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PERIODIC GROUPS COVERED BY TRANSITIVE SUBGROUPS OF FINITARY PERMUTATIONS OR BY IRREDUCIBLE SUBGROUPS OF FINITARY TRANSFORMATIONS

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ABSTRACT. Let $\mathfrak{X}$ be either the class of all transitive groups of finitary permutations, or the class of all periodic irreducible finitary linear groups. We show that almost primitive $\mathfrak{X}$-groups are countably recognizable, while totally imprimitive $\mathfrak{X}$-groups are in general not countably recognizable. In addition we derive a structure theorem for groups all of whose countable subsets are contained in totally imprimitive $\mathfrak{X}$-subgroups. It turns out that totally imprimitive p-groups in the class $\mathfrak{X}$ are countably recognizable.

1. INTRODUCTION

A class $\mathfrak{G}$ of groups is said to be *countably recognizable* if every group, all of whose countable subsets are contained in countable $\mathfrak{G}$-subgroups, is itself a $\mathfrak{G}$-group. Many examples of such classes are discussed in [21, Section 8.3]. In the present work we are concerned with countable recognizability for transitive finitary permutation groups and for periodic irreducible finitary linear groups. Recall that a group is said to be a *finitary permutation group* if it is isomorphic to a subgroup of $\mathrm{FSym}(\Omega)$, the group of those permutations of some set Ω which fix all but finitely many elements. Correspondingly, a group is said to be *finitary* $\mathbb{F}$*-linear* if it is isomorphic to a subgroup of $\mathrm{FGL}_{\mathbb{F}}(V)$, the group of those invertible $\mathbb{F}$-linear transformations of some $\mathbb{F}$-vector space V which act like the identity on a subspace of finite codimension in V. Note that $\mathrm{FSym}(\Omega)$ is periodic and finitary linear on the natural $\mathbb{F}$-module $\mathbb{F}\Omega$. A survey about features of finitary linear groups is given in [20].

By a classical theorem of A. I. Mal′cev [14, Theorem IV], a group is linear of degree n if and only if all its finitely generated subgroups are linear of degree n. Later, it was shown by J. I. Hall [4, Theorem 1] that simple finitary linear groups are countably recognizable. This result was extended by the authors to periodic primitive finitary linear groups [13, Theorem A]. Here we made use of Hall's classification of the non-linear periodic simple finitary linear groups [6], [5], and of the fact that the commutator subgroup of a non-linear periodic primitive finitary linear group is simple [13, Theorem B]. In their recent paper [7], K. K. Hickin

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and R. E. Phillips derived partial results about countable recognizability in certain classes of totally imprimitive finitary permutation groups. In the present article we shall complete all these investigations with the following general treatment.

Notation. *Throughout, let $\mathfrak{X}$ be either the class of all transitive groups of finitary permutations, or the class of all periodic irreducible finitary linear groups.*

An *ω-cover* of the group G is a family of countable subgroups of G such that each countable subset of G is contained in a subgroup from this family. The notions *totally imprimitive* and *almost primitive*, which apply to transitive groups of finitary permutations as well as to irreducible groups of finitary transformations, are explained in Section 2. Note that totally imprimitive $\mathfrak{X}$-groups are always countable (see Section 2).

Theorem A. *Almost primitive $\mathfrak{X}$-groups are countably recognizable.*

Theorem B. *There exist uncountable groups which admit neither a faithful transitive finitary permutation representation nor a faithful irreducible finitary linear representation, although they contain an ω-cover of totally imprimitive $\mathfrak{X}$-subgroups.*

In fact, we shall construct a huge variety of such groups, even with prescribed composition factors. On the other hand, we shall complement Theorem B with a very precise structure theorem for groups which contain an ω-cover of totally imprimitive $\mathfrak{X}$-subgroups.

Theorem C. *Let G be a group with an ω-cover of totally imprimitive $\mathfrak{X}$-subgroups. Then G embeds as a transitive resp. irreducible subgroup into the unrestricted wreath product $W = S \operatorname{Wr}_\omega \operatorname{FSym}(\omega)$, where S is a finite group resp. an irreducible subgroup of $\operatorname{GL}_{\mathbb{K}}(U)$ for some finite-dimensional vector space U over an algebraically closed field $\mathbb{K}$. Here, W acts naturally on the set $S \times \omega$ resp. on the vector space $U \otimes \omega$. Further, the following three properties hold.*

(a) The image $G\pi$ of the projection π of G into the top group of W is a totally imprimitive subgroup of $\operatorname{FSym}(\omega)$.

(b) G is contained in $W_0 \cdot C$, where W_0 denotes the restricted wreath product $S \operatorname{wr}_\omega \operatorname{FSym}(\omega)$, and where C is the group of all base group functions $f : \omega \to \zeta_1(S)$ which, for every proper $G\pi$-block Ω_0 in ω, are constant on all but finitely many $G\pi$-translates of Ω_0, resp. act like a scalar on all but finitely many G-translates of $U \otimes \Omega_0$.

(c) In particular, G' is countable, being a subgroup of W_0, and $C \cap G$ is an abelian normal subgroup of countable index in G.

The properties recorded in Theorem C are reflected in the examples constructed to show Theorem B. In Section 7 we shall also derive sufficient conditions which ensure countable recognizability in the totally imprimitive case. As an amazing consequence we obtain

Theorem D. *Every p-group with an ω-cover of totally imprimitive $\mathfrak{X}$-subgroups is a totally imprimitive $\mathfrak{X}$-group.*

In particular, no p-group in the class $\mathfrak{X}$ is $L_{\omega_1,\omega}$-equivalent to an uncountable group.

[11, Example 3.4] shows, that finitary permutation p-groups and unipotent finitary linear p-groups are not countably recognizable. In fact, the direct product of the above example and any of the groups constructed in the proof of Theorem B has no faithful finitary linear representation at all, although each of its countable subgroups is a group of finitary permutations! (Cf. Section 5.4.) For any pair of primes $p \neq q$ we even obtain periodic locally solvable $\{p, q\}$-groups with this property. Therefore just the following question remains open.

Question. *Are finitary linear p-groups over fields of characteristic $\neq p$ countably recognizable?*

Note that such groups are subdirect products of irreducible finitary linear p-groups over a fixed field of coprime characteristic. In his recent work [24], B. A. F. Wehrfritz shows that the answer to the Question is affirmative for hypercentral groups of central height ω.

2. Background Information

This section is a résumé of well-known properties and techniques which we intend to use in our proofs without further reference. Note first that every periodic group of finitary permutations or finitary linear transformations is locally finite.

2.1. Finitary Permutation Groups (see [17], [18]). Let G be a transitive subgroup of $\mathrm{FSym}(\Omega)$, where Ω is infinite. The *support* $\mathrm{supp}_\Omega A$ of a subset A of G is the set of all those elements from Ω which are actually moved by A. Since G contains just elements of finite support, any proper G-block in Ω is finite. Hence one of the following three cases occurs.

(1) Ω contains no proper G-block, that is, G is *primitive*. By a well-known theorem of H. Wielandt [26, Satz 9.4], the group G is then either $\mathrm{FSym}(\Omega)$ or the alternating group $\mathrm{Alt}(\Omega)$.

(2) There exists a maximal proper G-block Ω_0 in Ω. Then the induced action of G on the corresponding system Δ of imprimitivity in Ω is primitive, whence G is said to be *almost primitive*. If we identify Ω in the right way with $\Omega_0 \times \Delta$, then G becomes a subgroup of the *restricted* wreath product $W = S_0 \,\mathrm{wr}_\Delta\, \mathrm{FSym}(\Delta)$, where S_0 denotes the subgroup of $\mathrm{Sym}(\Omega_0)$ induced from the setwise stabilizer of Ω_0 in G. We shall frequently replace the action of S_0 on Ω_0 by the right regular action of S_0 on itself. In this way, G and W become almost primitive subgroups of $\mathrm{FSym}(S_0 \times \Delta)$.

(3) There are proper G-blocks in Ω, but no maximal ones. Then Ω is countable as the union of an ascending chain $\{\Omega_n\}_{n\in\omega}$ of proper G-blocks. In particular, G is countable too. In this case G is said to be *totally imprimitive*. Let $\Delta_0 = \Omega_0$, and for every $n \in \omega$, let Δ_{n+1} be the system of imprimitivity in Ω_{n+1} containing the block Ω_n. If we identify Ω in the right way with the restricted product

$$\{(\Lambda_n)_{n\in\omega} \in \textstyle\prod_{n\in\omega} \Delta_n \mid \Lambda_n = \Omega_n \text{ for all but finitely many } n \in \omega\},$$

and if we let S_n denote the subgroup of $\mathrm{Sym}(\Delta_n)$ induced from the setwise stabilizer of Ω_n in G, then G becomes a subgroup of the *restricted* wreath product $\mathrm{wr}_{n\in\omega} S_n$. This wreath product is the direct limit of the iterated wreath products $G_n = S_0 \,\mathrm{wr} \cdots \mathrm{wr}\, S_n$ with respect to the canonical embeddings of G_n onto the component of the base group of G_{n+1} corresponding to Ω_n.

In all three cases, the derived subgroup G' of G has non-trivial intersection with every proper normal subgroup of G, and G' is the unique minimal transitive normal subgroup of G. In particular, G' is perfect. In modification of the usual terminology we call G *almost primitive* if it is not totally imprimitive.

2.2. Finitary Linear Groups (see [19], [15])**.** Let G be a periodic irreducible subgroup of $\mathrm{FGL}_{\mathbb{F}}(V)$, where $\dim_{\mathbb{F}} V$ is infinite. Suppose that G is imprimitive. Any proper G-block V_0 in V is finite-dimensional, and G permutes the corresponding system Δ of imprimitivity transitively and finitarily. If we identify V in the right way with the $\mathbb{F}$-vector space $V_0 \otimes \Delta = \bigoplus_{\delta \in \Delta} V_0 \otimes \delta$, then G becomes a subgroup of the restricted wreath product $S_0 \operatorname{wr}_\Delta \mathrm{FSym}(\Delta)$, where S_0 denotes the subgroup of $\mathrm{GL}_{\mathbb{F}}(V_0)$ induced from the normalizer of V_0 in G. Note that S_0 acts irreducibly on V_0.

Depending on the way in which G acts on Δ, the group is said to be *almost primitive*, respectively *totally imprimitive*. The unipotent radical of G is trivial, and every normal subgroup of G is completely reducible. Moreover, G' has non-trivial intersection with every proper normal subgroup of G, and G' is the unique minimal irreducible normal subgroup of G. In particular, G' is perfect. In the totally imprimitive case, the countability of Δ entails that $\dim_{\mathbb{F}} V$ is countable, and G is countable from [23, 1.19] and the theorem of A. E. Zalesskii [28] and D. J. Winter [27], applied to V_0. In modification of the usual terminology, we say that G is *almost primitive* if it is either an almost primitive finitary linear group in the usual sense, or a primitive group of finitary permutations.

We also note that every non-trivial normal subgroup N of G either is irreducible, or the N-homogeneous components in V are finite-dimensional: In the case when there is more than one N-homogeneous component in V, these components form proper G-blocks and must therefore be finite-dimensional. In the other case the unique N-homogeneous component is a direct sum of finitely many copies of an infinite-dimensional irreducible $\mathbb{F}N$-module, since N acts finitarily on V. But G acts finitarily on V too, and therefore the component merely consists of one copy of the irreducible $\mathbb{F}N$-module.

2.3. Ultraproducts (see [5], Appendix)**.** A set Φ of subsets of the set I is said to be a *filter* on I if the following conditions are satisfied:

(i) $\emptyset \notin \Phi$,
(ii) $X, Y \in \Phi$ imply $X \cap Y \in \Phi$, and
(iii) $X \subseteq Y \subseteq I$ and $X \in \Phi$ implies $Y \in \Phi$.

It is a straightforward consequence of Zorn's Lemma that every filter on I is contained in an *ultrafilter* on I, that is, a filter Φ with the additional property that for every $X \subseteq I$, either $X \in \Phi$ or $I - X \in \Phi$.

Suppose now that $\{S_i \mid i \in I\}$ is a family of non-empty sets. Then every ultrafilter Φ on I gives rise to an equivalence relation $\sim_\Phi$ on the cartesian product $\prod_{i\in I} S_i$ via

$$(s_i)_{i\in I} \sim_\Phi (t_i)_{i\in I} \qquad \text{if and only if} \qquad \{i \in I \mid s_i = t_i\} \in \Phi.$$

The set of equivalence classes $(\prod_{i\in I} S_i)/\Phi$ is called an *ultraproduct*. If for every $i \in I$ we are given a group G_i of automorphisms of the vector space V_i over the field $\mathbb{K}_i$, then $(\prod_{i\in I} G_i)/\Phi$ becomes a group of automorphisms of the vector space $(\prod_{i\in I} V_i)/\Phi$ over the field $(\prod_{i\in I} \mathbb{K}_i)/\Phi$, where all the operations in question are defined componentwise. Correspondingly, every ultraproduct $(\prod_{i\in I} G_i)/\Phi$ of permutation groups $G_i \leq \mathrm{Sym}(\Omega_i)$ acts componentwise on the set $(\prod_{i\in I} \Omega_i)/\Phi$. Moreover, there is the famous theorem of Łoś [2, 5.2.1] with the consequence that common properties of all factors of an ultraproduct, which can be formulated by first order sentences, are inherited by the ultraproduct itself [2, 5.2.2].

3. Covers

Proposition 3.1. *Every almost primitive resp. totally imprimitive $\mathfrak{X}$-group has an ω-cover of almost primitive resp. totally imprimitive $\mathfrak{X}$-subgroups.*

Proof. It suffices to consider an uncountable and hence almost primitive $\mathfrak{X}$-group G. The primitive permutation groups $\mathrm{Alt}(\Omega)$ and $\mathrm{FSym}(\Omega)$ are obviously covered by countable subgroups of the same kind.

Now let G be an almost primitive finitary permutation group. Then $G \leq S_0 \,\mathrm{wr}_\Delta\, \mathrm{FSym}(\Delta)$, where S_0 is finite. The wreath product acts on $\Omega = S_0 \times \Delta$. Consider a countably infinite subgroup X of G. Let Δ_X be the smallest countable subset of Δ such that $\mathrm{supp}_\Omega X \subseteq S_0 \times \Delta_X$. Since G acts highly transitively on Δ, we can find for all $\lambda_1, \lambda_2 \in S_0 \times \Delta_X$ some $g \in G$ such that $\lambda_1 g = \lambda_2$ and $\mathrm{supp}_\Omega g \subseteq S_0 \times \Delta_X$. Moreover, for every $\pi \in \mathrm{Alt}(\Delta_X)$ there exists $h \in G$ such that h induces π on Δ_X and satisfies $\mathrm{supp}_\Omega h \subseteq S_0 \times \Delta_X$. Together with X, all these elements g and h generate a countable subgroup of G, which acts almost primitively on $S_0 \times \Delta_X$.

The periodic almost primitive finitary linear groups can be handled in the same way. □

In the converse direction, we shall make extensive use of the following elementary observation.

Lemma 3.2. *Let the ω-cover $\mathfrak{C}$ of the group G be the union of subsets $\mathfrak{C}_n$ $(n \in \omega)$. Then one of the families $\mathfrak{C}_n$ forms an ω-cover of G.*

Proof. Assume that the assertion is false. Then there are countable subgroups X_n $(n \in \omega)$ of G such that no subgroup from $\mathfrak{C}_n$ contains X_n. But now the countable subgroup $\langle X_n \mid n \in \omega \rangle$ cannot be contained in any subgroup from the ω-cover $\mathfrak{C}$, a contradiction. □

We note an immediate consequence of Lemma 3.2 and Theorem A.

Corollary 3.3. *Periodic irreducible finitary linear groups that are not totally imprimitive are countably recognizable.*

Proof. Suppose that the periodic group G has an ω-cover $\mathfrak{C}$ of such groups. From Lemma 3.2, we may assume that $\mathfrak{C}$ consists exclusively of groups of precisely one of the following types: linear groups, non-linear primitive groups, almost primitive groups. By Theorem A, we need to consider the first two cases only. Non-linear primitive groups have been treated in [13, Theorem A]. If every $X \in \mathfrak{C}$ is a periodic irreducible linear group, then G is linear by Mal′cev's Theorem. From Lemma 3.2 we may assume that the characteristic of the underlying fields is independent of $X \in \mathfrak{C}$. Since every $X \in \mathfrak{C}$ has trivial unipotent radical, the same holds for G. Therefore G is countable by [23, 1.19] and [28], [27]. In particular, $G \in \mathfrak{C}$ is an irreducible linear group. □

4. Almost Primitive Groups

In this section we shall prove Theorem A.

Lemma 4.1. *Every infinite primitive finitary permutation group admits just one non-trivial transitive finitary permutation representation, namely the natural one.*

Proof. Let $\mathrm{Alt}(\Delta) \leq G \leq \mathrm{FSym}(\Delta)$, and suppose that G acts as a transitive finitary permutation group on the infinite set Ω. Since G' is infinite and simple, we have $G' = \mathrm{Alt}(\Omega)$. From [19, 3.3.1] or [3, Corollary 3.5], the complex vector space $V = [\mathbb{C}\Omega, G]$ is an irreducible $\mathbb{C}G$-module. Hence [3, Theorem B] yields $V \cong_{\mathbb{C}G} [\mathbb{C}\Delta, G]$. It remains to show that the action of every element $g \in G$ on Ω is determined by its action on V. But we can choose $\omega_g \in \Omega - \mathrm{supp}_\Omega\, g$. Then the action of g on the $\mathbb{C}$-basis $\{\omega - \omega_g \mid \omega \in \Omega - \{\omega_g\}\}$ of V is similar to the action of g on $\Omega - \{\omega_g\}$. □

Lemma 4.2. *Let V_0 be a finite-dimensional $\mathbb{F}$-vector space, and let Δ be an infinite set. Consider the canonical finitary linear representation of the restricted wreath product $W = \mathrm{GL}_\mathbb{F}(V_0)\,\mathrm{wr}_\Delta\,\mathrm{FSym}(\Delta)$ on the vector space $V = V_0 \otimes \Delta$. Let G be a subgroup of W such that the induced action of G on Δ is primitive, and such that the normalizer in G of $V_0 \otimes \delta$ induces an irreducible subgroup of $\mathrm{GL}_\mathbb{F}(V_0 \otimes \delta)$ for each $\delta \in \Delta$. Then G acts irreducibly on V.*

Proof. Consider a non-zero vector $v = \sum_{\delta \in \Delta_0} v_\delta \otimes \delta$ in V, where Δ_0 is a finite subset of Δ. Fix some $\delta_0 \in \Delta_0$ with $v_{\delta_0} \neq 0$. Let S be the normalizer in G of $V_0 \otimes \delta_0$, and let S_0 be a finite subset of S such that $\mathbb{F}S_0$ and $\mathbb{F}S$ induce the same subspace of $\mathrm{End}_\mathbb{F}(V_0 \otimes \delta_0)$ on $V_0 \otimes \delta_0$. Without loss we may assume that S_0 fixes every vector in $V_0 \otimes (\Delta - \Delta_0)$. Since G acts highly transitively on Δ, we can find $h \in G$ such that h normalizes $V_0 \otimes \delta_0$, and such that the permutation of Δ induced from h moves $\Delta_0 - \{\delta_0\}$ into $\Delta - \Delta_0$. But then $V_0 \otimes \delta_0 \leq [v, S_0^h]$, and hence $V \leq [v, G]$. □

Theorem 4.3. *Periodic almost primitive finitary linear groups are countably recognizable.*

Proof. Suppose that the group G has an ω-cover $\mathfrak{C}$ of subgroups which are almost primitive finitary linear groups on some infinite-dimensional vector spaces. Each $X \in \mathfrak{C}$ is an irreducible subgroup of the *restricted* wreath product $W_X = S_X\,\mathrm{wr}_{\Delta_X}\,\mathrm{FSym}(\Delta_X)$, where $S_X \leq \mathrm{GL}_{\mathbb{F}_X}(V_{X,0})$ for some finite- (possibly zero-) dimensional $\mathbb{F}_X$-vector space $V_{X,0}$, and where the image of the canonical homomorphism $\pi_X : X \to \mathrm{FSym}(\Delta_X)$ contains $\mathrm{Alt}(\Delta_X)$. The wreath product W_X acts naturally on the vector space $V_X = V_{X,0} \otimes \Delta_X$, and the irreducible subgroup S_X of $\mathrm{GL}_{\mathbb{F}_X}(V_{X,0})$ is induced from the normalizer of $V_{X,0} \otimes \delta_{X,0}$ in X, for some fixed $\delta_{X,0} \in \Delta_{X,0}$. Each $g \in X$ has the form $g = f_{X,g} \cdot g\pi_X$ for some $f_{X,g} : \Delta_X \to S_X$. From [23, 1.19] we may assume that the fields $\mathbb{F}_X$ are algebraically closed.

For $X \in \mathfrak{C}$, let $P_X = \{Y \in \mathfrak{C} \mid X \leq Y\}$. Since the infinite simple group $\mathrm{Alt}(\Delta_X)$ contains finite p-subgroups of arbitrary large solubility lengths, [23, 9.20 and 9.4] imply that $\mathrm{Alt}(\Delta_X)$ cannot be a composition factor of the base group of W_Y for any $Y \in P_X$. In particular, $K_{X,Y} = X \cap \ker \pi_Y \leq \ker \pi_X$. Let $K_X = \bigcap_{Y \in P_X} K_{X,Y}$. Note that $K_{X_1} \leq K_{X_2}$ whenever $X_1 \leq X_2$. Therefore $K = \bigcup_{X \in \mathfrak{C}} K_X$ is a normal subgroup in G with $K \cap X = K_X$. Let $\pi : G \to G/K$ denote the canonical epimorphism. Consider a finite subgroup F of G'. Choose $X \in \mathfrak{C}$ such that $F \leq X'$ and $K \cap F = F \cap \ker \pi_X$. Let $\Xi = \mathrm{supp}_{\Delta_X} F\pi_X$. Then $F\pi_X \leq X'\pi_X = \mathrm{Alt}(\Delta_X)$, and so F is contained in a finite subgroup $\overline{F}$ of X such that $\overline{F}\pi_X = \mathrm{Alt}(\Xi)$. This shows that $G'\pi$ has a Kegel cover with alternating quotients. Hence [10, Theorem 4.1] yields that $G'\pi$ is simple.

Next, consider a finite subgroup F of G with $(F \cap G')\pi \neq 1$. Let Φ be an ultrafilter on $\mathfrak{C}$ which contains the sets P_X $(X \in \mathfrak{C})$. From Lemma 3.2 we may assume that there exists $m \in \omega$ such that $|\mathrm{supp}_{\Delta_X} F\pi_X| = m$ for all $X \in \mathfrak{C}$. Now $G\pi \leq (\prod_{X\in\mathfrak{C}} \mathrm{FSym}(\Delta_X))/\Phi$ via $g\pi = (g\pi_X)_{X\in\mathfrak{C}}$ for all $g \in G$; here $g\pi_X = 1$ whenever $g \notin X \in \mathfrak{C}$. The ultraproduct acts faithfully on the set $\Sigma = (\prod_{X\in\mathfrak{C}} \Delta_X)/\Phi$. From Łoś Theorem, $|\mathrm{supp}_\Sigma F\pi| = m$. Since $\mathrm{FSym}(\Sigma)$ is a normal subgroup of $\mathrm{Sym}(\Sigma)$, the simplicity of $G'\pi$ enforces that $(G'F)\pi \leq \mathrm{FSym}(\Sigma)$. Let Δ be an orbit of $(G'F)\pi$ in Σ, on which $G'\pi$ acts non-trivially. The pointwise stabilizer $Z\pi$ in $(G'F)\pi$ of Δ is a normal subgroup, which intersects $G'\pi$ trivially, and which must therefore commute with $G'\pi$. Assume that $Z\pi$ contains a non-trivial element $z\pi$. Choose $X \in \mathfrak{C}$ such that $z \in X$ and $z\pi_X \neq 1$. Because $[z, X'] \leq [z, G'] \cap X \leq K \cap X \leq \ker \pi_X$ we obtain $[z\pi_X, X'\pi_X] = 1$. This contradiction shows that $(G'F)\pi$ acts faithfully on Δ. However, the group $G'\pi$ acts transitively on Δ. But then $G'\pi = \mathrm{Alt}(\Delta)$ and $|(G'F)\pi : G'\pi| \leq 2$. Since F was an arbitrary finite subgroup of G, we conclude that $|G\pi : G'\pi| \leq 2$, and that $\mathrm{Alt}(\Delta) \leq G\pi \leq \mathrm{FSym}(\Delta)$ for some orbit Δ of $G\pi$ in the ultraproduct Σ.

In the sequel we choose F such that $(G'F)\pi = G\pi$. We may assume without loss that $F \leq X$ for all $X \in \mathfrak{C}$, and that there exists $m \in \omega$ such that $|\mathrm{supp}_{\Delta_X} F\pi_X| = m$ for all $X \in \mathfrak{C}$ (Lemma 3.2). From Łoś Theorem, $|\mathrm{supp}_\Delta F\pi| \leq m$.

Consider some $X \in \mathfrak{C}$. Note that $X\pi_X$ is an image of $X\pi$, with kernel $K_{X,X}\pi$. Since $X'\pi_X$ cannot occur as a composition factor in a subdirect product of finite groups, there exists an infinite orbit Γ_X of $X\pi$ in Δ. Since $X\pi_X$ is non-abelian, $K_{X,X}\pi$ cannot act transitively on Γ_X. From Lemma 4.1 the group $X\pi_X$ acts naturally on the set of orbits of $K_{X,X}\pi$ in Γ_X. Let ν be the size of such an orbit. Then

$$\nu \cdot |\mathrm{supp}_{\Gamma_X} F\pi| \;\leq\; \nu \cdot m \;=\; \nu \cdot |\mathrm{supp}_{\Delta_X} F\pi_X| \;\leq\; |\mathrm{supp}_{\Gamma_X} F\pi|.$$

It follows that $\nu = 1$, and that $K_{X,X}\pi$ acts trivially on Γ_X. Now $X\pi_X$ acts naturally on Γ_X. Moreover, $|\mathrm{supp}_{\Gamma_X} F\pi| \leq |\mathrm{supp}_{\Delta_X} F\pi_X|$ enforces that Γ_X is the unique infinite $X\pi$-orbit in Δ. This allows us to identify Δ_X with the subset Γ_X. Let Λ be a countable subset of Δ. Then there is a countable subgroup Z of G such that $Z\pi = \mathrm{Alt}(\Lambda)$. If $X \in \mathfrak{C}$ contains Z, then the infinite alternating group $Z\pi$ cannot be a composition factor of the base group of W_X. Hence $Z \cap \ker \pi_X = Z \cap K$ and $\Lambda \subseteq \Delta_X$. In particular, $\Delta = \bigcup_{X\in\mathfrak{C}} \Delta_X$ and $\Delta_X \subseteq \Delta_Y$ for $X \leq Y$.

We now form the ultraproduct $S = (\prod_{X\in\mathfrak{C}} S_X)/\Phi$, and define a homomorphism $\tau : G \longrightarrow W = S \,\mathrm{Wr}_\Delta\, \mathrm{FSym}(\Delta)$ by $g\tau = f_g \cdot g\pi$, where $f_g : \Delta \to S$ is given via

$$(\delta)f_g \;=\; ((\delta)f_{X,g})_{X\in\mathfrak{C}} \quad \text{for all} \quad \delta \in \Delta;$$

here $(\delta)f_{X,g} = 1$ whenever $g \notin X \in \mathfrak{C}$ or $\delta \notin \Delta_X$. The wreath product W acts on $V = V_0 \otimes \Delta$, where $V_0 = (\prod_{X\in\mathfrak{C}} V_{X,0})/\Phi$ is a vector space over the field $\mathbb{F} = (\prod_{X\in\mathfrak{C}} \mathbb{F}_X)/\Phi$. From Lemma 3.2 we may assume that there exists $n \in \omega$ such that $\dim_{\mathbb{F}_X} V_{X,0} = n$ for all $X \in \mathfrak{C}$. By Łoś Theorem, $\dim_{\mathbb{F}} V_0 = n$ too. We now show that τ is injective.

For any finite subset $H \subseteq X \in \mathfrak{C}$ we let $\Delta_{X,H}$ denote the smallest subset of Δ_X such that H fixes each vector in $V_{X,0} \otimes (\Delta_X - \Delta_{X,H})$. Note that $\Delta_{X,H}$ is finite. Consider a finite subset $\Lambda \subseteq \Delta_X$, where $X \in \mathfrak{C}$ is fixed. There exists a finite subgroup $H \leq X$ of smallest possible order such that $H\pi_X = \mathrm{Alt}(\Lambda)$. Then every proper normal subgroup of H is contained in $\ker \pi_X$. Consequently, if Λ has been chosen large enough, then H cannot have a non-trivial linear representation of degree $\leq n$ ([23, 9.20 and 9.4]). This shows that $\Delta_{Y,H} = \mathrm{supp}_{\Delta_Y} H\pi_Y \subseteq \mathrm{supp}_\Delta H\pi$ for all $Y \in P_X$.

Assume that there exists $1 \neq g \in X \cap \ker \tau$. Choose Λ and H as above such that $\Delta_{X,g} \subseteq \Lambda$ and $\Delta_{X,g} h \cap \Delta_{X,g} = \emptyset$ for some $h \in H$. Then $[g,h] \neq 1$. For every $Y \in P_X$, this enforces $(\delta) f_{Y,g} \neq 1$ for some $\delta \in \mathrm{supp}_\Delta H\pi$. By [5, Lemma B.1], there exists $\delta \in \mathrm{supp}_\Delta H\pi$ such that $\{Y \in P_X \mid (\delta) f_{Y,g} \neq 1\} \in \Phi$. But then $(\delta) f_g \neq 1$ and $g\tau \neq 1$. This contradiction shows that τ is an embedding.

We shall show next that $G\tau \leq \mathrm{FGL}_{\mathbb{F}}(V)$. To this end, consider $g \in G$ and $\delta \in \Delta$ with $(\delta) f_g \neq 1$. Choose $X \in \mathfrak{C}$ such that $g \in X$ and $\delta \in \Delta_{X,g}$. Assume that there exists $Y \in P_X$ satisfying $\delta \notin \Delta_{Y,g}$. Then $\delta \notin \mathrm{supp}_\Delta g\pi$. Choose $\Lambda \subseteq \Delta_X$ and $H \leq X$ as above such that $\Lambda \cap (\Delta_{X,g} \cup \Delta_{Y,g}) = \{\delta\}$. Since $\mathrm{Alt}(\Delta_Y) \leq Y\pi_Y$, there exists $t \in Y$ such that $\Delta_X \cap \mathrm{supp}_\Delta t\pi = \emptyset$ and $\Delta_{Y,H}(t\pi) \cap \Delta_{Y,g} = \emptyset$. From the action of $\langle g, H^t \rangle \tau$ on $V_0 \otimes \Delta_X$ we see that $[g, H^t] \neq 1$. On the other hand, the subgroup $\langle g, H^t \rangle$ of Y satisfies $[g, H^t] = 1$ from its action on V_Y. This contradiction shows that $\delta \in \Delta_{Y,g}$ for every $Y \in P_X$. Since the join of every countable subfamily of the ω-cover $\mathfrak{C}$ is contained in a subgroup from $\mathfrak{C}$, we can now find for every countable subset Γ_0 of $\Gamma = \{\delta \in \Delta \mid (\delta) f_g \neq 1\}$ some $X \in \mathfrak{C}$ such that $g \in X$ and $\Gamma_0 \subseteq \Delta_{X,g}$. Since $\Delta_{X,g}$ is finite, Γ must be finite too, that is, $g\tau \in \mathrm{FGL}_{\mathbb{F}}(V)$. In particular, $G\tau$ is contained in the *restricted* wreath product $S \,\mathrm{wr}_\Delta\, \mathrm{FSym}(\Delta)$.

The ultraproduct $\mathbb{F}$ of the algebraically closed fields $\mathbb{F}_X$ is algebraically closed too. By [23, 1.17], each S_X contains precisely n^2 elements which are linearly independent over $\mathbb{F}_X$. By Łoś Theorem, S too contains precisely n^2 elements which are linearly independent over $\mathbb{F}$. Hence S is an irreducible subgroup of $\mathrm{GL}_{\mathbb{F}}(V_0)$ by [23, 1.18]. Moreover, the n^2 independent elements in S are still induced from the normalizer in $G\tau$ of $V_0 \otimes \delta_0$ for some fixed δ_0. Now Lemma 4.2 ensures that $G\tau$ acts irreducibly on V. Since G contains an infinite alternating section, $G\tau$ must act almost primitively on V. □

Theorem 4.4. *Almost primitive finitary permutation groups are countably recognizable.*

Proof. Suppose that the group G has an ω-cover $\mathfrak{C}$ of subgroups which are almost primitive finitary permutation groups on some infinite sets. Each $X \in \mathfrak{C}$ is a transitive subgroup of the *restricted* wreath product $W_X = S_X \,\mathrm{wr}_{\Delta_X}\, \mathrm{FSym}(\Delta_X)$, acting on $\Omega_X = S_X \times \Delta_X$. Here the finite (possibly trivial) group S_X is induced from the setwise stabilizer in G of the block $S_X \times \{\delta_X\}$, for some fixed $\delta_X \in \Delta_X$. Moreover, the image of the canonical homomorphism $\pi_X : X \to \mathrm{FSym}(\Delta_X)$ contains $\mathrm{Alt}(\Delta_X)$. From Lemma 3.2 we may assume that the groups S_X $(X \in \mathfrak{C})$ are all isomorphic. We consider S_X as a linear group. Then we can go through the proof of Theorem 4.3. Finally we obtain an embedding $\tau : G \longrightarrow W = S \,\mathrm{wr}_\Delta\, \mathrm{FSym}(\Delta)$. This wreath product acts as a finitary permutation group on $\Omega = S \times \Delta$. And the ultraproduct $S = (\prod_{X \in \mathfrak{C}} S_X)/\Phi$ is induced from the setwise stabilizer in G of the block $S \times \{\delta\}$, for some fixed $\delta \in \Delta$. Moreover, the image of the canonical homomorphism $\pi : G \to \mathrm{FSym}(\Delta)$ contains $\mathrm{Alt}(\Delta)$. It is easy to see that G acts transitively on Ω. Since G contains an infinite alternating section, G must act almost primitively on Ω. □

5. Counterexamples

In this section we shall construct uncountable groups with an ω-cover of totally imprimitive $\mathfrak{X}$-subgroups. These so-called *counterexamples* will provide a proof of

Theorem B. K. Hickin and R. Phillips have shown in [7] that totally imprimitive finitary permutation groups G with the property $|NG'/G'| < \infty$ for every intransitive normal subgroup N are countably recognizable. However they were not able to decide whether every totally imprimitive finitary permutation group G has this property. The cornerstone of our construction will be examples of totally imprimitive finitary permutation groups G in which this property fails, and which are therefore of independent interest.

Definition. In the sequel, a totally imprimitive finitary permutation group G will be called *slim* if it contains an intransitive normal subgroup N such that NG'/G' is infinite.

5.1. Construction of Slim Groups. Let $\mathbb{F}_p$ denote the prime field in characteristic $p>0$, and let ω denote the first infinite ordinal. Choose non-trivial finite groups S_n $(n\in\omega)$ and non-trivial faithful irreducible $\mathbb{F}_pS_n$-modules C_n $(n\in\omega)$. Let S_n act on the $\mathbb{F}_p$-vector space $B_n = \langle b_n\rangle \oplus C_n$ via $[b_n, S_n] = 0$. Let X_n be the group generated by S_n and the stability group of the chain $0 < C_n < B_n$. Let X_n act on the $\mathbb{F}_p$-vector space $A_n = \langle a_n\rangle \oplus B_n$ via $[a_n, X_n] = 0$. The group ring $\mathbb{F}_pX_n$ contains a projective indecomposable $\mathbb{F}_pX_n$-module P_n such that head and socle of P_n are isomorphic to C_n ([8, VII.10.3/10.9/11.6]). Since P_n is also an injective $\mathbb{F}_pX_n$-module ([8, VII.7.8]), and since B_n is indecomposable, there exists an embedding $\varphi_n : B_n \to P_n$. Since C_n is a non-trivial $\mathbb{F}_pX_n$-module, we can extend φ_n to an embedding $\varphi_n : A_n \to \mathbb{F}_pX_n$ via $a_n\varphi_n = \sum_{x\in X_n} x$. Note that $A_n\varphi_n$ is contained in the augmentation ideal $\kappa(\mathbb{F}_pX_n)$. In the sequel we shall suppress φ_n and identify A_n with its image in $\kappa(\mathbb{F}_pX_n)$.

Next, let $W_0 = X_0$ and $W_n = W_{n-1} \operatorname{wr} X_n$ for all $n\geq 1$. Then $W_m = W_{n-1} \operatorname{wr} Q_{n,m}$ for all $m\geq n$, where $Q_{n,m} = X_n \operatorname{wr} \cdots \operatorname{wr} X_m$. We also let $W = \bigcup_{n\in\omega} W_n$ and $W = W_{n-1} \operatorname{wr} Q_n$ for $Q_n = \bigcup_{m\geq n} Q_{n,m}$. We shall now recursively construct $\mathbb{F}_pW_n$-submodules $\overline{B}_n \leq \overline{A}_n$ of the base group of $C_p \operatorname{wr} W_n$ and an $\mathbb{F}_pX_n$-monomorphism $\psi_n : A_n \to \overline{A}_n$ such that $B_n\psi_n \leq \overline{B}_n$ and $\overline{A}_n = \overline{B}_n \oplus \langle a_n\psi_n\rangle$ and $[\overline{A}_n, W_n] \leq \overline{B}_n$.

For $n=0$ we choose $\overline{B}_0 = B_0 \leq A_0 = \overline{A}_0$ and $\psi_0 = id$. Because A_0 is contained in the permutation module $\mathbb{F}_pX_0$, the semidirect product $B_0 \rtimes W_0$ can be viewed as a transitive subgroup of $C_p \operatorname{wr} W_0$. Suppose next that $\overline{A}_n, \overline{B}_n, \psi_n$ have been constructed for some n. Consider the action of $W_{n+1} = W_n \operatorname{wr} X_{n+1}$ on $\overline{A}_n^{X_{n+1}} = \bigoplus_{x\in X_{n+1}} \overline{A}_n^x \leq C_p \operatorname{wr} W_{n+1}$. Let K_n be the kernel of the trace map

$$\overline{A}_n^{X_{n+1}} \ni (a_x)_{x\in X_{n+1}} \longmapsto \sum_{x\in X_{n+1}} a_x \in \overline{A}_n.$$

Clearly $\langle a_n\psi_n\rangle^{X_{n+1}} \cong \mathbb{F}_pX_{n+1}$ and $K_n \cap \langle a_n\psi_n\rangle^{X_{n+1}} \cong \kappa(\mathbb{F}_pX_{n+1})$ as $\mathbb{F}_pX_{n+1}$-modules. We can therefore find an $\mathbb{F}_pX_{n+1}$-monomorphism $\psi_{n+1} : A_{n+1} \to K_n \cap \langle a_n\psi_n\rangle^{X_{n+1}}$ and choose

$$\overline{B}_{n+1} = \overline{B}_n^{X_{n+1}} \oplus B_{n+1}\psi_{n+1} \leq \overline{B}_n^{X_{n+1}} \oplus A_{n+1}\psi_{n+1} = \overline{A}_{n+1}.$$

It remains to show that $\overline{B}_{n+1}$ and $\overline{A}_{n+1}$ are W_{n+1}-invariant. Clearly they are X_{n+1}-invariant. Moreover $[A_{n+1}\psi_{n+1}, W_n] \leq [\langle a_n\psi_n\rangle, W_n] \leq [\overline{A}_n, W_n] \leq \overline{B}_n$, whence $[\overline{A}_{n+1}, W_n] \leq \overline{B}_n \leq \overline{B}_{n+1}$. This completes the recursion.

Now $B = \bigcup_{n\in\omega} \overline{B}_n$ is an $\mathbb{F}_pW$-submodule of the base group of $C_p \operatorname{wr} W$ such that $G = B \rtimes W$ is a totally imprimitive subgroup of $C_p \operatorname{wr} W$. In the next section we shall show that BG'/G' is infinite, which of course means that the group G is slim. In the sequel we shall suppress ψ_n and identify A_n with its image in $\overline{A}_n$.

5.2. Additional Properties of the Group G.

5.2.1. $[a_n, W_n] = 0$ for all n.

Proof. Clearly $[a_0, W_0] = [a_0, X_0] = 0$. And if $[a_n, W_n] = 0$, then $a_{n+1} \in A_{n+1} \leq \langle a_n \rangle^{X_{n+1}}$ is centralized by $W_n^{X_{n+1}}$. On the other hand, $[a_{n+1}, X_{n+1}] = 0$ by choice of a_{n+1}. Altogether, $[a_{n+1}, W_{n+1}] = 0$. □

This implies

5.2.2. $[B_{n+1}, W_n^W] \leq [\langle a_n \rangle^{X_{n+1}}, W_n^{X_{n+1}}] = 0$,

and so

5.2.3. $[\overline{B}_{n+1}, W_{n+1}] \;=\; \left([\overline{B}_n, W_n]^{X_{n+1}} + (K_n \cap \overline{B}_n^{X_{n+1}})\right) \;\oplus\; [B_{n+1}, X_{n+1}]$.

In particular,

5.2.4. $[\overline{B}_{n+1}, W_{n+1}] \cap \overline{B}_n = [\overline{B}_n, W_n]$ *and* $[B, W] = \bigcup_{n \in \omega} [\overline{B}_n, W_n]$.

5.2.5. *G is a slim group.*

Proof. Since $BG'/G' \cong B/B \cap G' \cong B/[B, W]$, it suffices to show that $p^{n+1} = |\overline{B}_n/[\overline{B}_n, W_n]|$ for all n. But $|\overline{B}_0/[\overline{B}_0, W_0]| = p$, and induction yields

$$\begin{aligned} |\overline{B}_{n+1}/[\overline{B}_{n+1}, W_{n+1}]| &= |\overline{B}_n^{X_{n+1}}/([\overline{B}_n, W_n]^{X_{n+1}} + (K_n \cap \overline{B}_n^{X_{n+1}}))| \cdot |B_{n+1}/C_{n+1}| \\ &= |\overline{B}_n/[\overline{B}_n, W_n]| \cdot p \;=\; p^{n+2}. \end{aligned}$$

□

5.2.6. $C_B(W_n^W) = \bigoplus_{k \geq 1} B_{n+k}^{Q_{n+k+1}}$ *for all n.*

Proof. It suffices to calculate that $C_{\overline{B}_m}(W_n^W) = B_{n+1}^{Q_{n+2,m}} \oplus \cdots \oplus B_m^{Q_{m+1,m}}$ for $m \geq n$ (where $Q_{k,\ell} = 1$ for $k > \ell$). Consider the case $n = m$ first. Clearly $C_{\overline{B}_0}(W_0^W) = C_{\overline{B}_0}(W_0) = 0$. And if $C_{\overline{B}_n}(W_n^W) = 0$, then $C_{\overline{B}_n^{X_{n+1}}}(W_{n+1}^W) \leq C_{\overline{B}_n^{X_{n+1}}}(W_n^W) = 0$, whence

$$\begin{aligned} C_{\overline{B}_{n+1}}(W_{n+1}^W) &\cong C_{\overline{B}_{n+1}}(W_{n+1}^W)\overline{B}_n^{X_{n+1}}/\overline{B}_n^{X_{n+1}} \\ &\leq C_{\overline{B}_{n+1}/\overline{B}_n^{X_{n+1}}}(X_{n+1}) \;\cong\; C_{B_{n+1}}(X_{n+1}) \;=\; 0. \end{aligned}$$

Suppose now that $C_{\overline{B}_m}(W_n^W)$ has the desired form for some $m \geq n$. Then 5.2.2 gives

$$\begin{aligned} C_{\overline{B}_{m+1}}(W_n^W) &= (C_{\overline{B}_m}(W_n^W))^{X_{m+1}} \;\oplus\; B_{m+1} \\ &= (B_{n+1}^{Q_{n+2,m}} \;\oplus\; \cdots \;\oplus\; B_m^{Q_{m+1,m}})^{X_{m+1}} \;\oplus\; B_{m+1}^{Q_{m+2,m+1}} \\ &= B_{n+1}^{Q_{n+2,m+1}} \;\oplus\; \cdots \;\oplus\; B_{m+1}^{Q_{m+2,m+1}}. \end{aligned}$$

□

It is evident now that the centralizers $C_B(W_n^W)$ $(n \in \omega)$ form a strictly descending chain in B. Since G is finitary, their intersection is trivial. Let C denote the completion of B with respect to the residual system $\{C_B(W_n^W) \mid n \in \omega\}$. Let C_0 be the set of all $c \in C$ satisfying

$$c \in \bigcap_{m \geq n} (\overline{B}_n \oplus B_{n+1} \oplus \cdots \oplus B_m \oplus C_C(W_m^W)) \qquad \text{for some} \quad n = n(c) \in \omega.$$

Obviously C_0 is a group of cardinality $2^{\aleph_0}$.

5.2.7. *C_0 is an $\mathbb{F}_p W$-module.*

Proof. If $c \in C_0$ and $w \in W$, then there exists $n \in \omega$ such that $w \in W_n$ and $c = b_m + d_m$ for all $m \geq n$, where $b_m \in \overline{B}_n = \overline{B}_n^w$ and $d_m \in B_{n+1} + \cdots + B_m + C_C(W_m^W) \leq C_C(W_n^W) \leq C_C(w)$. Then $(b_m + d_m)^w \in \overline{B}_n + B_{n+1} + \cdots + B_m + C_C(W_m^W)$ for all $m \geq n$, and so $c^w = \bigcap_{m \geq n} (b_m + d_m)^w + C_C(W_m^W) \in C_0$. □

It follows from 5.2.6 that

5.2.8. $\bigcap_{n\in\omega} ([B, W] + C_B(W_n^W)) = \bigcap_{n\in\omega} ([B, W] \oplus \bigoplus_{k\geq 1} \langle b_{n+k}\rangle) = [B, W]$.

This also implies

5.2.9. $[B, W] = \bigcap_{n\in\omega}([B, W] + C_C(W_n^W))$.

Remark 5.2.10. We can choose for S_n any finite simple group $\neq C_p$. Then G has its composition factors isomorphic to C_p and S_n $(n \in \omega)$. In particular, choosing all S_n cyclic of prime order $q \neq p$ yields a locally solvable $\{p, q\}$-group G. One can also obtain slim p-groups by choosing all X_n elementary abelian of order p^2, and replacing A_n by the last but one non-trivial term of the Loewy series of $\mathbb{F}_p X_n$. Slim p-groups will however not lead to counterexamples as in Section 5.3, since totally imprimitive p-groups are countably recognizable by Theorem D (the centralizers $C_B(W_n^W)$ behave differently).

5.3. Construction of Counterexamples. We shall show, in this section, that the uncountable group $C_0 \rtimes W$ has an ω-cover of subgroups which are isomorphic to totally imprimitive subgroups of the wreath product $C_p \operatorname{wr} W$ containing W. Let Ω be the set on which W acts canonically. By [19, 3.3.3] the group W acts irreducibly on $\mathbb{C}\Omega$. Since C_p has a non-trivial one-dimensional complex representation, every subgroup of $C_p \operatorname{wr} W$ containing W has a faithful irreducible finitary linear representation on $\mathbb{C}\Omega$ too. Therefore $C_0 \rtimes W$ will be a counterexample which proves Theorem B.

Consider a countable subgroup H of $C_0 \rtimes W$ containing $G = B \rtimes W$. Then $[C_0, W] = [B, W] \leq H$. We choose H such that $|H \cap C_0 : [B, W]|$ is infinite; that is, $H \cap C_0 = [B, W] \oplus E$, where $E = \bigoplus_{i\geq 1} \langle c_i \rangle$. It suffices to embed such a group H onto a transitive subgroup of $C_p \operatorname{wr} W$.

Let Ω_n be the non-trivial orbit of W_n in Ω. We shall recursively construct embeddings $\alpha_k : H \longrightarrow C_p \operatorname{Wr}_\Omega W$ $(k \geq 0)$ such that the following hold:

– $\alpha_k|_W = id_W$, and $(H \cap C_0)\alpha_k$ lies in the base group of $C_p \operatorname{Wr}_\Omega W$;

– for every $c \in H \cap C_0$ there exists $n \in \omega$ such that the base group function $c\alpha_k : \Omega \to C_p$ is constant on each of the orbits of the pointwise stabilizer of Ω_n in W;

– $c_j \alpha_k = c_j \alpha_{k-1}$ for all $j < k$, and $c_k \alpha_k$ is finitary (note that $c_1\alpha_k, \ldots, c_k\alpha_k$ are finitary then).

Let us begin with the construction of α_0. Let Ψ be an ultrafilter on ω containing all cofinite subsets of ω. By Łoś Theorem, $(\prod_{m\in\omega} C_p)/\Psi \cong C_p$. Therefore a W-homomorphism

$$\alpha_0 : C_0 \longrightarrow ((\prod_{m\in\omega} C_p)/\Psi)\ \mathrm{Wr}_\Omega\ W \;=\; C_p \,\mathrm{Wr}_\Omega\, W$$

is given via

$$\begin{aligned} (\nu)(c\alpha_0) \;&=\; ((\nu)d_m)_{m\in\omega} && \text{for all } \nu\in\Omega, \text{ whenever} \\ c \;=\; \bigcap_{m\geq n} &(d_m + C_C(W_m^W)) && \text{with} \quad d_m \in \overline{B}_n + B_{n+1} + \cdots + B_m. \end{aligned}$$

α_0 is injective: If $c\in\ker\alpha_0$, then $\{m\in\omega \mid m\geq n \text{ and } (\Omega_\ell)d_m = 0\}\in\Psi$ for all $\ell\in\omega$, whence $c \in \bigcap_{\ell\in\omega} C_C(W_\ell^W) = 0$. Thus α_0 gives rise to an embedding $\alpha_0 : H \longrightarrow C_p \,\mathrm{Wr}_\Omega\, W$ via $\alpha_0|_W = id_W$. The second of the above requirements for α_0 is satisfied by choice of C_0.

Suppose now that α_k has been constructed for some k. We shall provide α_{k+1}. Choose $n=n(k)$ such that

(a) $\operatorname{supp} c_j\alpha_k \subseteq C_p\times\Omega_n$ for all $j\leq k$,

(b) $c_{k+1}\alpha_k$ is constant on the orbits of the pointwise stabilizer of Ω_n in W, and

(c) $c_1,\dots,c_{k+1}$ are independent modulo $[B,W] + C_{H\cap C_0}(W_n^W)$ (this condition can be realized because of 5.2.9 above).

Let U be the setwise stabilizer of Ω_n in W, let T be a right transversal of U in W containing 1, and let $T_0\subseteq T$ be a set of representatives of the double cosets UwU $(w\in W)$ containing 1.

Lemma 5.3.1. *The elements* $t(c_j\alpha_k)t^{-1}$ $(t\in T_0,\ j\leq k+1)$ *are independent modulo* $([B,U] + C_{H\cap C_0}(W_n^W))\alpha_k$.

Proof. Suppose that

$$\sum_{t\in T_0}\sum_{j\leq k+1} \lambda_{t,j}\, t(c_j\alpha_k)t^{-1} \;\equiv\; 0 \quad \text{modulo } ([B,U] + C_{H\cap C_0}(W_n^W))\alpha_k$$

for certain $\lambda_{t,j}\in\mathbb{F}_p$. Since we calculate modulo $C(W_n^W)$, condition (b) allows us to assume that $\operatorname{supp} c_{k+1}\alpha_k \subseteq C_p\times\Omega_n$, and hence $\operatorname{supp} t(c_j\alpha_k)t^{-1} \subseteq C_p\times\Omega_n t^{-1}$ for all $t\in T_0$, $j\leq k+1$. Since T_0 represents the double cosets UwU $(w\in W)$, the sets $\Omega_n t_1^{-1}$ and $\Omega_n t_2^{-1}$ lie in different U-orbits for $t_1\neq t_2$ from T_0. Hence

$$\sum_{j\leq k+1} \lambda_{t,j}\, t(c_j\alpha_k)t^{-1} \;\equiv\; 0 \quad \text{modulo } ([B,U] + C_{H\cap C_0}(W_n^W))\alpha_k \quad \text{for every } t\in T_0.$$

Conjugation with t and application of α_k^{-1} yields

$$\sum_{j\leq k+1} \lambda_{t,j}\, c_j \;\equiv\; 0 \quad \text{modulo} \quad [B,W] + C_{H\cap C_0}(W_n^W) \quad \text{for every } t\in T_0.$$

And condition (c) implies $\lambda_{t,j} = 0$ for all $t\in T_0$, $j\leq k+1$. □

Let D be the group of all constant functions $\Omega_n \to C_p$. Because of Lemma 5.3.1 we can find a homomorphism $\varphi_k : (H \cap C_0)\alpha_k \cdot U \longrightarrow D$ such that

(i) $([B,U] + C_{H\cap C_0}(W_n^W))\alpha_k\varphi_k \;=\; 0,$

(ii) $c_j\alpha_k\varphi_k \;=\; 0$ for $j \le k+1,$

(iii) $(t(c_j\alpha_k)t^{-1})\varphi_k \;=\; t\,(c_j\alpha_k)^{-1}t^{-1}|_{\Omega_n}$ for $1 \ne t \in T_0,\ j \le k+1,$ and

(iv) $U\varphi_k \;=\; 0.$

The choice in (iv) is possible, since $(H \cap C_0)\alpha_k \cdot U \equiv (H \cap C_0)\alpha_k \times U$ modulo $[B,U]\alpha_k$. A derivation δ_k of $H\alpha_k$ into the base group of $C_p \operatorname{Wr}_\Omega W$ is now given via $W\delta_k = 0$ and

$$c\alpha_k\delta_k|_{\Omega_n\tau} \;=\; \tau^{-1}(\tau(c\alpha_k)\tau^{-1})\varphi_k\,\tau \qquad \text{for all } c \in H \cap C_0 \text{ and } \tau \in T.$$

We define the group homomorphism $\alpha_{k+1} : H \longrightarrow C_p \operatorname{Wr}_\Omega W$ via $h\alpha_{k+1} = h\alpha_k \cdot h\alpha_k\delta_k$ for all $h \in H$. We show that α_{k+1} is an embedding: Suppose that $h \in \ker\alpha_{k+1}$. Then $h \in H \cap C_0$, and $h\alpha_k = -(h\alpha_k\delta_k)$ is constant on each of the blocks $\Omega_n\tau$ $(\tau \in T)$. It follows that $h\alpha_k \in (C_{H\cap C_0}(W_n^W))\alpha_k \trianglelefteq H\alpha_k$. But then $\tau(h\alpha_k)\tau^{-1} \in \ker\varphi_k$ for all $\tau \in T$, whence $h\alpha_k = -(h\alpha_k\delta_k) = 0$. Thus, α_{k+1} is injective.

For every $1 \ne \tau \in T$ there is a unique $1 \ne t \in T_0$ such that $\tau U = tU$. Then

$$\tau\,(c_j\alpha_k)\tau^{-1} \;\equiv\; t\,(c_j\alpha_k)t^{-1} \quad \text{modulo } [B,U]\alpha_k \le \ker\varphi_k \quad \text{for all } j \le k+1.$$

Moreover, (a) and (b) imply that $c_j\alpha_k$ is constant on $\Omega_n tU$, whence

$$\begin{aligned}(\tau\,(c_j\alpha_k)\tau^{-1})\varphi_k \;&=\; (t\,(c_j\alpha_k)t^{-1})\varphi_k\\ &=\; t\,(c_j\alpha_k)^{-1}t^{-1}|_{\Omega_n} \;=\; \tau\,(c_j\alpha_k)^{-1}\tau^{-1}|_{\Omega_n}\end{aligned}$$

for all $j \le k+1$. It follows that $c_j\alpha_{k+1} = c_j\alpha_k$ for $j \le k$, and that $\operatorname{supp} c_{k+1}\alpha_{k+1} \subseteq C_p \times \Omega_n$. Moreover, the definition of δ_k and the corresponding property of α_k imply directly that for every $c \in H \cap C_0$ there exists $m \ge n$ such that $c\alpha_{k+1}$ is constant on each of the orbits of the pointwise stabilizer of Ω_m in W. The construction of α_{k+1} is complete.

Finally, we choose the ultrafilter Ψ on ω as above and define a homomorphism

$$\alpha : H \;\longrightarrow\; ((\prod_{k\in\omega} C_p)/\Psi)\ \operatorname{Wr}_\Omega\ W \;=\; C_p \operatorname{Wr}_\Omega W$$

diagonally; that is, $\alpha|_W = id_W$ and

$$(\nu)(c\alpha) \;=\; ((\nu)(c\alpha_k))_{k\in\omega} \qquad \text{for all } c \in H \cap C_0 \text{ and } \nu \in \Omega.$$

Here $E\alpha$ is finitary, since $\operatorname{supp} c_k\alpha = \operatorname{supp} c_k\alpha_k$ for all k. And $[B,W]\alpha$ is finitary too, since $\operatorname{supp}[b,w]\alpha_k \subseteq C_p \times \operatorname{supp}_\Omega w$ for all $b \in B$, $w \in W$, $k \in \omega$. It follows that $H\alpha$ is a totally imprimitive subgroup of the restricted wreath product $C_p \operatorname{wr}_\Omega W$.

It remains to show that α is injective. Clearly $\ker\alpha \le H \cap C_0$. Assume that $1 \ne c \in \ker\alpha$. Then $1 \ne [c,w] \in [B,W] \cap \ker\alpha$ for some $w \in W$. But $\operatorname{supp}[c,w]\alpha_k$ is uniformly bounded for all $k \in \omega$, whence $[c,w]\alpha \ne 1$ (see [5, Lemma B.1]). Thus α is the desired embedding.

5.4. Finitary Linear Representations of the Counterexamples. Although the counterexample $Z = C_0 \rtimes W \cong (C_0 \rtimes W)\alpha_0 \leq C_p \operatorname{Wr}_\Omega W$ is not irreducible finitary linear, we note that the wreath product has a faithful finitary linear representation on the $\mathbb{F}_p$-vector space with basis $\Omega \cup \{x\}$ via $[x, C_p \operatorname{Wr}_\Omega W] = 0$ and

$$(\nu)(fw) \;=\; ((\nu)f)\cdot x \;+\; (\nu)w \qquad \text{for all } \nu \in \Omega,\ w \in W \text{ and } f : \Omega \to C_p = \mathbb{F}_p^+.$$

However, Z has no faithful finitary linear representation over any field of characteristic $q \geq 0$ coprime to p: Otherwise the non-existence of a unipotent normal subgroup in Z would show, that Z is a subdirect product of irreducible finitary linear groups Z_i $(i \in I)$ over a field of characteristic q ([19, Proposition 1]). Since Z does not contain an infinite simple section, the groups Z_i are totally imprimitive and countable. Now Z' is a countable subgroup of the uncountable group Z. Therefore there exists a countable subset $J \subseteq I$ such that each group Z_i $(i \in I \setminus J)$ is abelian. Let K denote the kernel of the canonical projection of Z into $\prod_{j \in J} Z_j$. Then $[K, Z'] \leq K \cap Z' = 1$, whence $1 \neq K \leq C_Z(Z') = 1$. This contradiction shows that faithful finitary linear representations of Z are just possible over fields of characteristic p. But then the direct product $Z \times \operatorname{Tor}(\prod_{n \in \omega} C_{p^n})$ is a group which has no faithful finitary linear representation at all, although its countable subgroups are finitary permutation groups (cf. [11, Example 3.4]).

6. Totally Imprimitive Groups

This section is devoted to the proof of Theorem C. In the sequel, we will use the following standard notation for iterated commutators: $[a,_0 b] = a$ and $[a,_{k+1} b] = [[a,_k b], b]$ for all $k \in \omega$.

Lemma 6.1. *Every finite group satisfies the identity* $[x,_\ell y] = [x,_{\ell-k} y]$ *for suitable positive integers* $k < \ell$.

Proof. Let S be a finite group. For all $a, b \in S$, there exist positive integers $k_{a,b} < \ell_{a,b}$ such that $[a,_m b] = [a,_{m-k_{a,b}} b]$ for all $m \geq \ell_{a,b}$. Then S satisfies the required identity with $k = \prod_{a,b \in S} k_{a,b}$ and $\ell = \prod_{a,b \in S} \ell_{a,b}$. ☐

Lemma 6.2. *Let* $S = \langle g, h\rangle$ *be a group such that the conjugates* $h^{-j} g h^j$ *commute for* $0 \leq j \leq \ell$. *Then* $[g,_\ell h] \cdot [g,_{\ell-k} h]^{-1} \equiv g^{h^\ell} \mod \langle g, g^h, ..., g^{h^{\ell-1}}\rangle$ *for* $0 \leq k < \ell$.

Proof. Induction shows that $[g,_j h] \equiv g^{h^j} \mod \langle g, g^h, ..., g^{h^{(j-1)}}\rangle$ for $0 \leq j \leq \ell$. ☐

As a first step towards Theorem C we establish a counterpart to [7, Lemma 1] (see Proposition 6.6) for totally imprimitive finitary linear groups.

Proposition 6.3. *Let the periodic group G have an ω-cover of totally imprimitive finitary linear groups. Then the commutator subgroup G' of G is countable. In particular, G has countable conjugacy classes.*

Proof. Let $\mathfrak{C}$ be an ω-cover of subgroups of the group G such that each $X \in \mathfrak{C}$ is a totally imprimitive subgroup of $\mathrm{FGL}_{\mathbb{F}_X}(V_X)$ for some infinite-dimensional $\mathbb{F}_X$-vector space V_X. From Lemma 3.2 we may assume that the characteristic p of the fields $\mathbb{F}_X$ is independent of $X \in \mathfrak{C}$. Since the unipotent radical of every $X \in \mathfrak{C}$ is trivial, G has no non-trivial normal p-subgroup in the case when $p > 0$.

Consider a fixed $X \in \mathfrak{C}$. Let $P_X = \{Y \in \mathfrak{C} \mid X \leq Y\}$, and consider some $Y \in P_X$. From [1, Proposition 1], the group X cannot be a subdirect product of linear groups. Hence V_Y is an irreducible $\mathbb{F}_Y\langle X^Y\rangle$-module, and $Y' \leq \langle X^Y\rangle$. This holds for every $Y \in P_X$, whence $G' \leq \langle X^G\rangle$. Let $H = \langle X^G\rangle$. We now show that H is countable.

Let $\{X_n\}_{n\in\omega}$ be an ascending chain of finite subgroups with union X. By Lemma 3.2, we can recursively choose integers d_n $(n \in \omega)$ and a descending chain $\{\mathfrak{C}_n\}_{n\in\omega}$ of ω-subcovers of $\mathfrak{C}$ such that

$$\dim_{\mathbb{F}_Y}[V_Y, X_n] \leq d_n \qquad \text{for all } Y \in \mathfrak{C}_n,\ n \in \omega.$$

Let Φ_n be an ultrafilter on $\mathfrak{C}_\mathfrak{n}$ which contains the set $P_Y \cap \mathfrak{C}_n$ for every $Y \in \mathfrak{C}_n$. For each $n \in \omega$, the group G embeds into the ultraproduct $K_n = (\prod_{Y\in\mathfrak{C}_n} Y)/\Phi_n$ via $g = (g_Y)_{Y\in\mathfrak{C}_n}$ for all $g \in G$; here $g_Y = g$ if $g \in Y$, and $g_Y = 1$ otherwise. From Łoś Theorem, K_n acts as a transformation group on the vector space $W_n = (\prod_{Y\in\mathfrak{C}_n} V_Y)/\Phi_n$ in such a way, that

$$\dim[W_n, X_m] \leq d_m \quad \text{for all} \quad m \leq n.$$

Let Ψ be an ultrafilter on ω containing all cofinite subsets of ω. Then G embeds diagonally into the ultraproduct $K = (\prod_{n\in\omega} K_n)/\Psi$, which acts as a transformation group on the vector space $W = (\prod_{n\in\omega} W_n)/\Psi$ in such a way, that

$$\dim[W, X_n] \leq d_n \quad \text{for all} \quad n \in \omega.$$

In particular, $H \leq \mathrm{FGL}(W)$. Let $\mathfrak{S}$ be an H-composition series in W. As in the proof of [16, Theorem B(vi)], each of the normal closures $\langle X_n^G\rangle = \langle X_n^{G'}\rangle = \langle X_n^H\rangle$ $(n \in \omega)$ acts non-trivially on just finitely many factors of $\mathfrak{S}$. Therefore H acts non-trivially on just countably many factors of $\mathfrak{S}$. Since the characteristic of the field $\mathbb{F}$ associated with W is still p, the unipotent radical of H is trivial. Therefore H is a subdirect product of the countably many groups induced from H on the above factors of $\mathfrak{S}$. Since we want to prove countability of H, it suffices to consider the action of H on one of these factors; that is, we may assume without loss that H acts irreducibly on W. If $\dim_{\mathbb{F}} W$ is finite, then H is countable from [23, 1.19] and [28], [27]. Suppose now that $\dim_{\mathbb{F}} W$ is infinite. Since no $Y \in \mathfrak{C}$ contains an infinite alternating section, the same holds for H. Hence H is a totally imprimitive subgroup of $\mathrm{FGL}_{\mathbb{F}}(W)$. In particular, H is countable. □

Our next result corresponds to [7, Theorem 4].

Proposition 6.4. *Totally imprimitive periodic finitary linear groups without non-trivial abelian normal subgroups are countably recognizable.*

Proof. Let the periodic group G have an ω-cover of totally imprimitive finitary linear groups which do not contain a non-trivial abelian normal subgroup. By Proposition 6.3, the derived subgroup of G is countable, and hence an irreducible subgroup of some group in the ω-cover. Thus G' is a totally imprimitive subgroup of $\mathrm{FGL}_{\mathbb{F}}(V)$ for some infinite-dimensional $\mathbb{F}$-vector space V. From [12] we may choose $\mathbb{F}$ algebraically closed. Consider a non-trivial finite subgroup F of G'. Choose a proper G'-block V_0 in V such that F fixes each $V_0 g \neq V_0$ $(g \in G')$ elementwise. Then $C_{G'}(F^G)$ must leave each $V_0 g$ $(g \in G')$ invariant. From [23, 1.17] the normalizer in G' of V_0 contains a finite subgroup E such that $\mathbb{F}E$ induces $\mathrm{End}_{\mathbb{F}}(V_0)$ on V_0. From Schur's Lemma ([23, 1.2]), the centralizer $C_{G'}(\langle E, F\rangle^G)$ acts like a scalar on each $V_0 g$ $(g \in G')$. The hypothesis now yields $C_{G'}(\langle E, F\rangle^G) = 1$, whence $C_G(\langle E, F\rangle^G) = 1$ too. Since G has countable conjugacy classes (Proposition 6.3), the centralizer $C_G(\langle E,F\rangle^G)$ has countable index in G. And so G must be countable. □

Theorem 6.5. *Theorem C holds for the class $\mathfrak{X}$ of all irreducible periodic finitary linear groups.*

Proof. Let $\mathfrak{C}$ be an ω-cover of the group G such that each $X\in\mathfrak{C}$ is a totally imprimitive subgroup of $\mathrm{FGL}_{\mathbb{F}_X}(V_X)$ for some infinite-dimensional $\mathbb{F}_X$-vector space V_X. From Lemma 3.2 we may assume that $\chi=\operatorname{char}\mathbb{F}_X$ is independent of $X\in\mathfrak{C}$. And [12] allows us to choose for $\mathbb{F}_X$ the algebraic closure $\mathbb{F}$ of the prime field in characteristic χ. By Proposition 6.3, the commutator subgroup G' is countable. We may therefore assume that $G'=X'$ for all $X\in\mathfrak{C}$.

From Proposition 6.4 we may assume that G contains a non-trivial elementary-abelian normal p-subgroup N for some prime p. Since every $X\in\mathfrak{C}$ has trivial unipotent radical, we see that $p\neq\chi$. Fix $1\neq z\in N\cap G'$. For every $X\in\mathfrak{C}$ there exists an X-block $V_{X,0}$ in V_X such that z *acts just on* $V_{X,0}$, that is, z centralizes every $V_{X,0}x\neq V_{X,0}$ $(x\in X)$. As N is elementary-abelian, z has just finitely many G-conjugates which act just on $V_{X,0}$ [23, 2.2]. Since z has just countably many G-conjugates, Lemma 3.2 allows us to assume that $d=\dim_{\mathbb{F}}V_{X,0}$ and the finite set of those G-conjugates of z which act just on $V_{X,0}$ are independent of $X\in\mathfrak{C}$. Let F be their join, and let $F=F_0$, F_1, ... be the G-conjugates of F. From Schur's Lemma [23, 1.2], the elementary-abelian p-group $\langle F^G\rangle$ acts like a scalar on each of its homogeneous components. Therefore the G-blocks

$$V_{X,i} \;=\; \{v\in V_X \mid [v,F_i]\neq 0\}\cup\{0\} \qquad (i\in\omega)$$

form a system of imprimitivity in V_X. Let $V_0=\mathbb{F}^d$ and $V=V_0\otimes\omega$. An identification of $V_{X,i}$ with $V_0\otimes i$ leads to an identification of V_X with V, and to an embedding σ_X of X into the *restricted* wreath product $\mathrm{GL}_{\mathbb{F}}(V_0)\,\mathrm{wr}_\omega\mathrm{FSym}(\omega)$, which acts on V. For each $g\in X\in\mathfrak{C}$ we have $g\sigma_X=f_{X,g}\pi_g$ for some $f_{X,g}:\omega\to\mathrm{GL}_{\mathbb{F}}(V_0)$, where $\pi_g\in\mathrm{FSym}(\omega)$ is given via $F_{i\pi_g}=F_i^g$, independently of $X\in\mathfrak{C}$. We let π denote the corresponding homomorphism $g\mapsto\pi_g$ of G into $\mathrm{FSym}(\omega)$.

Now ω contains just countably many proper $G\pi$-blocks Ω_n $(n\in\omega)$. The normalizer in $G'\sigma_X$ of $V_0\otimes\Omega_n$ acts irreducibly on $V_0\otimes\Omega_n$, and $\mathbb{F}$ is algebraically closed. Hence [23, 1.17] and Lemma 3.2 yield a sequence $\{E_n\}_{n\in\omega}$ of finite subgroups of G', and a descending chain $\{\mathfrak{C}_n\}_{n\in\omega}$ of ω-subcovers of $\mathfrak{C}$ such that $\mathbb{F}(E_n\sigma_X)$ induces $\mathrm{End}_{\mathbb{F}}(V_0\otimes\Omega_n)$ on $V_0\otimes\Omega_n$ for all $X\in\mathfrak{C}_n$, $n\in\omega$. Since V is countable, we may even assume that $\sigma_X|_{E_n}=\sigma_Y|_{E_n}$ for all $X,Y\in\mathfrak{C}_n$, $n\in\omega$.

Next, let Φ_n be an ultrafilter on $\mathfrak{C}_n$ which contains the set $P_X\cap\mathfrak{C}_n$ for every $X\in\mathfrak{C}$, where $P_X=\{Y\in\mathfrak{C}\mid X\leq Y\}$. From Łoś Theorem, the vector space $U_n=(\prod_{X\in\mathfrak{C}_n}V_0)/\Phi_n$ over the algebraically closed field $\mathbb{K}_n=(\prod_{X\in\mathfrak{C}_n}\mathbb{F})/\Phi_n$ has dimension d. And a group homomorphism τ_n of G into the *unrestricted* wreath product $W_n=\mathrm{GL}_{\mathbb{K}_n}(U_n)\,\mathrm{Wr}_\omega\,\mathrm{FSym}(\omega)$ is given by $g\tau_n=f_g^n\pi_g$ for all $g\in G$, where $f_g^n:\omega\to\mathrm{GL}_{\mathbb{K}_n}(U_n)$ is defined via

$$(i)f_g^n \;=\; ((i)f_{X,g})_{X\in\mathfrak{C}_n} \quad \text{for all} \quad i\in\omega;$$

here $f_{X,g}=1$ whenever $g\notin X\in\mathfrak{C}_n$. We shall show next that each τ_n is actually an embedding.

Assume that $\ker\tau_n\neq 1$. Clearly $\ker\tau_n$ is not a χ-group. Choose $1\neq g\in G'\cap\ker\tau_n$ of prime order $q\neq\chi$. Then $\pi_g=1$, and $g\sigma_X$ $(X\in\mathfrak{C}_n)$ normalizes each block $V_0\otimes i$ $(i\in\omega)$. The elementary-abelian q-subgroups of $\mathrm{GL}_{\mathbb{F}}(V_0\otimes i)$

have ranks bounded by d ([23, 2.2]). Choose positive integers $k<\ell$ such that the holomorph of the elementary-abelian q-group of rank d satisfies the identity $[x,_\ell y] = [x,_{\ell-k} y]$ (Lemma 6.1). Note that $\ell \geq d$. From [17, Lemma 3.2] there exists $h \in G'$ such that the elements $h^{-j}gh^j$ $(0 \leq j \leq \ell)$ commute, since they act just on X-blocks of V_X with pairwise trivial intersections, for some $X \in \mathfrak{C}$. Lemma 6.2 yields $[g,_\ell h] \neq [g,_{\ell-k} h]$. Let $\omega_0 = \mathrm{supp}_\omega \pi_h$. For fixed $X \in \mathfrak{C}$ and $i \in \omega - \omega_0$, let η_i denote the projection of $\langle g, h\rangle$ into $\mathrm{GL}_{\mathbb{F}}(V_{X,i})$. Since $(h^{-k}gh^k)\eta_i \in A_i = \langle h^{-j}gh^j \mid 0 \leq j < k\rangle\eta_i$ for some $k \leq d$, the elementary-abelian q-group A_i is $h\eta_i$-invariant. By choice of k and ℓ we have $([g,_\ell h] \cdot [g,_{\ell-k} h]^{-1})\eta_i = 1$. It follows that $([g,_\ell h] \cdot [g,_{\ell-k} h]^{-1})\sigma_X$ acts just on $V_0 \otimes \omega_0$, for all $X \in \mathfrak{C}_n$. Hence, for every $X \in \mathfrak{C}$ there is some $i \in \omega_0$ such that $(i)f_{X,g} \neq 1$. By [5, Lemma B.1] there exists $i \in \omega_0$ such that $\{X \in \mathfrak{X} \mid (i)f_{X,g} \neq 1\} \in \Phi$. But then $(i)f_g^n \neq 1$, and hence $g\tau_n \neq 1$, a contradiction. Thus, τ_n is an embedding of G into W_n.

Let Ψ be an ultrafilter on ω containing all cofinite subsets of ω. Again, $U = (\prod_{n\in\omega} U_n)/\Psi$ is a d-dimensional vector space over the algebraically closed field $\mathbb{K} = (\prod_{n\in\omega} \mathbb{K}_n)/\Psi$, and an embedding $\tau : G \longrightarrow W = \mathrm{GL}_{\mathbb{K}}(U)\,\mathrm{Wr}_\omega\,\mathrm{FSym}(\omega)$ is given by $g\tau = f_g\pi_g$ for all $g \in G$, where

$$(i)f_g \;=\; ((i)f_g^n)_{n\in\omega} \quad \text{for all} \quad i \in \omega.$$

Since G does not contain an infinite alternating section, the image $G\pi$ of the projection of G into the top group of W is a totally imprimitive subgroup of $\mathrm{FSym}(\omega)$. It remains to show that $G\tau$ acts irreducibly on $U \otimes \omega$, and that the base group functions f_g $(g \in G)$ act like scalars on $U \otimes \Delta$ for almost all $G\pi$-translates Δ of every Ω_n $(n \in \omega)$.

First of all we identify $\mathbb{F}$ diagonally with a subfield of each $\mathbb{K}_n$ $(n \in \omega)$ and of $\mathbb{K}$. Similarly we can identify $\mathbb{K}_n \otimes V_0$, resp. $\mathbb{K} \otimes V_0$, with a $\mathbb{K}_n$- resp. $\mathbb{K}$-subspace of U_n, resp. U. In this way all the embeddings σ_X $(X \in \mathfrak{C})$ map into each W_n $(n \in \omega)$ and into W. From our choice of E_n we see that $\tau_n|_{E_n} = \sigma_X|_{E_n}$ for all n and all $X \in \mathfrak{C}_n$, whence $\tau|_{E_n} = \sigma_X|_{E_n}$. Since $\dim_{\mathbb{K}} \mathbb{K}(E_n\tau)|_{U\otimes\Omega_n} = (d \cdot |\Omega_n|)^2$, we conclude that $\mathbb{K}(E_n\tau)$ induces $\mathrm{End}_{\mathbb{K}}(U \otimes \Omega_n)$ on $U \otimes \Omega_n$. In particular, $G\tau$ is an irreducible subgroup of W.

Now consider fixed $g \in G$, $n \in \omega$. Choose $X \in \mathfrak{C}_n$ containing g. Then Ω_n is contained in a proper $G\pi$-block $\overline{\Omega}_n$ of ω such that $\langle E_n, g\rangle\sigma_X$ acts trivially on $U \otimes (\omega - \overline{\Omega}_n)$. Note that $\overline{\Omega}_n$ contains just finitely many $G\pi$-translates of Ω_n. Consider a $G\pi$-translate Δ of Ω_n which lies in $\omega - \overline{\Omega}_n$. Then there exists $h \in G'$ such that $E_n^h\sigma_X$ and $E_n^h\tau$ act trivially on $U \otimes \overline{\Omega}_n$, and such that $\mathbb{K}(E_n^h\sigma_X)$ and $\mathbb{K}(E_n^h\tau)$ induce $\mathrm{End}_{\mathbb{K}}(U \otimes \Delta)$ on $U \otimes \Delta$. Since $g\sigma_X$ commutes with $E_n^h\sigma_X$, also $g\tau$ commutes with $E_n^h\tau$. But $\mathrm{supp}_\omega \pi_g \subseteq \overline{\Omega}_n$. Therefore $f_g|_{U\otimes\Delta} \in \mathrm{GL}_{\mathbb{K}}(U \otimes \Delta)$ commutes with $\mathbb{K}(E_n^h\tau)|_{U\otimes\Delta} = \mathrm{End}_{\mathbb{K}}(U \otimes \Delta)$. We thus obtain that f_g acts like a scalar on $U \otimes \Delta$. □

The permutation group version of Theorem C will be proved similarly to the transformation group version. For the convenience of the reader we first give a proof of the crucial [7, Lemma 1].

Proposition 6.6 ([7, Lemma 1])**.** *Let the group G have an ω-cover of totally imprimitive finitary permutation groups. Then the commutator subgroup G' of G is countable.*

Proof. Let $\mathfrak{C}$ be an ω-cover of subgroups of the group G such that each $X \in \mathfrak{C}$ is a totally imprimitive subgroup of $\mathrm{FSym}(\Omega_X)$ for some infinite set Ω_X. For fixed $X \in \mathfrak{C}$, let $P_X = \{Y \in \mathfrak{C} \mid X \leq Y\}$, and consider some $Y \in P_X$. From [25, Theorem 1], the group X cannot be a subdirect product of finite groups. Hence $\langle X^Y \rangle$ acts transitively on Ω_X, and $Y' \leq \langle X^Y \rangle$. This holds for every $Y \in P_X$, whence $G' \leq \langle X^G \rangle$. Therefore it suffices to show that G has countable conjugacy classes.

Consider elements $g, h \in G$. If $g, h \in X \in \mathfrak{C}$, then $\mathrm{supp}_{\Omega_X} \langle g, h \rangle$ is contained in a finite X-block in Ω_X, whence h centralizes all but finitely many X-conjugates of g. An application of Lemma 3.2 shows that h centralizes all but finitely many G-conjugates of g. In particular, G acts finitarily on the conjugacy class g^G. Since no $X \in \mathfrak{C}$ has an infinite alternating section, the same holds for G. But then G must act as a totally imprimitive finitary permutation group on g^G, whence g^G is countable. □

Theorem 6.7. *Theorem C holds for the class $\mathfrak{X}$ of all transitive finitary permutation groups.*

Proof. Let $\mathfrak{C}$ be an ω-cover of the group G such that each $X \in \mathfrak{C}$ is a totally imprimitive subgroup of $\mathrm{FSym}(\Omega_X)$ for some infinite set Ω_X. By Proposition 6.6, the commutator subgroup G' is countable. We may therefore assume that $G' = X'$ for all $X \in \mathfrak{C}$.

Fix $1 \neq z \in G'$. For every $X \in \mathfrak{C}$ there exists an X-block $\Omega_{X,0}$ in Ω_X such that z fixes $\Omega_X - \Omega_{X,0}$ pointwise. Since G' is countable, Lemma 3.2 allows us to assume that $d = |\Omega_{X,0}|$ and the finite set of those G-conjugates of z that fix $\Omega_X - \Omega_{X,0}$ pointwise are independent of $X \in \mathfrak{C}$. Let F be their join, and let $F = F_0, F_1, \ldots.$ be the G-conjugates of F. For every G-translate Δ of $\Omega_{X,0}$ there exists a unique i such that $\mathrm{supp}_{\Omega_X} F_i \subseteq \Delta$; we denote Δ by $\Omega_{X,i}$. Let $\Gamma = \{1, \ldots, d\}$ and $\Omega = \Gamma \times \omega$. An identification of $\Omega_{X,i}$ with $\Gamma \times i$ leads to an identification of Ω_X with Ω, and to an embedding $\sigma_X : X \longrightarrow \mathrm{Sym}(\Gamma)\,\mathrm{wr}_\omega \mathrm{FSym}(\omega)$, where the wreath product acts on Ω. For each $g \in X \in \mathfrak{C}$ we have $g\sigma_X = f_{X,g}\pi_g$ for some $f_{X,g} : \omega \to \mathrm{Sym}(\Gamma)$, where $\pi_g \in \mathrm{FSym}(\omega)$ is given via $F_{i\pi_g} = F_i^g$, independently of $X \in \mathfrak{C}$. We let π denote the corresponding homomorphism $g \mapsto \pi_g$ of G into $\mathrm{FSym}(\omega)$.

Now ω contains just countably many proper $G\pi$-blocks Ω_n $(n \in \omega)$. The setwise stabilizer $S_{X,n}\sigma_X$ in $G'\sigma_X$ of $\Gamma \times \Omega_n$ acts transitively on $\Gamma \times \Omega_n$. Hence Lemma 3.2 yields a sequence $\{E_n\}_{n\in\omega}$ of finite subgroups of G', and a descending chain $\{\mathfrak{C}_n\}_{n\in\omega}$ of ω-subcovers of $\mathfrak{C}$ such that $E_n\sigma_X$ induces the action of $S_{X,n}\sigma_X$ on $\Gamma \times \Omega_n$, for all $X \in \mathfrak{C}_n$, $n \in \omega$. Since Ω is countable, we may even assume that $\sigma_X|_{E_n} = \sigma_Y|_{E_n}$ for all $X, Y \in \mathfrak{C}_n$, $n \in \omega$.

In exactly the same way as in the proof of Theorem 6.5 we can now invoke ultraproduct constructions to obtain an embedding $\tau : G \longrightarrow S\,\mathrm{Wr}_\omega\,\mathrm{FSym}(\Omega)$ with the desired properties; here S is the subgroup of $\mathrm{Sym}(\Gamma \times \Omega_0)$ induced from $E_0\sigma_X$. □

7. Some Sufficient Conditions

This section consists of a brief discussion of sufficient conditions for countable recognizability of totally imprimitive $\mathfrak{X}$-groups. It shows how Theorem C can be applied in particular instances. The following observation is straightforward from Theorem C and generalizes conditions given in [7] for the permutation group case.

Proposition 7.1. *The class of all totally imprimitive $\mathfrak{X}$-groups G that satisfy $|NG'/G'| < \infty$ for every elementary-abelian normal subgroup N is countably recognizable.*

Proof. Assume that there exists an uncountable group G with an ω-cover $\mathfrak{C}$ of totally imprimitive $\mathfrak{X}$-subgroups satisfying the property in the statement of the proposition. Then G has an uncountable abelian normal subgroup N by Theorem C. We can replace G by $G'N$ and choose N elementary-abelian. Lemma 3.2 allows us to assume that the finite value $s = |(N \cap X)G'/G'|$ is independent of $X \in \mathfrak{C}$. But then $|NG'/G'| = s$ too, in contradiction to the uncountability of N. □

Remark 7.2. Among the totally imprimitive $\mathfrak{X}$-groups G satisfying $|NG'/G'| < \infty$ for every elementary-abelian normal subgroup N are not only those without abelian normal subgroups, but also full wreath products $L \operatorname{wr} F_1 \operatorname{wr} \cdots \operatorname{wr} F_n \operatorname{wr} \cdots$, where the F_n are finite groups, and where L is a finite resp. irreducible linear group. In fact, for every elementary-abelian normal subgroup N of such a full wreath product G there exists $n \in \omega$ such that N is contained in $\langle A^G \rangle$, where A is an elementary-abelian normal subgroup of the unipotent-free linear group $L_n = L \operatorname{wr} F_1 \operatorname{wr} \cdots \operatorname{wr} F_n$; and it is well-known that $\langle A^G \rangle G'/G' \cong A/[A, L_n]$ is finite.

Our most spectacular result, however, concerns p-groups.

Proof of Theorem D. Assume that there exists an uncountable p-group G with an ω-cover of totally imprimitive $\mathfrak{X}$-subgroups. Then G has an uncountable abelian normal subgroup N by Theorem C. We can replace G by $G'N$ and choose N elementary-abelian. Now we refer to the notation introduced in the proofs of Theorems 6.5 and 6.7, and we choose $\Omega_0 = \{0\}$. In addition, Lemma 3.2 allows us to assume that $\sigma_X|_F = \sigma_Y|_F$ for all $X, Y \in \mathfrak{C}$.

We consider the transformation group case first. Since $F\sigma_X$ acts just on $V_0 \otimes 0$, it is normalized by $E_0\sigma_X$ $(X \in \mathfrak{C}_0)$. Because G is a p-group, the centralizer $C_F(E_0)$ must be non-trivial. Let T be a right transversal in G' of the setwise stabilizer in G' of $V_0 \otimes 0$. For all $t \in T, X \in \mathfrak{C}_0$, the group algebra $\mathbb{F}(E_0^t\sigma_X)$ induces $\operatorname{End}_{\mathbb{F}}(V_0 \otimes 0\pi_t)$ on $V_0 \otimes 0\pi_t$. Therefore, $(C_{N\cap X}(E_0^t))\sigma_X$ acts like $(C_F(E_0)^t)\sigma_X$ on $V_0 \otimes 0\pi_t$. It follows that

$$C_{N\cap X}(E_0^G) \ \leq \ \mathrm{Dr}_{t\in T}\ (C_F(E_0))^t \ \leq \ C_F(E_0)\cdot[F, G'] \ \leq \ F\cdot G'$$

for all $X \in \mathfrak{C}_0$. And so $C_N(E_0^G)$ becomes countable. On the other hand, the conjugacy classes in G are countable, so $C_N(E_0^G)$ has countable index in N. This contradicts the uncountability of N.

We now consider the permutation group case. From Lemma 3.2 we may assume that the subgroup of $\operatorname{Sym}(\Gamma \times 0)$ induced from $E_0\sigma_X$ is independent of $X \in \mathfrak{C}_0$, so that it can be identified with S. In this way the embeddings σ_X $(X \in \mathfrak{C}_0)$ and τ_0 map into the wreath product $W = S \operatorname{Wr}_\omega \operatorname{FSym}(\omega)$. The augmentation ideal of the group algebra $\mathbb{C}S$ is a direct sum of non-trivial irreducible $\mathbb{C}S$-submodules $M_1, \ldots, M_r$, and there is a faithful action of W on the $\mathbb{C}$-vector space $\bigoplus_{i=1}^r (M_i \otimes \omega)$. In particular, W is a subdirect product of the wreath products $Z_i = \mathrm{GL}_{\mathbb{C}}(M_i) \operatorname{Wr}_\omega \operatorname{FSym}(\omega)$ $(1 \leq i \leq r)$, and the projection $\rho_j : W \to Z_j$ must map $N\tau_0$ onto an uncountable subgroup of Z_j for some fixed j. Let $\eta = \tau_0\rho_j$. Since the conjugacy classes in $G\eta$ are countable too, it again suffices to show that $C_{N\eta}((E_0^G)\eta)$ is countable. To this end we can copy the argument applied in the transformation group case: We may assume that $F\eta$ is non-trivial. Then $F\eta$ acts just on $M_j \otimes 0$ and is therefore normalized by $E_0\eta$. Thus $C_{F\eta}(E_0\eta)$ is non-trivial. And since $E_0\tau_0$ induces the action of S on $S\times 0$, the group $E_0\eta$ acts irreducibly on $M_j\otimes 0$, whence $C_{(N\cap X)\eta}((E_0^t)\eta)$ acts like $C_{F\eta}(E_0\eta)^{t\eta}$ on $M_j\otimes 0\pi_t$, for all $X\in\mathfrak{C}_0$, $t\in G'$. ☐

Another observation deals with the coprime situation.

Proposition 7.3. *Suppose that in Theorem C the orders of the elements in $C\cap G$ are coprime to the orders of the elements in $G/C\cap G$. Then G is a totally imprimitive $\mathfrak{X}$-group.*

Proof. Assume that there exists an uncountable group G as in Theorem C such that the elements in $N = C\cap G$ have orders coprime to those of the elements in G/N. Without loss we may assume that $G = G'N$, and that N is an elementary-abelian p-group for some prime p. Since N is uncountable, we can find $z\in N-[N,G]$. In order to copy the argument in the proof of Theorem D, we just need to show that $C_F(E_0)\neq 1$. But because of the coprime situation we have $F=[F,E_0]\times C_F(E_0)$; and since $z\in F-[F,E_0]$, we see that $C_F(E_0)\neq 1$. ☐

We shall conclude with a quite interesting condition on stabilizers of blocks.

Proposition 7.4. *Consider the situation in Theorem C, and let H denote the setwise stabilizer in G' of $S\times 0$ resp. $U\otimes 0$. If $\mathrm{Hom}_{\mathbb{Z}}(H,\zeta_1(S))$ is countable, then G is a totally imprimitive $\mathfrak{X}$-group.*

Proof. We refer to the notation introduced in Theorem C and in the proofs of Theorems 6.5/6.7, and we choose $\Omega_0=\{0\}$. As in the proof of Theorem D, the embeddings σ_X $(X\in\mathfrak{C}_0)$ map into $W = S\,\mathrm{Wr}_\omega\,\mathrm{FSym}(\omega)$. Fix some $X\in\mathfrak{C}_0$. Let K denote the subgroup of W which consists of all base group functions $\omega\to\zeta_1(S)$. For every $Y\in\mathfrak{C}_0$, the difference map $\delta_{Y,X}$ of G' into the base group of W is defined via

$$g\delta_{Y,X} \;=\; (g\sigma_Y)^{-1}\cdot g\sigma_X \qquad \text{for all } \; g\in G'.$$

Consider $h\in E_0$ and $t\in G'$. Because $\sigma_X|_{E_0}=\sigma_Y|_{E_0}$, the element $t\delta_{Y,X}$ centralizes $E_0\sigma_X$. Hence

$$(h^{t^{-1}})\sigma_X \;=\; (h\sigma_X)^{(t\sigma_X)^{-1}} \;=\; (h\sigma_X)^{t\delta_{Y,X}\cdot(t\sigma_X)^{-1}} \;=\; (h\sigma_Y)^{(t\sigma_Y)^{-1}} \;=\; (h^{t^{-1}})\sigma_Y,$$

that is, $\sigma_X|_{\langle E_0^{G'}\rangle}=\sigma_Y|_{\langle E_0^{G'}\rangle}$. Since $E_0^t\sigma_X$ induces the action of S on $S\times 0\pi_t$, resp. since $\mathbb{K}(E_0^t\sigma_X)$ induces the action of $\mathrm{End}_{\mathbb{K}}(U)$ on $U\otimes 0\pi_t$, we obtain $G'\delta_{Y,X}\le K$. Thus $\delta_{Y,X}$ is a derivation of G' into the $G\pi$-module K, that is,

$$(gh)\delta_{Y,X} \;=\; (g\delta_{Y,X})^{h\pi}\cdot h\delta_{Y,X} \qquad \text{for all } \; g,h\in G'.$$

Consider $\zeta_1(S)$ as a trivial $\mathbb{Z}H$-module. Now $H\pi$ is the stabilizer of the point 0 in the transitive subgroup $G'\pi$ of $\mathrm{FSym}(\omega)$, and so an isomorphism

$$\varphi : K\longrightarrow \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G',\zeta_1(S))$$

is given via

$$(g)(f\varphi) = (0\pi_g)f \qquad \text{for all } \; f\in K,\, g\in G'.$$

Therefore Shapiro's Lemma ([9, Section 6.3]) yields $\mathrm{H}^1(G',K)\cong \mathrm{H}^1(H,\zeta_1(S))$. Recall that $\mathrm{H}^1(G',K)$ is isomorphic to the group of outer derivations $G'\to K$ ([22, 11.4.6]). And every derivation $H\to\zeta_1(S)$ is a homomorphism. By hypothesis there are just countably many such homomorphisms. Therefore Lemma 3.2 allows us to assume that there exists a fixed derivation $\delta : G' \to K$ such that $\delta_{Y,X} - \delta$ is an inner derivation for all $Y \in \mathfrak{C}_0 - \{X\}$. Fix $Y \in \mathfrak{C}_0 - \{X\}$. Then $\delta_{Z,Y} = \delta_{Z,X} - \delta_{Y,X}$ is an inner derivation for all $Z \in \mathfrak{C}_0 - \{X, Y\}$, that is, there exists $c_Z \in K$ such that

$$(g\sigma_Z)^{-1} \cdot g\sigma_Y \;=\; [g\pi, c_Z] \qquad \text{for all } \; g \in G'.$$

Define $\sigma : G \longrightarrow W_0 = S \operatorname{wr}_\omega \mathrm{FSym}(\omega)$ via $g\sigma = g\sigma_Z \cdot [g\pi, c_Z]$ whenever $g \in Z \in \mathfrak{C}_0 - \{X, Y\}$. Since $G'\sigma = G'\sigma_Y$ is an irreducible subgroup of W_0, it remains to show that σ is an embedding.

Suppose that $Z_1, Z_2 \in \mathfrak{C}_0 - \{X, Y\}$ satisfy $Z_1 \le Z_2$. Define $\sigma_i : Z_i \to W_0$ via $g\sigma_i = g\sigma_{Z_i} \cdot [g\pi, c_{Z_i}]$ for all $g \in Z_i$ $(i = 1, 2)$. Then $\sigma_1|_{G'} = \sigma_Y|_{G'} = \sigma_2|_{G'}$. But $G'\sigma_Y$ has trivial centralizer in W_0, whence $\sigma_1 = \sigma_2|_{Z_1}$. This shows that σ is a well-defined homomorphism. σ is also injective, since $G' \cap \ker \sigma = G' \cap \ker \sigma_Y = 1$. $\square$

In the permutation group case, countability of $\mathrm{Hom}_{\mathbb{Z}}(H, \zeta_1(S))$ in Proposition 7.4 is equivalent to finiteness of H/H^pH' for every prime p dividing the order of the finite group $\zeta_1(S)$.

Corollary 7.5. *The class of all totally imprimitive finitary permutation groups G satisfying the following condition is countably recognizable: Whenever S is the normalizer in G of a proper block, and whenever H is the setwise stabilizer in G of the same block, then H/H^pH' is finite for every prime p dividing the order of some element in $\zeta_1(S)$.*

Proof. Let G be a group with an ω-cover of subgroups satisfying the hypothesis. Again we refer to the notation introduced in Theorem 6.7. Let H_X $(X \in \mathfrak{C})$ denote the setwise stabilizer in X' of $S \times 0$. Lemma 3.2 allows us to assume that $|H_X/H_X^pH_X'|$ is independent of X for every prime dividing the order of $\zeta_1(S)$. The assertion now follows from Theorem C and Proposition 7.4. $\square$

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