\omega\right)=\frac{\lambda_2^l}{42}\left((21+5\sqrt{21})+(-3+\sqrt{21})\omega\right).\end{aligned}$$

The cases $j\neq 0$ can be treated in the same way. Table 6 lists the corresponding values.

Now set $\beta = \lambda_2^{1/3}$. Then

$$\max_{0\le j<7,\,0\le k<3} \beta^k \frac{|c'_{jk}| + |d'_{jk}|}{42} = \frac{3}{7}.$$

Hence, if $n = 2^k + \cdots$ and $S_{7,j}^{(m)}(\omega, n) = c_{nj} + \omega d_{nj}$, then

$$|c_{nj}| + |d_{nj}| \le \frac{3}{7}\frac{\beta^k}{1-\beta^{-1}} = 1.0534\ldots\beta^k.$$

If $n = 2^{3l} + 0\cdot 2^{3l-1} + \cdots$, then

$$c_{n0} - |d_{n0}| \ge \frac{\beta^{3l}}{42}\,(c'_{00} - |d'_{00}|) - 1.0535\beta^{3l-2} \ge 0.057\,\beta^{3l}.$$

Similarly, if $n = 2^{3l} + 2^{3l-1} + \cdots$, then

$$\begin{aligned} c_{n0} - |d_{n0}| &\ge \frac{\beta^{3l}}{42}\left(c'_{00} - |d'_{00}| + \beta^{-3}(-d'_{62} - |c'_{62} - d'_{62}|)\right) - 1.0535\beta^{3l-2} \\ &\ge 0.087\,\beta^{3l} \end{aligned}$$

If $n = 2^{3l+1} + \cdots$, then we have to distinguish more cases. In the case $n = 2^{3l+1} + 0\cdot 2^{3l} + 0\cdot 2^{3l-1} + \cdots$ we immediately obtain

$$c_{n0} - |d_{n0}| \ge \frac{\beta^{3l}}{42}\,(c'_{01} - |d'_{01}|) - 1.0535\beta^{3l-2} \ge 0.23\,\beta^{3l}.$$

If $n = 2^{3l+1} + 0\cdot 2^{3l} + 2^{3l-1} + \cdots$, then

$$c_{n0} - |d_{n0}| \ge \frac{\beta^{3l}}{42}\left(c'_{01} - |d'_{01}| + \beta^{-3}(-d'_{52} - |c'_{52} - d'_{52}|)\right) - 1.0535\beta^{3l-2} \ge 0.23\,\beta^{3l}.$$

Furthermore, if $n = 2^{3l+1} + 2^{3l} + \cdots$, then

$$c_{n0} - |d_{n0}| \ge \frac{\beta^{3l}}{42}\,(c'_{01} - |d'_{01}| - d'_{50} - |c'_{50} - d'_{50}|)) - 1.0535\beta^{3l-1} \ge 0.16\,\beta^{3l}.$$

Finally, the case $n = 2^{3l+2} + \cdots$ can be treated in the same way. Hence

$$c_{n0} - |d_{n0}| \ge c\lambda_2^{(\log n)/(3\log 2)},$$

and consequently (46). □

Similarly to the first part of Theorem 1, we are also able to provide infinitely many examples for phenomena of type (**N1**) for parity $r = 3$.

Theorem 7. *Suppose that $r = 3$ and that q is an odd multiple of 7. Then* (**N1**) *and* (**N2**) *hold.*

The essential part of the proof is to identify the largest eigenvalue. This will be done in the following lemma.

Lemma 10. *Suppose that q is a positive odd integer. Then any eigenvalue*

$$\lambda_l(\omega) = \prod_{m=0}^{s-1}\left(1 + \omega\zeta_q^{l2^m}\right)$$

of $\mathbf{M}(\omega)$ is bounded by $|\lambda_l(\omega)| < ((5+\sqrt{21})/2)^{s/3}$ or $\lambda_l(\omega) = ((5+\sqrt{21})/2)^{s/3}$.

The case $\lambda_l(\omega) = ((5+\sqrt{21})/2)^{s/3}$ appears if and only if $q \equiv 0 \bmod 7$ and either $l \equiv 3q/7 \bmod q$ or $l \equiv 5q/7 \bmod q$ or $l \equiv 6q/7 \bmod q$.

Proof. Let $\lambda_l(\omega) = \prod_{m=0}^{s-1}\left(1+\omega\zeta_q^{l2^m}\right)$ be an eigenvalue of $\mathbf{M}(\omega)$. If $l \equiv 3q/7 \bmod q$ or $l \equiv 5q/7 \bmod q$ or $l \equiv 6q/7 \bmod q$, then $\lambda_l(\omega) = ((5+\sqrt{21})/2)^{s/3}$.

In the remaining cases we use the following partition: M_1, $M_2 = M_1+1$, $M_3 = M_1+2$, M_4, $M_5 = M_4+1$, $M_6 = M_4-1$, M_7 of $\{0,1,\dots,s-1\}$. M_1 consists of those m such that $\arg(\zeta_q^{l2^m}) \in (-4\pi/7, -2\pi/7)$ and M_4 of those m which are not contained in M_2 and satisfy $\arg(\zeta_q^{l2^m}) \in (-8\pi/7, -4\pi/7)$. Set

$$\begin{aligned} f(x) &= 8\left|\cos\left(\frac{x}{2}+\frac{\pi}{3}\right)\cos\left(x+\frac{\pi}{3}\right)\cos\left(2x+\frac{\pi}{3}\right)\right|, \\ g(x) &= 8\left|\cos\left(\frac{x}{2}+\frac{\pi}{3}\right)\cos\left(x+\frac{\pi}{3}\right)\cos\left(\frac{x}{4}-\frac{\pi}{3}\right)\right|. \end{aligned}$$

Then $f(-2\pi/7) = (5+\sqrt{21})/2$ and

$$f(x) = \left|(1+\omega e^{ix})(1+\omega e^{2ix})(1+\omega e^{4ix})\right| < f(-2\pi/7)$$

for $x \in (-4\pi/7, -2\pi/7)$. Hence

$$\prod_{m\in M_1\cup M_2\cup M_3}\left|1+\omega\zeta_q^{l2^m}\right| < \left(\frac{5+\sqrt{21}}{2}\right)^{|M_1|}.$$

Similarly, $g(x) < f(-2\pi/7)$, $x \in (-8\pi/7, -4\pi/7)$, implies

$$\prod_{m\in M_4\cup M_5\cup M_6}\left|1+\omega\zeta_q^{l2^m}\right| < \left(\frac{5+\sqrt{21}}{2}\right)^{|M_4|}.$$

Finally, $|1+\omega e^{ix}| < f(-2\pi/7)^{1/3}$, $x \in (-4\pi/7, 6\pi/7)$, provides

$$|1+\omega\zeta_q^{l2^m}| < \left(\frac{5+\sqrt{21}}{2}\right)^{1/3}$$

for all $m \in M_7$, which completes the proof of Lemma 10. □

Now the proof of Theorem 7 is almost the same as the proof of Proposition 6. Therefore we will not give the details here.

Next, let $r = 4$. Here we prove.

Proposition 7. *We have*

$$\arg\left(S_{15,0}(i,n)\right) \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \quad \textit{for almost all } n \ge 0. \tag{47}$$

It is easy to verfy that Proposition 7 implies Theorem 6 for $r = 4$. Since (47) is equivalent to

$$A_{15,0;4,0}(n) - A_{15,0;4,2}(n) > |A_{15,0;4,1}(n) - A_{15,0;4,3}(n)|, \tag{48}$$

we have $A_{15,0;4,0}(n) > A_{15,0;4,2}(n)$. By Theorem 1 ($q = 15$) we also know that

$$A_{15,0;4,0}(n) + A_{15,0;4,2}(n) > A_{15,0;4,1}(n) + A_{15,0;4,3}(n). \tag{49}$$

Let $\{k,l\} = \{1,3\}$ and suppose that $A_{15,0;4,k}(n) \ge A_{15,0;4,l}(n)$. Then (48) and (49) imply

$$A_{15,0;4,0}(n) > A_{15,0;4,k}(n) \ge A_{15,0;4,l}(n),$$

and consequently (45).

Table 7.1

j	c'_{j0}	d'_{j0}	c'_{j1}	d'_{j1}
0	8	0	$8+\sqrt{15}$	1
1	1	$\sqrt{15}$	1	$8+\sqrt{15}$
2	1	$\sqrt{15}$	$1-\sqrt{15}$	$1+\sqrt{15}$
3	-2	0	$-2-\sqrt{15}$	1
4	1	$\sqrt{15}$	1	$-2+\sqrt{15}$
5	-4	0	$4-\sqrt{15}$	1
6	-2	0	-2	-4
7	1	$-\sqrt{15}$	1	$-2-\sqrt{15}$
8	1	$\sqrt{15}$	$1+\sqrt{15}$	$1+\sqrt{15}$
9	-2	0	$-2-\sqrt{15}$	1
10	-4	0	-4	-2
11	1	$-\sqrt{15}$	1	$-4-\sqrt{15}$
12	-2	0	$-2+\sqrt{15}$	1
13	1	$-\sqrt{15}$	1	$-2-\sqrt{15}$
14	1	$-\sqrt{15}$	$1+\sqrt{15}$	$1-\sqrt{15}$

It should also be mentioned that $\Re(S_{15,0}(i,n)) > 0$ for almost all n is also sufficient to prove (45). By (6) we have

$$A_{15,0;4,m}(n) = \frac{1}{4}\sum_{l=0}^{3} i^{-lm} S_{15,0}(i^l, n).$$

Hence $\Re(S_{15,0}(i,n)) > 0$ implies $A_{15,0;4,0}(n) > A_{15,0;4,2}(n)$. Furthermore, by Theorem 1 $S_{15,0}(-1,n) \gg n^{\frac{\log 3}{\log 4}}$. Consequently we also have

$$A_{15,0;4,0}(n) > \max(A_{15,0;4,1}(n), A_{15,0;4,3}(n))$$

for sufficienty large n.

Proof of Proposition 7. The computation of the eigenvalues of $\mathbf{M}(i)$ can be worked out explicitly:

$$\begin{aligned}
\lambda_1 &= \lambda_{\{1,2,4,8\}} = (1+i\zeta_15)(1+i\zeta_15^2)(1+i\zeta_15^4)(1+i\zeta_15^8) \\
&= 2-(\zeta_15^3+\zeta_15^6+\zeta_15^9+\zeta_15^{12})-(\zeta_15^5+\zeta_15^{10})+i\sum_{j=1}^{14}\left(\frac{j}{15}\right)\zeta_15^j \\
&= 4-\sqrt{15},
\end{aligned}$$

where $\left(\frac{\cdot}{15}\right)$ denotes the Jacobi-Kronecker symbol. The other eigenvalues are given by $\lambda_2 = \lambda_{\{14,7,11,13\}} = 4+\sqrt{15}$, $\lambda_3 = \lambda_{\{3,6,9,12\}} = 1$, $\lambda_4 = \lambda_{\{5,10\}} = -1$, and by $\lambda_5 = \lambda_{\{0\}} = -4$. Hence the largest eigenvalue is λ_2. Now we can proceed as in the proof of Proposition 6. We just reproduce a table (Tables 7.1 and 7.2) for c'_{jk} and d'_{jk} defined by

$$S_{15,j}^{(m)}(i,2^k) = \frac{\lambda_2^{[\frac{k}{4}]}}{30}(c'_{jk}+id'_{jk}). \quad \square$$

Table 7.2

j	c'_{j2}	d'_{j2}	c'_{j3}	d'_{j3}
0	$10+2\sqrt{15}$	2	$16+4\sqrt{15}$	2
1	$\sqrt{15}$	$9+2\sqrt{15}$	$3+\sqrt{15}$	$9+3\sqrt{15}$
2	$-\sqrt{15}$	$9+2\sqrt{15}$	1	$14+3\sqrt{15}$
3	$-10-2\sqrt{15}$	2	$-9-2\sqrt{15}$	$2+\sqrt{15}$
4	$-\sqrt{15}$	-1	$-2-\sqrt{15}$	$9+2\sqrt{15}$
5	$-9-\sqrt{15}$	$-1-\sqrt{15}$	$-14-\sqrt{15}$	$-1+2\sqrt{15}$
6	$-\sqrt{15}$	-3	$-9-\sqrt{15}$	$-3-3\sqrt{15}$
7	0	$-6-2\sqrt{15}$	-2	$-16-4\sqrt{15}$
8	$5+\sqrt{15}$	$-1+\sqrt{15}$	$6+\sqrt{15}$	-1
9	0	2	$1+\sqrt{15}$	$-3-\sqrt{15}$
10	$-5-\sqrt{15}$	$-1-\sqrt{15}$	$-2-\sqrt{15}$	-1
11	0	$-6-2\sqrt{15}$	$6+2\sqrt{15}$	$-6-2\sqrt{15}$
12	$\sqrt{15}$	-3	1	$2+\sqrt{15}$
13	$5+\sqrt{15}$	$-1-\sqrt{15}$	$3+\sqrt{15}$	$-1-\sqrt{15}$
14	$\sqrt{15}$	-1	1	$-6-\sqrt{15}$

Finally, let us consider the cases $r=5$ and $r=6$. In the case $r=5$ it suffices to show that

$$\Re\left(S_{31,0}(\zeta_5,n)\right) > \Re\left(\zeta_5^{-m}S_{31,0}(\zeta_5,n)\right) \qquad (m\not\equiv 0 \bmod 5),$$

which can be checked by considering the largest eigenvalue $\lambda_{-1}(\zeta_5)$ and similar calculations as above. (Again a simple computer program assists us.)

The case $r=6$ is interesting because (45) can be deduced from Theorems 1 and 7. By (6)

$$A_{63,0;6,m}(n) = \frac{1}{6}\sum_{l=0}^{5}\zeta_6^{-lm}S_{63,0}(\zeta_6^l,n).$$

By Theorem 7, $\arg(S_{63,0}(\zeta_6^2,n))\in(-\pi/3,\pi/3)$. Thus, for sufficiently large n,

$$A_{63;0;6,0}(n) > \max(A_{63;0;6,2}(n),A_{63;0;6,4}(n)),$$

since the largest eigenvalue of $\mathbf{M}(\zeta_6^2)$ is larger than the largest eigenvalue of $\mathbf{M}(\zeta_6)$. Furthermore, by Theorem 1 $S_{63,0}(-1,n)\gg n^{\frac{\log 3}{\log 4}}$, and consequently

$$A_{63;0;6,0}(n) > \max(A_{63;0;6,1}(n),A_{63;0;6,3}(n),A_{63;0;6,5}(n))$$

for sufficientely large n.

Therefore we have provided a complete answer for the case $q=2^r-1$ with respect to (**N1**). However, the situation is much more delicate when we consider (**N2**) instead of (**N1**).

Theorem 8. *We have*

$$R_{127,0;7,0}(n) > 0 \quad \textit{for almost all } n\ge 0.$$

This means that (**N2**) holds for $r=7$ although (**N1**) fails. (We do not give a detailed proof. We only want to mention that it suffices to show that $\Re\left(S_{127,0}(\zeta_7,n)\right) > 0$.) Therefore it might be possible that (**N2**) holds for all $r\ge 2$. But again the answer is negative.

Theorem 9. *There are infinitely many $r \geq 2$ such that*

$$R_{2^r-1,0;r,0}(n) < 0 \quad \text{for infinitely many } n \geq 0.$$

Sketch of the Proof. It is sufficient to show that there are infinitely many $r \geq 2$ such that the eigenvalue $\lambda_l(\zeta_r^m)$, $0 < l < 2^r - 1$, $0 < m < r$, of largest modulus $|\lambda_m|$ is negative. In what follows we will indicate that if there exists a positive integer m_r such that $|\sqrt{r}/\pi + C - m_r| < 1/4$ (where C a real constant and $r \geq r_0$ is sufficiently large), then $\lambda_1(\zeta_r^{-m_r})$ is the eigenvalue of largest modulus

$$|\lambda_m| = |\lambda_1(\zeta_r^{-m_r})| \sim \frac{2^r e^{-1/2}}{2\pi\sqrt{r}}.$$

Since

$$\arg(\lambda_1(\zeta_r^{-m_r})) = \sum_{j=0}^{r-1} \pi \left(\frac{2^j}{2^r - 1} - \frac{m_r}{r} \right) = \pi(1 - m_r),$$

we have $\operatorname{sgn}(\lambda_1(\zeta_r^{-m_r})) < 0$ if m_r is even. Obviously, this case occurs infinitely many times.

We use the fact that the digit expansion of $x_0 = l/(2^r - 1) = \sum_{k\geq 1} c_k 2^{-k}$ is periodic, i.e. $c_{k+r} = c_k$, and that $x_j = \sum_{k\geq 1} c_{k+j} 2^{-k}$ satisfies $\zeta_{2^r-1}^{l2^j} = e^{2\pi i x_j}$. (Compare with the proof of Proposition 5.) By considering several subcases it turns out that if l is unbounded, then

$$|\lambda_l(\zeta_r^m)| = o(2^r r^{-1/2}).$$

Conversely, if l is bounded, then

$$\max_m |\lambda_l(\zeta_r^m)| \sim \frac{2^{r-1} e^{-1/2}}{l\pi\sqrt{r}},$$

in which the maximum is attained for $|m| \sim \sqrt{r}/\pi$. Therefore $l = 1$, $|m| \sim \sqrt{r}/\pi$ is the only relevant case. (Since $|1 + \zeta_r^m \zeta_q^{2^j}| < |1 + \zeta_r^{-m} \zeta_q^{2^j}|$ $(m > 0)$, we may also assume that $m \sim -\sqrt{r}/\pi$.) A more detailed analysis shows that the maximum value of $|\lambda_1(\zeta_q^m)|$ is attained for

$$m = -\frac{\sqrt{r}}{\pi} - C + \mathcal{O}(r^{-1/2}),$$

in which m is assumed to be a continuous real parameter and C is a computable constant. Furthermore, if $\sqrt{r}/\pi + C$ is near to an integer m_r, e.g. $|\sqrt{r}/\pi + C - m_r| < 1/4$, and if r is sufficiently large, then $|\lambda_m| = |\lambda_1(\zeta_q^{-m_r})|$. □

We finish this section on higher parities with an analogue to Theorem 2.

Theorem 10. *For any $r > 1$ there exists a constant $C_r > 0$ such that for any $t \geq 1$ primes $q \in \mathbf{P}_t$ satisfying* (**N1**) *or* (**N2**) *are bounded by*

$$q \leq C_r t^4 \log^4 t.$$

For the proof we can use a similar procedure as above. Instead of (40) we need the following formula.

Proposition 8. *Suppose that p is an odd prime and $s = \mathrm{ord}_p(2)$. If $y \in \mathbf{C}$ has modulus $|y| = 1$, then for any $\mathbf{l} \in L$*

$$\Re\left(\sum_{l\in\mathbf{l}} \frac{1}{(1+y\zeta_p^l)(1+y\zeta_p^{2l})}\right) = \frac{s}{2} - \frac{1}{4}\sum_{l\in\mathbf{l}} \frac{1}{1-\Re\zeta_p^{2l}}\left(1+2\frac{\cos(\arg y/2)}{\cos((\arg y)/2+\arg\zeta_p^l)}\right). \tag{50}$$

Proof. From

$$\begin{aligned} s &= \sum_{l\in\mathbf{l}} \frac{1+y\zeta_p^l+y\zeta^{2l}+y^2\zeta^{3l}}{(1+y\zeta_p^l)(1+y\zeta_p^{2l})} \\ &= \sum_{l\in\mathbf{l}} \frac{1}{(1+y\zeta_p^l)(1+y\zeta_p^{2l})} + \sum_{l\in\mathbf{l}} \frac{1}{(1+\overline{y}\zeta_p^{-l})(1+\overline{y}\zeta_p^{-2l})} \\ &\quad + \sum_{l\in\mathbf{l}} \frac{y\zeta_p^l(1+\zeta_p^l}{(1+y\zeta_p^l)(1+y\zeta_p^{2l})} \end{aligned}$$

we obtain

$$\Re\left(\sum_{l\in\mathbf{l}} \frac{1}{(1+y\zeta_p^l)(1+y\zeta_p^{2l})}\right) = \frac{s}{2} - \frac{1}{2}\sum_{l\in\mathbf{l}} \frac{y\zeta_p^l(1+\zeta_p^l}{(1+y\zeta_p^l)(1+y\zeta_p^{2l})} = \frac{s}{2} - \frac{1}{2}S(y),$$

where the mapping $y \mapsto S(y)$, $y \neq -\zeta_p^{-l}$, is continuous. In particular,

$$S(-1) = -\sum_{l\in\mathbf{l}} \frac{\zeta_p^l}{(1-\zeta_p^l)^2} = \frac{1}{2}\sum_{l\in\mathbf{l}} \frac{1}{1-\Re\zeta_p^l}.$$

By using a partial fraction expansion it follows that $S(y)$, $y \neq 1$, can be represented by

$$\begin{aligned} S(y) &= \frac{1-y}{1+y}\sum_{l\in\mathbf{l}} \frac{1}{(1+y\zeta_p^l)} - \frac{1-y}{1+y}\sum_{l\in\mathbf{l}} \frac{1}{1+y\zeta_p^{2l}} + \frac{2y}{1+y}\sum_{l\in\mathbf{l}} \frac{\zeta_p^l}{1+y\zeta_p^{2l}} \\ &= \frac{2y}{1+y}\sum_{l\in\mathbf{l}} \frac{1}{\zeta_p^{-l}+y\zeta_p^l}. \end{aligned}$$

Since $S(-1)$ is finite, it follows that

$$\sum_{l\in\mathbf{l}} \frac{1}{\zeta_p^{-l}-\zeta_p^l} = 0$$

and consequently

$$\begin{aligned}
S(y) &= \frac{2y}{1+y}\sum_{l\in\mathbf{l}}\left(\frac{1}{\zeta_p^{-l}+y\zeta_p^l}-\frac{1}{\zeta_p^{-l}-\zeta_p^l}\right)\\
&= -\Re\left(\sum_{l\in\mathbf{l}}\frac{y\zeta_p^l}{(\zeta_p^{-l}+y\zeta_p^l)(\zeta_p^{-l}-\zeta_p^l)}\right)\\
&= -\sum_{l\in\mathbf{l}}\frac{\zeta_p^{-l}-y\zeta_p^l}{(\zeta_p^{-l}+y\zeta_p^l)(\zeta_p^{-l}-\zeta_p^l)}\\
&= S(-1)-\sum_{l\in\mathbf{l}}\left(\frac{\zeta_p^{-l}-y\zeta_p^l}{(\zeta_p^{-l}+y\zeta_p^l)(\zeta_p^{-l}-\zeta_p^l)}-\frac{\zeta_p^{-l}+\zeta_p^l}{(\zeta_p^{-l}-\zeta_p^l)^2}\right)\\
&= S(-1)-\sum_{l\in\mathbf{l}}\frac{1+y}{(\zeta_p^{-l}-\zeta_p^l)^2(\zeta_p^{-l}+y\zeta_p^l)}\\
&= \frac{1}{2}\sum_{l\in\mathbf{l}}\frac{1}{1-\Re\zeta_p^l}+\sum_{l\in\mathbf{l}}\frac{1}{1-\Re\zeta_p^{2l}}\frac{1+y}{\zeta_p^{-l}+y\zeta_p^l},
\end{aligned}$$

which proves (50). □

The essential difference between the proofs of Theorem 2 and Theorem 10 is that you have to take into account the sign of

$$\frac{\cos(m\pi/r)}{\cos(m\pi/r+\arg\zeta_p^l)}.$$

Let $\mathbf{l}^-$ denote the set of those $l\in\mathbf{l}$ such that this sign is negative. Then it is an easy exercise to show that

$$\sum_{l\in\mathbf{l}^-}\frac{1}{1-\Re\zeta_p^{2l}}\frac{\cos(m\pi/r)}{\cos(m\pi/r+\arg\zeta_l)}=\mathcal{O}_r(p\log p).$$

You only have to verify that $1-\Re\zeta_p^{2l}>c_r$ for $l\in\mathbf{l}^-$ and that $\arg\zeta_l$ is different for different $l\in\mathbf{l}$. Hence, if $p>C_r(t\log p)^4$ (for a sufficiently large constant $C_r>0$), then

$$\frac{1}{8\pi 1^2}\frac{p^{3/2}}{t^2\log p}>\frac{p}{2t}+\mathcal{O}_r(p\log p),$$

which implies that $\Re(S_{p,0}(\zeta_r^m,2^{2as-2}))<0$ for sufficiently large a.

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Department of Geometry, Technical University of Vienna, Wiedner Hauptstrasse 8-10, A-1040 Vienna, Austria
E-mail address: michael.drmota@tuwien.ac.at

Institute of Mathematics, Warsaw University, Banacha 2, 02-097 Warsaw, Poland
E-mail address: skalba@mimuw.edu.pl