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ON CHOW MOTIVES OF 3-FOLDS

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Abstract. Let k be a field of characteristic zero. For every smooth, projective k-variety Y of dimension n which admits a connected, proper morphism $f : Y \to S$ of relative dimension one, we construct idempotent correspondences (projectors) $\pi_{ij}(Y) \in CH^n(Y \times Y, \mathbb{Q})$ generalizing a construction of Murre. If $n = 3$ and the transcendental cohomology group $H^2_{\mathrm{tr}}(Y)$ has the property that $H^2_{\mathrm{tr}}(Y, \mathbb{C}) = f^* H^2_{\mathrm{tr}}(S, \mathbb{C}) + \mathrm{Im}(f^* H^1(S, \mathbb{C}) \otimes H^1(Y, \mathbb{C}) \to H^2_{\mathrm{tr}}(Y, \mathbb{C}))$, then we can construct a projector $\pi_2(Y)$ which lifts the second Künneth component of the diagonal of Y. Using this we prove that many smooth projective 3-folds X over k admit a Chow-Künneth decomposition $\Delta = p_0 + ... + p_6$ of the diagonal in $CH^3(X \times X, \mathbb{Q})$.

0. Introduction

Definition. Let $X \in Sm(k)$ be a smooth, projective $k-$variety of dimension d. We say that X has a **Chow-Künneth decomposition**, if there exist projectors $p_0, p_1, ..., p_{2d}$ in $CH^d(X \times X) \otimes \mathbb{Q}$ such that the following properties hold:

(1) $p_j \circ p_i = \delta_{ij} \cdot p_i$.

(2) $\Delta = \sum p_i$.

(3) In étale cohomology the p_i induce the $(2d - i, i)-$th Künneth component of the diagonal.

We say that X has a **Murre decomposition**, if additionally the following properties hold:

(4) $p_0, ..., p_{j-1}$ and $p_{2j+1}, .., p_{2d}$ act trivially on $CH^j(X) \otimes \mathbb{Q}$.

(5) If we put $F^0 CH^j(X) \otimes \mathbb{Q} = CH^j(X) \otimes \mathbb{Q}$ and inductively $F^k CH^j(X) \otimes \mathbb{Q} := Ker(p_{2j+1-k} \mid_{F^{k-1}})$, then this descending filtration is intrinsic.

(6) Always $F^1 CH^j(X) \otimes \mathbb{Q} = CH^j_{hom}(X) \otimes \mathbb{Q}$.

Finally a Chow-Künneth decomposition $\{p_i(X)\}$ of X is called **special**, if the following properties are satisfied: $p_\ell(X)$ is supported on $V_\ell \times X$ for every $0 \leq \ell \leq d - 1$, where $V_\ell \subset X$ is a closed subset of dimension ℓ and $p_{d-\ell} = p^{\mathrm{tr}}_{d+\ell}$ for all ℓ.

Conjecture (J. Murre)**.** *Every smooth projective variety has a Murre decomposition.*

Parts of (1)–(6) have been proved for curves [K], [M], surfaces [Mu2], products of a curve and a surface [Mu1], uniruled 3-folds [dAMS], elliptic modular 3-folds [GM],

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abelian varieties [Sh],[B],[DM],[Kü] certain moduli spaces [dBR] and generalized flag varieties [Kö].

In this paper we want to generalize the results of [dAMS], where certain correspondences were constructed in order to obtain a Chow-Künneth decomposition (see [Mu1]) of a smooth uniruled projective 3-fold X over $\mathbb{C}$:

Theorem 1 ([dAMS])**.** *Let X be a uniruled complex projective 3-fold. Then X admits a special Chow-Künneth decomposition with properties (4) and (6) of a Murre decomposition.*

This construction worked mainly, because - in the case where X is birational to a conic bundle - a certain blow-up Y of X has the property that there is a morphism $f : Y \to S$ such that the general fiber is a smooth rational curve.

The main observation in this paper is that one can in certain cases allow that the generic fiber is of positive genus: let X be a smooth, projective 3-fold over a field k of characteristic zero. Let $\sigma : Y \to X$ be a blow-up and assume that there is a connected proper morphism $f : Y \to S$, generically smooth of relative dimension one onto a smooth surface S. Let $i : Z \hookrightarrow Y$ be a smooth divisor finite over S. Then there exist projectors (=idempotent correspondences) in $Y \times Y$,

$$\pi_{i0} := \frac{1}{m}(i \times 1)_*(h \times f)^*\pi_i(S),$$
$$\pi_{i2} := \frac{1}{m}(1 \times i)_*(f \times h)^*\pi_i(S),$$
$$\pi_{i1} := \frac{1}{\deg(\epsilon)}(f \times f)^*(\pi_i(S)) \cdot D.$$

for $0 \leq i \leq 4$ which can be combined to form a Chow-Künneth decomposition of Y (and therefore later on X) if the following condition is satisfied:

Definition. Let $f : Y \to S$ be as above. We say that f **decomposes** $H^2_{\mathrm{tr}}(Y)$, if $H^2_{\mathrm{tr}}(Y) = f^*H^2_{\mathrm{tr}}(S, \mathbb{C}) + \mathrm{Im}(f^*H^1(S, \mathbb{C}) \otimes H^1(Y, \mathbb{C}) \to H^2_{\mathrm{tr}}(Y))$.

Here we defined $H^2_{\mathrm{tr}}(Y) := H^2(Y, \mathbb{C})/NS(Y) \otimes \mathbb{C}$. This notion is a generalization of isotriviality and means that the cohomology classes in $H^2_{\mathrm{tr}}(Y, \mathbb{C})$ behave somewhat like the situation where Y is the product of S with some curve. In this paper we therefore prove the following generalization of theorem 1:

Theorem 2. *Let X be a smooth projective 3-fold over a field of characteristic 0 such that a blow-up Y of X has a fibration $f : Y \to S$ which decomposes $H^2_{tr}(Y)$. Then X admits a special Chow-Künneth decomposition.*

Examples. Let S be a smooth projective surface. Consider a smooth ample hypersurface Y in $S \times S_1$, where S_1 is a smooth, projective surface with $H^2_{\mathrm{tr}}(S_1) = 0$. Let $f = pr_1$ be the first projection. By the Lefschetz theorems, $f = pr_1 : Y \to S$ decomposes $H^2_{\mathrm{tr}}(Y)$. Cyclic branched coverings of $S \times C$ (C a smooth curve) also have this property.

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1. New correspondences

In this section we generalize the construction of J. Murre from [Mu2]. Let S be an $n-1$-dimensional smooth projective variety and assume that there exist a Chow-Künneth decomposition for S given by projectors $\pi_0(S), \dots, \pi_{2n-2}(S)$ satisfying $\pi_i(S)^{\mathrm{tr}} = \pi_{2n-2-i}(S)$. We assume that $f : Y \longrightarrow S$ is a proper, connected morphism generically smooth of relative dimension one. Let $Z \subset Y$ be a smooth divisor and assume that the morphism $h = f|_Z : Z \longrightarrow S$ has finite fibers.

From now on suppose that the ground field k is big enough such that Z and f are defined over k. This is not a serious restriction by [Sch, 1.17]. Let e be a $k-$rational point in S, $F = f^{-1}(e)$, p a $k-$rational point in $F \cap Z$ and $m = (Z \cdot F)$. There are natural morphisms

$$Pic^0(Y) \longrightarrow Pic^0(F) \cong Alb(F) \longrightarrow Alb(Y)$$

which do not induce, in general, an isogeny. Nevertheless, if one takes $K = ker\ (Alb(Y) \longrightarrow Alb(S))$, this maps does induce an isogeny $\epsilon : K^\vee \to K$ which has a factorization

$$\epsilon : K^\vee = Pic^0(Y)/f^*Pic^0(S) \to Pic^0(F) \cong Alb(F) \to K.$$

Here the point is that the restriction map $H^1(Y)/f^*H^1(S) \to H^1(F)$ is injective in our situation. This can be seen as follows: Let $\Sigma \subset S$ be the non-smooth locus of f and $T := f^{-1}(\Sigma)$. Using the localization sequence on Y and S we obtain a commutative diagram

$$\begin{array}{ccccccc} 0 & \to & H^1(S) & \to & H^1(S \setminus \Sigma) & \to & H^2_\Sigma(S) \\ & & \downarrow f^* & & \downarrow f^* & & \cap \\ 0 & \to & H^1(Y) & \to & H^1(Y \setminus T) & \to & H^2_T(Y) \end{array}$$

which, together with the degeneration of the Leray spectral sequence over $S \setminus \Sigma$, shows that $H^1(Y)/f^*H^1(S) \subset H^0(S \setminus \Sigma, R^1 f_* \mathbb{C}) \subset H^1(F)$ maps isomorphically to the invariant part of $H^1(F)$.

One has an inverse isogeny $\epsilon^{\mathrm{tr}} : K \to K^\vee$. Now we can define a map $Y \to K$ by composing the Albanese map $alb : Y \to Alb(Y)$ with base point p with the projection $Alb(Y) \to K$, which is induced by the multisection $Z \subset Y$. Thus we obtain a map

$$Y \times Y \to K \times K \xrightarrow{id \times \epsilon^{\mathrm{tr}}} K \times K^\vee$$

and we let D be the pullback of the normalized Poincaré divisor. This divisor D will play an essential role in what follows and has the following properties:

$$(*) : D \cap \{p\} \times Y = Y \times \{p\} \cap D = 0, \quad (f \times 1)_* D = 0.$$

The last property follows from the fact that the degree of D over $S \times Y$ is zero by definition of D.

Lemma 1. *With the notation above, one has*

$$(1 \times f \times 1)_*(D \times Y \cap Y \times D) = deg(\epsilon)(pr_{13}^{Y \times S \times Y})^* D.$$

Proof. By the theorem of the cube it is enough to show that the left hand side and the right hand side coincide when restricted to $Y \times S \times \{p\}$, $Y \times \{e\} \times Y$ and $\{p\} \times S \times Y$.

For the right hand side one has

$$(pr_{13}^{Y\times S\times Y})^* D \cap \{p\} \times S \times Y = (pr_{13}^{Y\times S\times Y})^*(D \cap \{p\} \times Y) = 0,$$

while for the left hand side one gets

$$\begin{aligned}&(1 \times f \times 1)_*(D \times Y \cap Y \times D) \cap \{p\} \times S \times Y \\ &\quad = (1 \times f \times 1)_* \left[(D \cap \{p\} \times Y) \times Y \cap Y \times D\right] = 0\end{aligned}$$

by $(*)$. In a similar fashion

$$(pr_{13}^{Y\times S\times Y})^* D \cap Y \times S \times \{p\} = (pr_{13}^{Y\times S\times Y})^*(D \cap Y \times \{p\}) = 0$$

and

$$\begin{aligned}&(1 \times f \times 1)_*(D \times Y \cap Y \times D) \cap Y \times S \times \{p\} \\ &\quad = (1 \times f \times 1)_* \left[Y \times (D \cap Y \times \{p\}) \cap D \times Y\right] = 0\end{aligned}$$

by $(*)$. Finally the right hand side restricted to $Y \times \{e\} \times Y$ is simply $\deg(\epsilon) \cdot D$ whereas the left hand side we abbreviate by E for a moment. Observe that E also is a pull back from $K \times K$, by the following base change diagram (using [Fu, 1.7])

$$\begin{array}{ccccc} Y \times F \times Y & \to & K \times F \times K & \hookrightarrow & K \times K \times K \\ \downarrow & & \downarrow & & \\ Y \times \{e\} \times Y & \to & K \times \{e\} \times K & & \end{array}$$

By the universal property of Poincaré divisors (see [Mum]) it is sufficient to check that E acts as a correspondence $E : K \to K^\vee$ in the same way as $\deg(\epsilon) \cdot D$ does. Let W be a zero cycle such that its Albanese image is in K. Then, by a computation in the following diagram

$$\begin{array}{ccc} Y \times F \times Y & \xrightarrow{1\times i\times 1} & Y \times Y \times Y \\ {\scriptstyle 1\times f\times 1}\downarrow & & \downarrow {\scriptstyle pr_3} \\ Y \times \{e\} \times Y & \xrightarrow{pr_3} & Y \end{array}$$

we finally compute

$$E(W) = pr_{2*}^{Y\times Y}(W \times Y \cap E)$$

$$= pr_{3*}^{Y\times Y\times Y}[D \times Y \cap Y \times D \cap W \times Y \times Y \cap Y \times F \times Y]$$

$$= pr_{2*}^{Y\times Y} pr_{23*}^{Y\times Y\times Y}[D \times Y \cap W \times Y \times Y \cap Y \times D \cap Y \times F \times Y]$$

$$= pr_{2*}^{Y\times Y}[\epsilon^{tr}(W) \times Y \cap F \times Y \cap D]$$

$$= pr_{2*}^{Y\times Y}[\deg(\epsilon) \cdot W \times Y \cap D] = \deg(\epsilon) D(W).$$

In the above computation we used the following fact: the composition of the following maps and correspondences equals $\epsilon \circ \epsilon^{tr} = \deg(\epsilon) \cdot Id$:

$$K \hookrightarrow Alb(Y) \xrightarrow{D} Pic^0(Y) \to K^\vee \to Pic^0(F) \cong Alb(F) \to K.$$

Hence E and $\deg(\epsilon) \cdot D$ act identically on K. That proves the assertion. $\square$

For $j = 0, 1, 2$ and $0 \leq i \leq 2n-2$ one defines a cycle π_{ij} as follows:

$$\pi_{i0} := \frac{1}{m}(i \times 1)_*(h \times f)^*\pi_i(S),$$
$$\pi_{i2} := \frac{1}{m}(1 \times i)_*(f \times h)^*\pi_i(S),$$
$$\pi_{i1} := \frac{1}{\deg(\epsilon)}(f \times f)^*(\pi_i(S)) \cdot D.$$

Proposition 1. *The π_{ij} satisfy the following properties:*
(1) $\pi_{ik} \circ \pi_{jk} = \delta_{ij}\pi_{ik}$, in particular all π_{ik} are projectors.
(2) $\pi_{i1} \circ \pi_{j0} = 0$.
(3) $\pi_{i2} \circ \pi_{j1} = 0$.
(4) $\pi_{i2} \circ \pi_{j0} = 0$.

Proof. If no π_{i1} is involved, the result follows easily as in [dAMS]. On the other hand

$$\begin{aligned}
&\deg(\epsilon)^2(\pi_{i1} \circ \pi_{j1}) \\
&= (pr_{13}^{Y\times Y\times Y})_*[(f \times f \times f)^*(\pi_j(S) \times S \cap S \times \pi_i(S)) \cap D \times Y \cap Y \times D] \\
&= (pr_{13})_*[(f \times 1 \times f)^*(\pi_j(S) \times S \cap S \times \pi_i(S)) \cap (1 \times f \times 1)_*(D \times Y \cap Y \times D)] \\
&= \deg(\epsilon)^2 \cdot \delta_{ij}\pi_{i1}
\end{aligned}$$

since $(1 \times f \times 1)_*(D \times Y \cap Y \times D) = \deg(\epsilon)(pr_{13}^{Y\times S\times Y})^*D$ by lemma 1.

For $\pi_{i1} \circ \pi_{j0}$ (or $\pi_{j2} \circ \pi_{i1}$), one needs that $(f \times 1)_*D = 0$ by $(*)$. □

Proposition 2. *$\pi_{02}, \pi_{11}, \pi_{20}$ satisfy the following additional properties:*
(1) $\pi_{20} = id$ on $f^*H^2(S, \mathbb{Q})$.
(2) $\pi_{02} = id$ on $\mathbb{Q} \cdot [Z]$ and $\pi_{02} = 0$ on $f^*H^2_{tr}(S, \mathbb{Q})$.
(3) $\pi_{11} = 0$ on $f^*H^2(S, \mathbb{Q})$ and $\pi_{11} = id$ on

$$Im(f^*H^1(S, \mathbb{Q}) \otimes H^1(Y, \mathbb{Q}) \to H^2_{tr}(Y, \mathbb{Q})/Imf^*H^2(S, \mathbb{Q})).$$

(4) If furthermore $f : Y \to S$ decomposes $H^2_{tr}(Y)$, then $\pi_2 := \pi_{20} + \pi_{02} + \pi_{11} - \pi_{20}\pi_{02} - \pi_{11}\pi_{02} - \pi_{20}\pi_{11} + \pi_{20}\pi_{11}\pi_{02}$ is a projector that acts as the identity on $H^2_{tr}(Y, \mathbb{Q})$.

Proof. (1) and (2) are proven in [dAMS]. (3): On $f^*H^2(S, \mathbb{Q})$ we have by (1) that $\pi_{20} = id$ therefore $\pi_{11} = \pi_{11} \circ \pi_{20} = 0$ on $f^*H^2(S, \mathbb{Q})$ by proposition 1. To prove the rest of (3) we have to make an explicit computation. Let

$$f^*\alpha \wedge \beta \in \mathrm{Im}(f^*H^1(S, \mathbb{Q}) \otimes H^1(Y, \mathbb{Q}) \to H^2_{\mathrm{tr}}(Y, \mathbb{Q}))$$

be representing differential forms. Then (as currents)

$$\pi_{11}(f^*\alpha \wedge \beta) = \frac{1}{\deg(\epsilon)} pr_{2*}^{Y\times Y}[(f \times f)^*\pi_1(S) \cap D \cap (f^*\alpha \wedge \beta) \times Y]$$

$$= \frac{1}{\deg(\epsilon)} pr_{2*}^{S\times Y}[(1 \times f)^*(\pi_1(S) \cap \alpha \times S) \cap (f \times 1)_*(D \cap \beta \times Y)]$$

by the projection formula. Now let $\gamma := (f \times 1)_*(D \cap \beta \times Y)$, a 1-current on $S \times Y$. By the Künneth formula, we obtain $\gamma = pr_1^*(\gamma_1) + pr_2^*(\gamma_2)$. Hence we get

$$\begin{aligned}\pi_{11}(f^*\alpha \wedge \beta) &= \frac{1}{\deg(\epsilon)} pr_{2*}^{S\times Y}[(1 \times f)^*(\pi_1(S) \cap (\alpha \wedge \gamma_1) \times Y)] \\ &\quad + \frac{1}{\deg(\epsilon)} pr_{2*}^{S\times Y}[(1 \times f)^*(\pi_1(S) \cap \alpha \times Y)] \wedge \gamma_2 \\ &= \frac{1}{\deg(\epsilon)}[f^*\pi_1(S)(\alpha \wedge \gamma_1) + f^*\alpha \wedge \gamma_2]\end{aligned}$$

since $\pi_1(S)$ is the identity on $H^1(S)$.

But the first term is zero, since $\pi_1(S)$ acts trivially on $H^2(S)$. Therefore we have shown that $\pi_{11}(f^*\alpha \wedge \beta) = \frac{1}{\deg(\epsilon)} f^*\alpha \wedge \gamma_2$. It remains to prove that

$$\beta - \frac{1}{\deg(\epsilon)}\gamma_2 \in f^*H^1(S, \mathbb{Q}).$$

Recall that $H^1(Y)/f^*H^1(S) \hookrightarrow H^1(F)$ is an injection. Thus we have only to show that after restriction to $F \times F$ one has

$$\frac{1}{\deg(\epsilon)}(f \times 1)_*(D|_{F\times F} \cap \delta \times F) = \delta$$

for all $\delta \in H^1(K) \subset H^1(F, \mathbb{Q})$. However on $F \times F$ the map $(f \times 1)_*$ equals pr_{2*} and the assertion follows from the fact that $\frac{1}{\deg(\epsilon)} pr_{2*}(D|_{F\times F} \cap pr_1^*\delta) = \delta$, which in turn is a consequence of the fact that the projector $\frac{1}{\deg(\epsilon)} D|_{F\times F} \in CH^1(F \times F, \mathbb{Q})$ induces by construction the projection map $H^1(F) \to H^1(K) \subset H^1(F)$. This proves (3).

Finally (4) is a formal consequence of (1)-(3). If a, b, c are elements of a ring R with $a^2 = a, b^2 = b, c^2 = c, ba = ca = cb = 0$ and we set $p = a + b + c - ab - ac - bc + abc$, then $p^2 = p$. If furthermore H is a vector space where a, b, c act as endomorphisms and preserve a subspace $A \subset H$, then p acts as the identity on H, whenever $a = id$ on A, $b = c = 0$ on A and $b = id$ on H/A. We apply this to $a = \pi_{20}, b = \pi_{11}, c = \pi_{02}$ and $H = H^2_{\mathrm{tr}}(Y, \mathbb{Q})$, $A = \mathrm{Im}(f^*H^2(S, \mathbb{Q}))$. □

2. Proof of theorem 2

Let X be a smooth projective n-fold over a field k of characteristic zero together with a rational point $e \in X(k)$. By [Sch, 1.17] we may again assume that k is algebraically closed. First we need two lemmas about blow-ups on X.

Lemma 2. *Let $\varphi : Y \to X$ be a consecutive blow up of X along several smooth centers W_j of dimension ≤ 1 and $q \in CH^n(Y \times Y)$ be a projector which is supported in $H \times Y$ for some effective irreducible divisor $H \subset Y$ and acts trivially or as the identity on $H^{2n-2}(Y, \mathbb{Q})$. Then $p := (\varphi \times \varphi)_* q \in CH^n(X \times X)$ is a projector and for all $\alpha \in CH^*(X)$ on has $p(\alpha) = \varphi_* q(\varphi^*(\alpha))$. Moreover if one has two orthogonal projectors q_1, q_2 on Y satisfying the same properties, then the corresponding projectors p_1 and p_2 are again orthogonal.*

Proof. By induction on the number of blow-ups we may assume that there is just one blow-up along a smooth subvariety $W \subset X$. Let $E \subset Y$ be the exceptional divisor and $T \subset \varphi(H) \times X$ the algebraic subset $T := \varphi(H) \times W \cup (\varphi(H) \cap W) \times X$.

We have to show that the cycle $\mathfrak{C} := p \circ p - p \in CH_n(\varphi(H) \times X)$ is trivial. Consider the localization sequence

$$CH_n(T) \xrightarrow{j_*} CH_n(\varphi(H) \times X) \to CH_n(U) \to 0$$

where $U := \varphi(H) \times X \setminus T$. Since q is a projector, the image of $\mathfrak{C}$ in $CH_n(U)$ vanishes. Therefore $\mathfrak{C} = j_* Q$. If $\dim(W) = 0$, we have $Q = 0$ by dimension reasons. If $\dim(W) = 1$, one gets that

$$Q = a(\varphi(H) \times W) + b((\varphi(H) \cap W) \times X)$$

with $a, b \in \mathbb{Z}$.

But for every cohomology class $\alpha \in H^{2n-2}(X, \mathbb{Q})$, $p(\alpha) = \varphi_* q(\varphi^*(\alpha))$. If q is zero or the identity on $H^{2n-2}(X, \mathbb{Q})$, this implies that $p(\alpha)$ is zero or equal to α. In both cases $\mathfrak{C}(\alpha) = 0$ for every $\alpha \in H^{2n-2}(X, \mathbb{Q})$ or $\alpha \in H^0(X, \mathbb{Q})$. Therefore $a = b = 0$ and p is a projector.

If q_1, q_2 are two orthogonal projectors, then one concludes in a similar way that $p_1 \circ p_2$ is supported on T and therefore it vanishes by the same argument as above. □

Let $\pi_i(S)$ be a special Chow-Künneth decomposition for S. Keeping the notation $\pi_{ij}(Y) = \pi_{ij}$ of the previous section, we have a projector

$$\pi_2 := \pi_{20} + \pi_{02} + \pi_{11} - \pi_{20}\pi_{02} - \pi_{11}\pi_{02} - \pi_{20}\pi_{11} + \pi_{20}\pi_{11}\pi_{02}$$

on Y by propositions 1 and 2.

Now we choose a Zariski open subset $U \subset Y$ such that $f : U \to f(U)$ is contained in the smooth part of f. Let $T := Y \setminus U \overset{\iota}{\hookrightarrow} Y$ be the complement. From now on we also will assume that (after further blow-ups of Y) we are in the situation where $T = \bigcup T_i$ is a divisor with strict normal crossings. Additionally we choose U small enough so that $f : Z \cap U \to f(U)$ is étale.

The Leray spectral sequence for f on U

$$H^k(f(U), R^\ell f_* \mathbb{C}) \Longrightarrow H^{k+\ell}(U, \mathbb{C})$$

degenerates at E_2 and each term has a mixed Hodge structure by [Sa] whose lowest weight piece $W_0 H^k(f(U), R^\ell f_* \mathbb{C})$ converges to

$$W_0 H^{k+\ell}(U, \mathbb{C}) = \mathrm{Im}(H^{k+\ell}(Y, \mathbb{C}) \to H^{k+\ell}(U, \mathbb{C})).$$

Lemma 3. *The projectors $\pi_{ij}|_{U \times U}$ act trivially on $W_0 H^k(f(U), R^\ell f_* \mathbb{C})$ if $i \neq k$ or $j \neq \ell$. In particular by the Leray spectral sequence π_{11} acts trivially on $W_0 H^m(U, \mathbb{C})$ if $m \neq 2$.*

Proof. Let

$$\mathfrak{D}_U^m = \bigoplus_{r+s=m} \mathfrak{D}_U^{r,s}$$

be the sheaf of complex valued $m-$currents on U. The sequences

$$0 \to f^* \mathfrak{D}^1_{f(U)} \to \mathfrak{D}^1_U \to \mathfrak{D}^1_{U/f(U)} \to 0$$

are canonically split due to the assumption that $Z \cap U$ is étale over $f(U)$. Therefore we get a canonical decomposition

$$\mathfrak{D}_U^m = \bigoplus_{a+b+c+d=m} f^* \mathfrak{D}^{a,b}_{f(U)} \wedge \mathfrak{D}^{c,d}_{U/f(U)}$$

and any element of $H^i(f(U), R^j f_*\mathbb{C})$ may be represented by a sum of currents of the form $\omega = f^*\alpha \wedge \beta \in f^*\mathfrak{D}^{a,b}_{f(U)} \wedge \mathfrak{D}^{c,d}_{U/f(U)}$ with $a+b=i, c+d=j$.

As long as we only consider cohomology of weight zero, which is the image of the cohomology of the compactification, all direct images are defined and therefore currents behave well under correspondences. Every subvariety W of codimension r has an $(r,r)-$current δ_W associated to it. If $\omega = f^*\alpha \wedge \beta \in f^*\mathfrak{D}_f(U)^{a,b} \wedge \mathfrak{D}^{c,d}_{U/f(U)}$, then $\pi_{i0}(\omega) = \frac{1}{m} i_* h^* \pi_i(S)[\alpha \wedge h_*(\beta|_Z)]$. $\beta|_Z = 0$ unless $c = d = 0$ (Z being finite over S). This proves that $\pi_{i0} = 0$ on every group $H^k(f(U), R^\ell f_*\mathbb{C})$ for $\ell \geq 1$ and acts as δ_{ik} on $H^k(f(U), f_*\mathbb{C})$, because $h_*(1) = m \in \mathfrak{D}^{0,0}_{f(U)}$.

Similarly $\pi_{i2}(\omega) = h^*\pi_i(S)(\alpha \wedge f_*\beta)$ (see [dAMS, thm. 3.3.]). This is nonzero in cohomology only when $c = d = 1$ and $a + b = i$, proving the assertion for π_{i2}.

In the last case

$$\pi_{i1}(\omega) = pr_{2*}^{f(U)\times U}[(1\times f)^*(pr_1^*(\alpha) \wedge \delta_{\pi_i(S)}) \wedge (f\times 1)_*(\delta_D \wedge pr_1^*(\beta))]$$

The last term $(f\times 1)_*(\delta_D \wedge pr_1^*(\beta))$ is zero unless $c+d=1$ by a type argument, in which case one gets

$$\pi_{i1}(\omega) = pr_{2*}^{f(U)\times U}[(1\times f)^*(pr_1^*(\alpha) \wedge \delta_{\pi_i(S)})] \wedge \gamma = f^*\pi_i(S)(\alpha) \wedge \gamma$$

with $\gamma \in \mathfrak{D}^{c,d}_{U/f(U)}$. This is zero if $a + b \neq i$. □

Lemma 4. *π_2 operates trivially on $H^3(Y, \mathbb{C})$.*

Proof. Take $\beta \in H^3(Y, \mathbb{Q})$ and check that $\pi_{20}(\beta) = \pi_{02}(\beta) = 0$ by using the formulas $\pi_{j0}(\beta) = \frac{1}{m} f^*\pi_j(S)(h_*(\beta \cap Z))$ and $\pi_{j2}(\beta) = \frac{1}{m} i_* h^* \pi_j(S)(f_*\beta)$ and the fact that $\pi_i(S)$ satisfies property (3) of a Chow-Künneth decomposition.

It remains to show that $\pi_{11}(\beta) = 0$. Look at the long exact sequence for cohomology with supports (with T, U as defined above):

$$\ldots \to H^2(U, \mathbb{C}) \to H^3_T(Y, \mathbb{C}) \xrightarrow{\iota_*} H^3(Y, \mathbb{C}) \xrightarrow{j^*} H^3(U, \mathbb{C}) \to \ldots .$$

By the purity of the Hodge structure $H^3(Y, \mathbb{C})$, the image of ι_* is generated by the image of $\bigoplus H^1(T_i, \mathbb{C})$ under the sum of the Gysin maps.

Lemma 4 follows therefore from two assertions:

(1) $\pi_{11}(\beta)$ is an element of $\mathrm{Im}(H^3_T(Y, \mathbb{C}) \xrightarrow{\iota_*} H^3(Y, \mathbb{C}))$.

(2) π_{11} acts trivially on the image of every $H^1(T_i, \mathbb{C})$ in $H^3(Y, \mathbb{C})$.

Ad (1): By Chow's moving lemma, we may assume that π_{11} intersects $T \times Y$ properly. This implies that $j^*\pi_{11}(\beta) = (\pi_{11}|_{U\times U})(j^*(\beta))$ which vanishes by lemma 3. Therefore $\pi_{11}(\beta) \in \mathrm{Im}(H^3_T(Y, \mathbb{C}) \xrightarrow{\iota_*} H^3(Y, \mathbb{C}))$.

Ad (2): It is even enough to show that π_{11} is zero on the image of all maps

$$Pic^0(T_i) \xrightarrow{\iota_*} J^2(Y)$$

to Griffiths' intermediate Jacobian. Namely if we know that

$$\tau = \pi_{11} : J^2(Y) \to J^2(Y)$$

is the zero map, then the continuous lifting

$$\bar{\tau} : H^3(Y, \mathbb{C})/F^2 \to H^3(Y, \mathbb{C})/F^2$$

also has to vanish. By using complex conjugation this in turn implies that the resulting map on $H^3(Y)$ is trivial.

Now consider a class $\beta \in Pic^0(T)$ (without loss of generality we may assume that T is irreducible). Then, using the projector $\pi_1(S) = (C \times S) \cap D_S$ of [Mu2], we obtain

$$\pi_{11}(\iota_*\beta) = pr_{2*}[(f \times f)^*(C \times S \cap D_S) \cap D \cap (\iota_*\beta \times Y)]$$

$$= pr_{2*}^{T\times Y}[(g \times f)^*(C \times S \cap V \times S \cap D_S) \cap (\iota \times 1)^*(D) \cap (\beta \times Y)]$$

Here $V := f(T) \overset{j}{\hookrightarrow} S$ sits in the commutative diagram

$$\begin{array}{ccc} T & \overset{\iota}{\hookrightarrow} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ V & \overset{j}{\hookrightarrow} & S \end{array}$$

C can be chosen to intersect V properly in a finite set, so

$$\pi_{11}(\iota_*\beta) = pr_{2*}^{T\times Y}[(\sum_{Q\in C\cap V} g^*Q \times f^*W(Q)) \cap (\iota \times 1)^*(D) \cap (\beta \times Y)]$$

where $W(Q) \in Pic^0(S)$ and hence by suitable rewriting

$$\pi_{11}(\iota_*\beta) = \sum_{P\in g^*(C\cap V)\cap|\beta|} pr_{2*}[a_P P \times (W(P) \cdot D(P))]$$

for some classes $W(P), D(P) \in Pic^0(Y)$ and $a_P \in \mathbb{Z}$. The lemma follows then from the fact that the cup product map

$$Pic^0(Y) \times Pic^0(Y) \to J^2(Y)$$

in Deligne cohomology is zero by [EV]. □

Remark. The above argument can also be applied to the projector $\pi_1(S)$ as defined in [Sch], since one only needs to add a term of the form $\frac{1}{2}\pi_3 \circ \pi_1$, with the one of [Mu2], therefore a similar argument shows that this new term also acts as zero.

Lemma 5. *Assume that $f : Y \to S$ decomposes $H^2_{tr}(Y)$. Then for a suitable choice of curves $\ell_i \subset Y$ $(1 \le i \le r = rkNS(Y) \otimes \mathbb{Q})$ and a matrix $B = (b_{ij}) \in Mat(r \times r, \mathbb{Q})$, the modified projector*

$$p_2(Y) = \pi_2(Y) + \sum b_{i,j}(\ell_i \times T_j) - \sum b_{i,j}(\ell_i \times T_j) \circ \pi_2(Y)$$

acts as the identity on $H^2(Y, \mathbb{Q})$.

Proof. By proposition 2 we know that $\pi_2(Y)$ acts as the identity on $H^2_{\mathrm{tr}}(Y, \mathbb{Q})$. Let $A \in \mathrm{Mat}(r \times r, \mathbb{Q})$ be the matrix describing the action of $\pi_2(Y)$ on $NS(Y) \otimes \mathbb{Q}$. By the Hodge conjecture on Y, there exist suitable curves ℓ_i whose cohomology classes form a Poincaré dual vector space to the image of $NS(Y)\otimes\mathbb{Q}$. Let $M \in \mathrm{Mat}(r\times r, \mathbb{Q})$ be the invertible intersection matrix $M = (m_{i,j}) := (\ell_1, ..., \ell_r)^T(T_1, ..., T_r)$ and define B to be $M^{-1}(\mathbf{1} - A)$. Now the rest of the proof proceeds as in [dAMS]: $p_2(Y)$ acts via the matrix $MB + A + BA$ on $NS(Y) \otimes \mathbb{Q}$. Now $\pi_2^2 = \pi_2$ and we get $A^2 = A$ and therefore $BA = 0$. By definition of B, we obtain that $MB + A + BA = M(M^{-1}(\mathbf{1} - A)) + A = \mathbf{1}$. To show that p_2 is a projector, let us write $p_2 = \pi_2 + \beta - \beta\pi_2$. Note that $\beta\beta = \beta$, since $BMB = B$. From $BA = 0$ we deduce that $\pi_2\beta = 0$. Therefore $p_2 \circ p_2 = \pi_2^2 + \beta^2 + \beta\pi_2\beta\pi_2 + \pi_2\beta - \pi_2\beta\pi_2 + \beta\pi_2 - \beta\beta\pi_2 - \beta\pi_2\pi_2 - \beta\pi_2\beta = \pi_2 + \beta - \beta\pi_2 = p_2$ is a projector. □

As a corollary we obtain

Theorem 2. *Let X be a smooth projective 3-fold over a field of characteristic* 0 *such that a blow-up Y of X has a fibration $f : Y \to S$ which decomposes $H^2_{tr}(Y)$. Then X admits a special Chow-Künneth decomposition.*

Proof. Choose a blow-up $\varphi : Y \to X$ such that Y is a smooth projective 3-fold with an H^2-decomposing fibration $f : Y \to S$ of relative dimension one. S admits a special Chow-Künneth decomposition by [Mu1]. We may also assume that there exists a divisor $Z \subset Y$ which is finite over S. By extending the ground field k we may assume that Z and f are defined over k and use [Sch, 1.17.] to reduce back to the smaller field by taking the norm.

Now take the orthogonal set of projectors p_0, p_1, p_5, p_6 as defined in [Mu2] or [Sch], p_2 as in lemma 5 and $p_4 = p_2^{\mathrm{tr}}$ on Y. Apply the non-commutative Gram-Schmidt process (see proof of proposition 2) by modifying only p_2 and p_4. By lemmas 4 and 5 the new projectors, again denoted by $p_0, p_1, p_2, p_4, p_5, p_6$ have properties (1) and (3) of a Chow-Künneth decomposition for Y, since the orthogonality guarantees the vanishing of p_i on $H^j(Y)$ for $i \neq 3, j$. Finally we define

$$p_3 := \Delta_Y - \sum_{i \neq 3} p_i.$$

Then the set $\{p_i\}$ satisfies properties (1)-(3) of a special Chow-Künneth decomposition for X. □

Final remarks. It is easy to see that parts of properties (4)-(6) are satisfied for our choice of projectors. But (5) and (6) for $CH^2(X)$ remain open.

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