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ON BC TYPE BASIC HYPERGEOMETRIC ORTHOGONAL POLYNOMIALS

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ABSTRACT. The five parameter family of Koornwinder's multivariable analogues of the Askey-Wilson polynomials is studied with four parameters generically complex. The Koornwinder polynomials form an orthogonal system with respect to an explicit (in general complex) measure. A partly discrete orthogonality measure is obtained by shifting the contour to the torus while picking up residues. A parameter domain is given for which the partly discrete orthogonality measure is positive. The orthogonality relations and norm evaluations for multivariable q-Racah polynomials and multivariable big and little q-Jacobi polynomials are proved by taking suitable limits in the orthogonality relations for the Koornwinder polynomials. In particular new proofs of several well-known q-analogues of the Selberg integral are obtained.

1. INTRODUCTION

In [25] Macdonald introduced a remarkable family of multivariable (q-)orthogonal polynomials associated with root systems (the so-called Macdonald polynomials). The Macdonald polynomials interpolate various families of special functions associated with groups, such as spherical functions on real semisimple Lie groups ($q = 1$), spherical functions on semisimple p-adic groups ($q = 0$) and characters of compact semisimple Lie groups. They also arise as spherical functions on certain quantum symmetric spaces (see for instance [29], [31]).

Cherednik (see for instance [8], [9], [10]) has shown that these polynomials are closely related to certain representations of affine Hecke algebras. Many properties of the polynomials can be derived by this approach while they were first untractable or could only be proved for very special parameter values from the (quantum) group interpretation. In particular the explicit evaluations for the quadratic norms of the Macdonald polynomials, which were conjectured by Macdonald [25], have been proved by Cherednik [8] using this approach.

In this paper we will consider families of multivariable q-orthogonal polynomials associated with the non-reduced root system BC. The starting point is the five parameter family of Koornwinder polynomials [24], which contains the three parameter families of BC type Macdonald polynomials and which reduces in the

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one-variable setting to the well-known four parameter family of Askey-Wilson polynomials [6]. Koornwinder [24] proved two basic properties of these polynomials, namely that they are mutually orthogonal with respect to an (explicit) absolutely continuous measure, and that they are diagonalized by an (explicit) second order q-difference equation. Van Diejen [12] evaluated the quadratic norms for a four parameter subfamily of the Koornwinder polynomials using so-called Pieri formulas. The affine Hecke algebraic approach can be applied to this five parameter family of Koornwinder polynomials (see [28], [30] and [32]), which in particular yields a proof of van Diejen's quadratic norm evaluations for the full five parameter family of Koornwinder polynomials (see [32]).

The one-variable Askey-Wilson polynomials contain various interesting families of basic hypergeometric orthogonal polynomials as special cases or as limit cases (these families are collected in the Askey tableau, see [6], [23]). In particular the Askey tableau contains the families of q-Racah polynomials [5], big q-Jacobi polynomials [2] and little q-Jacobi polynomials [1]. These three families were recently introduced and studied in the multivariable setting.

The multivariable q-Racah polynomials [14] can be considered as Koornwinder polynomials for which the parameters satisfy a particular truncation condition. In [14] it was shown that they are orthogonal with respect to a finite, discrete orthogonality measure. The multivariable big and little q-Jacobi polynomials [33] can be considered as limit cases of rescaled Koornwinder polynomials in which some of the parameters tend to infinity (see [36]). In [33] it was shown that they are orthogonal with respect to infinite discrete measures (which can be expressed in terms of multidimensional Jackson integrals).

The derivation of the orthogonality relations for these three limit cases followed the same line of arguments as in the case of the Macdonald polynomials [25] and the Koornwinder polynomials [24]. The quadratic norm evaluations for the multivariable q-Racah polynomials were obtained in [14] by use of Pieri formulas (see [13] for the constant term identity, i.e. the quadratic norm evaluation of the unit polynomial). In this paper we prove the orthogonality relations and norm evaluations for these three limit cases of the Koornwinder polynomials by extending the limit relations to the level of the orthogonality measures in a suitable, weak sense.

We proceed as follows. In section 2 the orthogonality relations [24] for the Koornwinder polynomials and the corresponding quadratic norm evaluations [12], [32] are extended to the case that four of the five parameters are generically complex. The extended orthogonality measure is an absolutely continuous complex measure with weight function identical to the weight function considered by Koornwinder [24], but with a suitably deformed integration contour. If all the four parameters have moduli < 1, then the n-torus T^n may be chosen as integration contour and Koornwinder's [24] orthogonality measure is recovered.

In section 3 a residue calculus is developed for the complex orthogonality measure of section 2. The orthogonality relations and norm evaluations for the Koornwinder polynomials can be reformulated with respect to partly discrete orthogonality measures using this calculus.

In section 4 the limit from Koornwinder polynomials to multivariable q-Racah polynomials is studied on the level of the orthogonality measures. A suitable partly discrete orthogonality measure for the Koornwinder polynomials is rescaled such that certain common poles of the completely discrete weights become zeros for the continuous parts of the orthogonality measure. These zeros cause the vanishing

of the continuous parts of the orthogonality measure in the limit. We end up with a finite discrete measure, which is easily recognized as the measure of the multivariable q-Racah polynomials. We obtain new proofs of the orthogonality relations and norm evaluations for the multivariable q-Racah polynomials, simply by taking the limit in the corresponding results for the Koornwinder polynomials. See [37] for the use of this idea in the one-variable case.

In section 5 the residue calculus of section 3 is used to deform the integration contour of the orthogonality measure for the Koornwinder polynomials (cf. section 2) to the n-torus T^n. We obtain a partly discrete orthogonality measure which is positive for a large parameter domain. In particular the parameter values which occur in the limits from Koornwinder polynomials to multivariable big and little q-Jacobi polynomials lie in this parameter domain.

In sections 6 and 7 these two limits are taken in the (suitably rescaled) positive, partly discrete orthogonality measure for the Koornwinder polynomials. The partly continuous contributions (which are supported on subtori of T^n) disappear in these limits because the corresponding weight functions tend to zero, while the support of the completely discrete part of the measure blows up to an infinite set. We end up with the orthogonality measures of the multivariable little and big q-Jacobi polynomials. The orthogonality relations and norm evaluations for the multivariable little and big q-Jacobi polynomials are obtained by taking limits of the corresponding results for the Koornwinder polynomials. The rigorous proofs of the limits of the orthogonality measures are postponed to section 8 and 9.

The constant term identities which are obtained as special cases of the quadratic norm evaluations for the multivariable little and big q-Jacobi polynomials reduce to well-known q-analogues of the Selberg integral. The constant term identity for the multivariable little q-Jacobi polynomials (Corollary 6.5) is known as the Askey-Habsieger-Kadell formula (see [4],[19],[21]) and was proved in full generality by Aomoto [3]. The constant term identity for the multivariable big q-Jacobi polynomials with one of the parameters discrete (Corollary 7.7) was conjectured by Askey [4] and proved by Evans [15]. The constant term identity in the general form (Corollary 7.6) is equivalent to Tarasov's and Varchenko's identity [38, Theorem (E.10)].

The method of proving the orthogonality relations and norm evaluations for the one-variable q-Racah, big and little q-Jacobi polynomials by considering them as limit cases of the Askey-Wilson polynomials was discussed in detail in [37]. Although the computations are more involved in the multivariable setting, the techniques we employ here are essentially the same as in the one-variable case.

Notations and conventions. We write $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. Empty sums are equal to 0; empty products are equal to 1. In order to keep the notations transparent, we will omit in formulas the dependence on parameters if the dependence is clear from the context. In this paper n denotes the rank of the BC type root system (i.e. the multivariable orthogonal polynomials which we study in this paper depend on n variables $z = (z_1, \dots, z_n)$). We use the convention that a function $h(z)$ for which a definition with respect to the n variables $z = (z_1, \dots, z_n)$ is given should be read with $n = m$ if it follows from the context that $z \in \mathbb{C}^m$. If $h(z)$ appears in formulas with $z \in \mathbb{C}^m$ and $m = 0$, then $h(z)$ should be read as 1.

Throughout this paper, we fix a $q \in (0,1)$. Parameters a, b and t are related to α, β and τ by $a = q^\alpha$, $b = q^\beta$ and $t = q^\tau$, and we choose $\alpha, \beta, \tau \in \mathbb{R}$ if a, b respectively t are positive real. We use this correspondence also formally for $b = 0$ (i.e. $\beta = \infty$) with the obvious interpretation of the formulas.

2. Koornwinder polynomials for generic parameter values

We first introduce some notations. The q-shifted factorial is given by

$$(x;q)_y := \frac{(x;q)_\infty}{(xq^y;q)_\infty}, \quad (x;q)_\infty := \prod_{j=0}^{\infty} \left(1 - q^j x\right), \tag{2.1}$$

provided that $xq^y \notin \{q^{-k}\}_{k \in \mathbb{N}_0}$. For $y = k \in \mathbb{N}_0$, we have $(x;q)_k = \prod_{i=0}^{k-1}(1 - xq^i)$, which is well defined for all $x \in \mathbb{C}$. We write furthermore

$$(x_1, \ldots, x_m; q)_y := \prod_{j=1}^{m} (x_j;q)_y \tag{2.2}$$

for products of q-shifted factorials.

Let S be the group of permutations of the set $\{1, \ldots, n\}$ and $W = S \ltimes \{\pm 1\}^n$ the Weyl group of type BC_n. Let $z_1, \ldots, z_n$ be independent variables; then W acts in a natural way on the algebra $A := \mathbf{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$. We denote A^W for the subalgebra of W-invariant functions in A. A basis for A^W is given by the monomials $\{m_\lambda \,|\, \lambda \in \Lambda\}$, where $\Lambda := \{\lambda \in \mathbb{N}_0^n \,|\, \lambda_1 \geq \ldots \geq \lambda_n\}$, and

$$m_\lambda(z) := \sum_{\mu \in W\lambda} z^\mu$$

with $z^\mu = z_1^{\mu_1} \ldots z_n^{\mu_n}$. The W-orbit of $\lambda \in \Lambda \subset \mathbb{Z}^n$ is with respect to the natural action of W on $\mathbb{Z}^n$.

We write $\underline{t} = (t_0, t_1, t_2, t_3)$ for the four tuple of parameters t_0, t_1, t_2, t_3. We assume in this section that $\underline{t} \in V$, where $V \subset \mathbb{C}^4$ is the following parameter domain.

Definition 2.1. Let V be the set of parameters $\underline{t} \in (\mathbb{C}^*)^4$ for which

$$\#\{\arg(t_i), \arg(t_i^{-1}) \,|\, i = 0,1,2,3\} = 8$$

and for which $t_0 t_1 t_2 t_3 \notin \mathbb{R}_{\geq 1}$. Here $\arg(u) \in [0, 2\pi)$ is the argument of $u \in \mathbb{C}^*$ and $\mathbb{R}_{\geq 1} := \{r \in \mathbb{R} \,|\, r \geq 1\}$.

For $\underline{t} \in V$ we define $\alpha_i^\pm = \alpha_i^\pm(\underline{t})$ by

$$\alpha_i^\pm := \frac{\arg(t_i^{\pm 1})}{2\pi}, \qquad i \in \{0,1,2,3\}. \tag{2.3}$$

Note that $\alpha_i^\pm \neq 0, 1/2$ and that $\alpha_i^- = 1 - \alpha_i^+$ for all i.

The (in general complex) measure which we will introduce in this section is supported on a suitably deformed n-torus $C^n \subset \mathbb{C}^n$, where $C \subset \mathbb{C}$ is the following deformation of the unit circle T.

Definition 2.2. We call a continuous rectifiable Jordan curve $C = \phi_C([0,1]) \subset \mathbb{C}$ a deformed circle if C has a parametrization ϕ_C of the form

$$\phi_C(x) = r_C(x) e^{2\pi i x} \ (x \in [0,1]), \quad r_C : [0,1] \to (0, \infty) \tag{2.4}$$

and if C is invariant under inversion, i.e. $C^{-1} := \{z^{-1} \,|\, z \in C\} = C$. For $\underline{t} \in V$, we call a deformed circle C a $\underline{t}$-contour if the four parameters t_0, t_1, t_2, t_3 are in the interior of C.

For a deformed circle $C = \phi_C([0,1])$ the radial function r_C satisfies $r_C(1-x) = (r_C(x))^{-1}$ since $C = C^{-1}$. Since a deformed circle C is by definition a closed contour, we furthermore have that $r_C(0) = r_C(1/2) = r_C(1) = 1$. Note that the unit circle T is a deformed circle with $r_T \equiv 1$. If $\underline{t} \in V$ and C is a $\underline{t}$-contour, then the radial function r_C satisfies the extra conditions

$$r_C(\alpha_i^+) > |t_i|, \qquad i \in \{0, \dots, 3\},$$

since t_i is in the interior of C for all i. In particular the unit circle T is a $\underline{t}$-contour if $|t_i| < 1$ for all $i \in \{0, \dots, 3\}$. We will use the convention that a deformed circle C is counterclockwise oriented (i.e. has the orientation induced from the parametrization ϕ_C) when we integrate over C.

Let $t \in (0,1)$, $\underline{t} \in V$ and let C be a deformed circle such that $t_i q^j \notin C$ for $i \in \{0, \dots, 3\}$ and $j \in \mathbb{N}_0$. Let $d\nu(z;\underline{t};t)$ be the measure on C^n given by

$$d\nu(z;\underline{t};t) := \Delta(z;\underline{t};t)\frac{dz}{z} \tag{2.5}$$

for $z \in C^n$ with $\frac{dz}{z} := \frac{dz_1}{z_1} \dots \frac{dz_n}{z_n}$ and with weight function $\Delta(z;\underline{t};t)$ given by

$$\Delta(z;\underline{t};t) = \left(\prod_{j=1}^{n} w_c(z_j;\underline{t})\right)\delta(z;t), \tag{2.6}$$

with $w_c(x;\underline{t})$ given by

$$w_c(x;\underline{t}) := \frac{\left(x^2, x^{-2}; q\right)_\infty}{\left(t_0x, t_0x^{-1}, t_1x, t_1x^{-1}, t_2x, t_2x^{-1}, t_3x, t_3x^{-1}; q\right)_\infty} \tag{2.7}$$

and $\delta(z;t)$ given by

$$\delta(z;t) := \prod_{1 \le i < j \le n} \left(z_iz_j, z_i^{-1}z_j, z_iz_j^{-1}, z_i^{-1}z_j^{-1}; q\right)_\tau. \tag{2.8}$$

The factor $w_c(x;\underline{t})$ is exactly the weight function which occurs in the continuous part of the orthogonality measure for the one-variable Askey-Wilson polynomials [6]. The interaction factor $\delta(z;t)$ is only present in the multivariable setting (i.e. when $n > 1$), so in particular the measure is independent of the deformation parameter t when $n = 1$.

The measure $d\nu(.;\underline{t};t)$ on C^n is well defined, since the poles of $\Delta(.;\underline{t};t)$ do not lie on the integration domain C^n. Indeed, the poles of the weight function $\Delta(z;\underline{t};t)$ lie on hyperplanes

$$z_i = t_m q^j \quad \text{or} \quad z_i = t_m^{-1}q^{-j} \tag{2.9}$$

with $m \in \{0, \dots, 3\}$, $j \in \mathbb{N}_0$ and $i \in \{1, \dots, n\}$ (the poles coming from $w_c(z_i)$) and on hypersurfaces

$$z_k^{\epsilon_k} z_l^{\epsilon_l} = t^{-1}q^{-j} \tag{2.10}$$

with $1 \le k \ne l \le n$, $j \in \mathbb{N}_0$ and $\epsilon_k, \epsilon_l \in \{-1, 1\}$ (the poles coming from $\delta(z;\underline{t})$). We have $z \notin C^n$ for a pole z of the form (2.9) since $C^{-1} = C$ and the assumption that $t_iq^j \notin C$ ($i \in \{0,1,2,3\}, j \in \mathbb{N}_0$) and it follows from the definition of a deformed circle C that $z \notin C^n$ for a pole z of the form (2.10).

In this section we will study orthogonal polynomials related to the complex measure $\left(C^n, d\nu(.;\underline{t};t)\right)$ where C is an arbitrary $\underline{t}$-contour. We first show that the

measure $d\nu(.;\underline{t};t)$ is independent of the $\underline{t}$-contour C when integrating against W-invariant Laurent polynomials. In order to obtain this result, we first define specific subsets of $(\mathbb{C}^*)^n$ on which the interaction factor $\delta(.;t)$ is analytic.

Let C, $\mathfrak{C}$ be deformed circles, with parametrization given by $\phi_C(x) = r_C(x)e^{2\pi ix}$, respectively $\phi_{\mathfrak{C}}(x) = r_{\mathfrak{C}}(x)e^{2\pi ix}$. Let $A^+(C,\mathfrak{C})$ be the open subset

$$A^+(C,\mathfrak{C}) := \{x \in [0,1] \,|\, r_C(x) > r_{\mathfrak{C}}(x)\} \subset (0,1). \tag{2.11}$$

Set

$$\Omega(C,\mathfrak{C}) := \Omega^+(C,\mathfrak{C}) \cup \Omega^+(\mathfrak{C},C) \cup C, \tag{2.12}$$

where $\Omega^+(C,\mathfrak{C})$ is given by

$$\Omega^+(C,\mathfrak{C}) := \bigcup_{x\in A^+(C,\mathfrak{C})} \{y(x)e^{2\pi ix} \,|\, r_C(x) \geq y(x) \geq r_{\mathfrak{C}}(x)\}. \tag{2.13}$$

The following properties of $\Omega(C,\mathfrak{C}) \subset \mathbb{C}^*$ follow easily from the definitions:

(i) $\Omega(C,\mathfrak{C}) = \Omega(\mathfrak{C},C)$.

We will use, in view of (i), the notation $\Omega = \Omega(C,\mathfrak{C})$ when it is clear from the context which pair of contours C, $\mathfrak{C}$ is meant.

(ii) $\Omega^{-1} = \Omega$.

(iii) The contour C can be deformed homotopically to $\mathfrak{C}$ within Ω.

We call $\Omega \subset \mathbb{C}^*$ *the domain associated with the pair* $(C,\mathfrak{C})$. We write $\mathcal{O}_W(\Omega^n)$ for the ring of W-invariant functions f which are analytic on Ω^n. We have now the following crucial lemma.

Lemma 2.3. *Let $t \in (0,1)$ and let C, $\mathfrak{C}$ be deformed circles satisfying the condition $t\big(r_C(x)\big) < r_{\mathfrak{C}}(x)$ for all $x \in A^+(C,\mathfrak{C})$. Then $\delta(.;t) \in \mathcal{O}_W(\Omega^n)$.*

Proof. Let C, $\mathfrak{C}$ be deformed circles satisfying $t\big(r_C(x)\big) < r_{\mathfrak{C}}(x)$ for all $x \in A^+(C,\mathfrak{C})$. Let $z \in (\mathbb{C}^*)^n$ such that $z_k^{\epsilon_k} z_l^{\epsilon_l} = t^{-1}q^{-j}$ for some $j \in \mathbb{N}_0$, some $k \neq l$ and some $\epsilon_k, \epsilon_l \in \{\pm 1\}$. Write $\beta_k := \arg(z_k)/2\pi$ and $\beta_l := \arg(z_l)/2\pi$. For the proof of the lemma it suffices to show that either $z_k \notin \Omega$ or $z_l \notin \Omega$.

As an example, let us check that either $z_k \notin \Omega$ or $z_l \notin \Omega$ when $\beta_k \in A^+(C,\mathfrak{C})$ and $z_k z_l = t^{-1}q^{-j}$ for some $j \in \mathbb{N}_0$. We then have $\beta_k = 1 - \beta_l$ and $\beta_l \in A^+(\mathfrak{C},C)$, so in particular $r_C(\beta_k) = r_C(\beta_l)^{-1}$, $r_{\mathfrak{C}}(\beta_k) = r_{\mathfrak{C}}(\beta_l)^{-1}$. Suppose that $z_l \in \Omega$. Then

$$|z_k| = t^{-1}q^{-j}|z_l^{-1}| \geq q^{-j}t^{-1}r_{\mathfrak{C}}(\beta_k) > q^{-j}r_C(\beta_k) \geq r_C(\beta_k);$$

hence $z_k \notin \Omega$. All the other cases are checked similarly. □

Lemma 2.4. *Let $\underline{t} \in V$, $t \in (0,1)$ and $f \in A^W$. Then*

$$\iint_{z\in C^n} f(z)d\nu(z;\underline{t};t) \tag{2.14}$$

is independent of the choice of $\underline{t}$-contour C.

Proof. Write $N_f(C)$ for the integral (2.14). We have to show that $N_f(C) = N_f(\mathfrak{C})$ for arbitrary pairs of $\underline{t}$-contours $(C,\mathfrak{C})$.

Let $\mathcal{L}$ be the collection of pairs of $\underline{t}$-contours $(C,\mathfrak{C})$ for which $A^+(C,\mathfrak{C})$ is a finite disjoint union of open intervals and for which $t\big(r_C(x)\big) < r_{\mathfrak{C}}(x)$ for all $x \in A^+(C,\mathfrak{C})$. Fix a pair $(C,\mathfrak{C}) \in \mathcal{L}$ and let Ω be the associated domain. Since the four parameters t_0, t_1, t_2, t_3 are in the interior of C and $\mathfrak{C}$, we have $w_c(.;\underline{t}) \in \mathcal{O}_{\{\pm 1\}}(\Omega)$,

and by Lemma 2.3 we have $\delta(.;t) \in \mathcal{O}_W(\Omega^n)$. So Cauchy's Theorem implies that $N_f(C) = N_f(\mathfrak{C})$.

Suppose now that $(C, \mathfrak{C})$ is an arbitrary pair of $\underline{t}$-contours. Then there exists a finite sequence of $\underline{t}$-contours $C_0, C_1, \dots, C_s$ such that $C_0 = \mathfrak{C}$, $C_s = C$ and such that $(C_i, C_{i-1}) \in \mathcal{L}$ for all $i \in \{1, \dots, s\}$. It follows that $N_f(C) = N_f(\mathfrak{C})$. $\square$

We define for parameters $\underline{t} \in V$ and $t \in (0,1)$ a symmetric bilinear form $\big(.,.\big)_{\underline{t},t}$ on A^W by

$$\big(f,g\big)_{\underline{t},t} := \frac{1}{(2\pi i)^n} \iint\limits_{z \in C^n} f(z)g(z)d\nu(z;\underline{t};t), \quad f,g \in A^W, \tag{2.15}$$

where C is an arbitrary $\underline{t}$-contour. The bilinear form (2.15) is independent of the choice of $\underline{t}$-contour C by Lemma 2.4. An important tool for studying orthogonal polynomials with respect to the bilinear form $\big(.,.\big)_{\underline{t},t}$ is an explicit second order q-difference operator $D = D_{\underline{t},t}$ which preserves the algebra A^W and which is symmetric with respect to the bilinear form $\big(.,.\big)_{\underline{t},t}$. The second order q-difference operator D was introduced by Koornwinder [24] and it is explicitly given by

$$D := \sum_{j=1}^{n} \left(\phi_j^+(z)(T_j^+ - \mathrm{Id}) + \phi_j^-(z)(T_j^- - \mathrm{Id})\right), \tag{2.16}$$

where $T_j^{\pm}$ is the $q^{\pm 1}$-shift in the jth coordinate,

$$\left(T_j^{\pm} f\right)(z) := f(z_1, \dots, z_{j-1}, q^{\pm 1} z_j, z_{j+1}, \dots, z_n),$$

and the functions $\phi_j^+(z;\underline{t};t)$ and $\phi_j^-(z;\underline{t};t)$ are given by

$$\phi_j^+(z;\underline{t};t) := \frac{\prod_{i=0}^{3}(1-t_i z_j)}{(1-z_j^2)(1-qz_j^2)} \prod_{l \neq j} \frac{(1-tz_l z_j)\left(1-tz_l^{-1}z_j\right)}{(1-z_l z_j)\left(1-z_l^{-1}z_j\right)},$$

$$\phi_j^-(z;\underline{t};t) := \phi_j^+(z^{-1};\underline{t};t),$$

where we have used the notation $z^{-1} = (z_1^{-1}, \dots, z_n^{-1})$. The BC type dominance order on Λ is defined by

$$\mu \leq \lambda \quad \Leftrightarrow \quad \sum_{j=1}^{i} \mu_j \leq \sum_{j=1}^{i} \lambda_j \quad (i = 1, \dots, n) \tag{2.17}$$

for $\lambda, \mu \in \Lambda$. Koornwinder proved the following triangularity property of D (see [24, Lemma 5.2] and the remark after [36, Proposition 4.1]).

Proposition 2.5. *Let $\lambda \in \Lambda$. For arbitrary $\underline{t} \in \mathbb{C}^4$ and $t \in \mathbb{C}$ we have*

$$Dm_\lambda = \sum_{\mu \leq \lambda} E_{\lambda,\mu} m_\mu \tag{2.18}$$

with $E_{\lambda,\mu}(\underline{t};t) \in \mathbb{C}$ depending polynomially on the parameters $\underline{t}$ and t. The leading term $E_{\lambda,\lambda}(\underline{t};t)$ will be denoted by $E_\lambda(\underline{t};t)$ and is given by

$$E_\lambda(\underline{t};t) := \sum_{j=1}^{n} \left(q^{-1}t_0t_1t_2t_3t^{2n-j-1}(q^{\lambda_j}-1) + t^{j-1}(q^{-\lambda_j}-1)\right). \tag{2.19}$$

In particular D preserves the algebra A^W. The other property of D which we already mentioned is the symmetry of D with respect to the bilinear form $\big(.,.\big)$, i.e.

$$\big(D_{\underline{t},t}f,g\big)_{\underline{t},t}=\big(f,D_{\underline{t},t}g\big)_{\underline{t},t},\quad f,g\in A^W, \tag{2.20}$$

for parameters $\underline{t}\in V$ and $t\in(0,1)$. Koornwinder [24, Lemma 5.3] proved (2.20) for parameters $\underline{t}$ with $|t_i|<1$ (then the unit circle T can be chosen as $\underline{t}$-contour). By Proposition 2.5, (2.20) follows for $\underline{t}\in V$ by analytic continuation.

We define explicit expressions $\mathcal{N}(\lambda;\underline{t};t)$ for $\lambda\in\Lambda$ by

$$\mathcal{N}(\lambda;\underline{t};t):=2^n n!\mathcal{N}^+(\lambda;\underline{t};t)\mathcal{N}^-(\lambda;\underline{t};t) \tag{2.21}$$

where $\mathcal{N}^+(\lambda):=\mathcal{N}^+(\lambda;\underline{t};t)$ is given by

$$\begin{aligned}\mathcal{N}^+(\lambda):=&\prod_{i=1}^n\frac{\big(q^{2\lambda_i-1}t^{2(n-i)}t_0t_1t_2t_3;q\big)_\infty}{\big(q^{\lambda_i-1}t^{n-i}t_0t_1t_2t_3,q^{\lambda_i}t^{n-i}t_0t_1,q^{\lambda_i}t^{n-i}t_0t_2,q^{\lambda_i}t^{n-i}t_0t_3;q\big)_\infty}\\&\cdot\prod_{1\le j<k\le n}\frac{\big(q^{\lambda_j+\lambda_k-1}t^{2n-j-k}t_0t_1t_2t_3,q^{\lambda_j-\lambda_k}t^{k-j};q\big)_\infty}{\big(q^{\lambda_j+\lambda_k-1}t^{2n-j-k+1}t_0t_1t_2t_3,q^{\lambda_j-\lambda_k}t^{k-j+1};q\big)_\infty},\end{aligned} \tag{2.22}$$

and $\mathcal{N}^-(\lambda):=\mathcal{N}^-(\lambda;\underline{t};t)$ is given by

$$\begin{aligned}\mathcal{N}^-(\lambda):=&\prod_{i=1}^n\frac{\big(q^{2\lambda_i}t^{2(n-i)}t_0t_1t_2t_3;q\big)_\infty}{\big(q^{\lambda_i+1}t^{n-i},q^{\lambda_i}t^{n-i}t_1t_2,q^{\lambda_i}t^{n-i}t_1t_3,q^{\lambda_i}t^{n-i}t_2t_3;q\big)_\infty}\\&\cdot\prod_{1\le j<k\le n}\frac{\big(q^{\lambda_j+\lambda_k}t^{2n-j-k}t_0t_1t_2t_3,q^{\lambda_j-\lambda_k+1}t^{k-j};q\big)_\infty}{\big(q^{\lambda_j+\lambda_k}t^{2n-j-k-1}t_0t_1t_2t_3,q^{\lambda_j-\lambda_k+1}t^{k-j-1};q\big)_\infty}.\end{aligned} \tag{2.23}$$

The following theorem extends the results of Koornwinder [24] (the orthogonality relations for the Koornwinder polynomials) and van Diejen [12], Sahi [32] (the quadratic norm evaluations for the Koornwinder polynomials) to parameters $\underline{t}\in V$ (in [24], [12] and [32] these results were obtained for a parameter domain such that $|t_i|\le 1$ for all i).

Theorem 2.6. *Let $\underline{t}\in V$ and $t\in(0,1)$. There exists a unique linear basis $\{P_\lambda(.;\underline{t};t)\}_{\lambda\in\Lambda}$ of A^W such that*

$$P_\lambda(.;\underline{t};t)=m_\lambda+\sum_{\mu<\lambda}c_{\lambda,\mu}(\underline{t},t)m_\mu,\quad c_{\lambda,\mu}(\underline{t},t)\in\mathbb{C}, \tag{2.24}$$

$$\big(P_\lambda(.;\underline{t};t),P_\mu(.;\underline{t};t)\big)_{\underline{t},t}=0\quad \text{if } \mu\neq\lambda. \tag{2.25}$$

Furthermore, $P_\lambda(.;\underline{t};t)$ is an eigenfunction of $D_{\underline{t},t}$ with eigenvalue $E_\lambda(\underline{t};t)$ and we have the explicit evaluation formula

$$\big(P_\lambda(.;\underline{t};t),P_\lambda(.;\underline{t};t)\big)_{\underline{t},t}=\mathcal{N}(\lambda;\underline{t};t) \tag{2.26}$$

for the quadratic norms of the polynomials P_λ.

Definition 2.7. The W-invariant Laurent polynomial $P_\lambda(.;\underline{t};t)$ is called the Koornwinder polynomial of degree $\lambda\in\Lambda$.

We end this section with a sketch of the proof of Theorem 2.6 using the techniques of Koornwinder [24] and van Diejen [12]. For more details, we refer the reader to these two papers.

We fix arbitrary $0\neq\nu\in\Lambda$. It is sufficient to prove the existence and uniqueness of a set of W-invariant Laurent polynomials $\{P_\lambda(.;\underline{t};t)\}_{\lambda\le\nu}$ satisfying (2.24) and

(2.25) for $\lambda, \mu \leq \nu$ and to prove the remaining assertions of the theorem for the polynomials $\{P_\lambda(.;\underline{t};t)\}_{\lambda\leq\nu}$.

We first define the polynomials $\{P_\lambda(.;\underline{t};t)\}_{\lambda\leq\nu}$ for a dense parameter domain $U_\nu \subset V\times(0,1)$. The subset U_ν is by definition the set of parameters $(\underline{t},t) \in V\times(0,1)$ such that $E_\mu(\underline{t};t) \neq E_\lambda(\underline{t};t)$ for all $\lambda,\mu \leq \nu$, $\lambda \neq \mu$. Note that $U_\nu \subset V \times (0,1)$ is open and dense since the eigenvalues $\{E_\lambda(\underline{t};t)\}_{\lambda\in\Lambda}$ are mutually different as polynomials in the parameters $\underline{t},t$.

The polynomials $P_\lambda(.;\underline{t};t) \in A^W$ for $(\underline{t},t) \in U_\nu$ and $\lambda \leq \nu$ are defined by

$$P_\lambda(.;\underline{t};t) := \left(\prod_{\mu<\lambda} \frac{D_{\underline{t},t} - E_\mu(\underline{t};t)}{E_\lambda(\underline{t};t) - E_\mu(\underline{t};t)}\right) m_\lambda \tag{2.27}$$

(cf. [25], [36]). It is an easy consequence of the triangularity of D (cf. Proposition 2.5) and of the Cayley-Hamilton Theorem that $P_\lambda(.;\underline{t};t)$ (2.27) is the unique function of the form (2.24) which is an eigenfunction of $D_{\underline{t},t}$ with eigenvalue $E_\lambda(\underline{t};t)$. The polynomials $\{P_\lambda(.;\underline{t};t)\}_{\lambda\leq\nu}$ (2.27) satisfy the orthogonality relations (2.25) for parameters values $(\underline{t},t) \in U_\nu$ since D is symmetric with respect to $\big(.,.\big)$ and $E_\mu(\underline{t};t) \neq E_{\mu'}(\underline{t};t)$ for $\mu,\mu' \leq \lambda$, $\mu \neq \mu'$.

Next we establish the quadratic norm evaluations (2.26) for $\{P_\lambda(.;\underline{t};t)\}_{\lambda\leq\nu}$ with $(\underline{t},t) \in U_\nu$. In the special case $\lambda = 0$, (2.26) reduces to

$$\begin{aligned}\big(1,1\big)_{\underline{t},t} &= \mathcal{N}(0;\underline{t};t)\\ &= 2^n n! \prod_{i=1}^n \frac{\big(t, t^{2n-i-1}t_0t_1t_2t_3;q\big)_\infty}{\big(q,t^{n-i+1};q\big)_\infty \prod_{0\leq j<k\leq 3}\big(t^{n-i}t_jt_k;q\big)_\infty}\end{aligned} \tag{2.28}$$

which was proved by Gustafson [17] for parameters $t \in (0,1)$ and $\underline{t} \in \mathbb{C}^4$ with $|t_i| < 1$ (since then the unit circle T can be chosen as $\underline{t}$-contour). The second equality follows by a straightforward computation (see also [26]). By analytic continuation, (2.28) is valid for parameters $t \in (0,1)$ and $\underline{t} \in V$.

For general λ, van Diejen [12] proved explicit Pieri formulas for the renormalized Koornwinder polynomials

$$p_\lambda(.;\underline{t};t) := c(\lambda;\underline{t};t)P_\lambda(.;\underline{t};t), \quad c(\lambda;\underline{t};t) := \frac{\mathcal{N}^+(0)}{\mathcal{N}^+(\lambda)} \prod_{j=1}^n (t_0t^{n-j})^{\lambda_j}. \tag{2.29}$$

The renormalization constant $c(\lambda;\underline{t};t)$ is a rational expression in the parameters $\underline{t},t$. The Pieri formulas give explicit expressions for the coefficients $d_\lambda^{(r)}(\mu;\underline{t};t)$ in the expansions

$$E_r(z;\underline{t};t)p_\lambda(z;\underline{t};t) = \sum_{\mu\leq\lambda+(1^n)} d_\lambda^{(r)}(\mu;\underline{t};t)p_\mu(z;\underline{t};t), \quad r \in \{1,\dots,n\}, \tag{2.30}$$

where $\{E_r(z;\underline{t};t)\}_{r=1}^n$ are explicit algebraic generators of the algebra of W-invariant Laurent polynomials A^W and where $(1^n) := (1,\dots,1) \in \Lambda$ is the nth fundamental weight (see [12] for the explicit formulas, or [14, Appendix B] where the notations are closer to the ones used in this paper). Van Diejen [12] proved the Pieri formulas for a four parameter family of Koornwinder polynomials. Sahi [32] proved that van Diejen's formulas are in fact valid for the full five parameter family of Koornwinder polynomials using the affine Hecke-algebraic approach. In particular, (2.30) may be viewed as an identity in the algebra of W-invariant Laurent polynomials over the quotient field $\mathbb{C}(\underline{t},t)$, where $\underline{t},t$ are considered as indeterminates.

The Pieri formulas and the orthogonality relations for the renormalized Koornwinder polynomials with real parameters t_i and $|t_i| < 1$ allowed van Diejen (cf. [12, Theorem 4]) to reduce the norm computation for arbitrary λ to the case $\lambda = 0 \in \Lambda$. Gustafson's evaluation (2.28) then completes the evaluation for general λ. By taking a dense subset of the parameter domain U_ν if necessary, exactly the same reduction can now be done for the norm evaluations of the polynomials $\{p_\lambda(.;\underline{t};t)\}_{\lambda\leq\nu}$ for parameters $(\underline{t},t) \in U_\nu$. The extension of Gustafson's result (2.28) then completes the proof of (2.26) for $\{P_\lambda(.;\underline{t};t)\}_{\lambda\leq\nu}$ and $(\underline{t},t) \in U_\nu$.

The polynomials $\{P_\lambda(.;\underline{t};t)\}_{\lambda\leq\nu}$ with $(\underline{t},t) \in U_\nu$ are uniquely characterized by (2.24) and (2.25) for $\lambda,\mu \leq \nu$, since their quadratic norms (2.26) are non-zero. Indeed, the functions

$$(\underline{t},t) \mapsto \mathcal{N}^{\pm}(\lambda;\underline{t};t): \quad V \times (0,1) \to \mathbb{C} \tag{2.31}$$

are well defined, continuous functions which do not have zeros on the domain $V \times (0,1)$. This is immediately clear except for $\mathcal{N}^+(\lambda)$ with $\lambda \in \Lambda$ and $\lambda_n = 0$. But then the expression for $\mathcal{N}^+(\lambda)$ can be simplified, similarly as the simplification of the expression for $\mathcal{N}(0)$ in (2.28), from which it follows that $\mathcal{N}^+(\lambda;\underline{t};t)$ is a well defined, continuous function of $(\underline{t},t) \in V \times (0,1)$ without zeros.

The proof of the theorem can now be finished by extending these results to parameter values $(\underline{t},t) \in V \times (0,1)$ using a continuity argument, as follows. The Koornwinder polynomial P_λ satisfies the following Gram-Schmidt formula:

$$P_\lambda(z;\underline{t};t) = m_\lambda(z) - \sum_{\mu<\lambda} \frac{\big(m_\lambda, P_\mu(.;\underline{t};t)\big)_{\underline{t},t}}{\mathcal{N}(\mu;\underline{t};t)} P_\mu(z;\underline{t};t) \tag{2.32}$$

for $(\underline{t},t) \in U_\nu$ and $\lambda \leq \nu$. By induction, we conclude from (2.32) that the coefficients $c_{\lambda,\mu} : U_\nu \to \mathbb{C}$ in (2.24) uniquely extend to continuous functions $c_{\lambda,\mu} : V \times (0,1) \to \mathbb{C}$ for all $\mu < \lambda \leq \nu$. Hence existence and uniqueness of $\{P_\lambda(.;\underline{t};t)\}_{\lambda\leq\nu}$ as well as the other assertions follow now by continuity for all $(\underline{t},t) \in V \times (0,1)$. This completes the proof of the theorem.

Remark 2.8. For $n = 1$ the polynomials $\{P_\lambda(z;\underline{t}) \,|\, \lambda \in \mathbb{N}_0\}$ are independent of t and are known as (monic, one-variable) Askey-Wilson polynomials. Theorem 2.6 reduces to the orthogonality relation and quadratic norm evaluation stated in [6, Theorem 2.3]. The polynomials $P_\lambda(z;\underline{t})$ ($\lambda \in \mathbb{N}_0$) can then be given explicitly in terms of the basic hypergeometric series

$${}_{s+1}\phi_s\left(\begin{matrix} a_1,\dots,a_{s+1} \\ b_1,\dots,b_s \end{matrix};q,z\right) = \sum_{m=0}^{\infty} \frac{(a_1,\dots,a_{s+1};q)_m}{(b_1,\dots,b_s,q;q)_m} z^m$$

as

$$P_\lambda(z;\underline{t}) = \frac{(t_0t_1,t_0t_2,t_0t_3;q)_\lambda}{t_0^\lambda(t_0t_1t_2t_3q^{\lambda-1};q)_\lambda}\, {}_4\phi_3\left(\begin{matrix} q^{-\lambda},\ q^{\lambda-1}t_0t_1t_2t_3,\ t_0z,\ t_0z^{-1} \\ t_0t_1,\ t_0t_2,\ t_0t_3 \end{matrix};q,q\right) \tag{2.33}$$

(see [6] for details). The renormalized Askey-Wilson polynomial $p_\lambda(z;\underline{t})$ (2.29) is then exactly the ${}_4\phi_3$ part of (2.33).

Remark 2.9. The renormalization constant $c(\lambda;\underline{t};t)$ (2.29) is easily seen to be regular and non-zero at $(\underline{t},t) \in V \times (0,1)$ for all $\lambda \in \Lambda$. Hence the renormalized Koornwinder polynomials $\{p_\lambda(z;\underline{t};t)\}_{\lambda\in\Lambda}$ form an orthogonal basis of A^W with respect to the bilinear form $\big(.,.\big)_{\underline{t},t}$ for all parameter values $(\underline{t},t) \in V \times (0,1)$.

3. Residue calculus for the orthogonality measure $d\nu$

In this section we develop a residue calculus for integrals of the form

$$\frac{1}{(2\pi i)^n}\iint\limits_{z\in C^n} f(z)d\nu(z) = \frac{1}{(2\pi i)^n}\iint\limits_{z\in C^n} f(z)\Delta(z)\frac{dz}{z}, \quad f\in\mathcal{O}_W(\Omega^n), \tag{3.1}$$

when C^n is shifted to $\mathfrak{C}^n$, where $\Omega\subset\mathbb{C}^*$ is the domain associated with the pair $(C,\mathfrak{C})$ and $(C,\mathfrak{C})$ is a so-called (n,t_0)-residue pair, which is defined as follows.

Definition 3.1. Let $\underline{t}=(t_0,t_1,t_2,t_3)\in V$ (V as given by Definition 2.1). A pair of contours $(C,\mathfrak{C})$ is called a (n,t_0)-residue pair if C and $\mathfrak{C}$ are deformed circles satisfying the following three properties.

(i) The subset $A^+(C,\mathfrak{C})$ (2.11) is an open interval for which $\alpha_0^+\in A^+(C,\mathfrak{C})$ but $\alpha_i^\pm\notin A^+(C,\mathfrak{C})$ $(i=1,2,3)$;

(ii) $t_iq^r\notin C\cup\mathfrak{C}$ for $r\in\mathbb{N}_0$ and $i\in\{0,\dots,3\}$;

(iii) $t_0t^pq^r\notin\mathfrak{C}$ for $p\in\{-1,\dots,n-1\}$ and $r\in\mathbb{Z}$.

We will see in this section that the poles which are picked up when deforming C^n to $\mathfrak{C}^n$ in (3.1) for a (n,t_0)-residue pair $(C,\mathfrak{C})$ only depend on q, t and t_0. We therefore fix in this section $t\in(0,1)$ and $t_1,t_2,t_3\in\mathbb{C}^*$ such that $\#\{\alpha_i^+,\alpha_i^-\mid i=1,2,3\}=6$ ($\alpha_i^\pm$ given by (2.3)) and simplify the notations by omitting the dependence on these parameters. For instance, we will write $w_c(x;t_0)$ instead of $w_c(x;\underline{t})$, etc. Due to the symmetry of $\Delta(z;\underline{t};t)$ in the four parameters t_0, t_1, t_2 and t_3, all the results on the residue calculus for (n,t_0)-residue pairs can be reformulated for (n,t_i)-residue pairs with $i\in\{0,1,2,3\}$ arbitrary, by relabelling the parameters $\underline{t}$.

We define now the measures $d\nu_r(z;t_0)$ $(r=1,\dots,n)$ which will appear when the contour C^n in (3.1) is shifted to $\mathfrak{C}^n$. First we need to introduce some more notations.

We set

$$P(r):=\{\lambda\in\mathbb{N}_0^r\mid\lambda_1\le\lambda_2\le\ldots\le\lambda_r\} \tag{3.2}$$

and we set $\lambda_0:=0$ for arbitrary $\lambda\in P(r)$. We write $\rho_i=\rho_i(t_0;t):=t_0t^{i-1}$ for $i\in\mathbb{Z}$ and, for $\lambda\in P(r)$,

$$\rho q^\lambda:=(\rho_1q^{\lambda_1},\rho_2q^{\lambda_2},\dots,\rho_rq^{\lambda_r})=(t_0q^{\lambda_1},t_0tq^{\lambda_2},\dots,t_0t^{r-1}q^{\lambda_r}). \tag{3.3}$$

Define $D(r)=D(r;C,\mathfrak{C};t_0)$ for $r=1,\dots,n$ by

$$D(r):=\{\rho q^\lambda\mid\lambda\in P(r)\text{ and }r_C(\alpha_0^+)>|\rho_iq^{\lambda_i}|>r_{\mathfrak{C}}(\alpha_0^+)\quad(i=1,\dots,r)\}. \tag{3.4}$$

Observe that, for $\omega\in D(r)$, we have $\omega_i\in\operatorname{int}(\Omega)$ for all i, where $\operatorname{int}(\Omega)$ is the interior of Ω. For $\rho q^\lambda\in D(r)$, we set

$$\Delta^{(d)}(\rho q^\lambda;t_0):=\left(\prod_{j=1}^r w_d\big(\rho_jq^{\lambda_j};\rho_jq^{\lambda_{j-1}}\big)\right)\delta_d\big(\rho q^\lambda\big) \tag{3.5}$$

with

$$w_d(\rho_jq^{\lambda_j};\rho_jq^{\lambda_{j-1}}):=\operatorname{res}_{x=\rho_jq^{\lambda_j}}\left(\frac{w_c(x;\rho_jq^{\lambda_{j-1}},t_1,t_2,t_3)}{x}\right) \tag{3.6}$$

where w_c is given by (2.7) and with interaction factor

$$\delta_d(\rho q^\lambda) := \prod_{1\leq k<l\leq r} \frac{\big(\rho_k^{-1}\rho_l q^{\lambda_l-\lambda_k}, \rho_k^{-1}\rho_l^{-1}q^{-\lambda_k-\lambda_l};q\big)_\tau}{\big(\rho_k\rho_l q^{\lambda_{k-1}+\lambda_l}, \rho_k\rho_l^{-1}q^{\lambda_{k-1}-\lambda_l};q\big)_{\lambda_k-\lambda_{k-1}}}. \tag{3.7}$$

The discrete parts of the measure which will appear after deforming C^n to $\mathfrak{C}^n$ in (3.1) involve the weights $\Delta^{(d)}$ (3.5). Observe that for $r=1$ and $\rho q^\lambda = t_0 q^i \in D(1)$, we have $\Delta^{(d)}(t_0q^i;t_0) = w_d(t_0q^i;t_0)$.

The weight function $w_c\big(x;\rho_j q^{\lambda_{j-1}}\big)/x$ has a simple pole at $x=\rho_j q^{\lambda_j}$ since $\lambda_j-\lambda_{j-1}\in\mathbb{N}_0$ and $\underline{t}\in V$; hence $w_d(\rho_j q^{\lambda_j};\rho_j q^{\lambda_{j-1}})$ (3.6) is non-zero. An explicit expression for w_d can be given using the fact that for $\underline{\tau}=(\tau_0,\tau_1,\tau_2,\tau_3)\in V$ and for $i\in\mathbb{N}_0$, the discrete weight

$$w_d(\tau_0q^i;\tau_0) = w_d\left(\tau_0q^i;\tau_0;\tau_1,\tau_2,\tau_3\right) = \operatorname{res}_{x=\tau_0q^i}\left(\frac{w_c(x;\underline{\tau})}{x}\right) \tag{3.8}$$

can be explicitly given by the formula

$$\begin{aligned} w_d(\tau_0q^i;\tau_0) =& \frac{\big(\tau_0^{-2};q\big)_\infty}{\big(q,\tau_0\tau_1,\tau_1/\tau_0,\tau_0\tau_2,\tau_2/\tau_0,\tau_0\tau_3,\tau_3/\tau_0;q\big)_\infty} \\ &\cdot\frac{\big(\tau_0^2,\tau_0\tau_1,\tau_0\tau_2,\tau_0\tau_3;q\big)_i}{\big(q,\tau_0q/\tau_1,\tau_0q/\tau_2,\tau_0q/\tau_3;q\big)_i}\frac{\big(1-\tau_0^2q^{2i}\big)}{\big(1-\tau_0^2\big)}\left(\frac{q}{\tau_0\tau_1\tau_2\tau_3}\right)^i \end{aligned} \tag{3.9}$$

(see [6, Theorem 2.4] or [18, (7.5.22)] to avoid a small misprint).

For $\rho q^\lambda\in D(r)$ and $z\in\mathfrak{C}^{n-r}$, we set

$$d\nu_r(\rho q^\lambda,z;t_0) = \Delta_r(\rho q^\lambda,z;t_0)\frac{dz}{z} \tag{3.10}$$

with weight function $\Delta_r(\rho q^\lambda,z;t_0)$ given by

$$\Delta_r(\rho q^\lambda,z;t_0) := \Delta^{(d)}\big(\rho q^\lambda;t_0\big)\Delta(z;t_0)\delta_c\big(\rho q^\lambda;z\big) \tag{3.11}$$

where $\Delta(z;t_0)$ is the weight function (2.6) in the variables $z=(z_1,\dots,z_{n-r})$ and where $\delta_c(\rho q^\lambda;z)$ is an interaction factor given by

$$\delta_c(\rho q^\lambda;z) := \prod_{\substack{1\leq k\leq r\\ 1\leq l\leq n-r}} \big(\rho_k q^{\lambda_k}z_l, \rho_k q^{\lambda_k}z_l^{-1}, \rho_k^{-1}q^{-\lambda_k}z_l, \rho_k^{-1}q^{-\lambda_k}z_l^{-1};q\big)_\tau. \tag{3.12}$$

In particular we have $\Delta_n(\rho q^\lambda;t_0)=\Delta^{(d)}(\rho q^\lambda;t_0)$ for $\rho q^\lambda\in D(n)$. The measure $d\nu_r(\rho q^\lambda,z;t_0)$ is well defined on $D(r)\times\mathfrak{C}^{n-r}$ since the denominator of $\Delta_r(\rho q^\lambda,z;t_0)$ is non-zero by properties **(ii)** and **(iii)** of the (n,t_0)-residue pair $(C,\mathfrak{C})$ (Definition 3.1). We call $d\nu_r$ the rth *measure associated with the* (n,t_0)*-residue pair* $(C,\mathfrak{C})$.

Proposition 3.2. *Let $(C,\mathfrak{C})$ be a (n,t_0)-residue pair and let $\Omega=\Omega(C,\mathfrak{C})$ be the associated domain. Let $d\nu_r$ be the rth measure associated with $(C,\mathfrak{C})$. Then*

$$\begin{aligned} \frac{1}{(2\pi i)^n}\iint_{z\in C^n} f(z)d\nu(z) &= \frac{1}{(2\pi i)^n}\iint_{z\in\mathfrak{C}^n} f(z)d\nu(z) \\ &+\sum_{r=1}^n \frac{2^r\big(n-r+1\big)_r}{(2\pi i)^{n-r}}\sum_{\omega\in D(r)}\iint_{z\in\mathfrak{C}^{n-r}} f(\omega,z)d\nu_r(\omega,z) \end{aligned} \tag{3.13}$$

for $f\in\mathcal{O}_W(\Omega^n)$, where $\big(u\big)_r := \prod_{i=0}^{r-1}(u+i)$ is the shifted factorial.

In the next lemma we will give the proof of Proposition 3.2 for (n,t_0)-residue pair $(C,\mathfrak{C})$ such that the interaction factor $\delta(.;t)$ is analytic on $\big(\Omega(C,\mathfrak{C})\big)^n$. We use in this lemma the notations of Definition 2.2, so in particular we write $\phi_C(x) = r_C(x)e^{2\pi i x}$ for the parametrization of a deformed circle C.

Lemma 3.3. *Suppose that $(C,\mathfrak{C})$ is a (n,t_0)-residue pair such that*

$$t\big(r_C(x)\big) < r_{\mathfrak{C}}(x), \qquad \forall x \in A^+(C,\mathfrak{C}). \tag{3.14}$$

Then (3.13) *is valid.*

Proof. Fix a (n,t_0)-residue pairs $(C,\mathfrak{C})$ satisfying the extra condition (3.14). We will prove by induction on $l \in \{0,\dots,n\}$ that

$$\begin{aligned}\frac{1}{(2\pi i)^n}\iint\limits_{z\in C^l\times\mathfrak{C}^{n-l}} f(z)\Delta(z;t_0)\frac{dz}{z} =& \frac{1}{(2\pi i)^n}\iint\limits_{z\in\mathfrak{C}^n} f(z)\Delta(z;t_0)\frac{dz}{z}\\ &+\frac{2l}{(2\pi i)^{n-1}}\sum_{i\in I}\iint\limits_{z\in\mathfrak{C}^{n-1}} f(t_0q^i,z)\Delta_1(t_0q^i,z;t_0)\frac{dz}{z}\end{aligned} \tag{3.15}$$

for $f\in\mathcal{O}_W(\Omega^n)$, where

$$I = \{i\in\mathbb{N}_0 \,|\, r_C(\alpha_0^+) > |t_0q^i| > r_{\mathfrak{C}}(\alpha_0^+)\}. \tag{3.16}$$

Then (3.13) is the special case $l=n$ in (3.15), since $D(r)=\emptyset$ for $r>1$ by (3.14).

For $l=0$, (3.15) is trivial. Let $l\in\{1,\dots,n\}$. Since $\delta(.;t)\in\mathcal{O}_W(\Omega^n)$ by Lemma 2.3, we can shift C to $\mathfrak{C}$ for the first variable z_1 in the left-hand side of (3.15) and we obtain, by (3.8), by the W-invariance of $\Delta(z)$ and by Cauchy's Theorem

$$\begin{aligned}\frac{1}{(2\pi i)^n}\iint\limits_{z\in C^l\times\mathfrak{C}^{n-l}} f(z)\Delta(z;t_0)\frac{dz}{z} =& \frac{1}{(2\pi i)^n}\iint\limits_{z\in C^{l-1}\times\mathfrak{C}^{n-l+1}} f(z)\Delta(z;t_0)\frac{dz}{z}\\ &+\frac{2}{(2\pi i)^{n-1}}\sum_{i\in I}\iint\limits_{z\in C^{l-1}\times\mathfrak{C}^{n-l}} f(t_0q^i,z)\Delta_1(t_0q^i,z;t_0)\frac{dz}{z}\end{aligned} \tag{3.17}$$

for $f\in\mathcal{O}_W(\Omega^n)$. Here we have used that the residue at $z_1 = t_0^{-1}q^{-i}$ $(i\in I)$ of $\Delta(z_1,z';t_0)/z_1$ is equal to $-\Delta_1(t_0q^i,z';t_0)$ since

$$\operatorname{res}_{x=t_0^{-1}q^{-i}}\left(\frac{w_c(x;\underline{t})}{x}\right) = -w_d(t_0q^i;t_0;t_1,t_2,t_3).$$

The weight function $\Delta_1(t_0q^i,z;t_0)$ in (3.17) can be rewritten as

$$\Delta_1(t_0q^i,z;t_0) = h(t_0q^i,z;t_0)\Delta(z;tt_0q^i) \tag{3.18}$$

with

$$h(t_0q^i,z;t_0) = w_d(t_0q^i;t_0)\prod_{s=1}^{n-1}\frac{\big(t_0^{-1}q^{-i}z_s,t_0^{-1}q^{-i}z_s^{-1};q\big)_\tau}{\big(t_0z_s,t_0z_s^{-1};q\big)_i}. \tag{3.19}$$

This follows by interchanging the factor $\big(t_0z_s,t_0z_s^{-1};q\big)_\infty$ in the denominator of $w_c(z_s;t_0)$ with the factor $\big(tt_0q^iz_s,tt_0q^iz_s^{-1};q\big)_\infty$ in the denominator of $\delta_c(t_0q^i;z)$ for $s=1,\dots,n-1$. We have $\Delta(.;tt_0q^i)\in\mathcal{O}_W(\Omega^{n-1})$ for $i\in I$ by (3.14) and

Lemma 2.3. We claim that $h(t_0q^i,.;t_0)\in\mathcal{O}_W(\Omega^{n-1})$ for $i\in I$. Indeed, it is sufficient to check that the map

$$x\mapsto\frac{\big(t_0^{-1}q^{-i}x,t_0^{-1}q^{-i}x^{-1};q\big)_\infty}{\big(t_0x,t_0x^{-1};q\big)_i\big(tt_0^{-1}q^{-i}x,tt_0^{-1}q^{-i}x^{-1};q\big)_\infty}\tag{3.20}$$

is analytic on Ω when $i\in I$. The zeros of the factor $\big(t_0x,t_0x^{-1};q\big)_i$ in the denominator are compensated by zeros in the numerator. Next, we check that $\big(tt_0^{-1}q^{-i}x;q\big)_\infty$ is non-zero for $x\in\Omega$ and $i\in I$. Now $\big(tt_0^{-1}q^{-i}x;q\big)_\infty=0$ iff $x=t^{-1}q^{i-m}t_0$ for some $m\in\mathbb{N}_0$. In particular, we must have $\arg(x)=\arg(t_0)=2\pi\alpha_0^+$. Since $i\in I$, we have, for $m\in\mathbb{N}_0$,

$$|t^{-1}q^{i-m}t_0|>t^{-1}r_{\mathfrak{C}}(\alpha_0^+)q^{-m}\geq t^{-1}r_{\mathfrak{C}}(\alpha_0^+)>r_C(\alpha_0^+),$$

where the last inequality is obtained from the extra condition (3.14). Since $x\in\Omega$ with $\arg(x)=2\pi\alpha_0^+$ implies that $r_{\mathfrak{C}}(\alpha_0^+)\leq|x|\leq r_C(\alpha_0^+)$, we conclude that $\big(tt_0^{-1}q^{-i}x;q\big)_\infty\neq0$ for $x\in\Omega$ and $i\in I$. Since $\Omega^{-1}=\Omega$, we then also have $\big(tt_0^{-1}q^{-i}x^{-1};q\big)_\infty\neq0$ for $x\in\Omega$ and $i\in I$. Thus the map given by (3.20) is analytic on Ω if $i\in I$. In particular,

$$f(t_0q^i,.)\Delta_1(t_0q^i,.;t_0)\in\mathcal{O}_W(\Omega^{n-1})$$

for $i\in I$, so we obtain by Cauchy's Theorem and (3.17)

$$\begin{aligned}\frac{1}{(2\pi i)^n}\iint\limits_{z\in C^l\times\mathfrak{C}^{n-l}}f(z)\Delta(z;t_0)\frac{dz}{z}&=\frac{1}{(2\pi i)^n}\iint\limits_{z\in C^{l-1}\times\mathfrak{C}^{n-l+1}}f(z)\Delta(z;t_0)\frac{dz}{z}\\&\quad+\frac{2}{(2\pi i)^{n-1}}\sum_{i\in I}\iint\limits_{z\in\mathfrak{C}^{n-1}}f(t_0q^i,z)\Delta_1(t_0q^i,z;t_0)\frac{dz}{z}\end{aligned}\tag{3.21}$$

for $f\in\mathcal{O}_W(\Omega^n)$. Then (3.15) follows by applying the induction hypotheses on the integral over $C^{l-1}\times\mathfrak{C}^{n-l+1}$ in (3.21). □

Lemma 3.3 can be used to prove Proposition 3.2 inductively. The following definition will be used to formulate the induction hypotheses.

Definition 3.4. Let $(C,\mathfrak{C})$ be a (n,t_0)-residue pair and let $A^+(C,\mathfrak{C})$ (2.11) be the associated open interval. A sequence of closed contours $(C_0,\dots,C_s)$ is called a (n,t_0)-resolution for $(C,\mathfrak{C})$ if the contours C_l are deformed circles satisfying the following four conditions (we write r_l for the (radial) functions r_{C_l} in the parametrization ϕ_{C_l} of C_l):

(i) $C_0=\mathfrak{C}$ and $C_s=C$;
(ii) $r_l(x)=r_{\mathfrak{C}}(x)=r_C(x)$ for $x\notin A^+(C,\mathfrak{C})\cup A^+(\mathfrak{C},C)$ and $l\in\{0,\dots,s\}$;
(iii) $t\big(r_{l+1}(x)\big)<r_l(x)<r_{l+1}(x)$ for $x\in A^+(C,\mathfrak{C})$ and $l\in\{0,\dots,s-1\}$;
(iv) $t_0t^pq^r\notin C_l$ for $p\in\{-1,\dots,n-1\}$, $r\in\mathbb{Z}$ and $l\in\{1,\dots,s-1\}$.

We call s the *length* of the resolution.

Observe that there exists a (n,t_0)-resolution for every (n,t_0)-residue pair $(C,\mathfrak{C})$. If $(C_0,\dots,C_s)$ is a (n,t_0)-resolution for a (n,t_0)-residue pair $(C,\mathfrak{C})$, then (C_l,C_{l-1}) is a (n,t_0)-residue pair satisfying the extra condition (3.14) used to prove Lemma 3.3 ($l\in\{1,\dots,s\}$). We will prove now Proposition 3.2 by induction on the length of the resolution.

Proof of Proposition 3.2. Suppose that, for all $n \in \mathbb{N}$ and all $t_0 \in \mathbb{C}^*$ with $\underline{t} = (t_0, t_1, t_2, t_3) \in V$, Proposition 3.2 has been proved for (n, t_0)-residue pairs which have a (n, t_0)-resolution of length $\leq s-1$, where $s \geq 2$.

Fix arbitrary $n \in \mathbb{N}$ and $t_0 \in \mathbb{C}^*$ such that $\underline{t} = (t_0, t_1, t_2, t_3) \in V$. The induction step is clear for $n = 1$, so we may assume that $n > 1$. Let $(C, \mathfrak{C})$ be a (n, t_0)-residue pair with a (n, t_0)-resolution $(C_0, \ldots, C_s)$ of length s. It suffices to prove (3.13) for the (n, t_0)-residue pair $(C, \mathfrak{C})$. We write $\Omega^{(l)}$ and $\Omega_{(l)}$ for the domains associated with the (n, t_0)-residue pairs (C_l, C_{l-1}) and $(C_l, \mathfrak{C})$ respectively ($l \in \{1, \ldots, s\}$). Note that $\Omega_{(1)} \subset \Omega_{(2)} \subset \ldots \subset \Omega_{(s)} = \Omega$ where Ω is the domain associated with the (n, t_0)-residue pair $(C, \mathfrak{C})$. By (3.15) and (3.18), we have

$$\begin{aligned}\frac{1}{(2\pi i)^n} \iint\limits_{z \in C^n} f(z)\Delta(z; t_0)\frac{dz}{z} =& \frac{1}{(2\pi i)^n} \iint\limits_{z \in (C_{s-1})^n} f(z)\Delta(z; t_0)\frac{dz}{z} \\ &+ \frac{2n}{(2\pi i)^{n-1}} \sum_{i \in I_s} \iint\limits_{z \in (C_{s-1})^{n-1}} f_i(z)\Delta(z; tt_0q^i)\frac{dz}{z}\end{aligned} \tag{3.22}$$

for $f \in \mathcal{O}_W(\Omega^n)$, where

$$I_s := \{i \in \mathbb{N}_0 \,|\, r_s(\alpha_0^+) > |t_0q^i| > r_{s-1}(\alpha_0^+)\} \tag{3.23}$$

and $f_i(z) := f(t_0q^i, z)h(t_0q^i, z; t_0)$ with h given by (3.19). We will apply the induction hypotheses on all the terms in the right-hand side of (3.22). For the integral over $\big(C_{s-1}\big)^n$ note that $(C_0, \ldots, C_{s-1})$ is a (n, t_0)-resolution of length $s-1$ for the (n, t_0)-residue pair $(C_{s-1}, \mathfrak{C})$. Hence, by the induction hypotheses,

$$\begin{aligned}\frac{1}{(2\pi i)^n} \iint\limits_{z \in (C_{s-1})^n} f(z)\Delta(z; t_0)\frac{dz}{z} &= \frac{1}{(2\pi i)^n} \iint\limits_{z \in \mathfrak{C}^n} f(z)\Delta(z; t_0)\frac{dz}{z} \\ &+ \sum_{r=1}^{n} \frac{2^r\big(n-r+1\big)_r}{(2\pi i)^{n-r}} \sum_{\omega \in D(r; C_{s-1}, \mathfrak{C}; t_0)} \iint\limits_{z \in \mathfrak{C}^{n-r}} f(\omega, z)\Delta_r(\omega, z; t_0)\frac{dz}{z}\end{aligned} \tag{3.24}$$

for all $f \in \mathcal{O}_W(\Omega^n)$.

Now fix an $i \in I_s$. We have seen in the proof of Lemma 3.3 that $h(t_0q^i, .; t_0) \in \mathcal{O}_W((\Omega^{(s)})^{n-1})$. In fact it follows from the proof that $h(t_0q^i, .; t_0) \in \mathcal{O}_W(\Omega^{n-1})$. In particular we have $f_i \in \mathcal{O}_W((\Omega_{(s-1)})^{n-1})$ for $f \in \mathcal{O}_W(\Omega^n)$. Observe furthermore that $(tt_0q^i, t_1, t_2, t_3) \in V$ since $\arg(tt_0q^i) = \arg(t_0)$ and that $(C_0, \ldots, C_{s-1})$ is a $(n-1, tt_0q^i)$-resolution of length $s-1$ for the $(n-1, tt_0q^i)$-residue pair $(C_{s-1}, \mathfrak{C})$. So we can apply the induction hypotheses to all the terms in the second line of (3.22), and we obtain

$$\begin{aligned}\frac{2n}{(2\pi i)^{n-1}} \iint\limits_{z \in (C_{s-1})^{n-1}} f_i(z)\Delta(z; tt_0q^i)\frac{dz}{z} &= \frac{2n}{(2\pi i)^{n-1}} \iint\limits_{z \in \mathfrak{C}^{n-1}} f_i(z)\Delta(z; tt_0q^i)\frac{dz}{z} \\ + \sum_{r=2}^{n} \frac{2^r\big(n-r+1\big)_r}{(2\pi i)^{n-r}} &\sum_{\omega \in D(r-1; C_{s-1}, \mathfrak{C}; tt_0q^i)} \iint\limits_{z \in \mathfrak{C}^{n-r}} f_i(\omega, z)\Delta_{r-1}(\omega, z; tt_0q^i)\frac{dz}{z}\end{aligned} \tag{3.25}$$

for $f \in \mathcal{O}_W(\Omega^n)$ and $i \in I_s$.

Substitution of (3.24) and (3.25) in the right-hand side of (3.22) completes the proof of (3.13), since

$$D(1;C,\mathfrak{C};t_0)=D(1;C_{s-1},\mathfrak{C};t_0)\cup\{t_0q^i\}_{i\in I_s},$$
$$D(r;C,\mathfrak{C};t_0)=D(r;C_{s-1},\mathfrak{C};t_0)\cup\bigcup_{i\in I_s}\big\{(t_0q^i,\omega)\,|\,\omega\in D(r-1;C_{s-1},\mathfrak{C};tt_0q^i)\big\}$$

disjoint unions ($r\in\{2,\dots,n\}$) and

$$\begin{aligned} f_i(z)\Delta(z;tt_0q^i)&=f(t_0q^i,z)\Delta_1(t_0q^i,z;t_0),\\ f_i(\omega,z)\Delta_{r-1}(\omega,z;tt_0q^i)&=f(t_0q^i,\omega,z)\Delta_r(t_0q^i,\omega,z;t_0)\end{aligned}\tag{3.26}$$

for $i\in I_s$, $r\in\{2,\dots,n\}$ and $\omega\in D(r-1;C_{s-1},\mathfrak{C};tt_0q^i)$. □

4. Limit transitions to multivariable q-Racah polynomials

Multivariable q-Racah polynomials are Koornwinder polynomials for which the parameter values $(\underline{t},t)$ satisfy a particular truncation condition. They are orthogonal with respect to a finite, discrete measure and their quadratic norms are explicitly known (see [14] and [13]).

In this section we derive the orthogonality relations and norm evaluations for the multivariable q-Racah polynomials directly from the orthogonality relations and norm evaluations for the Koornwinder polynomials (cf. Theorem 2.6) using the residue calculus of the previous section.

For $\lambda\in P(r)$ we set

$$\begin{aligned}\Delta^{qR}\big(\rho q^\lambda;\underline{t};t\big):=&\prod_{i=1}^r\left(\frac{\big(q\rho_i^2;q\big)_{2\lambda_i}}{\big(\rho_i^2;q\big)_{2\lambda_i}(q^{-1}t_0t_1t_2t_3t^{2i-2})^{\lambda_i}}\prod_{j=0}^3\frac{\big(t_j\rho_i;q\big)_{\lambda_i}}{\big(qt_j^{-1}\rho_i;q\big)_{\lambda_i}}\right)\\ &\cdot\prod_{1\le k<l\le r}\frac{\big(q\rho_k\rho_l,t\rho_k\rho_l;q\big)_{\lambda_k+\lambda_l}\big(q\rho_k^{-1}\rho_l,t\rho_k^{-1}\rho_l;q\big)_{\lambda_l-\lambda_k}}{\big(t^{-1}q\rho_k\rho_l,\rho_k\rho_l;q\big)_{\lambda_k+\lambda_l}\big(t^{-1}q\rho_k^{-1}\rho_l,\rho_k^{-1}\rho_l;q\big)_{\lambda_l-\lambda_k}}\end{aligned}\tag{4.1}$$

where $\rho_i:=t_0t^{i-1}$ and we set, for $r\in\mathbb{N}_0$,

$$\begin{aligned}K_r(\underline{t};t):=&\prod_{i=1}^r\frac{\big(\rho_i^{-2};q\big)_\infty}{\big(q,\rho_it_1,\rho_i^{-1}t_1,\rho_it_2,\rho_i^{-1}t_2,\rho_it_3,\rho_i^{-1}t_3;q\big)_\infty}\\ &\cdot\prod_{1\le k<l\le r}\big(\rho_k^{-1}\rho_l,\rho_k^{-1}\rho_l^{-1};q\big)_\tau\\ =&\prod_{i=1}^r\frac{\big(\rho_i^{-2},t,t_0^{-2}t^{2-i-r};q\big)_\infty}{\big(q,\rho_it_1,\rho_i^{-1}t_1,\rho_it_2,\rho_i^{-1}t_2,\rho_it_3,\rho_i^{-1}t_3,t^i,t_0^{-2}t^{2-2i};q\big)_\infty}.\end{aligned}\tag{4.2}$$

The discrete weights $\Delta^{(d)}(\rho q^\lambda;\underline{t};t)$ (3.5) can now be rewritten as follows.

Proposition 4.1. *For $\lambda\in P(r)$ we have*

$$\Delta^{(d)}(\rho q^\lambda;\underline{t};t)=K_r(\underline{t};t)\Delta^{qR}(\rho q^\lambda;\underline{t};t),$$

where $\rho_i=t_0t^{i-1}$.

Proof. We rewrite every factor $\big(xq^m;q\big)_y$ in the explicit expression of the discrete weight $\Delta^{(d)}(\rho q^\lambda;\underline{t};t)$ in which $m\in\mathbb{Z}$ only depends on λ as a quotient of infinite products using (2.1). Then we replace the factors of the form $\big(cq^m;q\big)_\infty$ by

$\big(c;q\big)_\infty\big(c;q\big)_m^{-1}$ if $m\in\mathbb{N}_0$, respectively by $\big(c;q\big)_\infty(-c)^{-m}q^{-\binom{1-m}{2}}\big(qc^{-1};q\big)_{-m}$ if $m\in-\mathbb{N}$. For the case $m\in-\mathbb{N}$, we used here the formula

$$\big(q^{1-l}x;q\big)_l=(-x)^lq^{-\binom{l}{2}}\big(x^{-1};q\big)_l,\quad l\in\mathbb{N}. \tag{4.3}$$

Using this method we obtain, for $j\in\{1,\dots,r\}$,

$$\begin{aligned}&\prod_{i=1}^{j-1}\frac{1}{\big(\rho_i\rho_jq^{\lambda_{i-1}+\lambda_j},\rho_i\rho_j^{-1}q^{\lambda_{i-1}-\lambda_j};q\big)_{\lambda_i-\lambda_{i-1}}}\\&\quad=(-1)^{\lambda_{j-1}}t^{(j-1)\lambda_j}q^{\lambda_{j-1}-\binom{\lambda_j-\lambda_{j-1}}{2}+\binom{\lambda_j}{2}}\frac{\big(q;q\big)_{\lambda_j-\lambda_{j-1}}\big(t_0\rho_j;q\big)_{\lambda_j}}{\big(\rho_j^2;q\big)_{\lambda_j+\lambda_{j-1}}\big(qt_0^{-1}\rho_j;q\big)_{\lambda_j}}\\&\quad\cdot\prod_{i=1}^{j-1}\frac{\big(t\rho_i\rho_j;q\big)_{\lambda_i+\lambda_j}\big(q\rho_i^{-1}\rho_j;q\big)_{\lambda_j-\lambda_i}}{\big(\rho_i\rho_j;q\big)_{\lambda_i+\lambda_j}\big(qt^{-1}\rho_i^{-1}\rho_j;q\big)_{\lambda_j-\lambda_i}}t^{\lambda_i-\lambda_j}.\end{aligned}\tag{4.4}$$

Using (4.4) and applying the same method to the explicit expression for the weight w_d (3.9) give

$$\begin{aligned}w_d(\rho_jq^{\lambda_j};\rho_jq^{\lambda_{j-1}})&\prod_{i=1}^{j-1}\frac{1}{\big(\rho_i\rho_jq^{\lambda_{i-1}+\lambda_j},\rho_i\rho_j^{-1}q^{\lambda_{i-1}-\lambda_j};q\big)_{\lambda_i-\lambda_{i-1}}}\\&=\frac{\big(\rho_j^{-2};q\big)_\infty}{\big(q,\rho_jt_1,\rho_j^{-1}t_1,\rho_jt_2,\rho_j^{-1}t_2,\rho_jt_3,\rho_j^{-1}t_3;q\big)_\infty}\\&\quad\cdot\frac{\big(q\rho_j^2;q\big)_{2\lambda_j}}{\big(\rho_j^2;q\big)_{2\lambda_j}(q^{-1}t_0t_1t_2t_3)^{\lambda_j}}\prod_{k=0}^{3}\frac{\big(t_k\rho_j;q\big)_{\lambda_j}}{\big(qt_k^{-1}\rho_j;q\big)_{\lambda_j}}\\&\quad\cdot\prod_{i=1}^{j-1}\frac{\big(t\rho_i\rho_j;q\big)_{\lambda_i+\lambda_j}\big(q\rho_i^{-1}\rho_j;q\big)_{\lambda_j-\lambda_i}}{\big(\rho_i\rho_j;q\big)_{\lambda_i+\lambda_j}\big(qt^{-1}\rho_i^{-1}\rho_j;q\big)_{\lambda_j-\lambda_i}}t^{\lambda_i-\lambda_j}\end{aligned}\tag{4.5}$$

for $j\in\{1,\dots,r\}$. Now again applying the above mentioned method gives

$$\begin{aligned}&\prod_{i=1}^{j-1}\big(\rho_i^{-1}\rho_jq^{\lambda_j-\lambda_i},\rho_i^{-1}\rho_j^{-1}q^{-\lambda_i-\lambda_j};q\big)_\tau\\&\quad=\prod_{i=1}^{j-1}\big(\rho_i^{-1}\rho_j,\rho_i^{-1}\rho_j^{-1};q\big)_\tau\frac{\big(t\rho_i^{-1}\rho_j;q\big)_{\lambda_j-\lambda_i}\big(q\rho_i\rho_j;q\big)_{\lambda_i+\lambda_j}}{\big(\rho_i^{-1}\rho_j;q\big)_{\lambda_j-\lambda_i}\big(qt^{-1}\rho_i\rho_j;q\big)_{\lambda_i+\lambda_j}}t^{-\lambda_i-\lambda_j}\end{aligned}\tag{4.6}$$

for $j\in\{1,\dots,r\}$. Now the proposition follows by multiplying (4.5) and (4.6) and taking the product over $j\in\{1,\dots,r\}$. □

In the remainder of this section we fix a $N\in\mathbb{N}$. In the next theorem the orthogonality relations for the Koornwinder polynomials are given when the parameters $(\underline{t},t)$ satisfy the truncation condition $t^{n-1}t_0t_3=q^{-N}$. We will formulate the theorem with the parameters considered as indeterminates. Set $F:=\mathbb{C}(\underline{t},t)$, $\mathbb{F}:=\mathbb{C}(t_0,t_1,t_2,t)$ and $\underline{t}_N:=(t_0,t_1,t_2,t^{1-n}t_0^{-1}q^{-N})$. Let A_F^W, respectively $A_{\mathbb{F}}^W$, be the algebra of W-invariant Laurent polynomials over the field F, respectively $\mathbb{F}$. We define the Koornwinder polynomial $P_\lambda(.;\underline{t};t)\in A_F^W$ of degree $\lambda\in\Lambda$ over the

field F by

$$P_\lambda(.;\underline{t};t) := \left(\prod_{\mu<\lambda} \frac{D_{\underline{t},t} - E_\mu(\underline{t},t)}{E_\lambda(\underline{t},t) - E_\mu(\underline{t};t)}\right) m_\lambda. \tag{4.7}$$

The Koornwinder polynomial $P_\lambda(.;\underline{t};t) \in A^W$ of degree λ as defined in Theorem 2.6 (see also Definition 2.7) can be reobtained from (4.7) by specializing the parameters $(\underline{t},t)$ to values in $V \times (0,1)$.

Note that $P_\lambda(.;\underline{t}_N;t) \in A^W_{\mathbb{F}}$ is well defined since the eigenvalues $\{E(\lambda;\underline{t}_N;t)\}_{\lambda\in\Lambda}$ (2.19) are mutually different as elements in $\mathbb{C}[t_0,t_1,t_2,t]$.

Definition 4.2. We call $\{P_\lambda(.;\underline{t}_N;t)\}_{\lambda\in\Lambda_N} \subset A^W_{\mathbb{F}}$ with $\Lambda_N := \{\lambda \in \Lambda \,|\, \lambda_1 \leq N\}$ the multivariable (BC type) q-Racah polynomials.

Let $\lambda \in P(n)$. Then the weight $\Delta^{qR}(\rho q^\lambda;\underline{t}_N;t) \in \mathbb{F}$ is well defined (Δ^{qR} given by (4.1)) and it is non-zero if and only if $\lambda_n \leq N$, due to the factor $\big(\rho_n t_3;q\big)_{\lambda_n}$ in the numerator of $\Delta^{qR}(\rho q^\lambda;\underline{t};t)$. So the bilinear form

$$\langle f,g\rangle_{qR,\underline{t}_N,t} := \sum_{\lambda\in P(n)} f(\rho q^\lambda)g(\rho q^\lambda)\Delta^{qR}(\rho q^\lambda;\underline{t}_N;t), \quad f,g \in A^W_{\mathbb{F}}, \tag{4.8}$$

takes its values in the field $\mathbb{F}$. Let $\mathcal{N}^{qR}(\lambda;\underline{t};t)$ for $\lambda \in \Lambda$ be given by

$$\mathcal{N}^{qR}(\lambda;\underline{t};t) := \frac{\mathcal{N}(\lambda;\underline{t};t)}{K_n(\underline{t};t)2^n n!} \tag{4.9}$$

where $\mathcal{N}(\lambda)$ (2.21) is the expression for the quadratic norms of the Koornwinder polynomial P_λ. Substitution of the explicit expressions of $\mathcal{N}(\lambda)$ and K_n in (4.9) yields that $\mathcal{N}^{qR}(\lambda;\underline{t}_N;t) \in \mathbb{F}$ and that $\mathcal{N}^{qR}(\lambda;\underline{t}_N;t)$ is non-zero if and only if $\lambda \in \Lambda_N$.

Theorem 4.3. *The multivariable q-Racah polynomials $P_\lambda(.;\underline{t}_N;t)$ ($\lambda \in \Lambda_N$) are mutually orthogonal with respect to $\langle .,.\rangle_{qR,\underline{t}_N,t}$ and their quadratic norms are given by*

$$\langle P_\lambda(.;\underline{t}_N;t), P_\lambda(.;\underline{t}_N;t)\rangle_{qR,\underline{t}_N,t} = \mathcal{N}^{qR}(\lambda;\underline{t}_N;t), \qquad \lambda \in \Lambda_N. \tag{4.10}$$

Proof. Let $\tilde{V} \subset (\mathbb{C}^*)^4$ be the set of parameters $\underline{t} \in (\mathbb{C}^*)^4$ for which $t_0t_1t_2t_3 \in \mathbb{C}\setminus\mathbb{R}$. Note that there exists an open dense subset $I_N \subset (0,1)$ such that $E_\lambda(\underline{t};t) \neq E_\mu(\underline{t};t)$ for all $\underline{t} \in \tilde{V}$, $t \in I_N$ and all $\lambda,\mu \in \Lambda_N$ with $\lambda \neq \mu$.

Fix $t_0,t_1,t_2 \in \mathbb{C}^*$ such that $\#\{\arg(t_i),\arg(t_i^{-1}) \,|\, i=0,1,2\} = 6$ and $t \in I_N$. Then $\underline{t}_N \in \tilde{V}$ and there exists a sequence $\{t_{3,i}\}_{i\in\mathbb{N}_0} \subset \mathbb{C}^*$ converging to $t^{1-n}t_0^{-1}q^{-N}$ such that $\underline{t}_i := (t_0,t_1,t_2,t_{3,i}) \in V \cap \tilde{V}$ for all i (V given in Definition 2.1). By considering a subsequence if necessary, we may assume that there exist (n,t_0)-residue pairs $(C_i,\mathfrak{C})$ where C_i is a $\underline{t}_i$-contour and where $\mathfrak{C}$ is a deformed circle such that the sequences $\{t_1q^j,t_2q^j,t_{3,i}q^j\}_{j\in\mathbb{N}_0}$ are in the interior of $\mathfrak{C}$ for all i and such that $t^{n-1}t_0q^N$ is in the exterior of $\mathfrak{C}$. Then we obtain from Theorem 2.6, Proposition 3.2 and Proposition 4.1 that

$$\begin{aligned}\frac{\mathcal{N}(\lambda;\underline{t}_i;t)}{K_n(\underline{t}_i;t)}\delta_{\lambda,\mu} &= \frac{1}{(2\pi i)^n}\iint_{z\in\mathfrak{C}^n}\big(P_\lambda P_\mu\big)(z;\underline{t}_i;t)\frac{\Delta(z;\underline{t}_i;t)}{K_n(\underline{t}_i;t)}\frac{dz}{z}\\ &+\sum_{r=1}^n \frac{2^r\big(n-r+1\big)_r}{(2\pi i)^{n-r}}\sum_{\omega\in D(r)}\iint_{z\in\mathfrak{C}^{n-r}}\big(P_\lambda P_\mu\big)(\omega,z;\underline{t}_i;t)\frac{\Delta_r(\omega,z;\underline{t}_i;t)}{K_n(\underline{t}_i;t)}\frac{dz}{z}\end{aligned} \tag{4.11}$$

where $\delta_{\lambda,\mu}$ is the Kronecker-delta and $D(r) = D(r; C_i, \mathfrak{C}; t_0; t)$ (3.4) (which is independent of i). By (3.11) and Proposition 4.1 we have

$$\Delta_r(\omega, z; \underline{t}_i; t) = K_r(\underline{t}_i; t)\Delta^{qR}(\omega; \underline{t}_i; t)\Delta(z; \underline{t}_i; t)\delta_c(\omega; z). \tag{4.12}$$

After substitution of (4.12) in the right-hand side of (4.11) for all r, it follows from the bounded convergence theorem that the limit $i \to \infty$ may be pulled through the integrals in the right-hand side of (4.11). Only the completely discrete part survives the limit $i \to \infty$ in the equality (4.11) since

$$\lim_{i\to\infty} \frac{K_r(\underline{t}_i; t)}{K_n(\underline{t}_i; t)} = 0, \qquad 0 \le r < n,$$

by the factor $\big(\rho_n t_3; q\big)_\infty$ in the denominator of $K_n(\underline{t}; t)$. The theorem follows now for the specialized parameter values t_0, t_1, t_3, t from the fact that

$$\{\rho q^\lambda \mid \lambda \in P(n),\ \lambda_n \le N\} \subset D(n)$$

and the fact that $\Delta^{qR}(\rho q^\lambda; \underline{t}_N; t) = 0$ for $\lambda \in P(n)$ with $\lambda_n > N$. It is now clear that the theorem also holds over the field $\mathbb{F}$. □

The constant term identity can be simplified as follows.

Corollary 4.4. *We have the summation formula*

$$\langle 1, 1\rangle_{qR,\underline{t}_N,t} = \prod_{i=1}^{n} \frac{\big(qt_0^2 t^{2n-i-1}, qt_1^{-1}t_2^{-1}t^{i-n}; q\big)_N}{\big(qt_0t_1^{-1}t^{n-i}, qt_0t_2^{-1}t^{n-i}; q\big)_N}. \tag{4.13}$$

Proof. First note that by (2.28), (4.2) and (4.9) we have the explicit formula

$$\mathcal{N}^{qR}(0; \underline{t}; t) = \prod_{i=1}^{n} \frac{\big(t_0t_1t_2t_3t^{2n-i-1}, t_0^{-1}t_1t^{1-i}, t_0^{-1}t_2t^{1-i}, t_0^{-1}t_3t^{1-i}; q\big)_\infty}{\big(t_0^{-2}t^{1+i-2n}, t_1t_2t^{i-1}, t_1t_3t^{i-1}, t_2t_3t^{i-1}; q\big)_\infty}. \tag{4.14}$$

Then (4.13) follows by substitution of $t_3 = t_0^{-1}t^{1-n}q^{-N}$ in (4.14) and by applying formula (4.3) repeatedly (see also [13, section 3]). □

The second order q-difference operator $D_{\underline{t}_N,t}$ (2.16) diagonalizes the q-Racah polynomials $\{P_\lambda(.; \underline{t}_N; t)\}_{\lambda\in\Lambda_N}$. By Theorem 4.3 we conclude that $D_{\underline{t}_N,t}$ is symmetric with respect to $\langle .,.\rangle_{qR,\underline{t}_N,t}$. In [14] the symmetry of $D_{\underline{t}_N,t}$ was proved by direct calculations and the orthogonality relations for the multivariable q-Racah polynomials were proved using the symmetry of $D_{\underline{t}_N,t}$. Furthermore, in [14] the quadratic norms of the q-Racah polynomials were expressed in terms of the quadratic norm of the unit polynomial by studying Pieri formulas for the q-Racah polynomials. The constant term identity (4.13) was recently proved by van Diejen [13, Theorem 3] by truncating a multivariable analogue of Roger's ${}_6\phi_5$-series [13, Theorem 2], which in turn is closely related to an Aomoto-Ito type sum (cf. [3], [20]) for the non-reduced root system BC_n. The proofs of the summation formulas in [13] are based on a multiple ${}_6\psi_6$ summation formula of Gustafson.

In the one-variable case it is known that the Askey-Wilson integral can be rewritten as an infinite sum of residues for some parameter region by shifting the contour over four infinite sequences of poles (see [6, Theorem 2.1]). More generally one can ask the question whether a completely discrete orthogonality measure for the Koornwinder polynomials can be obtained by pulling the $\underline{t}$-contours over certain infinite sequence of poles in the orthogonality relations of the Koornwinder polynomials (Theorem 2.6).

Strong indications in that direction can be found in Gustafson's paper [16] and the recent paper of Tarasov and Varchenko [38] where contours in multidimensional integrals are shifted over infinite sequences of poles in order to arrive at (purely discrete) multidimensional Jackson integrals. Another strong indication is the fact that the Macdonald polynomials are orthogonal with respect to Aomoto-Ito type (cf. [3], [20]) weight functions (see Cherednik [11]). Since the B, C and D type Macdonald polynomials can be obtained from the Koornwinder polynomials by suitable specialization of the parameters, we thus have orthogonality relations for these subfamilies of the Koornwinder polynomials with respect to infinite discrete measures (and the corresponding discrete weights are directly related to (4.1), see [13]).

In this paper we will not consider the above-mentioned questions, but instead look at the implications of the residue calculus for certain limit cases of the Koornwinder polynomials (the so-called multivariable big and little q-Jacobi polynomials). In order to study these limit cases we first need to consider the Koornwinder polynomials for yet another parameter domain. This will be the subject of the next section.

5. Koornwinder polynomials with positive orthogonality measure

In this section we will consider the Koornwinder polynomials for parameters $\underline{t}$ in the following parameter domain.

Definition 5.1. Let V_K be the set of parameters $\underline{t} = (t_0, t_1, t_2, t_3)$ which satisfy the following conditions:

(1) The parameters t_0, t_1, t_2, t_3 are real, or if complex, then they appear in conjugate pairs.

(2) $t_k t_l \notin \mathbb{R}_{\geq 1}$ for all $0 \leq k < l \leq 3$.

Note that parameters $\underline{t} \in V_K$ satisfy the following properties:

(A) $t_i \in \mathbb{R}$ if $|t_i| \geq 1$;

(B) There are at most two parameters with modulus ≥ 1. If there are two, then one is positive and the other is negative.

We will show that the Koornwinder polynomials are orthogonal with respect to a positive, partly discrete orthogonality measure when $t \in (0,1)$ and $\underline{t} \in V_K$ by shifting the contour C^n in the integral

$$\frac{1}{(2\pi i)^n} \iint\limits_{z \in C^n} P_\lambda(z) P_\mu(z) \Delta(z) \frac{dz}{z} \tag{5.1}$$

to the n-torus T^n for a specific parameter domain $V_0 \subset V$ (here C is a $\underline{t}$-contour (Definition 2.2) and V is the parameter domain given in Definition 2.1). We then obtain a partly discrete orthogonality measure which turns out to be well defined and positive for parameter values $\underline{t} \in V_K$. Orthogonality relations for parameter values $\underline{t} \in V_K$ with respect to this positive, partly discrete orthogonality measure can then be derived by suitable continuity arguments.

The parameter domain V_0 is defined as follows.

Definition 5.2. Let V_0 be the set of parameters $\underline{t} \in V$ for which

(i) at most two parameters have modulus > 1;

(ii) $t_i t^j q^p \notin T$ for $i \in \{0, \ldots, 3\}$, $j \in \{-1, \ldots, n-1\}$ and $p \in \mathbb{Z}$.

Fix $t \in (0,1)$, $\underline{t} \in V_0$ and $0 \le i \ne j \le 3$ such that $|t_k| < 1$ for $k \ne i, j$. We write $\rho_p^{(i)} := t^{p-1} t_i$ respectively $\rho_p^{(j)} := t^{p-1} t_j$ for $p \in \mathbb{Z}$. Define for $r \in \mathbb{N}$ a finite discrete set $D_i(r) = D_i(r; \underline{t}; t) \subset \mathbb{C}^r$ by

$$D_i(r) := \{\rho^{(i)} q^\mu \mid \mu \in P(r),\ |\rho_r^{(i)} q^{\mu_r}| > 1\} \tag{5.2}$$

and similarly for $D_j(r)$, where $P(r)$ is given by (3.2) and

$$\rho^{(i)} q^\mu = (\rho_1^{(i)} q^{\mu_1}, \dots, \rho_r^{(i)} q^{\mu_r})$$

for $\mu \in P(r)$. Observe that $D_i(r) = \emptyset$ if $|t_i| < 1$. Furthermore, we write $F(r) = F(r; \underline{t}; t) \subset \mathbb{C}^r$ for the disjoint union

$$F(r) := \bigcup_{\substack{l+m=r \\ l,m \in \mathbb{N}_0}} D_i(l) \times D_j(m), \qquad r \in \{1, \dots, n\}. \tag{5.3}$$

We use here the convention that $D_i(l) \times D_j(m) = \emptyset$ if $l > 0$ and $D_i(l) = \emptyset$ or if $m > 0$ and $D_j(m) = \emptyset$, and that $D_i(0) \times D_j(m) = D_j(m)$, $D_i(l) \times D_j(0) = D_i(l)$. Let $\omega \in F(r)$ and $z \in T^{n-r}$ and set

$$d\nu_r^K(\omega, z; \underline{t}; t) := \Delta_r^K(\omega, z; \underline{t}; t) \frac{dz}{z} \tag{5.4}$$

with weight function $\Delta_r^K(\omega, z)$ for $\omega = (\vartheta, \zeta) \in D_i(l) \times D_j(m)$ given by

$$\Delta_r^K(\omega, z; \underline{t}; t) := \Delta^{(d)}(\vartheta; t_i) \Delta^{(d)}(\zeta; t_j) \Delta(z; \underline{t}; t) \delta_c(\vartheta; \zeta, z) \delta_c(\zeta; z) \tag{5.5}$$

where $\Delta^{(d)}$ is given by (3.5) and δ_c is given by (3.12). For the special case $l = 0$, respectively $m = 0$, (5.5) simplifies to

$$\Delta_r^K(\zeta, z; \underline{t}; t) = \Delta^{(d)}(\zeta; t_j) \Delta(z; \underline{t}; t) \delta_c(\zeta; z) = \Delta_r(\zeta, z; t_j), \tag{5.6}$$

respectively

$$\Delta_r^K(\vartheta, z; \underline{t}; t) = \Delta^{(d)}(\vartheta; t_i) \Delta(z; \underline{t}; t) \delta_c(\vartheta; z) = \Delta_r(\vartheta, z; t_i), \tag{5.7}$$

where Δ_r is given by (3.11). We obtain from the residue calculus of section 3 the following lemma.

Lemma 5.3. *Let $t \in (0,1)$ and $\underline{t} \in V_0$. Let C be a $\underline{t}$-contour and $f \in A^W$. Then,*

$$\begin{aligned} \frac{1}{(2\pi i)^n} \iint_{z \in C^n} f(z) d\nu(z) &= \frac{1}{(2\pi i)^n} \iint_{z \in T^n} f(z) d\nu(z) \\ &+ \sum_{r=1}^n \frac{2^r (n-r+1)_r}{(2\pi i)^{n-r}} \sum_{\omega \in F(r)} \iint_{z \in T^{n-r}} f(\omega, z) d\nu_r^K(\omega, z). \end{aligned} \tag{5.8}$$

Proof. If $|t_k| < 1$ for all k, then we only have the completely continuous measure $d\nu$ on T^n in the right-hand side of (5.8) since $F(r) = \emptyset$. Since T is a $\underline{t}$-contour in this case, the lemma follows from Lemma 2.4.

Suppose that at most one parameter has modulus > 1. By the symmetry of $d\nu(z; \underline{t}; t)$ in the four parameters $\underline{t}$, we may assume that $|t_0| > 1$. By Lemma 2.4, we may assume that the $\underline{t}$-contour C satisfies the additional conditions that $A^+ := \{x \in [0,1] \mid r_C(x) > 1\}$ is an open interval and that $\alpha_0^+ \in A^+$ but $\alpha_i^\pm \notin A^+$ for $i = 1, 2, 3$ (here r_C is as in Definition 2.2, and α_i^+ is given by (2.3)). Then (C, T) is a (n, t_0)-residue pair since $\underline{t} \in V_0$ (Definition 5.2), and $D_0(l) = D(l; C, T; t_0)$

(3.4) since t_0 is in the interior of C. The lemma is then a direct consequence of Proposition 3.2 and (5.7).

Suppose now that two parameters have moduli >1. Without loss of generality, we may assume that $|t_0|>1$ and $|t_1|>1$ and that the $\underline{t}$-contour C satisfies the additional condition that

$$\{x\in[0,1]\,|\,r_C(x)>1\}=A_0^+\cup A_1^+$$

disjoint union, with A_i^+ open intervals such that $\alpha_i^+\in A_i^+$ and $\alpha_j^\pm\notin A_i^+$ for $j\neq i$ and $i=0,1$. Let $C':=\phi_{C'}([0,1])$ be the deformed circle with parametrization $\phi_{C'}(x)=r_{C'}(x)e^{2\pi ix}$ given by

$$r_{C'}(x):=r_C(x)\ \ (x\notin A_0^+\cup A_0^-),\quad r_{C'}(x):=1\ \ (x\in A_0^+\cup A_0^-),$$

where $A_0^-:=(1-\beta,1-\alpha)$ when $A_0^+=(\alpha,\beta)$. Then (C,C') is a (n,t_0)-residue pair, (C',T) is a (n,t_1)-residue pair and $D_0(l)=D(l;C,C';t_0)$, respectively $D_1(m)=D(m;C',T;t_1)$, since t_0 and t_1 are in the interior of C. Write Ω' for the domain associated with (C',T). Then $\delta_c(\vartheta;.)\in\mathcal{O}_W\big((\Omega')^{n-l}\big)$ for $\vartheta\in D_0(l)$; hence the lemma follows by applying Proposition 3.2 first to the (n,t_0)-residue pair (C,C'), and then to the (n,t_1)-residue pair (C',T). □

For $t=q^k$ with $k\in\mathbb{N}$, formula (5.8) can be rewritten as

$$\begin{aligned}&\frac{1}{(2\pi i)^n}\iint\limits_{z\in C^n}f(z)d\nu(z)\\&\quad=\sum_{r=0}^n\frac{2^r\binom{n}{r}}{(2\pi i)^{n-r}}\sum_{e_1,\dots,e_r}\sum_{\substack{z_i\in\{e_iq^l\}_{l=0}^{N_{e_i}}\\ i=1,\dots,r}}\iint\limits_{(z_{r+1},\dots,z_n)\in T^{n-r}}f(z)\delta(z;q^k)\\&\qquad\cdot\prod_{i=1}^r w_d(z_i;e_i)\prod_{j=r+1}^n w_c(z_j)\frac{dz_j}{z_j}\end{aligned}\tag{5.9}$$

for $f\in A^W$, where the sum is over $e_i\in\{t_j\,|\,|t_j|>1\}$ and N_{e_i} is the largest positive integer such that $|e_iq^{N_{e_i}}|>1$. This follows from the fact that $\delta(z;q^k)=0$ if $z_i=q^lz_j$ for some $i\neq j$ and some $l\in\{0,\dots,k-1\}$ and from the fact that

$$w_d(x;e_iq^l)=\big(e_ix,e_ix^{-1};q\big)_lw_d(x;e_i),\qquad x=e_iq^{l+m},m\in\mathbb{N}_0.$$

Remark 5.4. Observe that $\delta(.;t)\in A^W$ when $t=q^k$ with $k\in\mathbb{N}$. In particular, we will only encounter residues of the factor $\prod_{i=1}^n w_c(z_i)$ when deforming C^n for parameter $t=q^k$ ($k\in\mathbb{N}$) to the torus T^n in the left-hand side of (5.9). Consequently (5.9) can also be proved by induction on n using the residue calculus for the one-variable weight functions w_c and using the W-invariance of the integrand in the left-hand side of (5.9).

We define now bilinear forms $\langle.,.\rangle_{r,\underline{t},t}$ on A^W for $r\in\{0,\dots,n\}$, $\underline{t}\in V_0$ and $t\in(0,1)$ by

$$\begin{aligned}\langle f,g\rangle_0&:=\iint\limits_{z\in T^n}f(z)g(z)d\nu(z),\\ \langle f,g\rangle_r&:=\sum_{\omega\in F(r)}\iint\limits_{z\in T^{n-r}}f(\omega,z)g(\omega,z)d\nu_r^K(\omega,z),\quad r\in\{1,\dots,n\},\end{aligned}\tag{5.10}$$

for $f, g \in A^W$ and we set

$$\langle f, g\rangle_{\underline{t},t} := \sum_{r=0}^{n} \frac{2^r\big(n-r+1\big)_r}{(2\pi i)^{n-r}} \langle f, g\rangle_{r,\underline{t},t}, \quad f, g \in A^W. \tag{5.11}$$

In the following lemma we consider the symmetric bilinear form $\langle .,.\rangle_{\underline{t},t}$ for parameter values $(\underline{t}, t) \in V_K \times (0,1)$.

Lemma 5.5. *Let $t \in (0,1)$ and $\underline{t} \in V_K$.*

(i) *The bilinear form $\langle .,.\rangle_{\underline{t},t}$ is well defined;*

(ii) *The weight function $\Delta(z;\underline{t};t)$, respectively $\Delta_r^K(\omega, z;\underline{t};t)$, is positive for $z \in T^n$, respectively $(\omega, z) \in F(r) \times T^{n-r}$ $(r = 1, \dots, n)$.*

Proof. The discrete weights w_d (3.9) appearing as factors of the weight function $\Delta_r^K(\omega, z)$ for $r > 0$ are well defined and strictly positive. Indeed if $t_0t_1t_2t_3 = 0$, then the factors $(t_iq/t_j;q)_k\, t_j^k$ in the denominator of w_d (3.9) should be read as $\prod_{l=0}^{k-1}\big(t_j - t_iq^{l+1}\big)$. The factor $\delta(z;t) = |\delta_+(z;t)|^2$ is also well defined and positive for $z \in T^{n-r}$.

Without loss of generality we may assume that $|t_2|, |t_3| < 1$. Fix $\omega = (\vartheta, \zeta) \in F(r)$ with $\vartheta \in D_0(l)$ and $\zeta \in D_1(m)$ $(r = l + m)$. The factor $\delta_d(\vartheta)$, respectively $\delta_d(\zeta)$ (3.7), appearing in the discrete weights $\Delta^{(d)}(\vartheta;t_0)$, respectively $\Delta^{(d)}(\zeta;t_1)$, when $l > 0$, respectively $m > 0$, is well defined and strictly positive. Indeed, if $\vartheta \in D_0(l)$ and $l > 0$, then we have $|t_0| > 1$; hence $t_0 \in \mathbb{R}$. Then $\delta_d(\vartheta) > 0$ follows easily from the definition of the set $D_0(l)$ (5.2).

It remains to show that $h(z) := \big(\prod_{l=1}^{n-r} w_c(z_l;\underline{t})\big)\delta_c(\vartheta;\zeta,z)\delta_c(\zeta;z)$ is well defined and positive for $z \in T^{n-r}$. Let us check the case that both t_0 and t_1 have moduli ≥ 1, and that t_0 is positive real and t_1 negative real (see property **(B)** for parameters $\underline{t} \in V_K$). The case that at most one parameter has modulus ≥ 1 will then also be clear.

Rewrite the factor $\big(x^2, x^{-2};q\big)_\infty$ appearing in the numerator of $w_c(x;\underline{t})$ as

$$\big(x^2, x^{-2};q\big)_\infty = \big(x, -x, x^{-1}, -x^{-1};q\big)_\infty \big(qx^2, qx^{-2};q^2\big)_\infty.$$

Then it is sufficient to check that the factors of the form

$$h_0(x) := \frac{\big(x, x^{-1};q\big)_\infty}{\big(t_0x, t_0x^{-1};q\big)_\infty} \prod_{k=1}^{l} \big(\vartheta_kx, \vartheta_kx^{-1}, \vartheta_k^{-1}x, \vartheta_k^{-1}x^{-1};q\big)_\tau,$$
$$h_1(x) := \frac{\big(-x, -x^{-1};q\big)_\infty}{\big(t_1x, t_1x^{-1};q\big)_\infty} \prod_{k=1}^{m} \big(\zeta_kx, \zeta_kx^{-1}, \zeta_k^{-1}x, \zeta_k^{-1}x^{-1};q\big)_\tau$$

$(l, m \in \{0, \dots, n-1\})$ are well defined and positive for $x \in T$. Indeed, the remaining factors of $w_c(x;\underline{t})$ are easily seen to be well defined and positive since $|t_2|, |t_3| < 1$ and t_2, t_3 are both real or are a conjugate pair, while the remaining factors

$$\prod_{\epsilon_i,\epsilon_j=\pm1} \big(\vartheta_i^{\epsilon_i}\zeta_j^{\epsilon_j};q\big)_\tau, \qquad i \in \{1, \dots, l\},\ j \in \{1, \dots, m\},$$

of $\delta_c(\vartheta;\zeta,z)$ are well defined and positive since t_0 is positive real and t_1 is negative real. Now let $\lambda \in P(l)$ such that $\vartheta = \rho^{(0)}q^\lambda \in D_0(l)$. Then $h_0(x) = |h_0^+(x)|^2$ for

$x \in T$ with h_0^+ given by

$$\begin{aligned} h_0^+(x) :=& \frac{\big(x;q\big)_\infty}{\big(t_0x;q\big)_\infty} \prod_{k=1}^{l} \big(\vartheta_k x, \vartheta_k^{-1}x;q\big)_\tau \\ =& \frac{\big(x;q\big)_\infty}{\big(t\vartheta_l x;q\big)_\infty} \prod_{k=1}^{l} \frac{\big(\vartheta_k^{-1}x;q\big)_\tau}{\big(t\vartheta_{k-1}x;q\big)_{\lambda_k-\lambda_{k-1}}} \end{aligned}$$

where $\vartheta_0 := t^{-1}t_0$ and $\lambda_0 := 0$. It follows that $h_0^+(x)$ is well defined for $x \in T$, since the possible zero at $x = 1$ of the factor $\big(t\vartheta_l x;q\big)_\infty$ in the denominator can be compensated by the zero at $x = 1$ of the factor $\big(x;q\big)_\infty$. Similarly, one deals with $h_1(x)$. □

Let $A_{\mathbb{R}}^W$ be the $\mathbb{R}$-algebra of W-invariant Laurent polynomials in the variables $z_1, \dots, z_n$. We have the following corollary of Lemma 5.5.

Corollary 5.6. *Let $\underline{t} \in V_K$ and $t \in (0,1)$. Then the restriction of the bilinear form $\langle .,. \rangle_{\underline{t},t}$ to $A_{\mathbb{R}}^W \times A_{\mathbb{R}}^W$ maps into $\mathbb{R}$ and is positive definite.*

Proof. The monomials m_λ ($\lambda \in \Lambda$) are real valued on $F(r) \times T^{n-r}$ since $F(r) \subset \mathbb{R}^r$ by property **(A)** for parameters in V_K (Definition 5.1), so the assertion follows from Lemma 5.5**(ii)**. □

The following theorem defines the Koornwinder polynomials for parameters $\underline{t} \in V_K$ and $t \in (0,1)$ as a special choice of orthogonal basis for $A_{\mathbb{R}}^W$ with respect to the *positive definite* bilinear form $\langle .,. \rangle_{\underline{t},t} : A_{\mathbb{R}}^W \times A_{\mathbb{R}}^W \to \mathbb{R}$.

Theorem 5.7. *For parameters $(\underline{t},t) \in V_K \times (0,1)$ there exists a unique basis $\{P_\lambda(.;\underline{t};t)\}_{\lambda\in\Lambda}$ of $A_{\mathbb{R}}^W$ such that*

(i) *$P_\lambda(.;\underline{t};t) = m_\lambda + \sum_{\mu<\lambda} c_{\lambda,\mu}(\underline{t};t)m_\mu$, some $c_{\lambda,\mu}(\underline{t};t) \in \mathbb{R}$;*

(ii) *$\langle P_\lambda(.;\underline{t};t), P_\mu(.;\underline{t};t)\rangle_{\underline{t},t} = 0$ if $\mu \neq \lambda$.*

Furthermore, $P_\lambda(.;\underline{t};t)$ is an eigenfunction of $D_{\underline{t},t}$ with eigenvalue $E_\lambda(\underline{t};t)$ and we have the explicit evaluation formula

$$\langle P_\lambda(.;\underline{t};t), P_\lambda(.;\underline{t};t)\rangle_{\underline{t},t} = \mathcal{N}(\lambda;\underline{t};t), \quad \lambda \in \Lambda,$$

for the quadratic norms of the polynomials P_λ.

Proof. Fix $t \in (0,1)$ and $\underline{t} \in V_K$. Since $\langle .,. \rangle_{\underline{t},t}$ is positive definite on $A_{\mathbb{R}}^W$, there exists for $\lambda \in \Lambda$ a unique W-invariant Laurent polynomial $P_\lambda(.;\underline{t};t) \in A_{\mathbb{R}}^W$ satisfying (**i**) and the conditions $\langle P_\lambda(.;\underline{t};t), m_\mu\rangle_{\underline{t};t} = 0$ for all $\mu < \lambda$. Furthermore,

$$P_\lambda(z;\underline{t};t) = m_\lambda(z) - \sum_{\mu<\lambda} \frac{\langle m_\lambda, P_\mu(.;\underline{t};t)\rangle_{\underline{t},t}}{\langle P_\mu(.;\underline{t};t), P_\mu(.;\underline{t};t)\rangle_{\underline{t},t}} P_\mu(z;\underline{t};t), \quad \lambda \in \Lambda. \tag{5.12}$$

By Lemma 5.3, the polynomials $P_\lambda(z;\underline{t};t) = m_\lambda(z) + \sum_{\mu<\lambda} c_{\lambda,\mu}(\underline{t};t)m_\mu(z)$ for $\underline{t} \in V_0$ as defined in Theorem 2.6 satisfy the same formula (5.12). Fix $\underline{t} \in V_K \setminus V_K^-$, where V_K^- is the set of parameters $\underline{t} \in V_K$ such that $t_i = \pm t^{-m}q^{-s}$ for some $i \in \{0,\dots,3\}$, $m \in \{0,\dots,n-1\}$ and $s \in \mathbb{N}_0$. Let $\{\underline{t}_k\}_{k\in\mathbb{N}_0}$ be a sequence in V_0 converging to $\underline{t}$. Then, by the bounded convergence theorem,

$$\lim_{k\to\infty} \langle f,g\rangle_{\underline{t}_k,t} = \langle f,g\rangle_{\underline{t},t}, \quad \forall f,g \in A^W. \tag{5.13}$$

Indeed, by assuming $\underline{t} \notin V_K^-$, we have that $F(r;\underline{\tau};t) = F(r;\underline{t};t)$ for $\underline{\tau}$ in an open neighbourhood of $\underline{t}$ ($r = 1,\dots,n$) and that no zeros in the denominator of the

expression for $\Delta_r^K(\omega,.;\underline{t};t)$ ($\omega \in F(r)$, $r \in \{0,\dots,n-1\}$) occur which need to be compensated by zeros in the numerator (see the proof of Lemma 5.5). Hence the bounded convergence theorem may be applied at once.

By induction on λ we then obtain from (5.12) and (5.13) that

$$\lim_{k\to\infty} c_{\lambda,\mu}(\underline{t}_k;t) = c_{\lambda,\mu}(\underline{t};t), \quad \mu < \lambda, \tag{5.14}$$

where $c_{\lambda,\mu}$ are the expansion coefficients of P_λ with respect to the monomials m_μ ($\mu \in \Lambda$). By the residue calculus given in Lemma 5.3, we can reformulate Theorem 2.6 with respect to the bilinear form $\langle .,. \rangle_{\underline{\tau},t}$ for $\underline{\tau} \in V_0$. The theorem follows then for $\underline{t} \in V_K \setminus V_K^-$ by taking limits in the reformulated results using Proposition 2.5 and (5.14).

To prove the theorem for $\underline{t} \in V_K^-$, we use again a continuity argument. We treat here one typical example; the general case is derived similarly. We assume that $\underline{t} \in V_K^-$ with $t_0 = t^{-m}q^{-s}$ for some $m \in \{0,\dots,n-1\}$, $s \in \mathbb{N}_0$ and that $t_i \neq t^{-l}q^{-p}$ for all $i \in \{1,2,3\}$, $l \in \{0,\dots,n-1\}$ and $p \in \mathbb{N}_0$. Then there exists an $\epsilon > 0$ such that $(\tau_0,t_1,t_2,t_3) \in V_K \setminus V_K^-$ and $F(r;\tau_0,t_1,t_2,t_3;t) = F(r;\underline{t};t)$ for all $r \in \{1,\dots,n\}$ and all $\tau_0 \in \mathbb{R}_{>0}$ with $t_0 - \tau_0 < \epsilon$. We claim that

$$\lim_{\tau_0 \uparrow t_0} \langle f,g \rangle_{\tau_0,t_1,t_2,t_3,t} = \langle f,g \rangle_{\underline{t},t}, \qquad \forall f,g \in A^W. \tag{5.15}$$

We use the bounded convergence theorem. In Lemma 5.5 we have seen that zeros in the denominator of the expression for the weight function $\Delta_r^K(\omega,.;\underline{\tau};t)$ can occur when $\omega \in F(r;\underline{\tau};t)$ and $\underline{\tau} \in V_K^-$, and that these zeros can be compensated by zeros in the numerator. It follows, from the specific form of these compensated zeros and from the fact that the functions

$$h^{\pm}(u,x) := \begin{cases} \frac{(1\pm x)}{(1\pm ux)} & \text{if } \ u \neq 1, \\ 1 & \text{if } u = 1 \end{cases}$$

are bounded on $U \times T$ where $U \subset \mathbb{R}_{>0}$ is some open set containing 1, that the bounded convergence theorem may be applied in the limit (5.15). Now the theorem for the specific parameter values $\underline{t}$ follows by continuity arguments from (5.15). $\square$

For parameters $\underline{t} \in V_K$ with $|t_i| \leq 1$ the orthogonality measure is completely continuous (i.e. we have $\langle .,. \rangle = \langle .,. \rangle_0$) and coincides with Koornwinder's orthogonality measure [24]. In particular the orthogonality relations reduce to Koornwinder's orthogonality relations (see [24]) and the quadratic norm evaluations reduce to van Diejen's quadratic norm evaluations (see [12]) for parameter values $\underline{t} \in V_K$ with $|t_i| \leq 1$ for all i.

Theorem 5.7 for $n = 1$ reduces to the orthogonality relations and norm evaluations stated in [6, Theorem 2.5].

Theorem 5.7 implies that $D_{\underline{t},t}$ is symmetric with respect to $\langle .,. \rangle_{\underline{t},t}$. The symmetry of D and the orthogonality of the Koornwinder polynomials with respect to $\langle .,. \rangle$ have been proved by different methods in [35] for deformation parameter $t = q^k$ with $k \in \mathbb{N}$. In this special case we can rewrite the bilinear form $\langle .,. \rangle_{\underline{t},q^k}$ using (5.9)

and we obtain

$$\langle f,g\rangle = \sum_{r=0}^{n} \frac{2^r\binom{n}{r}}{(2\pi i)^{n-r}} \sum_{e_1,\dots,e_r} \sum_{\substack{z_i\in\{e_iq^l\}_{l=0}^{N_{e_i}}\\ i=1,\dots,r}} \iint_{(z_{r+1},\dots,z_n)\in T^{n-r}} f(z)g(z)\delta(z;q^k) \cdot \prod_{i=1}^{r} w_d(z_i;e_i) \prod_{j=r+1}^{n} w_c(z_j)\frac{dz_j}{z_j} \tag{5.16}$$

for $f,g\in A^W$, where the notation is as in formula (5.9).

6. Limit transitions to multivariable little q-Jacobi polynomials

In this section we consider a limit case of the Koornwinder polynomials with positive partly discrete orthogonality measure (Theorem 5.7) for which the continuous part of the orthogonality measure disappears while the completely discrete part of the orthogonality measure blows up to an infinite discrete measure.

We will obtain as limit the family of multivariable little q-Jacobi polynomials (previously introduced in [33]) which depends (besides on $q,t\in(0,1)$) on two parameters. The parameter domain for the little q-Jacobi polynomials is defined as follows.

Definition 6.1. Let V_L be the set of parameters (a,b) for which $a\in\big(0,1/q\big)$ and $b\in(-\infty,1/q\big)$.

For functions $f:\mathbb{C}\to\mathbb{C}$ and $u,v\in\mathbb{C}$, the Jackson q-integral of f over $[u,v]$ is defined by

$$\begin{aligned}\int_u^v f(x)d_qx &:= \int_0^v f(x)d_qx - \int_0^u f(x)d_qx,\\ \int_0^v f(x)d_qx &:= (1-q)\sum_{k=0}^{\infty} f(vq^k)vq^k,\end{aligned} \tag{6.1}$$

provided that the infinite sums are absolutely convergent. For a point $\xi\in(\mathbb{C}^*)^n$, the Jackson integral of f over the set

$$\langle\xi\rangle_n := \{\xi q^\nu \mid \nu\in P(n)\}, \tag{6.2}$$

where $\xi q^\nu := (\xi_1q^{\nu_1},\dots,\xi_nq^{\nu_n})$ and $P(n)$ is given by (3.2), is defined by

$$\iint_{\langle\xi\rangle_n} f(z)d_qz := (1-q)^n \sum_{\nu\in P(n)} f(\xi q^\nu)\prod_{i=1}^{n}\xi_iq^{\nu_i} \tag{6.3}$$

provided that the multisum is absolutely convergent. Note that, for special points $\xi=(\xi_1,\xi_1\gamma,\dots,\xi_1\gamma^{n-1})\in\big(\mathbb{C}^*\big)^n$, the multisum (6.3) can be expressed as an iterated Jackson integral by

$$\iint_{\langle\xi\rangle_n} f(z)d_qz = \int_{z_1=0}^{\xi_1}\int_{z_2=0}^{\gamma z_1}\dots\int_{z_n=0}^{\gamma z_{n-1}} f(z)d_qz_n\dots d_qz_1. \tag{6.4}$$

Let $A^S_{\mathbb{R}}$ be the $\mathbb{R}$-algebra of S-invariant polynomials in the variables $z_1,\dots,z_n$. An $\mathbb{R}$-basis for $A^S_{\mathbb{R}}$ is given by the set of monomials $\{\tilde m_\lambda\}_{\lambda\in\Lambda}$, where $\tilde m_\lambda(z) :=$

$\sum_{\mu\in S\lambda} z^\mu$. Define a symmetric bilinear form $\langle .,.\rangle_{L,t}^{a,b}$ on $A_{\mathbb{R}}^S$ for $t\in(0,1)$ and $(a,b)\in V_L$ by

$$\langle f,g\rangle_L := \iint\limits_{\langle\rho_L\rangle_n} f(z)g(z)\Delta^L(z)d_qz, \quad f,g\in A_{\mathbb{R}}^S, \tag{6.5}$$

where $\rho_{L,i} := t^{i-1}$ and where the weight function $\Delta^L(z)=\Delta^L(z;a,b;t)$ is given by

$$\Delta^L(z) := q^{-2\tau^2\binom{n}{3}}t^{-(\alpha+1)\binom{n}{2}}\left(\prod_{i=1}^n v_L(z_i)\right)\delta_{qJ}(z), \tag{6.6}$$

with

$$v_L(x;a,b) := \frac{\big(qx;q\big)_\infty}{\big(qbx;q\big)_\infty}x^\alpha, \tag{6.7}$$

and with interaction factor $\delta_{qJ}(z;t)$ given by

$$\delta_{qJ}(z;t) := \prod_{1\le i<j\le n}|z_i-z_j||z_i|^{2\tau-1}\big(qt^{-1}z_j/z_i;q\big)_{2\tau-1}. \tag{6.8}$$

The function v_L is exactly the weight function in the orthogonality measure of the one-variable little q-Jacobi polynomials [1]. The same bilinear form $\langle .,.\rangle_L$ was considered in [33, section 5], up to the positive constant $q^{-2\tau^2\binom{n}{3}}t^{-(\alpha+1)\binom{n}{2}}$. The weights $\Delta^L(z)$ in the bilinear form $\langle .,.\rangle_L$ are strictly positive for $z\in\langle\rho_L\rangle_n$ and $\langle f,g\rangle_L$, written out as a multidimensional infinite sum, is absolutely convergent for all $f,g\in A_{\mathbb{R}}^S$ (see [33]).

Definition 6.2. Let $t\in(0,1)$ and $(a,b)\in V_L$. The multivariable little q-Jacobi polynomials $\{P_\lambda^L(.;a,b;t)\}_{\lambda\in\Lambda}$ are by definition the unique symmetric polynomials which satisfy

(a) $P_\lambda^L = \tilde m_\lambda + \sum_{\mu<\lambda} c_{\lambda,\mu}^L\tilde m_\mu$ for certain $c_{\lambda,\mu}^L = c_{\lambda,\mu}^L(a,b;t)\in\mathbb{R}$;

(b) $\langle P_\lambda^L,\tilde m_\mu\rangle_L = 0$ for $\mu<\lambda$.

The following proposition establishes the link between Koornwinder polynomials with positive partly discrete orthogonality measure and the multivariable little q-Jacobi polynomials. We use the notation $|\lambda| := \sum_{i=1}^n\lambda_i$ for the length of a partition $\lambda\in\Lambda$.

Proposition 6.3. *Let $t\in(0,1)$, $(a,b)\in V_L$ and define for $\epsilon\in\mathbb{R}^*$*

$$\underline{t}_L(\epsilon) := \big(\epsilon^{-1}q^{\frac12}, -aq^{\frac12}, \epsilon bq^{\frac12}, -q^{\frac12}\big). \tag{6.9}$$

Then there exists a sequence of positive real numbers $\{\epsilon_k\}_{k\in\mathbb{N}_0}$ which converges to 0, such that

$$\begin{aligned}\lim_{k\to\infty}\left(\prod_{i=1}^n\big(-\epsilon_k^{-1}qt^{i-1},-\epsilon_k^{-1}qat^{i-1};q\big)_\infty\right)\big(\epsilon_kq^{-\frac12}\big)^{|\lambda|+|\mu|}\langle m_\lambda,m_\mu\rangle_{\underline{t}_L(\epsilon_k),t}\\ = 2^nn!\big(q;q\big)_\infty^{-2n}(1-q)^{-n}\langle\tilde m_\lambda,\tilde m_\mu\rangle_{L,t}^{a,b}\end{aligned} \tag{6.10}$$

for all $\lambda,\mu\in\Lambda$, where $\langle .,.\rangle_{\underline{t},t}$ is given by (5.11).

The proof of the proposition will be given in section 8. Observe that $\underline{t}_L(\epsilon)\in V_K$ for $\epsilon\in\mathbb{R}_{>0}$ sufficiently small, so $\langle .,.\rangle_{\underline{t}_L(\epsilon),t}$ is well defined and positive definite for $\epsilon>0$ sufficiently small by Lemma 5.5 and Corollary 5.6.

We will use Proposition 6.3 to prove that the multivariable little q-Jacobi polynomials are limit cases of the Koornwinder polynomials and to establish orthogonality relations and norm evaluations for the little q-Jacobi polynomials with respect to the scalar product $\langle .,. \rangle_L$ on $A^S_{\mathbb{R}}$.

The following definition of limit transitions between S-invariant Laurent polynomials will be used (cf. [36]). Let $f(.;u)$ $(u \in \mathbb{R}^*)$ and f be S-invariant Laurent polynomials in n variables $z_1, \ldots, z_n$. Then we write $\lim_{u\to 0} f(.;u) = f$ if $\lim_{u\to 0} f(z;u) = f(z)$ for all $z \in (\mathbb{R}^*)^n$. Observe that the $\mathbb{R}$-algebra of S-invariant Laurent polynomials has as $\mathbb{R}$-basis the set of monomials $\{\tilde{m}_\lambda(z)\}_{\lambda\in\tilde{\Lambda}}$, where $\tilde{\Lambda} := \{\lambda \in \mathbb{Z}^n \,|\, \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n\}$ and $\tilde{m}_\lambda(z) := \sum_{\mu\in S\lambda} z^\mu$. If $f(.;u) = \sum_{\lambda\in\tilde{\Lambda}} c_\lambda(u)\tilde{m}_\lambda$ $(u \in \mathbb{R}^*)$ and $f = \sum_{\lambda\in\tilde{\Lambda}} c_\lambda \tilde{m}_\lambda$ satisfy the additional condition that $\{\lambda \in \tilde{\Lambda} \,|\, c_\lambda(u) \neq 0\}$ is contained in some finite u-independent subset for $|u|$ sufficiently small, then $\lim_{u\to 0} f(.;u) = f$ iff $\lim_{u\to 0} c_\lambda(u) = c_\lambda$ for all $\lambda \in \tilde{\Lambda}$. Crucial in the limit from Koornwinder polynomials to multivariable little q-Jacobi polynomials is a limit from rescaled monomials $m_\lambda(z|u)$ to $\tilde{m}_\lambda(z)$, where the rescaled monomial $m_\lambda(z|u)$ for $u \in \mathbb{R}^*$ is the S-invariant Laurent polynomial given by

$$m_\lambda(z|u) := u^{|\lambda|} m_\lambda(u^{-1}z), \qquad \lambda \in \Lambda, \tag{6.11}$$

where $u^{-1}z := (u^{-1}z_1, \ldots, u^{-1}z_n)$. In terms of the basis $\{\tilde{m}_\mu\}_{\mu\in\tilde{\Lambda}}$, we have

$$m_\lambda(z|u) = \sum_{\mu\in\tilde{\Lambda}\cap W\lambda} d_{\lambda,\mu}(u)\tilde{m}_\mu(z), \qquad \lambda \in \Lambda,$$

with $d_{\lambda,\mu}(u)$ homogeneous of degree $|\lambda|-|\mu|$ and $d_{\lambda,\lambda}(u) \equiv 1$. Furthermore, $|\mu| \leq |\lambda|$ if $\mu \in W\lambda$ and $|\lambda| = |\mu|$ iff $\mu \in S\lambda$. Hence we obtain the limit transitions

$$\lim_{u\to 0} m_\lambda(z|u) = \tilde{m}_\lambda(z), \qquad \lambda \in \Lambda. \tag{6.12}$$

We express the quadratic norms of the multivariable little q-Jacobi polynomials in terms of functions $\mathcal{N}^+_{qJ}(\lambda) = \mathcal{N}^+_{qJ}(\lambda;a,b;t)$ and $\mathcal{N}^-_{qJ}(\lambda) = \mathcal{N}^-_{qJ}(\lambda;a,b;t)$ which are defined by

$$\begin{aligned}\mathcal{N}^+_{qJ}(\lambda) := \prod_{i=1}^n &\frac{\Gamma_q(\lambda_i+1+(n-i)\tau+\alpha+\beta)\Gamma_q(\lambda_i+1+(n-i)\tau+\alpha)}{\Gamma_q(2\lambda_i+1+2(n-i)\tau+\alpha+\beta)} \\ &\cdot \prod_{1\leq j<k\leq n}\Bigg(\frac{\Gamma_q(\lambda_j+\lambda_k+1+(2n-j-k+1)\tau+\alpha+\beta)}{\Gamma_q(\lambda_j+\lambda_k+1+(2n-j-k)\tau+\alpha+\beta)} \\ &\qquad\cdot\frac{\Gamma_q(\lambda_j-\lambda_k+(k-j+1)\tau)}{\Gamma_q(\lambda_j-\lambda_k+(k-j)\tau)}\Bigg),\end{aligned} \tag{6.13}$$

$$\begin{aligned}\mathcal{N}^-_{qJ}(\lambda) := \prod_{i=1}^n &\frac{\Gamma_q(\lambda_i+1+(n-i)\tau)\Gamma_q(\lambda_i+1+(n-i)\tau+\beta)}{\Gamma_q(2\lambda_i+2+2(n-i)\tau+\alpha+\beta)} \\ &\cdot \prod_{1\leq j<k\leq n}\Bigg(\frac{\Gamma_q(\lambda_j+\lambda_k+2+(2n-j-k-1)\tau+\alpha+\beta)}{\Gamma_q(\lambda_j+\lambda_k+2+(2n-j-k)\tau+\alpha+\beta)} \\ &\qquad\cdot\frac{\Gamma_q(\lambda_j-\lambda_k+1+(k-j-1)\tau)}{\Gamma_q(\lambda_j-\lambda_k+1+(k-j)\tau)}\Bigg),\end{aligned} \tag{6.14}$$

where

$$\Gamma_q(u) := \frac{(q;q)_{u-1}}{(1-q)^{u-1}}, \qquad u \notin -\mathbb{N}_0, \tag{6.15}$$

is the q-Gamma function. For $\lambda \in \Lambda$, $(a,b) \in V_L$ and $t \in (0,1)$ let $\mathcal{N}^L(\lambda) = \mathcal{N}^L(\lambda; a, b; t)$ be given by

$$\mathcal{N}^L(\lambda) := q^{\sum_{i=1}^n (\lambda_i + \alpha + 2(n-i)\tau)\lambda_i} \mathcal{N}^+_{qJ}(\lambda) \mathcal{N}^-_{qJ}(\lambda). \tag{6.16}$$

Observe that $\mathcal{N}^L(\lambda)$ is well defined and positive.

Theorem 6.4. *Let $(a,b) \in V_L$ and $t \in (0,1)$. There exists a sequence of positive real numbers $\{\epsilon_k\}_{k\in\mathbb{N}_0}$ which converges to 0, such that*

$$\lim_{k\to\infty} \left(q^{-\frac{1}{2}}\epsilon_k\right)^{|\lambda|} P_\lambda\big(q^{\frac{1}{2}}\epsilon_k^{-1} z; \underline{t}_L(\epsilon_k); t\big) = P^L_\lambda(z; a, b; t) \tag{6.17}$$

for all $\lambda \in \Lambda$. Furthermore, the polynomials $\{P^L_\lambda\}_{\lambda\in\Lambda}$ are orthogonal with respect to $\langle .,. \rangle_L$ and the quadratic norms of the little q-Jacobi polynomials are given by

$$\langle P^L_\lambda, P^L_\lambda \rangle_L = \mathcal{N}^L(\lambda), \qquad \lambda \in \Lambda. \tag{6.18}$$

Proof. We write

$$\begin{aligned} \big(q^{-\frac{1}{2}}\epsilon\big)^{|\lambda|} P_\lambda\big(z; \underline{t}_L(\epsilon); t\big) &= \sum_{\mu\le\lambda} c_{\lambda,\mu}(\epsilon)\big(q^{-\frac{1}{2}}\epsilon\big)^{|\mu|} m_\mu(z), \\ P^L_\lambda(z; a, b; t) &= \sum_{\mu\le\lambda} c^L_{\lambda,\mu} \tilde{m}_\mu(z) \end{aligned} \tag{6.19}$$

for the expansions of the Koornwinder polynomial and the multivariable little q-Jacobi polynomial in terms of monomials. In particular, we have $c_{\lambda,\lambda}(\epsilon) = c^L_{\lambda,\lambda} = 1$.

Let $\preceq$ be a total order on Λ such that $\mu \preceq \lambda$ if $\mu \le \lambda$. Let $\{\epsilon_k\}_{k\in\mathbb{N}_0}$ be a sequence in $\mathbb{R}_{>0}$ converging to 0 such that (6.10) is satisfied for all $\lambda, \mu \in \Lambda$. We will prove that

$$\lim_{k\to\infty} c_{\lambda,\mu}(\epsilon_k) = c^L_{\lambda,\mu}, \qquad \forall\, \mu \le \lambda, \tag{6.20}$$

and we will prove full orthogonality for the subset $\{P^L_\mu\}_{\mu\preceq\lambda}$ of multivariable little q-Jacobi polynomials by induction on $\lambda \in \Lambda$. The limit (6.17) is then an immediate consequence of (6.12), (6.19) and (6.20), and the quadratic norm evaluations (6.18) are then immediate consequences of the quadratic norm evaluations of the Koornwinder polynomials (cf. Theorem 5.7), Proposition 6.3, (6.20) and the observation that

$$\begin{aligned} \lim_{\epsilon\to 0} \left(\prod_{i=1}^n \big(-\epsilon^{-1} q t^{i-1}, -\epsilon^{-1} q a t^{i-1}; q\big)_\infty \right) (q^{-1/2}\epsilon)^{2|\lambda|} \mathcal{N}(\lambda; \underline{t}_L(\epsilon); t) \\ = 2^n n! \big(q;q\big)_\infty^{-2n} (1-q)^{-n} \mathcal{N}^L(\lambda; a, b; t). \end{aligned}$$

So it remains to prove the induction step (the case $\lambda = 0$ being trivial). For $\lambda \ne 0$, observe that $\underline{t}_L(\epsilon) \in V_K$ for $\epsilon > 0$ sufficiently small; hence by Theorem 5.7 we can write

$$\big(q^{-\frac{1}{2}}\epsilon\big)^{|\lambda|} P_\lambda\big(z; \underline{t}_L(\epsilon); t\big) = \big(q^{-\frac{1}{2}}\epsilon\big)^{|\lambda|} m_\lambda(z) - \sum_{\nu<\lambda} d_{\lambda,\nu}(\epsilon)\big(q^{-\frac{1}{2}}\epsilon\big)^{|\nu|} P_\nu\big(z; \underline{t}_L(\epsilon); t\big) \tag{6.21}$$

with

$$d_{\lambda,\nu}(\epsilon) := \frac{(q^{-\frac{1}{2}}\epsilon)^{|\lambda|+|\nu|}\langle m_\lambda, P_\nu(.;\underline{t}_L(\epsilon);t)\rangle_{\underline{t}_L(\epsilon),t}}{(q^{-\frac{1}{2}}\epsilon)^{2|\nu|}\langle P_\nu(.;\underline{t}_L(\epsilon);t), P_\nu(.;\underline{t}_L(\epsilon);t)\rangle_{\underline{t}_L(\epsilon),t}}$$

for $\epsilon > 0$ sufficiently small. By the induction hypotheses, we also have

$$P^L_\lambda(z;a,b;t) = \tilde{m}_\lambda(z) - \sum_{\nu<\lambda} d^L_{\lambda,\nu} P^L_\nu(z;a,b;t), \tag{6.22}$$

with

$$d^L_{\lambda,\nu} = \frac{\langle \tilde{m}_\lambda, P^L_\nu(.;a,b;t)\rangle^{a,b}_{L,t}}{\langle P^L_\nu(.;a,b;t), P^L_\nu(.;a,b;t)\rangle^{a,b}_{L,t}}.$$

It follows that, for $\mu < \lambda$,

$$c_{\lambda,\mu}(\epsilon) = -\sum_{\mu\leq\nu<\lambda} d_{\lambda,\nu}(\epsilon)c_{\nu,\mu}(\epsilon), \qquad c^L_{\lambda,\mu} = -\sum_{\mu\leq\nu<\lambda} d^L_{\lambda,\nu}c^L_{\nu,\mu} \tag{6.23}$$

for $\epsilon \in \mathbb{R}_{>0}$ sufficiently small. Again by the induction hypotheses and by Proposition 6.3, we obtain

$$\lim_{k\to\infty} d_{\lambda,\nu}(\epsilon_k) = d^L_{\lambda,\nu}, \qquad \forall\, \nu < \lambda. \tag{6.24}$$

So (6.20) follows from the induction hypotheses, (6.23) and (6.24). The orthogonality relations for $\{P^L_\mu\}_{\mu\preceq\lambda}$ now follow by taking limits in the orthogonality relations for the Koornwinder polynomials (Theorem 5.7). This completes the proof of the induction step. □

Full orthogonality of the multivariable little q-Jacobi polynomials was proved in [33] by means of an explicit second order q-difference operator D_L which diagonalizes the little q-Jacobi polynomials. The operator D_L and the corresponding eigenvalue equations can be obtained from (6.17) by taking the limit $k \to \infty$ in the equations

$$\big(q^{-1}\epsilon_k\big)^{|\lambda|}\big((D - E_\lambda)P_\lambda\big)(q^{\frac{1}{2}}\epsilon_k^{-1}z;\underline{t}_L(\epsilon_k);t) = 0, \qquad \lambda \in \Lambda, \tag{6.25}$$

where D is given by (2.16) and E_λ is given by (2.19) (see [33, section 3] for the formal computation of the limits of D and E_λ). In [36] it was shown that the formal computation of the limits of D and E_λ in [33, section 3] can be used to prove the limit transition (6.17) for generic $t \in (0,1)$. See also [34] for the special case that $t = q^k$, $k \in \mathbb{N}$.

The constant term identity for the little q-Jacobi polynomials can be rewritten as follows.

Corollary 6.5. *For $t \in (0,1)$ and $(a,b) \in V_L$, we have*

$$\langle 1,1\rangle^{a,b}_{L,t} = \prod_{j=1}^n \frac{\Gamma_q(\alpha+1+(j-1)\tau)\Gamma_q(\beta+1+(j-1)\tau)\Gamma_q(j\tau)}{\Gamma_q(\alpha+\beta+2+(n+j-2)\tau)\Gamma_q(\tau)}. \tag{6.26}$$

The constant term identity (6.26) has been studied extensively in the past 20 years. It was conjectured by Askey [4] for $t = q^k$, $k \in \mathbb{N}$ and proved in this case independently by Habsieger [19] and Kadell [21]. For arbitrary $t \in (0,1)$ the first proof appeared in Aomoto's paper [3] (see also [22] and [38] for alternative proofs).

7. Limit transitions to multivariable big q-Jacobi polynomials

In this section, we consider an other limit transition involving the Koornwinder polynomials with positive partly discrete orthogonality measure (Theorem 5.7) for which the continuous part of the orthogonality measure disappears while the completely discrete part of the orthogonality measure blows up to an infinite discrete measure. Instead of sending one parameter to infinity, as we did in the previous section, we will send now two parameters to infinity. We will obtain the four parameter family of multivariable big q-Jacobi polynomials (previously introduced in [33]) as limit case of the five parameter family of Koornwinder polynomials. The parameter domain V_B for the multivariable big q-Jacobi polynomials is defined as follows.

Definition 7.1. Let V_B be the set of parameters (a,b,c,d) for which $c,d>0$ and $a\in\big(-c/dq,1/q\big)$, $b\in\big(-d/cq,1/q\big)$ or $a=cu$, $b=-d\overline{u}$ with $u\in\mathbb{C}\setminus\mathbb{R}$.

Before defining the orthogonality measure for the big q-Jacobi polynomials we first need to introduce some more notations. We set

$$\langle\xi,\eta\rangle_n:=\bigcup_{j=0}^{n}\langle\xi\rangle_j\times\langle\eta\rangle_{n-j}\subset\mathbb{C}^n$$

where $\eta,\xi\in(\mathbb{C}^*)^n$ and $\langle\xi\rangle_n$ is defined by (6.2). Here we use the convention that $\langle\xi\rangle_n\times\langle\eta\rangle_0=\langle\xi\rangle_n$ and $\langle\xi\rangle_0\times\langle\eta\rangle_n=\langle\eta\rangle_n$. Let $\underline{c}=(c_0,\dots,c_n)\in(\mathbb{C}^*)^{n+1}$. Then we define the $\underline{c}$-weighted Jackson integral of f over the set $\langle\xi,\eta\rangle_n$ by

$$\begin{aligned}\iint\limits_{\langle\xi,\eta\rangle_n} f(z)d_q^{\underline{c}}z:=&\sum_{j=0}^{n}(-1)^{n-j}c_j\iint\limits_{z\in\langle\xi\rangle_j}\iint\limits_{w\in\langle\eta\rangle_{n-j}} f(z,w)d_qzd_qw\\ =&(1-q)^n\sum_{j=0}^{n}\sum_{\substack{\mu\in P(j)\\ \nu\in P(n-j)}}c_jf(\xi q^{\mu},\eta q^{\nu})\prod_{l=1}^{j}\xi_lq^{\mu_l}\prod_{m=1}^{n-j}(-\eta_mq^{\nu_m}),\end{aligned}\tag{7.1}$$

where the $j=0$, respectively $j=n$, term in (7.1) should be read as

$$(-1)^nc_0\iint\limits_{w\in\langle\eta\rangle_n}f(w)d_qw=(1-q)^n\sum_{\nu\in P(n)}c_0f(\eta q^{\nu})\prod_{m=1}^{n}\big(-\eta_mq^{\nu_m}\big),$$

respectively

$$c_n\iint\limits_{z\in\langle\xi\rangle_n}f(z)d_qz=(1-q)^n\sum_{\mu\in P(n)}c_nf(\xi q^{\mu})\prod_{l=1}^{n}\xi_lq^{\mu_l}.$$

If $\eta=(\eta_1,\eta_1\gamma',\dots,\eta_1(\gamma')^{n-1})$ and $\xi=(\xi_1,\xi_1\gamma,\dots,\xi_1\gamma^{n-1})$, then the $\underline{c}$-weighted Jackson integral over $\langle\xi,\eta\rangle_n$ can be rewritten as an iterated Jackson integral by

$$\begin{aligned}\iint\limits_{\langle\xi,\eta\rangle_n}f(z)d_q^{\underline{c}}z=\sum_{j=0}^{n}c_j\int_{z_1=0}^{\xi_1}\int_{z_2=0}^{\gamma z_1}\dots\int_{z_j=0}^{\gamma z_{j-1}}&\\ \cdot\int_{z_{j+1}=\eta_1}^{0}\int_{z_{j+2}=\gamma' z_{j+1}}^{0}\dots\int_{z_n=\gamma' z_{n-1}}^{0}&f(z)d_qz_n\dots d_qz_1.\end{aligned}$$

Define a symmetric bilinear form $\langle .,. \rangle_{B,t}^{a,b,c,d}$ on $A_{\mathbb{R}}^S$ for parameters $t \in (0,1)$ and $(a,b,c,d) \in V_B$ by

$$\langle f,g \rangle_B := \iint\limits_{\langle \rho_B, \sigma_B \rangle_n} f(z)g(z)\Delta^B(z) d_q^{\underline{c}_B} z, \quad f,g \in A_{\mathbb{R}}^S, \tag{7.2}$$

with $\rho_{B,i} := ct^{i-1}$, $\sigma_{B,i} := -dt^{i-1}$ and with weight function

$$\Delta^B(z) := \left(\prod_{i=1}^n v_B(z_i) \right) \delta_{qJ}(z), \tag{7.3}$$

where v_B is the weight function in the orthogonality measure for the one-variable big q-Jacobi polynomials [2],

$$v_B(x;a,b,c,d) := \frac{\big(qx/c, -qx/d; q\big)_\infty}{\big(qax/c, -qbx/d; q\big)_\infty} \tag{7.4}$$

and $\delta_{qJ}(z) = \delta_{qJ}(z;t)$ is given by (6.8). The weight $\underline{c}_B = \underline{c}_B(c,d;t)$ is of the form $c_{B,j} := c_B d_{B,j}$, with

$$d_{B,j} := \prod_{\substack{1 \leq k < m \leq n \\ k \leq j}} \Psi_t(-t^{n-m-k+1} d/c) \tag{7.5}$$

where $\Psi_t(x)$ is defined by

$$\Psi_t(x) := |x|^{2\tau-1} \frac{\theta(tx)}{\theta(qt^{-1}x)}, \tag{7.6}$$

with $\theta(x)$ the Jacobi theta function

$$\theta(x) := \big(q, x, qx^{-1}; q\big)_\infty, \tag{7.7}$$

and the constant c_B is defined by

$$c_B := \frac{\big(q;q\big)_\infty^n q^{-2\tau^2\binom{n}{3}} d^{-2\tau\binom{n}{2}-n} t^{-\binom{n}{2}}}{\prod_{i=1}^n \theta(-t^{1-i}c/d)}. \tag{7.8}$$

The positive constant c_B is not essential for the definition of $\langle .,. \rangle_B$. We have chosen to take this constant within the definition of $\langle .,. \rangle_B$ because it will simplify formulas and notations later on.

The Jacobi theta function satisfies the functional relation

$$\theta(q^k x) = (-x^{-1})^k q^{-\binom{k}{2}} \theta(x), \quad k \in \mathbb{N}_0. \tag{7.9}$$

This implies that Ψ_t is a quasi constant, i.e. $\Psi_t(qx) = \Psi_t(x)$. In particular, the weight $\underline{d}_B$ (7.5) is independent of a,b and quasi constant in the parameters c,d.

The bilinear form $\langle .,. \rangle_B$ is the same as the one defined in [33] up to the positive constant c_B (7.8). This is easily verified using the fact that $\langle .,. \rangle_B$ is defined as bilinear form on the space of *symmetric* polynomials.

The weight $\underline{c}_B$ in the definition of $\langle .,. \rangle_B$ is needed in order to obtain good asymptotic behaviour of the weights. To be more precise, let $j \in \{1,\ldots,n\}$, $\lambda \in P(j-1)$, $\mu \in P(n-j)$ and set $\lambda^{(l)} := (\lambda, l) \in P(j)$ for $l \geq \lambda_{j-1}$, respectively

$\mu^{(m)} := (\mu, m) \in P(n-j+1)$ for $m \geq \mu_{n-j}$. For $j = 1$, respectively $j = n$, this should be read as $\lambda^{(l)} = l \in P(1)$, respectively $\mu^{(m)} = m \in P(1)$. Define

$$\begin{aligned} z^+(l;\lambda,\mu) &:= \big(\rho_B q^{\lambda^{(l)}}, \sigma_B q^{\mu}\big) \in \langle \rho_B\rangle_j \times \langle \sigma_B\rangle_{n-j}, \qquad l \geq \lambda_{j-1}, \\ z^-(m;\lambda,\mu) &:= \big(\rho_B q^{\lambda}, \sigma_B q^{\mu^{(m)}}\big) \in \langle \rho_B\rangle_{j-1} \times \langle \sigma_B\rangle_{n-j+1}, \quad m \geq \mu_{n-j}. \end{aligned} \tag{7.10}$$

Then we have

$$\lim_{l\to\infty} z^+(l;\lambda,\mu) = (\rho_B q^{\lambda}, 0, \sigma_B q^{\mu}), \quad \lim_{m\to\infty} z^-(m;\lambda,\mu) = (\rho_B q^{\lambda}, \sigma_B q^{\mu}, 0)$$

(with the obvious conventions when $j = 1$, respectively $j = n$). We have now good asymptotic behaviour of the weights in the following sense.

Lemma 7.2. *Let $(a,b,c,d) \in V_B$ and $t \in (0,1)$. Then*

$$\lim_{l\to\infty} c_{B,j}\Delta^B(z^+(l;\lambda,\mu)) = \lim_{m\to\infty} c_{B,j-1}\Delta^B(z^-(m;\lambda,\mu)) \tag{7.11}$$

for all $\lambda \in P(j-1)$, $\mu \in P(n-j)$ and $j \in \{1,\dots,n\}$. The conditions (7.11) *for $\lambda \in P(j-1)$, $\mu \in P(n-j)$ and $j \in \{1,\dots,n\}$ characterize the weight $\underline{c}_B$ uniquely up to a multiplicative constant.*

Proof. See the proof of [33, Theorem 6.5]. □

It was proved in [33] that the weights $\Delta^B(z)$ in the bilinear form $\langle .,.\rangle_B$ are strictly positive for $z \in \langle \rho_B, \sigma_B\rangle_n$ and that $\langle f,g\rangle_B$, written out as a multidimensional infinite sum over $\langle \rho_B, \sigma_B\rangle_n$, is absolutely convergent for all $f,g \in A^S_{\mathbb{R}}$.

Definition 7.3. Let $t \in (0,1)$ and $(a,b,c,d) \in V_B$. The multivariable big q-Jacobi polynomials $\{P^B_\lambda(.;a,b,c,d;t)\}_{\lambda\in\Lambda}$ are by definition the unique symmetric polynomials which satisfy

(a) $P^B_\lambda = \tilde{m}_\lambda + \sum_{\mu<\lambda} c^B_{\lambda,\mu}\tilde{m}_\mu$ for certain constants $c^B_{\lambda,\mu} = c^B_{\lambda,\mu}(a,b,c,d;t) \in \mathbb{R}$;
(b) $\langle P^B_\lambda, \tilde{m}_\mu\rangle_B = 0$ for $\mu < \lambda$.

The following proposition is the analogue of Proposition 6.3 in case of the big q-Jacobi polynomials.

Proposition 7.4. *Let $t \in (0,1)$, $(a,b,c,d) \in V_B$ and define for $\epsilon \in \mathbb{R}^*$*

$$\underline{t}_B(\epsilon) := \big(\epsilon^{-1}(qc/d)^{\frac12}, -\epsilon^{-1}(qd/c)^{\frac12}, \epsilon a(qd/c)^{\frac12}, -\epsilon b(qc/d)^{\frac12}\big). \tag{7.12}$$

Then there exists a sequence of positive real numbers $\{\epsilon_k\}_{k\in\mathbb{N}_0}$ which converges to 0*, such that*

$$\begin{aligned} \lim_{k\to\infty} \left(\prod_{i=1}^n \big(-\epsilon_k^{-2}qt^{i-1};q\big)_\infty\right) \big(\epsilon_k(cd/q)^{\frac12}\big)^{|\lambda|+|\mu|} \langle m_\lambda, m_\mu\rangle_{\underline{t}_B(\epsilon_k),t} \\ = 2^n n!\big(q;q\big)_\infty^{-2n}(1-q)^{-n}\langle \tilde{m}_\lambda, \tilde{m}_\mu\rangle^{a,b,c,d}_{B,t} \end{aligned} \tag{7.13}$$

for all $\lambda,\mu \in \Lambda$, where $\langle .,.\rangle_{\underline{t},t}$ is given by (5.11).

The proof will be given in section 9. Note that $\underline{t}_B(\epsilon) \in V_K$ for $\epsilon \in \mathbb{R}_{>0}$ sufficiently small, so $\langle .,.\rangle_{\underline{t}_B(\epsilon),t}$ is well defined and positive definite for $\epsilon > 0$ sufficiently small by Lemma 5.5 and Corollary 5.6.

We can repeat now the arguments of the previous section to establish full orthogonality of the big q-Jacobi polynomials with respect to $\langle .,.\rangle_B$ and to calculate their

norms. For $\lambda \in \Lambda$, $(a,b,c,d) \in V_B$ and $t \in (0,1)$ let $\mathcal{N}^B(\lambda) = \mathcal{N}^B(\lambda; a,b,c,d;t)$ be given by

$$\begin{aligned}\mathcal{N}^B(\lambda) := (cd)^{|\lambda|} q^{\frac{1}{2}\sum_{i=1}^n (\lambda_i - 1 + 2(n-i)\tau)\lambda_i} \mathcal{N}^+_{qJ}(\lambda; a,b;t) \mathcal{N}^-_{qJ}(\lambda; a,b;t) \\ \cdot \prod_{i=1}^n \big(-q^{\lambda_i+1} b t^{n-i} c/d, -q^{\lambda_i+1} a t^{n-i} d/c; q\big)_\infty^{-1}\end{aligned} \tag{7.14}$$

where $\mathcal{N}^+_{qJ}$, respectively $\mathcal{N}^-_{qJ}$, is given by (6.13), respectively (6.14). Observe that $\mathcal{N}^B(\lambda)$ is well defined and positive.

Theorem 7.5. *Let $t \in (0,1)$ and $(a,b,c,d) \in V_B$. There exists a sequence of positive real numbers $\{\epsilon_k\}_{k\in\mathbb{N}_0}$ which converges to 0, such that*

$$\lim_{k\to\infty} \Big(\epsilon_k (cd/q)^{\frac{1}{2}}\Big)^{|\lambda|} P_\lambda\big((q/cd)^{\frac{1}{2}} \epsilon_k^{-1} z; \underline{t}_B(\epsilon_k); t\big) = P^B_\lambda(z; a,b,c,d;t) \tag{7.15}$$

for all $\lambda \in \Lambda$. Furthermore, the polynomials $\{P^B_\lambda\}_{\lambda\in\Lambda}$ are orthogonal with respect to $\langle .,. \rangle_B$ and the quadratic norms of the big q-Jacobi polynomials are given by

$$\langle P^B_\lambda, P^B_\lambda \rangle_B = \mathcal{N}^B(\lambda), \qquad \lambda \in \Lambda. \tag{7.16}$$

Proof. We have the limit

$$\begin{aligned}\lim_{\epsilon\to 0} \left(\prod_{i=1}^n \big(-\epsilon^{-2} q t^{i-1}; q\big)_\infty \right) \big((cd/q)^{\frac{1}{2}} \epsilon\big)^{2|\lambda|} \mathcal{N}(\lambda; \underline{t}_B(\epsilon); t) \\ = 2^n n! \big(q;q\big)_\infty^{-2n} (1-q)^{-n} \mathcal{N}^B(\lambda; a,b,c,d;t).\end{aligned}$$

The proof is now analogous to the proof of Theorem 6.4. □

Full orthogonality of the multivariable big q-Jacobi polynomials was proved in [33] by studying an explicit second order q-difference operator D_B which diagonalizes the big q-Jacobi polynomials. The operator D_B and the corresponding eigenvalue equations can be obtained from (7.15) by taking the limit $k \to \infty$ in the equations

$$\big(\epsilon_k (cd/q)^{\frac{1}{2}}\big)^{|\lambda|} \big((D - E_\lambda) P_\lambda\big)(\epsilon_k^{-1} (q/cd)^{\frac{1}{2}} z; \underline{t}_B(\epsilon_k); t) = 0, \qquad \lambda \in \Lambda, \tag{7.17}$$

where D is given by (2.16) and E_λ is given by (2.19) (see [33, section 3] for the easy computation). In [36] it was shown that the formal computation of the limits of D and E_λ in [33, section 3] can be used to prove the limit transition (7.15) for generic $t \in (0,1)$. See also [34] for the special case that $t = q^k$, $k \in \mathbb{N}$.

It follows from Theorem 7.5 that D_B is symmetric with respect to $\langle .,. \rangle_B$. In [33] the symmetry of D_B was established by direct calculations in which the asymptotic behaviour of the weight function Δ^B (see Lemma 7.2) plays a crucial role.

The quadratic norm evaluations of the multivariable big q-Jacobi polynomials for the special case $a = b = 0$, $c = 1$ and $t = q^k$ with $k \in \mathbb{N}$ were recently proved by Baker and Forrester [7, section 4.3] using Pieri formulas. In order to see that the quadratic norms of the big q-Jacobi polynomials in this special case are in agreement with the quadratic norm evaluations [7, (4.3)], one needs to use the evaluation formula for the Macdonald polynomials [27, (6.11)] together with [22, Proposition 3.2] and the computation preceding [33, Remark 5.4].

The constant term identity for the big q-Jacobi polynomials can be rewritten as follows.

Corollary 7.6. *Let $t \in (0,1)$ and $(a,b,c,d) \in V_B$. We have*

$$\langle 1,1\rangle_{B,t}^{a,b,c,d} = \prod_{j=1}^{n}\Bigg(\frac{\Gamma_q(\alpha+1+(j-1)\tau)\Gamma_q(\beta+1+(j-1)\tau)\Gamma_q(j\tau)}{\Gamma_q(\alpha+\beta+2+(n+j-2)\tau)\Gamma_q(\tau)} \cdot\big(-q^{\alpha+1+(j-1)\tau}d/c, -q^{\beta+1+(j-1)\tau}c/d;q\big)_\infty^{-1}\Bigg). \tag{7.18}$$

The q-Selberg integral (7.18) for $t=q^k$, $k\in\mathbb{N}$, reduces to the following evaluation formula.

Corollary 7.7. *Let $t=q^k$ with $k\in\mathbb{N}$ and $(a,b,c,d)\in V_B$. We have*

$$\int_{z_1=-d}^{c}\cdots\int_{z_n=-d}^{c}\prod_{1\le i<j\le n} z_i^{2k}\big(q^{1-k}\frac{z_j}{z_i};q\big)_{2k}\prod_{i=1}^{n}\frac{\big(qz_i/c,-qz_i/d;q\big)_\infty}{\big(q^{1+\alpha}z_i/c,-q^{1+\beta}z_i/d;q\big)_\infty}d_qz_i$$
$$= q^{k^2\binom{n}{3}-\binom{k}{2}\binom{n}{2}}\prod_{i=1}^{n}\Bigg(\frac{\Gamma_q(\alpha+1+(i-1)k)\Gamma_q(\beta+1+(i-1)k)\Gamma_q(ik+1)}{\Gamma_q(\alpha+\beta+2+(n+i-2)k)\Gamma_q(k+1)} \cdot\frac{\big(-d/c,-c/d;q\big)_\infty(cd)^{1+(i-1)k}}{\big(-q^{\alpha+1+(i-1)k}d/c,-q^{\beta+1+(i-1)k}c/d;q\big)_\infty(c+d)}\Bigg).$$

Proof. The bilinear form $\langle .,.\rangle_B$ differs from the one considered in [33, section 5] by the constant c_B (7.8), and c_B for $t=q^k$ ($k\in\mathbb{N}$) can be rewritten as

$$c_B = \frac{q^{\binom{k}{2}\binom{n}{2}-k^2\binom{n}{3}}(c+d)^n}{\big(-d/c,-c/d;q\big)_\infty^n(cd)^{n+\binom{n}{2}k}}, \qquad t=q^k,\ k\in\mathbb{N}.$$

This follows by a straightforward calculation using the relation $\theta(qx^{-1})=\theta(x)$, (7.9) and $\sum_{i=1}^n(i-1)^2 = 2\binom{n}{3}+\binom{n}{2}$. Hence the corollary follows from Corollary 7.6 and the computation preceding [33, Remark 5.4]. □

The constant term identity for the multivariable big q-Jacobi polynomials has appeared in the literature before. Corollary 7.7 was conjectured by Askey [4] and proved by Evans [15]. For arbitrary $t\in(0,1)$ the evaluation (7.18) is equivalent to Tarasov's and Varchenko's summation formula [38, Theorem (E.10)]. The proof of Tarasov and Varchenko is by computing residues for an A type generalization of Askey-Roy's q-beta integral. The equivalence of [38, Theorem (E.10)] with (7.18) can be seen by making the substitution of variables $p=q$, $x=t$, $a=-d$, $b=c$, $\alpha=-qb/d$, $\beta=qa/c$ and $l=n$ in [38, (E.10)] and by applying the formula

$$\Delta^B(\sigma_Bq^{\nu'},\rho_Bq^{\nu}) = \Delta^B(\rho_Bq^{\nu},\sigma_Bq^{\nu'})\prod_{\substack{1\le k\le j\\ 1\le m\le n-j}}\Psi_t(-t^{m-k}d/c)$$

where $\nu\in P(j)$ and $\nu'\in P(n-j)$ (here Ψ_t is given by (7.6)) .

8. Limit of the orthogonality measure (little q-Jacobi case)

In this section a proof of Proposition 6.3 is given for parameters $(a,b)\in V_L$ with $b\neq 0$. The condition $b\neq 0$ is not essential; we only make this assumption because the formulas are more transparent when we may divide by b.

Let $\epsilon\in\mathbb{R}_{>0}$ and set $\rho_{L,j}(\epsilon) := t_{L,0}(\epsilon)t^{j-1} = \epsilon^{-1}q^{\frac12}t^{j-1}$ for $j\in\mathbb{Z}$ (here $\underline{t}_L(\epsilon)$ is given by (6.9)). The parameter $t_{L,1}(\epsilon) = -aq^{\frac12}$ has modulus >1 for all ϵ if $a\in\big(q^{-\frac12},q^{-1}\big)$, so it can also give contributions to the discrete parts of the

symmetric form $\langle .,. \rangle_{\underline{t}_L(\epsilon),t}$ (5.11) in the limit (6.10). We therefore write $\sigma_{L,j} := t_{L,1}t^{j-1} = -aq^{\frac{1}{2}}t^{j-1}$ for $j \in \mathbb{Z}$. Then $F(r;\underline{t}_L(\epsilon);t)$ (5.3) for $\epsilon > 0$ sufficiently small is given by

$$F(r;\underline{t}_L(\epsilon);t) = \bigcup_{l+m=r} D_0(l;\underline{t}_L(\epsilon);t) \times D_1(m;\underline{t}_L(\epsilon);t) \subset \mathbb{C}^r$$

with the set $D_0(l;\underline{t}_L(\epsilon);t)$ (5.2) for $l > 0$ given by

$$\begin{aligned} D_0(l;\underline{t}_L(\epsilon);t) =& \{\rho_L(\epsilon)q^\nu \,|\, \nu \in P_L(l;\epsilon)\}, \\ P_L(l;\epsilon) :=& \{\nu \in P(l) \,|\, \quad |\rho_{L,0}(\epsilon)q^{\nu_l}| > 1\} \end{aligned} \tag{8.1}$$

and with the set $D_1(m;\underline{t}_L(\epsilon);t)$ (5.2) for $m > 0$ given by

$$D_1(m;\underline{t}_L(\epsilon);t) = \begin{cases} \{\sigma_L^m\} & \text{if } |\sigma_{L,m}| > 1, \\ \emptyset & \text{otherwise} \end{cases} \tag{8.2}$$

where $\sigma_L^m := (\sigma_{L,1},\dots,\sigma_{L,m})$. Note, in particular, that $D_1(m;\underline{t}_L(\epsilon);t)$ is independent of ϵ. Using the explicit definition of the symmetric form $\langle .,. \rangle$ (5.11) as given in section 4, as well as the definition for $m_\lambda(z|u)$ (6.11), we can write

$$\begin{aligned} &\left(\prod_{i=1}^n \big(-\epsilon^{-1}qt^{i-1}, -\epsilon^{-1}qat^{i-1};q\big)_\infty\right)\big(\epsilon q^{-1/2}\big)^{|\lambda|+|\mu|}\langle m_\lambda, m_\mu\rangle_{\underline{t}_L(\epsilon),t} \\ &= \sum_{r,l,m,\nu} \iint_{x\in T^{n-r}} \big(m_\lambda m_\mu\big)\big(\rho_L q^\nu, \epsilon q^{-\frac{1}{2}}\sigma_L^m, \epsilon q^{-\frac{1}{2}}x|\epsilon q^{-\frac{1}{2}}\big)\mathcal{W}^L_{l,m;r}(\nu,x;\epsilon)\frac{dx}{x} \end{aligned} \tag{8.3}$$

where the sum is over four tuples (r,l,m,ν) with $r \in \{0,\dots,n\}$, $l,m \in \mathbb{N}_0$ with $l+m=r$, and $\nu \in P(l)$ (the sum over $\nu \in P(l)$ should be ignored when $l=0$). The renormalized weights $\mathcal{W}^L_{l,m;r}(\nu,x;\epsilon)$ are given by $\mathcal{W}^L_{0,0;0}(-;x;\epsilon) := \Delta(x;\underline{t}_L(\epsilon);t)$ for $r=0$, and for $r=1,\dots,n$,

$$\begin{aligned} \mathcal{W}^L_{l,m;r}(\nu,x;\epsilon) =& \prod_{i=1}^n \big(-\epsilon^{-1}qt^{i-1}, -\epsilon^{-1}qat^{i-1};q\big)_\infty \\ & \cdot \frac{2^r\big(n-r+1\big)_r}{(2\pi i)^{n-r}}\Delta_r^K\big(\rho_L(\epsilon)q^\nu, \sigma_L^m, x;\underline{t}_L(\epsilon);t\big) \end{aligned}$$

if $\big(\rho_L(\epsilon)q^\nu, \sigma_L^m\big) \in F(r;\underline{t}_L(\epsilon);t)$ and zero otherwise. We split the renormalized weights in three parts:

$$\mathcal{W}^L_{l,m;r}(\nu,x;\epsilon) = \frac{2^r\big(n-r+1\big)_r}{(2\pi i)^{n-r}}\Delta^{KL}_{1,l}(\nu;\epsilon)\Delta^{KL}_{2,l,m}(\nu;\epsilon)\Delta^{KL}_{3,l,m}(\nu,x;\epsilon) \tag{8.4}$$

where $\Delta^{KL}_{1,l}$, $\Delta^{KL}_{2,l,m}$ and $\Delta^{KL}_{3,l,m}$ given by

$$\Delta^{KL}_{1,l}(\nu;\epsilon) := \left(\prod_{i=1}^l \big(-\epsilon^{-1}qt^{i-1}, -\epsilon^{-1}qat^{i-1};q\big)_\infty\right)\Delta^{(d)}\big(\rho_L(\epsilon)q^\nu;t_{L,0}(\epsilon)\big)$$

if $\nu \in P_L(l;\epsilon)$ and zero otherwise;

$$\begin{aligned} \Delta^{KL}_{2,l,m}(\nu;\epsilon) :=& \prod_{i=1}^m \big(-\epsilon^{-1}qt^{l+i-1}, -\epsilon^{-1}qat^{l+i-1};q\big)_\infty \\ & \cdot \Delta^{(d)}\big(\sigma_L^m;t_{L,1}(\epsilon)\big)\delta_c\big(\rho_L(\epsilon)q^\nu;\sigma_L^m\big) \end{aligned}$$

if $\nu \in P_L(l;\epsilon)$, $D_1(m;\underline{t}_L(\epsilon);t) \neq \emptyset$ and zero otherwise;

$$\Delta^{KL}_{3,l,m}(\nu,x;\epsilon) := \prod_{i=1}^{n-r}\big(-\epsilon^{-1}qt^{r+i-1},-\epsilon^{-1}qat^{r+i-1};q\big)_\infty \\ .\Delta(x;\underline{t}_L(\epsilon);t)\delta_c\big(\rho_L(\epsilon)q^\nu;x\big)\delta_c(\sigma_L^m;x)$$

if $\nu \in P_L(l;\epsilon)$, $D_1(m;\underline{t}_L(\epsilon);t) \neq \emptyset$ and zero otherwise, where $r = l+m$ and Δ is given by (2.6), $\Delta^{(d)}$ is given by (3.5) and δ_c is given by (3.12). The formula $\delta_c(z;u,v) = \delta_c(z;u)\delta_c(z;v)$ is used for obtaining (8.4). We have used for the definitions of $\Delta^{KL}_{1,l}$, $\Delta^{KL}_{2,l,m}$ and $\Delta^{KL}_{3,l,m}$ the obvious conventions when $l = 0$ or $m = 0$; for instance,

$$\Delta^{KL}_{2,0,0}(-;\epsilon) = 1, \quad \Delta^{KL}_{2,l,0}(\nu;\epsilon) = 1,$$
$$\Delta^{KL}_{2,0,m}(-;\epsilon) = \prod_{i=1}^{m}\big(-\epsilon^{-1}qt^{i-1},-\epsilon^{-1}qat^{i-1};q\big)_\infty\Delta^{(d)}\big(\sigma_L^m;t_{L,1}(\epsilon);t\big)$$

for l,m,ν such that $l,m > 0$, $\nu \in P(l;\epsilon)$ and $D(\sigma_L^m;t_{L,1}(\epsilon);t) \neq 0$.

We will use Lebesgue's dominated convergence theorem to pull a limit $\epsilon_k \downarrow 0$ in the right-hand side of (8.3) through the integration over $x \in T^{n-r}$ and through the infinite sum over $\nu \in P(l)$ for some sequence of positive real numbers $\{\epsilon_k\}_{k\in\mathbb{N}_0}$ converging to 0. For the application of Lebesgue's dominated convergence theorem we need certain estimates for the functions $\Delta^{KL}_{1,l}$, $\Delta^{KL}_{2,l,m}$ and $\Delta^{KL}_{3,l,m}$, which are given in the following lemma.

Lemma 8.1. *Keep the notations and conventions as above. In particular, let $l,m \in \mathbb{N}_0$ with $l+m \leq n$, and write $r := l+m$. Then there exists a sequence of positive real numbers $\{\epsilon_k\}_{k\in\mathbb{N}_0}$ which converges to 0, such that:*

(i) *If $l \in \mathbb{N}$, then for all $\nu \in P(l)$,*

$$\lim_{k\to\infty}\Delta^{KL}_{1,l}(\nu;\epsilon_k) = \big(q;q\big)_\infty^{-2l}\Delta^L(\rho_Lq^\nu;a,b;t)\prod_{i=1}^{l}\rho_{L,i}q^{\nu_i},$$

and there exists a $K \in \mathbb{R}_{>0}$ independent of $\nu \in P(l)$ such that

$$\sup_{k\in\mathbb{N}_0}|\Delta^{KL}_{1,l}(\nu;\epsilon_k)| \leq K\Delta^L\big(\rho_Lq^\nu;a,b;t\big)\prod_{i=1}^{l}\rho_{L,i}q^{\nu_i} \tag{8.5}$$

for all $\nu \in P(l)$.

(ii) *If $m \in \mathbb{N}$, then $\lim_{k\to\infty}\Delta^{KL}_{2,l,m}(\nu;\epsilon_k) = 0$ for all $\nu \in P(l)$ and*

$$\sup_{(\nu,k)\in P(l)\times\mathbb{N}_0}|\Delta^{KL}_{2,l,m}(\nu;\epsilon_k)| < \infty.$$

(iii) *If $r < n$, then $\lim_{k\to\infty}\Delta^{KL}_{3,l,m}(\nu,x;\epsilon_k) = 0$ for all $x \in T^{n-r}$, $\nu \in P(l)$ and*

$$\sup_{(\nu,x,k)\in P(l)\times T^{n-r}\times\mathbb{N}_0}|\Delta^{KL}_{3,l,m}(\nu,x;\epsilon_k)| < \infty.$$

Before giving a proof of Lemma 8.1, we first complete the proof of Proposition 6.3. Since the infinite sum

$$(1-q)^{-n}\langle 1,1\rangle^{a,b}_{L,t} = \sum_{\nu\in P(n)}\Delta^L(\rho_Lq^\nu;a,b;t)\prod_{i=1}^{n}\rho_{L,i}q^{\nu_i} \tag{8.6}$$

is absolutely convergent (cf. [33, proof of Proposition 6.1]) and

$$\sup_{\nu,\epsilon,x} |m_\lambda\big(\rho_L q^\nu, \epsilon q^{-\frac{1}{2}}\sigma_L^m, \epsilon q^{-\frac{1}{2}}x|\epsilon q^{-\frac{1}{2}}\big)| < \infty, \qquad \lambda \in \Lambda,$$

where the supremum is taken over triples (ν,ϵ,x) with $\nu \in P_L(l;\epsilon)$, $\epsilon \in \mathbb{R}_{>0}$ and $x \in T^{n-r}$, it follows by Lebesgue's dominated convergence theorem, (6.12), (8.3), (8.4) and Lemma 8.1 that

$$\begin{aligned}
&\lim_{k\to\infty}\left(\prod_{i=1}^n \big(-\epsilon_k^{-1}qt^{i-1}, -\epsilon_k^{-1}qat^{i-1};q\big)_\infty\right)\big(\epsilon_k q^{-\frac{1}{2}}\big)^{|\lambda|+|\mu|}\langle m_\lambda, m_\mu\rangle_{\underline{t}_L(\epsilon_k),t}\\
&\quad= \sum_{r,l,m,\nu}\iint_{x\in T^{n-r}} \lim_{k\to\infty}\big(m_\lambda m_\mu\big)\big(\rho_L q^\nu, \epsilon_k q^{-\frac{1}{2}}\sigma_L^m, \epsilon_k q^{-\frac{1}{2}}x|\epsilon_k q^{-\frac{1}{2}}\big)\mathcal{W}^L_{l,m;r}(\nu,x;\epsilon_k)\frac{dx}{x}\\
&\qquad= 2^n n!\big(q;q\big)_\infty^{-2n}\sum_{\nu\in P(n)}\big(\tilde m_\lambda \tilde m_\mu\big)(\rho_L q^\nu)\Delta^L(\rho_L q^\nu;a,b;t)\prod_{i=1}^n \rho_{L,i}q^{\nu_i}\\
&\qquad\qquad= 2^n n!(1-q)^{-n}\big(q;q\big)_\infty^{-2n}\langle \tilde m_\lambda, \tilde m_\mu\rangle^{a,b}_{L,t}
\end{aligned}$$

for some sequence of positive real numbers $\{\epsilon_k\}_{k\in\mathbb{N}_0}$ converging to 0, where the sum in the second line is over four tuples (r,l,m,ν) with $r \in \{0,\dots,n\}$, $l,m \in \mathbb{N}_0$ with $l+m=r$, and $\nu \in P(l)$. So for the proof of Proposition 6.3, it suffices to prove Lemma 8.1. We use the following elementary lemma.

Lemma 8.2 ([37, Lemma 3.1])**.** *For given $\epsilon_0 \in \mathbb{R}_{>0}$, we set $\epsilon_k := \epsilon_0 q^k$.*

(a) *Let $c \in \mathbb{C}$. For $\epsilon_0 \in \mathbb{R}_{>0}$ with $|c|\epsilon_0 \notin \{q^{-l}\}_{l\in\mathbb{N}_0}$ there exist positive constants $K^\pm > 0$ which only depend on ϵ_0 and $|c|$, such that $K^- \le |\big(c\epsilon_k;q\big)_\infty| \le K^+$ for all $k \in \mathbb{N}_0$. Furthermore, we have $\lim_{k\to\infty}\big(c\epsilon_k;q\big)_\infty = 1$.*

(b) *Let $a,b \in \mathbb{C}^*$, and set*

$$f_{\{l,m\}}(\epsilon;a,b) := \frac{\big(\epsilon^{-1}aq^{1-m};q\big)_m}{\big(\epsilon^{-1}bq^{1-m-l};q\big)_m}, \qquad l,m \in \mathbb{N}_0. \tag{8.7}$$

Let $\epsilon_0 \in \mathbb{R}_{>0}$ such that $\epsilon_0^{-1}|b| \notin \{q^k\}_{k\in\mathbb{N}_0}$. Then there exists a positive constant $K > 0$ which depends only on ϵ_0, $|a|$ and $|b|$, such that $|f_{\{l,m\}}(\epsilon_k;a,b)| \le K|q^l a/b|^m$ for all $k,l,m \in \mathbb{N}_0$. Furthermore, we have $\lim_{k\to\infty} f_{\{l,m\}}(\epsilon_k;a,b) = (q^l a/b)^m$.

(c) *Let $u_i, v_j \in \mathbb{C}^*$ for $i \in \{1,\dots,r\}$, $j \in \{1,\dots,s\}$ and assume that $r < s$, or that $r = s$ and $|u_1\dots u_r| < |v_1\dots v_r|$. Set*

$$g(\epsilon) := \frac{\big(\epsilon^{-1}u_1,\dots,\epsilon^{-1}u_r;q\big)_\infty}{\big(\epsilon^{-1}v_1,\dots,\epsilon^{-1}v_s;q\big)_\infty}. \tag{8.8}$$

Let $\epsilon_0 \in \mathbb{R}_{>0}$ such that $\epsilon_0^{-1}|v_j| \notin \{q^l\}_{l\in\mathbb{Z}}$ for $j \in \{1,\dots,s\}$. Then there exists a positive constant $K > 0$ which depends only on ϵ_0, $|u_i|$ and $|v_j|$, such that $\sup_{k\in\mathbb{N}_0}|g(\epsilon_k)| \le K$. Furthermore, we have $\lim_{k\to\infty} g(\epsilon_k) = 0$.

The proofs of **(b)** and **(c)** are based on the formula (4.3) for q-shifted factorials. See [37, Lemma 3.1] for details.

We proceed with the proof of Lemma 8.1. We use the notation $\epsilon_k := \epsilon_0 q^k$ for given $\epsilon_0 \in \mathbb{R}_{>0}$.

Proof of Lemma 8.1(**i**). By (3.5) one has

$$\Delta_{1,l}^{KL}(\nu;\epsilon) = \delta_d(\rho_L(\epsilon)q^\nu)\prod_{i=1}^{l}\big\{\big(-\epsilon^{-1}qt^{i-1},\ -\epsilon^{-1}qat^{i-1};q\big)_\infty \cdot w_d(\rho_{L,i}(\epsilon)q^{\nu_i};\rho_{L,i}(\epsilon)q^{\nu_{i-1}})\big\} \tag{8.9}$$

with δ_d given by (3.7) and w_d given by (3.9). By (3.7) and (6.9), we have

$$\delta_d(\rho_L(\epsilon)q^\nu) = F_1(\nu)G_1(\nu;\epsilon) \tag{8.10}$$

with

$$\begin{aligned} F_1(\nu) &:= \prod_{1\le i<j\le l}\frac{\big(t^{j-i}q^{\nu_j-\nu_i};q\big)_\tau}{\big(t^{i-j}q^{\nu_{i-1}-\nu_j};q\big)_{\nu_i-\nu_{i-1}}},\\ G_1(\nu;\epsilon) &:= \prod_{1\le i<j\le l}\frac{\big(\epsilon^2t^{2-i-j}q^{-\nu_i-\nu_j-1};q\big)_\tau}{\big(\epsilon^{-2}t^{i+j-2}q^{\nu_{i-1}+\nu_j+1};q\big)_{\nu_i-\nu_{i-1}}}, \end{aligned} \tag{8.11}$$

for $\nu\in P(l)$, where $\nu_0=0$. By applying (4.3) to the q-shifted factorials in the denominator of F_1 and using the formula

$$\sum_{i=1}^{l}(i-1)(l-i) = \binom{l}{3}, \tag{8.12}$$

we obtain

$$\begin{aligned} F_1(\nu) =&\delta_{qJ}(\rho_Lq^\nu)q^{-2\tau^2\binom{l}{3}}\prod_{j=1}^{l}\frac{\big(t^{j-1}q^{\nu_j+1};q\big)_\infty}{\big(q^{\nu_j-\nu_{j-1}+1};q\big)_\infty}t^{-2(l-j)\nu_j}\\ &\cdot\prod_{1\le i<j\le l}\big(-q^{\nu_j-\nu_i+1}t^{j-i}\big)^{\nu_i-\nu_{i-1}}q^{\binom{\nu_i-\nu_{i-1}}{2}}, \end{aligned} \tag{8.13}$$

where δ_{qJ} (6.8) is the interaction factor for the weight function of the little q-Jacobi polynomials. On the other hand, we have for $i=1,\dots,l$ by (3.9),

$$\big(-\epsilon^{-1}qt^{i-1},-\epsilon^{-1}qat^{i-1};q\big)_\infty w_d(\rho_{L,i}(\epsilon)q^{\nu_i};\rho_{L,i}(\epsilon)q^{\nu_{i-1}}) = I_{1,i}(\nu)J_{1,i}(\nu;\epsilon) \tag{8.14}$$

with

$$\begin{aligned} I_{1,i}(\nu) :=&\frac{\big(t^{i-1}q^{\nu_{i-1}+1}ab\big)^{\nu_{i-1}-\nu_i}}{\big(q,qbt^{i-1}q^{\nu_i};q\big)_\infty\big(q;q\big)_{\nu_i-\nu_{i-1}}}\\ =&v_L(\rho_{L,i}q^{\nu_i})\frac{a^{(1-i)\tau-\nu_i}\big(t^{i-1}q^{\nu_{i-1}+1}ab\big)^{\nu_{i-1}-\nu_i}}{\big(q,t^{i-1}q^{\nu_i+1};q\big)_\infty\big(q;q\big)_{\nu_i-\nu_{i-1}}} \end{aligned}$$

(here v_L (6.7) is the one-variable weight function of the little q-Jacobi polynomials) and with $J_{1,i}(\nu;\epsilon)$ given by

$$\begin{aligned} J_{1,i}(\nu;\epsilon) :=&\frac{\big(\epsilon^2t^{2-2i}q^{-2\nu_{i-1}-1};q\big)_\infty}{\big(\epsilon^2bt^{1-i}q^{-\nu_{i-1}},-\epsilon t^{1-i}q^{-\nu_{i-1}},-\epsilon at^{1-i}q^{-\nu_{i-1}};q\big)_\infty}\\ &\cdot\frac{\big(\epsilon^{-2}q^{1+2\nu_{i-1}}t^{2i-2};q\big)_{\nu_i-\nu_{i-1}}\big(-\epsilon^{-1}t^{i-1}q;q\big)_{\nu_{i-1}}\big(-\epsilon^{-1}at^{i-1}q;q\big)_{\nu_i}}{\big(\epsilon^{-2}b^{-1}t^{i-1}q^{\nu_{i-1}+1},-\epsilon^{-1}a^{-1}t^{i-1}q^{\nu_{i-1}+1};q\big)_{\nu_i-\nu_{i-1}}}\\ &\cdot\frac{\big(1-\epsilon^{-2}q^{1+2\nu_i}t^{2i-2}\big)}{\big(1-\epsilon^{-2}q^{1+2\nu_{i-1}}t^{2i-2}\big)}. \end{aligned}$$

So we have by (8.9), (8.10), (8.13) and (8.14) that $\Delta^{KL}_{1,l}(\nu;\epsilon) = M_1(\nu)N_1(\nu;\epsilon)$ for $\epsilon \in \mathbb{R}_{>0}$ and $\nu \in P_L(l;\epsilon)$ with

$$\begin{aligned} M_1(\nu) :=& F_1(\nu)\prod_{i=1}^{l} I_{1,i}(\nu) \\ =& \Delta^L(\rho_L q^\nu)\big(q;q\big)_\infty^{-2l} t^{\binom{l}{2}} \prod_{i=1}^{l}\big(t^{-2(l-i)}a^{-1}\big)^{\nu_i}\big(t^{i-1}q^{\nu_{i-1}+1}ab\big)^{\nu_{i-1}-\nu_i} \\ &\cdot \prod_{1\leq i<j\leq l}\big(-q^{\nu_j-\nu_i+1}t^{j-i}\big)^{\nu_i-\nu_{i-1}} q^{\binom{\nu_i-\nu_{i-1}}{2}} \end{aligned} \tag{8.15}$$

(Δ^L given by (6.6)) and with

$$N_1(\nu;\epsilon) := G_1(\nu;\epsilon)\prod_{i=1}^{l} J_{1,i}(\nu;\epsilon).$$

Now replace the factor $\big(-\epsilon^{-1}at^{i-1}q;q\big)_{\nu_i}$ in $J_{1,i}(\nu;\epsilon)$ by

$$\big(-\epsilon^{-1}at^{i-1}q^{\nu_{i-1}+1};q\big)_{\nu_i-\nu_{i-1}}\big(-\epsilon^{-1}at^{i-1}q;q\big)_{\nu_{i-1}}$$

for $i \in \{1,\dots,l\}$. Then $N_1(\nu;\epsilon)$ can explicitly be given by

$$N_1(\nu;\epsilon) = N_1^1(\nu;\epsilon)N_1^2(\nu;\epsilon)N_1^3(\nu;\epsilon) \tag{8.16}$$

with

$$\begin{aligned} N_1^1(\nu;\epsilon) :=& \prod_{i=1}^{l}\frac{\big(\epsilon^2 t^{2-2i}q^{-2\nu_{i-1}-1};q\big)_\infty}{\big(\epsilon^2 bt^{1-i}q^{-\nu_{i-1}}, -\epsilon t^{1-i}q^{-\nu_{i-1}}, -\epsilon at^{1-i}q^{-\nu_{i-1}};q\big)_\infty} \\ &\cdot \prod_{1\leq i<j\leq l}\big(\epsilon^2 t^{2-i-j}q^{-\nu_i-\nu_j-1};q\big)_\tau \end{aligned} \tag{8.17}$$

if $\nu \in P_L(l;\epsilon)$ and zero otherwise,

$$\begin{aligned} N_1^2(\nu;\epsilon) :=& \prod_{i=1}^{l}\Bigg(\frac{\big(\epsilon^{-2}q^{1+2\nu_{i-1}}t^{2i-2}, -\epsilon^{-1}at^{i-1}q^{1+\nu_{i-1}};q\big)_{\nu_i-\nu_{i-1}}}{\big(\epsilon^{-2}b^{-1}t^{i-1}q^{1+\nu_{i-1}}, -\epsilon^{-1}a^{-1}t^{i-1}q^{1+\nu_{i-1}};q\big)_{\nu_i-\nu_{i-1}}} \\ &\cdot\frac{\big(\epsilon^{-2}q^{1+2\nu_i}t^{2i-2};q\big)_1}{\big(\epsilon^{-2}q^{1+2\nu_{i-1}}t^{2i-2};q\big)_1}\Bigg) \end{aligned} \tag{8.18}$$

if $\nu \in P_L(l;\epsilon)$ and zero otherwise, and

$$N_1^3(\nu;\epsilon) = \frac{\prod_{i=1}^{l}\big(-\epsilon^{-1}t^{i-1}q, -\epsilon^{-1}at^{i-1}q;q\big)_{\nu_{i-1}}}{\prod_{1\leq i<j\leq l}\big(\epsilon^{-2}t^{i+j-2}q^{\nu_{i-1}+\nu_j+1};q\big)_{\nu_i-\nu_{i-1}}} \tag{8.19}$$

if $\nu \in P_L(l;\epsilon)$ and zero otherwise. For the factor N_1^1 we have, for generic $\epsilon_0 > 0$,

$$\lim_{k\to\infty} N_1^1(\nu;\epsilon_k) = 1, \qquad \nu \in P(l), \tag{8.20}$$

by Lemma 8.2**(a)**. For the factor N_1^2 we can use Lemma 8.2**(b)** to calculate the limit. We obtain, for generic $\epsilon_0 > 0$,

$$\lim_{k\to\infty} N_1^2(\nu;\epsilon_k) = \prod_{i=1}^{l}\big(q^{\nu_{i-1}+2}t^{i-1}a^2b\big)^{\nu_i-\nu_{i-1}} \tag{8.21}$$

for all $\nu \in P(l)$. As an example, let us calculate explicitly the limit of a factor of N_1^2, using Lemma 8.2**(b)**. Consider the factor

$$N_1^{i,2}(\nu;\epsilon) := \frac{\left(\epsilon^{-2}q^{1+2\nu_{i-1}}t^{2i-2};q\right)_{\nu_i-\nu_{i-1}}}{\left(\epsilon^{-2}q^{1+\nu_{i-1}}b^{-1}t^{i-1};q\right)_{\nu_i-\nu_{i-1}}} \tag{8.22}$$

of $N_1^2(\nu;\epsilon)$ for some $i \in \{1,\dots,l\}$. Then for generic $\epsilon_0 > 0$, we obtain by Lemma 8.2**(b)**

$$\begin{aligned}\lim_{k\to\infty} N_1^{i,2}(\nu;\epsilon_k) &= \lim_{k\to\infty} N_1^{i,2}(\nu;\epsilon_{k+\nu_l})\\ &= \lim_{k\to\infty} N_1^{i,2}(\nu;q^{\nu_l}\epsilon_k)\\ &= \lim_{k\to\infty} f_{\{\nu_{i-1},\nu_i-\nu_{i-1}\}}\left(\epsilon_{2\nu_l-\nu_i-\nu_{i-1}+2k};\epsilon_0^{-1}t^{2i-2},\epsilon_0^{-1}b^{-1}t^{i-1}\right)\\ &= \left(q^{\nu_{i-1}}t^{i-1}b\right)^{\nu_i-\nu_{i-1}}.\end{aligned} \tag{8.23}$$

The limits of the other factors of N_1^2 can be computed in a similar manner, which yield (8.21). Finally, we have, for generic $\epsilon_0 > 0$,

$$\begin{aligned}\lim_{k\to\infty} N_1^3(\nu;\epsilon_k) = &\prod_{i=1}^{l}\left(at^{2(i-1)}q^{\nu_{i-1}+1}\right)^{\nu_{i-1}}\\ &\cdot\prod_{1\le i<j\le l}\left(-t^{i+j-2}q^{\nu_{i-1}+\nu_j+1}\right)^{\nu_{i-1}-\nu_i}q^{-\binom{\nu_i-\nu_{i-1}}{2}}\end{aligned} \tag{8.24}$$

since

$$\sum_{i=1}^{l}\nu_{i-1} = \sum_{1\le i<j\le l}(\nu_i-\nu_{i-1}), \qquad \nu\in P(l). \tag{8.25}$$

We thus obtain for generic $\epsilon_0 > 0$ by (8.16), (8.20), (8.21) and (8.24)

$$\begin{aligned}\lim_{k\to\infty} N_1(\nu;\epsilon_k) = &\prod_{i=1}^{l}t^{(i-1)(\nu_i+\nu_{i-1})}q^{\nu_{i-1}\nu_i-\nu_{i-1}+2\nu_i}a^{2\nu_i-\nu_{i-1}}b^{\nu_i-\nu_{i-1}}\\ &\cdot\prod_{1\le i<j\le l}\left(-t^{i+j-2}q^{\nu_{i-1}+\nu_j+1}\right)^{\nu_{i-1}-\nu_i}q^{-\binom{\nu_i-\nu_{i-1}}{2}}\end{aligned} \tag{8.26}$$

for all $\nu \in P(l)$. By (8.15) and (8.26) we obtain, for generic $\epsilon_0 > 0$,

$$\begin{aligned}\lim_{k\to\infty}\Delta_{1,l}^{KL}(\nu;\epsilon_k) &= M_1(\nu)\lim_{k\to\infty}N_1(\nu;\epsilon_k)\\ &= \left(q;q\right)_\infty^{-2l}\Delta^L(\rho_L q^\nu)t^{\binom{l}{2}}q^{|\nu|}\\ &= \left(q;q\right)_\infty^{-2l}\Delta^L(\rho_L q^\nu)\prod_{i=1}^{l}\rho_{L,i}q^{\nu_i}\end{aligned}$$

for all $\nu \in P(l)$, where Δ^L is given by (6.6) and $|\nu| := \nu_1+\cdots+\nu_l$ for $\nu \in P(l)$.

To prove the estimate (8.5), we use the estimates of Lemma 8.2 **(a)** and **(b)** for (factors of) N_1. For N_1^1, we use Lemma 8.2**(a)** and the condition that $N_1^1(\nu;\epsilon) = 0$ if $\nu \notin P_L(l;\epsilon)$ to prove that $\sup_{\nu,k}|N_1^1(\nu;\epsilon_k)| < \infty$ for generic $\epsilon_0 > q^{\frac{1}{2}}$. Indeed, since $\nu \in P_L(l;\epsilon)$ implies $\epsilon < q^{\frac{1}{2}+\nu_l}$, we have, for $\epsilon_0 > q^{\frac{1}{2}}$,

$$\sup_{(\nu,k)\in P(l)\times\mathbb{N}_0}|N_1^1(\nu;\epsilon_k)| = \sup_{(\nu,k)\in P(l)\times\mathbb{N}_0}|N_1^1(\nu;\epsilon_k q^{\nu_l})| < \infty \tag{8.27}$$

by (8.17) and Lemma 8.2**(a)**.

For $N_1^2(\nu;\epsilon)$ we want to establish the estimate

$$\sup_{k\in\mathbb{N}_0}|N_1^2(\nu;\epsilon_k)|\leq K_1^2\prod_{i=1}^{l}\bigl(q^{\nu_{i-1}+2}t^{i-1}a^2|b|\bigr)^{\nu_i-\nu_{i-1}} \tag{8.28}$$

for generic $\epsilon_0>q^{\frac{1}{2}}$ with $K_1^2>0$ independent of $\nu\in P(l)$, in view of the limit (8.21). This can be done with the help of Lemma 8.2**(b)**. As an example, we consider the factor $N_1^{i,2}(\nu;\epsilon)$ (8.22). In view of the limit (8.23), we want to prove the estimate

$$\sup_{k\in\mathbb{N}_0}|N_1^{i,2}(\nu;\epsilon_k)|\leq K_1^{i,2}\bigl(q^{\nu_{i-1}}t^{i-1}|b|\bigr)^{\nu_i-\nu_{i-1}}$$

for generic $\epsilon_0>q^{\frac{1}{2}}$ with $K_1^{i,2}>0$ independent of $\nu\in P(l)$. This follows for generic $\epsilon_0>q^{\frac{1}{2}}$, using the fact that $N_1^{i,2}(\nu;\epsilon)=0$ if $\nu\notin P_L(l;\epsilon)$, by the estimates

$$\begin{aligned}\sup_{k\in\mathbb{N}_0}|N_1^{i,2}(\nu;\epsilon_k)|&=\sup_{k\in\mathbb{N}_0}|N_1^{i,2}(\nu;q^{\nu_l}\epsilon_k)|\\&=\sup_{k\in\mathbb{N}_0}|f_{\{\nu_{i-1},\nu_i-\nu_{i-1}\}}\bigl(\epsilon_{2\nu_l-\nu_i-\nu_{i-1}+2k};\epsilon_0^{-1}t^{2i-2},\epsilon_0^{-1}b^{-1}t^{i-1}\bigr)|\\&\leq\sup_{k\in\mathbb{N}_0}|f_{\{\nu_{i-1},\nu_i-\nu_{i-1}\}}\bigl(\epsilon_k;\epsilon_0^{-1}t^{2i-2},\epsilon_0^{-1}b^{-1}t^{i-1}\bigr)|\\&\leq K_1^{i,2}\bigl(q^{\nu_{i-1}}t^{i-1}|b|\bigr)^{\nu_i-\nu_{i-1}}\end{aligned}$$

with $K_1^{i,2}$ independent of ν by Lemma 8.2**(b)**. Estimates can be given for the other factors of N_1^2 in a similar manner.

For N_1^3, we want to prove that

$$\begin{aligned}\sup_{k\in\mathbb{N}_0}|N_1^3(\nu;\epsilon_k)|\leq&K_1^3\prod_{i=1}^{l}\bigl(at^{2(i-1)}q^{\nu_{i-1}+1}\bigr)^{\nu_{i-1}}\\&\cdot\prod_{1\leq i<j\leq l}\bigl(t^{i+j-2}q^{\nu_{i-1}+\nu_j+1}\bigr)^{\nu_{i-1}-\nu_i}q^{-\binom{\nu_i-\nu_{i-1}}{2}}\end{aligned} \tag{8.29}$$

for generic $\epsilon_0>q^{\frac{1}{2}}$ with $K_1^3>0$ independent of $\nu\in P(l)$, in view of the limit (8.24). This follows by straightforward estimates, using (4.3) and the fact that $N_1^3(\nu;\epsilon)=0$ if $\nu\notin P_L(l;\epsilon)$. Hence by (8.16), (8.27), (8.28) and (8.29) we have the estimate

$$\begin{aligned}\sup_{k\in\mathbb{N}_0}|N_1(\nu;\epsilon_k)|\leq&K_1\prod_{i=1}^{l}t^{(i-1)(\nu_i+\nu_{i-1})}q^{\nu_{i-1}\nu_i-\nu_{i-1}+2\nu_i}a^{2\nu_i-\nu_{i-1}}|b|^{\nu_i-\nu_{i-1}}\\&\cdot\prod_{1\leq i<j\leq l}\bigl(t^{i+j-2}q^{\nu_{i-1}+\nu_j+1}\bigr)^{\nu_{i-1}-\nu_i}q^{-\binom{\nu_i-\nu_{i-1}}{2}}\end{aligned} \tag{8.30}$$

for generic $\epsilon_0>q^{\frac{1}{2}}$, with $K_1>0$ independent of $\nu\in P(l)$, so in particular,

$$\sup_{k\in\mathbb{N}_0}|\Delta_{1,l}^{KL}(\nu;\epsilon_k)|=|M_1(\nu)|\sup_{k\in\mathbb{N}_0}|N_1(\nu;\epsilon_k)|\leq K\Delta^L(\rho_Lq^{\nu})\prod_{i=1}^{l}\rho_{L,i}q^{\nu_i}$$

with $K>0$ independent of $\nu\in P(l)$. This completes the proof of Lemma 8.1**(i)**. $\square$

*Proof of Lemma 8.1***(ii)**. Observe that $\delta_c(\rho_L(\epsilon);\sigma_L^m)$ (δ_c given by (3.12)) can be rewritten as

$$\begin{aligned}\delta_c(\rho_L(\epsilon)q^\nu;\sigma_L^m) = &\prod_{\substack{1\le i\le l\\ 1\le j\le m}} \left(-\epsilon a q^{-\nu_i}t^{j-i}, -\epsilon a^{-1}q^{-1-\nu_i}t^{2-i-j};q\right)_\tau \\ &\cdot \prod_{j=1}^m \frac{\left(-\epsilon^{-1}aqt^{j-1}, -\epsilon^{-1}a^{-1}t^{1-j};q\right)_\infty}{\left(-\epsilon^{-1}aqt^{l+j-1}, -\epsilon^{-1}a^{-1}t^{l-j+1};q\right)_\infty} \\ &\cdot \prod_{\substack{0\le i\le l-1\\ 1\le j\le m}} \frac{\left(-\epsilon^{-1}aq^{1+\nu_i}t^{l+j-1}, -\epsilon^{-1}a^{-1}q^{\nu_i}t^{l-j+1};q\right)_{\nu_{i+1}-\nu_i}}{\left(-\epsilon^{-1}aq^{1+\nu_i}t^{i+j-1}, -\epsilon^{-1}a^{-1}q^{\nu_i}t^{i-j+1};q\right)_{\nu_{i+1}-\nu_i}}\end{aligned} \tag{8.31}$$

where $\nu_0 = 0$. By the explicit expression for $\Delta^{(d)}$ (3.5) and for the parameters $\underline{t}_L(\epsilon)$ (6.9) we then obtain

$$\Delta^{KL}_{2,l,m}(\nu;\epsilon) = N_2^1 N_2^2(\nu;\epsilon) N_2^3(\epsilon) N_2^4(\nu;\epsilon) \tag{8.32}$$

with

$$N_2^1 := \prod_{i=1}^m \frac{\left(a^{-2}q^{-1}t^{2(1-i)};q\right)_\infty}{\left(q, aqt^{i-1}, a^{-1}t^{1-i};q\right)_\infty} \prod_{1\le i<j\le m} \left(t^{j-i}, a^{-2}t^{2-i-j}q^{-1};q\right)_\tau,$$

$$N_2^2(\nu;\epsilon) := \prod_{j=1}^m \frac{\prod_{i=1}^l \left(-\epsilon a q^{-\nu_i}t^{j-i}, -\epsilon a^{-1}q^{-1-\nu_i}t^{2-i-j};q\right)_\tau}{\left(-\epsilon a^{-1}bt^{1-j}, -\epsilon abqt^{j-1};q\right)_\infty}$$

if $\nu \in P_L(l;\epsilon)$ and zero otherwise,

$$N_2^3(\epsilon) := \prod_{j=1}^m \frac{\left(-\epsilon^{-1}t^{l+j-1}q;q\right)_\infty}{\left(-\epsilon^{-1}t^{l-j+1}a^{-1};q\right)_\infty},$$

$$N_2^4(\nu;\epsilon) := \prod_{\substack{0\le i\le l-1\\ 1\le j\le m}} \frac{\left(-\epsilon^{-1}aq^{1+\nu_i}t^{l+j-1}, -\epsilon^{-1}a^{-1}q^{\nu_i}t^{l-j+1};q\right)_{\nu_{i+1}-\nu_i}}{\left(-\epsilon^{-1}aq^{1+\nu_i}t^{i+j-1}, -\epsilon^{-1}a^{-1}q^{\nu_i}t^{i-j+1};q\right)_{\nu_{i+1}-\nu_i}}$$

if $\nu \in P_L(l;\epsilon)$ and zero otherwise. For generic $\epsilon_0 > q^{\frac{1}{2}}$ we have

$$\lim_{k\to\infty} N_2^2(\nu;\epsilon_k) = 1, \qquad \sup_{(\nu,k)\in P(l)\times\mathbb{N}_0} |N_2^2(\nu;\epsilon_k)| < \infty$$

by Lemma 8.2**(a)** and by the fact that $N_2^2(\nu;\epsilon) = 0$ if $\nu \notin P_L(l;\epsilon)$, we have

$$\lim_{k\to\infty} N_2^3(\epsilon_k) = 0, \quad \sup_{k\in\mathbb{N}_0} |N_2^3(\epsilon_k)| < \infty$$

by Lemma 8.2**(c)** since $0 < a < 1/q$, and we have

$$\lim_{k\to\infty} N_2^4(\nu;\epsilon_k) = t^{2m|\nu|}, \quad \sup_{k\in\mathbb{N}_0} |N_2^4(\nu;\epsilon_k)| \le K_2^4 t^{2m|\nu|} \le K_2^4$$

with $K_2^4 > 0$ independent of $\nu \in P(l)$ by Lemma 8.2**(b)** and by the fact that $N_2^4(\nu;\epsilon) = 0$ if $\nu \notin P_L(l;\epsilon)$. This completes the proof of Lemma 8.1**(ii)**. □

*Proof of Lemma 8.1***(iii)**. We first use the explicit formulas for the weight function Δ (2.6), (2.7), (2.8) and for δ_c (3.12) as well as the definition of $\underline{t}_L(\epsilon)$ (6.9) to give an explicit expression for $\Delta^{KL}_{3,l,m}(\nu,x;\epsilon)$. This explicit expression can be written as

$$\Delta^{KL}_{3,l,m}(\nu,x;\epsilon)=N_3^1(x)N_3^2(\nu,x;\epsilon)N_3^3(x;\epsilon)N_3^4(\nu,x;\epsilon)$$

with

$$N_3^1(x):=\delta(x;t)\delta_c(\sigma_L^m;x)\prod_{i=1}^{n-r}\frac{\left(x_i^2,x_i^{-2};q\right)_\infty}{\left(-q^{\frac12}x_i,-q^{\frac12}x_i^{-1},-q^{\frac12}ax_i,-q^{\frac12}ax_i^{-1};q\right)_\infty}, \tag{8.33}$$

if $D_1(m;\underline{t}_L(\epsilon);t)\neq 0$ or $m=0$ and zero otherwise, where δ is given by (2.8),

$$N_3^2(\nu,x;\epsilon):=\prod_{j=1}^{n-r}\frac{\prod_{i=1}^l\left(\epsilon q^{-\frac12-\nu_i}t^{1-i}x_j,\epsilon q^{-\frac12-\nu_i}t^{1-i}x_j^{-1};q\right)_\tau}{\left(\epsilon bq^{\frac12}x_j,\epsilon bq^{\frac12}x_j^{-1};q\right)_\infty} \tag{8.34}$$

if $\nu\in P_L(l;\epsilon)$ and zero otherwise,

$$N_3^3(x;\epsilon):=\prod_{i=1}^{n-r}\frac{\left(-\epsilon^{-1}qt^{r+i-1},-\epsilon^{-1}qat^{r+i-1},\epsilon^{-1}q^{\frac12}x_i,\epsilon^{-1}q^{\frac12}x_i^{-1};q\right)_\infty}{\left(\epsilon^{-1}q^{\frac12}x_i,\epsilon^{-1}q^{\frac12}x_i^{-1},\epsilon^{-1}q^{\frac12}t^lx_i,\epsilon^{-1}q^{\frac12}t^lx_i^{-1};q\right)_\infty} \tag{8.35}$$

and

$$N_3^4(\nu,x;\epsilon):=\prod_{\substack{0\leq i\leq l-1\\1\leq j\leq n-r}}\frac{\left(\epsilon^{-1}q^{\frac12+\nu_i}t^lx_j,\epsilon^{-1}q^{\frac12+\nu_i}t^lx_j^{-1};q\right)_{\nu_{i+1}-\nu_i}}{\left(\epsilon^{-1}q^{\frac12+\nu_i}t^ix_j,\epsilon^{-1}q^{\frac12+\nu_i}t^ix_j^{-1};q\right)_{\nu_{i+1}-\nu_i}} \tag{8.36}$$

if $\nu\in P_L(l;\epsilon)$ and zero otherwise, where $\nu_0=0$. Observe that N_3^1 is bounded on T^{n-r}. For N_3^2 it follows from Lemma 8.2**(a)** that $\lim_{k\to\infty}N_3^2(\nu,x;\epsilon_k)=1$ for all $\nu\in P(l)$, $x\in T^{n-r}$ and that

$$\sup_{(\nu,x,k)\in P(l)\times T^{n-r}\times\mathbb{N}_0}|N_3^2(\nu,x;\epsilon_k)|<\infty$$

for generic $\epsilon_0>q^{\frac12}$. By Lemma 8.2**(c)** and the fact that $0<a<1/q$, we have for generic $\epsilon_0>0$ that $\lim_{k\to\infty}N_3^3(x;\epsilon_k)=0$ for all $x\in T^{n-r}$ and

$$\sup_{(x,k)\in T^{n-r}\times\mathbb{N}_0}|N_3^3(x;\epsilon_k)|<\infty.$$

Finally, we can use Lemma 8.2**(b)** to prove that $\lim_{k\to\infty}N_3^4(\nu,x;\epsilon_k)=t^{2(n-r)|\nu|}$ for all $\nu\in P(l)$, $x\in T^{n-r}$ and that

$$\sup_{(\nu,x,k)\in P(l)\times T^{n-r}\times\mathbb{N}_0}|N_3^4(\nu,x;\epsilon_k)|<\infty$$

for generic $\epsilon_0>q^{\frac12}$. This completes the proof of Lemma 8.1**(iii)**. □

9. Limit of the orthogonality measure (big q-Jacobi case)

In the next lemma we give a new expression for the weight $\underline{c}_B$ which appears in the definition of the inner product $\langle .,.\rangle_B$.

Lemma 9.1. *The weight* $\underline{c}_B \in \left(\mathbb{C}^*\right)^{n+1}$ *can be rewritten as*

$$\begin{aligned} c_{B,j} =& \left(q;q\right)_\infty^n \prod_{i=1}^{j} \frac{\theta(-t^{i+j-n}c/d)}{\theta(-t^{1-i}d/c)\theta(-t^i c/d)} \prod_{i=1}^{n-j} \frac{1}{\theta(-t^{1-i}c/d)} \\ &.q^{-2\tau^2\left((n-j)\binom{j}{2}+\binom{j}{3}+\binom{n-j}{3}\right)} t^{-\binom{j}{2}-\binom{n-j}{2}} \\ &.c^{-2\tau\left(j(n-j)+\binom{j}{2}\right)-j} d^{-2\tau\binom{n-j}{2}+j-n} \end{aligned} \tag{9.1}$$

for $j \in \{0,\dots,n\}$.

Proof. For $j=0$, (9.1) follows from (7.8) since $c_{B,j} = c_B d_{B,j}$ with $d_{B,0}=1$. For $j \in \{1,\dots,n\}$, write $\tilde{c}_{B,j}$ for the right-hand side of (9.1); then by the explicit expression (7.5) for $d_{B,j}$ it remains to prove that

$$\frac{\tilde{c}_{B,j}}{\tilde{c}_{B,j-1}} = \prod_{m=j+1}^{n} \Psi_t(-t^{n-m-j+1}d/c)$$

for $j \in \{1,\dots,n\}$, with Ψ_t given by (7.6). This follows by a direct calculation. □

The remainder of this section is devoted to a proof of Proposition 6.3. We fix in this section $(a,b,c,d) \in V_B$ with $a,b \neq 0$. With slight modifications, the proof goes also through for $a=0$ or $b=0$.

For $\epsilon \in \mathbb{R}_{>0}$ we set $\rho_{B,j}(\epsilon) := t_{B,0}(\epsilon)t^{j-1} = (qc/d)^{\frac{1}{2}}\epsilon^{-1}t^{j-1}$ and $\sigma_{B,j}(\epsilon) := t_{B,1}(\epsilon)t^{j-1} = -(qd/c)^{\frac{1}{2}}\epsilon^{-1}t^{j-1}$ for $j \in \mathbb{Z}$, where $\underline{t}_B(\epsilon)$ is given by (7.12). Then for $\epsilon > 0$ sufficiently small, we have

$$F(r;\underline{t}_B(\epsilon);t) = \bigcup_{\substack{l+m=r \\ l,m\in\mathbb{N}_0}} D_0(l;\underline{t}_B(\epsilon);t) \times D_1(m;\underline{t}_B(\epsilon);t) \subset \mathbb{C}^r$$

where $F(r)$ is given by (5.3) and

$$\begin{aligned} D_0(l;\underline{t}_B(\epsilon);t) &= \{\rho_B(\epsilon)q^\nu \mid \nu \in P_B^{(0)}(l;\epsilon)\}, \\ P_B^{(0)}(l;\epsilon) &:= \{\nu \in P(l) \mid |\rho_{B,l}(\epsilon)q^{\nu_l}| > 1\}, \end{aligned} \tag{9.2}$$

if $l > 0$, respectively

$$\begin{aligned} D_1(m;\underline{t}_B(\epsilon);t) &= \{\sigma_B(\epsilon)q^\nu \mid \nu \in P_B^{(1)}(m;\epsilon)\}, \\ P_B^{(1)}(m;\epsilon) &:= \{\nu \in P(m) \mid |\sigma_{B,m}(\epsilon)q^{\nu_m}| > 1\} \end{aligned} \tag{9.3}$$

if $m > 0$. We write

(9.4)

$$\begin{aligned} &\left(\prod_{i=1}^{n}\left(-\epsilon^{-2}qt^{i-1};q\right)_\infty\right)\left((cd/q)^{\frac{1}{2}}\epsilon\right)^{|\lambda|+|\mu|}\langle m_\lambda, m_\mu\rangle_{\underline{t}_B(\epsilon),t} \\ &= \sum_{r,l,m,\nu,\nu'} \iint_{x\in T^{n-r}} \left(m_\lambda m_\mu\right)\left(\rho_B q^\nu, \sigma_B q^{\nu'}, (cd/q)^{\frac{1}{2}}\epsilon x|(cd/q)^{\frac{1}{2}}\epsilon\right)\mathcal{W}^B_{l,m;r}(\nu,\nu',x;\epsilon)\frac{dx}{x} \end{aligned}$$

where $\rho_{B,i} := ct^{i-1}$, $\sigma_{B,i} := -dt^{i-1}$ and $m_\lambda(z|u)$ is given by (6.11), and the sum is over five tuples (r,l,m,ν,ν') with $r \in \{0,\dots,n\}$, $l,m \in \mathbb{N}_0$ with $l+m=r$,

$\nu \in P(l)$, $\nu' \in P(m)$, and with the renormalized weight $\mathcal{W}^B_{l,m;r}(\nu,\nu',x;\epsilon)$ given by

$$\mathcal{W}^B_{l,m;r}(\nu,\nu',x;\epsilon) := \frac{2^r\big(n-r+1\big)_r}{(2\pi i)^{n-r}}\Delta^{KB}_{1,l,m}(\nu,\nu';\epsilon)\Delta^{KB}_{2,l,m}(\nu,\nu',x;\epsilon) \tag{9.5}$$

when $r = l+m$, with

$$\begin{aligned}\Delta^{KB}_{1,l,m}(\nu,\nu';\epsilon) := &\left(\prod_{i=1}^r\big(-\epsilon^{-2}qt^{i-1};q\big)_\infty\right)\delta_c\big(\rho_B(\epsilon)q^\nu;\sigma_B(\epsilon)q^{\nu'}\big)\\ &.\Delta^{(d)}\big(\rho_B(\epsilon)q^\nu;t_{B,0}(\epsilon)\big)\Delta^{(d)}\big(\sigma_B(\epsilon)q^{\nu'};t_{B,1}(\epsilon)\big)\end{aligned} \tag{9.6}$$

if $\nu \in P^{(0)}_B(l;\epsilon)$, $\nu' \in P^{(1)}_B(m;\epsilon)$ and zero otherwise,

$$\begin{aligned}\Delta^{KB}_{2,l,m}(\nu,\nu',x;\epsilon) := &\prod_{i=1}^{n-r}\big(-\epsilon^{-2}qt^{r+i-1};q\big)_\infty\\ &.\Delta(x;\underline{t}_B(\epsilon);t)\delta_c\big(\rho_B(\epsilon)q^\nu;x\big)\delta_c(\sigma_B\big(\epsilon)q^{\nu'};x\big)\end{aligned} \tag{9.7}$$

if $\nu \in P^{(0)}_B(l;\epsilon)$, $\nu' \in P^{(1)}_B(m;\epsilon)$ and zero otherwise, with δ_c given by (3.12). We use the obvious conventions when $l=0$, $m=0$ or $r=n$ (compare with the little q-Jacobi case in section 8). In particular, we have $\Delta^{KB}_{2,l,n-l}(\nu,\nu';\epsilon)=1$ for $\nu \in P^{(0)}_B(l;\epsilon)$, $\nu' \in P^{(1)}_B(n-l;\epsilon)$ and $l \in \{0,\dots,n\}$.

The following lemma will be used to pull a limit $\epsilon_k \downarrow 0$ in the right-hand side of (9.4) through the integration over $x \in T^{n-r}$ and through the infinite sums over $\nu \in P(l)$ and $\nu' \in P(m)$ for some sequence $\{\epsilon_k\}_{k\in\mathbb{N}_0}$ in $\mathbb{R}_{>0}$ converging to 0.

Lemma 9.2. *Keep the notations and conventions as above. Let $l,m \in \mathbb{N}_0$ with $l+m \in \{0,\dots,n\}$ and write $r := l+m$. Then there exists a sequence of positive real numbers $\{\epsilon_k\}_{k\in\mathbb{N}_0}$ which converges to 0, such that:*

(i) *For all $\nu \in P(l)$, $\nu' \in P(m)$ we have*

$$\lim_{k\to\infty}\Delta^{KB}_{1,l,m}(\nu,\nu';\epsilon_k) = \big(q;q\big)_\infty^{-2r}c_{B,l}\Delta^B(\rho_Bq^\nu,\sigma_Bq^{\nu'})\prod_{i=1}^l\rho_{B,i}q^{\nu_i}\prod_{j=1}^m|\sigma_{B,j}|q^{\nu'_j},$$

and there exists a $K \in \mathbb{R}_{>0}$ independent of $\nu \in P(l)$ and $\nu' \in P(m)$ such that

$$\sup_{k\in\mathbb{N}_0}|\Delta^{KB}_{1,l,m}(\nu,\nu';\epsilon_k)| \le Kc_{B,l}\Delta^B(\rho_Bq^\nu,\sigma_Bq^{\nu'})\prod_{i=1}^l\rho_{B,i}q^{\nu_i}\prod_{j=1}^m|\sigma_{B,j}|q^{\nu'_j}$$

for all $\nu \in P(l)$ and all $\nu' \in P(m)$, where $\Delta^B(z) = \Delta^B(z;a,b,c,d;t)$ is given by (7.3).

(ii) *If $r<n$, then $\lim_{k\to\infty}\Delta^{KB}_{2,l,m}(\nu,\nu',x;\epsilon_k)=0$ for all $\nu \in P(l)$, $\nu' \in P(m)$, $x \in T^{n-r}$ and*

$$\sup_{(k,\nu,\nu',x)\in\mathbb{N}_0\times P(l)\times P(m)\times T^{n-r}}|\Delta^{KB}_{2,l,m}(\nu,\nu',x;\epsilon_k)| < \infty.$$

The proof of Proposition 7.4 is now an easy consequence of Lemma 9.2 and Lebesgue's dominated convergence theorem. Indeed, the infinite sum

$$(1-q)^{-n}\langle 1,1\rangle_B = \sum_{(\nu,\nu',l)}c_{B,l}\Delta^B(\rho_Bq^\nu,\sigma_Bq^{\nu'})\prod_{i=1}^l\rho_{B,i}q^{\nu_i}\prod_{j=1}^{n-l}|\sigma_{B,j}|q^{\nu'_j}, \tag{9.8}$$

where the sum is taken over three tuples (ν,ν',l) with $\nu\in P(l)$ and $\nu'\in P(n-l)$ and $l\in\{0,\dots,n\}$, is absolutely convergent (cf. [36, proof of Proposition 6.1]). Since

$$\sup_{\nu,\nu',x,\epsilon}|m_\lambda\big(\rho_Bq^\nu,\sigma_Bq^{\nu'},(cd/q)^{\frac12}\epsilon x|(cd/q)^{\frac12}\epsilon\big)|<\infty,$$

where the supremum is taken over the four tuples (ν,ν',x,ϵ) with $\nu\in P_B^{(0)}(l;\epsilon)$, $\nu'\in P_B^{(1)}(m;\epsilon)$, $x\in T^{n-r}$ $(r=l+m)$ and $\epsilon>0$, we obtain, by Lebesgue's dominated convergence theorem, (6.12), (9.4), (9.8) and Lemma 9.2,

$$\begin{aligned}&\lim_{k\to\infty}\left(\prod_{i=1}^n\big(-\epsilon_k^{-2}qt^{i-1};q\big)_\infty\right)\big((cd/q)^{\frac12}\epsilon_k\big)^{|\lambda|+|\mu|}\langle m_\lambda,m_\mu\rangle_{\underline{t}_B(\epsilon_k),t}\\&=\frac{2^nn!}{\big(q;q\big)_\infty^{2n}}\sum_{l=0}^n\sum_{\substack{\nu\in P(l)\\ \nu'\in P(n-l)}}c_{B,l}\big(\tilde m_\lambda\tilde m_\mu\Delta^B\big)(\rho_Bq^\nu,\sigma_Bq^{\nu'})\prod_{i=1}^l\rho_{B,i}q^{\nu_i}\prod_{j=1}^{n-l}|\sigma_{B,j}|q^{\nu'_j}\\&=2^nn!(1-q)^{-n}\big(q;q\big)_\infty^{-2n}\langle\tilde m_\lambda,\tilde m_\mu\rangle_{B,t}^{a,b,c,d},\end{aligned}$$

for some sequence $\{\epsilon_k\}_{k\in\mathbb{N}_0}$ in $\mathbb{R}_{>0}$ converging to 0. So for the proof of Proposition 7.4, it suffices to prove Lemma 9.2.

Proof of Lemma 9.2. Using the explicit expressions for $\Delta^{(d)}$ (3.5), δ_c (3.12) and $\underline{t}_B(\epsilon)$ (7.12), we can write

$$\Delta^{KB}_{1,l,m}(\nu,\nu';\epsilon):=U_0(\nu,\nu';l,m)U_+(\epsilon;\nu,\nu';l,m)U_-(\epsilon;\nu,\nu';l,m)\tag{9.9}$$

with U_0,U_+, respectively U_-, the factor of $\Delta^{KB}_{1,l,m}$ consisting of products of q-shifted factorials of the form $\big(e;q\big)_s$, $\big(\epsilon^2e;q\big)_s$, respectively $\big(\epsilon^{-2}e;q\big)_s$ $(s\in\mathbb{N}_0\cup\{\infty\})$. By a straightforward computation, the factors U_0,U_+ and U_- can be explicitly given by

$$\begin{aligned}U_0(\nu,\nu';l,m):=&\Psi_0(\nu,l;a,b,c,d)\Psi_0(\nu',m;b,a,d,c)\\&\cdot\prod_{\substack{1\le i\le l\\1\le j\le m}}\big(-t^{i-j}q^{\nu_i-\nu'_j}c/d,-t^{j-i}q^{\nu'_j-\nu_i}d/c;q\big)_\tau\end{aligned}\tag{9.10}$$

if $(\nu,\nu')\in P_0(l;\epsilon)\times P_1(m;\epsilon)$, and zero otherwise, with

$$\begin{aligned}\Psi_0(\nu,l;a,b,c,d):=&F_1(\nu)\\&\cdot\prod_{i=1}^l\frac{1}{\big(q,-t^{1-i}q^{-\nu_{i-1}}d/c,at^{i-1}q^{\nu_{i-1}+1},-bt^{i-1}q^{\nu_{i-1}+1}c/d;q\big)_\infty}\\&\cdot\prod_{i=1}^l\frac{\big(at^{i-1}q^{\nu_{i-1}+1},-bt^{i-1}q^{\nu_{i-1}+1}c/d;q\big)_{\nu_i-\nu_{i-1}}}{\big(q,-t^{i-1}q^{\nu_{i-1}+1}c/d;q\big)_{\nu_i-\nu_{i-1}}\big(abt^{i-1}q^{\nu_{i-1}+1}\big)^{\nu_i-\nu_{i-1}}}\end{aligned}\tag{9.11}$$

where $\nu_0=0$ and $F_1(\nu)$ is given by (8.11), and

$$\begin{aligned}U_+(\epsilon;\nu,\nu';l,m):=&\Psi_+(\epsilon;\nu,l;a,b,c,d)\Psi_+(\epsilon;\nu',m;b,a,d,c)\\&\cdot\prod_{\substack{1\le i\le l\\1\le j\le m}}\big(-\epsilon^2t^{2-i-j}q^{-1-\nu_i-\nu'_j};q\big)_\tau\end{aligned}\tag{9.12}$$

if $(\nu,\nu')\in P_B^{(0)}(l;\epsilon)\times P_B^{(1)}(m;\epsilon)$ and zero otherwise, with

$$\begin{aligned}\Psi_+(\epsilon;\nu,l;a,b,c,d) &:= \prod_{i=1}^{l}\frac{\big(\epsilon^2t^{2(1-i)}q^{-2\nu_{i-1}-1}d/c;q\big)_\infty}{\big(\epsilon^2at^{1-i}q^{-\nu_{i-1}}d/c,-\epsilon^2bt^{1-i}q^{-\nu_{i-1}};q\big)_\infty}\\ &\quad\cdot\prod_{1\leq i<j\leq l}\big(\epsilon^2t^{2-i-j}q^{-\nu_i-\nu_j-1}d/c;q\big)_\tau\end{aligned}\tag{9.13}$$

and

$$\begin{aligned}U_-(\epsilon;\nu,\nu';l,m) &:= \Psi_-(\epsilon;\nu,l;a,b,c,d)\Psi_-(\epsilon;\nu',m;b,a,d,c)\\ &\quad\cdot\left(\prod_{j=1}^{m}\frac{\big(-\epsilon^{-2}qt^{l+j-1};q\big)_\infty}{\big(-\epsilon^{-2}qt^{j-1};q\big)_\infty}\right)\prod_{\substack{1\leq i\leq l\\1\leq j\leq m}}\big(-\epsilon^{-2}t^{i+j-2}q^{\nu_i+\nu'_j+1};q\big)_\tau\end{aligned}\tag{9.14}$$

if $(\nu,\nu')\in P_B^{(0)}(l;\epsilon)\times P_B^{(1)}(m;\epsilon)$ and zero otherwise, with

$$\begin{aligned}\Psi_-(\epsilon;\nu,l;a,b,c,d) &:= \prod_{i=1}^{l}\frac{\big(\epsilon^{-2}t^{2(i-1)}q^{2\nu_i+1}c/d;q\big)_1}{\big(\epsilon^{-2}t^{2(i-1)}q^{2\nu_{i-1}+1}c/d;q\big)_1}\\ &\quad\cdot\prod_{i=1}^{l}\frac{\big(\epsilon^{-2}t^{2(i-1)}q^{2\nu_{i-1}+1}c/d,-\epsilon^{-2}t^{i-1}q^{\nu_{i-1}+1};q\big)_{\nu_i-\nu_{i-1}}}{\big(\epsilon^{-2}a^{-1}t^{i-1}q^{\nu_{i-1}+1}c/d,-\epsilon^{-2}b^{-1}t^{i-1}q^{\nu_{i-1}+1};q\big)_{\nu_i-\nu_{i-1}}}\\ &\quad\cdot\frac{\prod_{i=1}^{l}\big(-\epsilon^{-2}qt^{i-1};q\big)_{\nu_{i-1}}}{\prod_{1\leq i<j\leq l}\big(\epsilon^{-2}t^{i+j-2}q^{\nu_{i-1}+\nu_j+1}c/d;q\big)_{\nu_i-\nu_{i-1}}}.\end{aligned}\tag{9.15}$$

For given $\epsilon_0\in\mathbb{R}^*$, we write $\epsilon_k:=\epsilon_0q^k$. Then for generic $\epsilon_0>0$ we have

$$\lim_{k\to\infty}U_+(\epsilon_k;\nu,\nu';l,m)=1\tag{9.16}$$

for all $(\nu,\nu')\in P(l)\times P(m)$ by Lemma 8.2**(a)**. By (8.25), we have for generic $\epsilon_0>0$

$$\begin{aligned}&\lim_{k\to\infty}\frac{\prod_{i=1}^{l}\big(-\epsilon_k^{-2}qt^{i-1};q\big)_{\nu_{i-1}}}{\prod_{1\leq i<j\leq l}\big(\epsilon_k^{-2}t^{i+j-2}q^{\nu_{i-1}+\nu_j+1}c/d;q\big)_{\nu_i-\nu_{i-1}}}\\ &=\prod_{i=1}^{l}q^{\binom{\nu_{i-1}+1}{2}}t^{(i-1)\nu_{i-1}}\prod_{1\leq i<j\leq l}\big(-t^{i+j-2}q^{\nu_{i-1}+\nu_j+1}c/d\big)^{\nu_{i-1}-\nu_i}q^{-\binom{\nu_i-\nu_{i-1}}{2}}\end{aligned}\tag{9.17}$$

for the factor of Ψ_- in the third line of (9.15). The factor of U_- in the second line of (9.14) can be rewritten as

$$\begin{aligned}&\left(\prod_{j=1}^{m}\frac{\big(-\epsilon^{-2}qt^{l+j-1};q\big)_\infty}{\big(-\epsilon^{-2}qt^{j-1};q\big)_\infty}\right)\prod_{\substack{1\leq i\leq l\\1\leq j\leq m}}\big(-\epsilon^{-2}t^{i+j-2}q^{\nu_i+\nu'_j+1};q\big)_\tau\\ &\qquad=\prod_{\substack{1\leq i\leq l\\1\leq j\leq m}}\frac{\big(-\epsilon^{-2}qt^{i+j-1};q\big)_{\nu_i+\nu'_j}}{\big(-\epsilon^{-2}qt^{i+j-2};q\big)_{\nu_i+\nu'_j}}.\end{aligned}\tag{9.18}$$

It follows then from (9.14), (9.15), (9.17), (9.18) and Lemma 8.2**(b)** that, for generic $\epsilon_0 > 0$,

$$\lim_{k\to\infty} U_-(\epsilon_k;\nu,\nu';l,m) = t^{m|\nu|+l|\nu'|}\Psi_-^\infty(\nu,l;a,b,c,d)\Psi_-^\infty(\nu',m;b,a,d,c) \tag{9.19}$$

with

$$\begin{aligned}\Psi_-^\infty(\nu,l;a,b,c,d) :=& \prod_{i=1}^{l}\left(t^{i-1}q^{\nu_{i-1}+2}ab\right)^{\nu_i-\nu_{i-1}} q^{\binom{\nu_{i-1}+1}{2}} t^{(i-1)\nu_{i-1}} \\ &\cdot \prod_{1\le i<j\le l}\left(-t^{i+j-2}q^{\nu_{i-1}+\nu_j+1}c/d\right)^{\nu_{i-1}-\nu_i} q^{-\binom{\nu_i-\nu_{i-1}}{2}}.\end{aligned} \tag{9.20}$$

We will now rewrite U_0 in the form

$$\begin{aligned}U_0(\nu,\nu';l,m) =& \left(q;q\right)_\infty^{-r}\prod_{i=1}^{l}\frac{\theta(-t^{i-m}c/d)}{\theta(-t^{1-i}d/c)\theta(-t^ic/d)}\prod_{j=1}^{m}\frac{1}{\theta(-t^{1-j}c/d)} \\ &\cdot C_0(\nu,\nu';l,m)\Delta^B(\rho_Bq^\nu,\sigma_Bq^{\nu'};a,b,c,d;t)\end{aligned} \tag{9.21}$$

and we determine the factor $C_0(\nu,\nu';l,m)$ explicitly. Using (7.9) and the formula $\theta(x)=\theta(qx^{-1})$ for the Jacobi theta function $\theta(x)$ (7.7), we can rewrite the factor $\Psi_0(\nu,l;a,b,c,d)$ (9.11) as

$$\begin{aligned}&\Psi_0(\nu,l;a,b,c,d) = F_1(\nu) \\ &\cdot\prod_{i=1}^{l}\frac{v_B(\rho_{B,i}q^{\nu_i};a,b,c,d)(t^{i-1}c/d)^{\nu_{i-1}}q^{\binom{\nu_{i-1}+1}{2}}}{\theta(-t^{1-i}d/c)\left(q^{\nu_i+1}t^{i-1};q\right)_\infty\left(q;q\right)_{\nu_i-\nu_{i-1}}(abt^{i-1}q^{\nu_{i-1}+1})^{\nu_i-\nu_{i-1}}}\end{aligned} \tag{9.22}$$

where v_B (7.4) is the one-variable weight function for the big q-Jacobi polynomials. Since $v_B(-dx;a,b,c,d)=v_B(dx;b,a,d,c)$, we obtain from (9.22)

$$\begin{aligned}&\Psi_0(\nu',m;b,a,d,c) = F_1(\nu') \\ &\cdot\prod_{j=1}^{m}\frac{v_B(\sigma_{B,j}q^{\nu'_j};a,b,c,d)(t^{j-1}d/c)^{\nu'_{j-1}}q^{\binom{\nu'_{j-1}+1}{2}}}{\theta(-t^{1-j}c/d)\left(q^{\nu'_j+1}t^{j-1};q\right)_\infty\left(q;q\right)_{\nu'_j-\nu'_{j-1}}(abt^{j-1}q^{\nu'_{j-1}+1})^{\nu'_j-\nu'_{j-1}}}.\end{aligned} \tag{9.23}$$

The factor $F_1(\nu)$ (8.11) of $\Psi_0(\nu,l;a,b,c,d)$ can be rewritten as

$$\begin{aligned}F_1(\nu) =& \delta_{qJ}(\rho_Bq^\nu)q^{-2\tau^2\binom{l}{3}}c^{-l(l-1)\tau}\prod_{i=1}^{l}\frac{\left(t^{i-1}q^{1+\nu_i};q\right)_\infty}{\left(q^{1+\nu_i-\nu_{i-1}};q\right)_\infty}t^{-2(l-i)\nu_i} \\ &\cdot\prod_{1\le i<j\le l}\left(-q^{\nu_j-\nu_i+1}t^{j-i}\right)^{\nu_i-\nu_{i-1}}q^{\binom{\nu_i-\nu_{i-1}}{2}}\end{aligned} \tag{9.24}$$

for $\nu\in P(l)$ where δ_{qJ} is given by (6.8). This follows from (8.13) since $\delta_{qJ}(\rho_Bq^\nu) = c^{l(l-1)\tau}\delta_{qJ}(\rho_Lq^\nu)$ for $\nu\in P(l)$ (here $\rho_{L,i}=t^{i-1}$). Similarly, we have, for the factor $F_1(\nu')$ of $\Psi_0(\nu',m;b,a,d,c)$,

$$\begin{aligned}F_1(\nu') =& \delta_{qJ}(\sigma_Bq^{\nu'})q^{-2\tau^2\binom{m}{3}}d^{-m(m-1)\tau}\prod_{j=1}^{m}\frac{\left(t^{j-1}q^{1+\nu'_j};q\right)_\infty}{\left(q^{1+\nu'_j-\nu'_{j-1}};q\right)_\infty}t^{-2(m-j)\nu'_j} \\ &\cdot\prod_{1\le i<j\le m}\left(-q^{\nu'_j-\nu'_i+1}t^{j-i}\right)^{\nu'_i-\nu'_{i-1}}q^{\binom{\nu'_i-\nu'_{i-1}}{2}},\end{aligned} \tag{9.25}$$

since $\delta_{qJ}(\sigma_Bq^{\nu'}) = d^{m(m-1)\tau}\delta_{qJ}(\rho_Lq^{\nu'})$ for $\nu'\in P(m)$.

Finally, we set, for $z=(z_1,\dots,z_r)$ with $r:=l+m$,

$$\tilde{\delta}^l_{qJ}(z):=\prod_{\substack{1\leq i\leq l\\ l+1\leq j\leq r}}|z_i-z_j||z_i|^{2\tau-1}\big(qt^{-1}z_j/z_i;q\big)_{2\tau-1}.$$

Then the factor of U_0 in the second line of (9.10) can be rewritten as

$$\begin{aligned}\prod_{\substack{1\leq i\leq l\\ 1\leq j\leq m}}&\big(-t^{i-j}q^{\nu_i-\nu'_j}c/d,-t^{j-i}q^{\nu'_j-\nu_i}d/c;q\big)_\tau\\ &=\tilde{\delta}^l_{qJ}(\rho_Bq^\nu,\sigma_Bq^{\nu'})t^{m|\nu|-l|\nu'|}\prod_{i=1}^l\frac{\theta(-t^{i-m}c/d)}{\theta(-t^ic/d)\big(ct^{i-1}q^{\nu_i}\big)^{2m\tau}}\end{aligned}\tag{9.26}$$

for $\nu\in P(l)$ and $\nu'\in P(m)$, since we have for $i\in\{1,\dots,l\}$, $j\in\{1,\dots,m\}$ that

$$\begin{aligned}&\big(-t^{i-j}q^{\nu_i-\nu'_j}c/d,-t^{j-i}q^{\nu'_j-\nu_i}d/c;q\big)_\tau\\ &\quad=\frac{\theta(-t^{i-j}c/d)\big(-t^{j-i-1}q^{1+\nu'_j-\nu_i}d/c;q\big)_\infty}{\theta(-t^{i-j+1}c/d)\big(-t^{j-i+1}q^{\nu'_j-\nu_i}d/c;q\big)_\infty}\big(1+t^{j-i}q^{\nu'_j-\nu_i}d/c\big)t^{\nu_i-\nu'_j}\end{aligned}\tag{9.27}$$

(formula (9.27) follows from a straightforward computation using (4.3), (7.9) and $\theta(x)=\theta(qx^{-1})$). Since

$$\delta_{qJ}(\rho_Bq^\nu,\sigma_Bq^{\nu'})=\delta_{qJ}(\rho_Bq^\nu)\delta_{qJ}(\sigma_Bq^{\nu'})\tilde{\delta}^l_{qJ}(\rho_Bq^\nu,\sigma_Bq^{\nu'})$$

for $\nu\in P(l)$ and $\nu'\in P(m)$, we obtain from (7.3), (9.10), (9.22), (9.23), (9.24), (9.25) and (9.26) that (9.21) holds with

$$\begin{aligned}C_0(\nu,\nu';l,m)=&t^{-m|\nu|-l|\nu'|}c^{-2lm\tau}q^{-2m\binom{l}{2}\tau^2}\\ &\cdot\hat{C}_0(\nu,l;a,b,c,d)\hat{C}_0(\nu',m;b,a,d,c)\end{aligned}\tag{9.28}$$

where

$$\begin{aligned}\hat{C}_0(\nu,l;a,b,c,d):=&\,q^{-2\tau^2\binom{l}{3}}c^{-2\binom{l}{2}\tau}\\ &\cdot\prod_{i=1}^l\big(t^{i-1}c/d\big)^{\nu_{i-1}}q^{\binom{\nu_{i-1}+1}{2}}t^{-2(l-i)\nu_i}\big(abt^{i-1}q^{\nu_{i-1}+1}\big)^{\nu_{i-1}-\nu_i}\\ &\cdot\prod_{1\leq i<j\leq l}\big(-q^{\nu_j-\nu_i+1}t^{j-i}\big)^{\nu_i-\nu_{i-1}}q^{\binom{\nu_i-\nu_{i-1}}{2}}.\end{aligned}\tag{9.29}$$

We have by Lemma 9.1 (with n in the right-hand side of (9.1) equal to $r=l+m$ in this situation) and by (9.9), (9.16), (9.19), (9.21) and (9.28) that for generic $\epsilon_0>0$,

$$\begin{aligned}\lim_{k\to\infty}\Delta^{KB}_{1,l,m}(\nu,\nu';\epsilon_k)&=U_0(\nu,\nu';l,m)\lim_{k\to\infty}U_-(\epsilon_k;\nu,\nu';l,m)\\ &=\big(q;q\big)_\infty^{-2r}c_{B,l}\Delta^B(\rho_Bq^\nu,\sigma_Bq^{\nu'};a,b,c,d;t)\\ &\quad\cdot E(\nu,l;a,b,c,d)E(\nu',m;b,a,d,c)\end{aligned}\tag{9.30}$$

for $\nu\in P(l)$ and $\nu'\in P(m)$, with

$$E(\nu,l;a,b,c,d):=q^{2\tau^2\binom{l}{3}}t^{\binom{l}{2}}c^{2\binom{l}{2}\tau+l}\hat{C}_0(\nu,l;a,b,c,d)\Psi^\infty_-(\nu,l;a,b,c,d).\tag{9.31}$$

By (9.20) and (9.29), we obtain

$$E(\nu,l;a,b,c,d)=c^lt^{\binom{l}{2}}q^{|\nu|}=\prod_{i=1}^l\rho_{B,i}q^{\nu_i},\quad \nu\in P(l).$$

In particular we have $E(\nu', m; b, a, d, c) = \prod_{j=1}^{m} |\sigma_{B,j}| q^{\nu'_j}$ for $\nu' \in P(m)$; hence by (9.30)

$$\lim_{k\to\infty} \Delta^{KB}_{1,l,m}(\nu, \nu'; \epsilon_k) = \big(q;q\big)_\infty^{-2r} c_{B,l} \Delta^B(\rho_B q^\nu, \sigma_B q^{\nu'}) \prod_{i=1}^{l} \rho_{B,i} q^{\nu_i} \prod_{j=1}^{m} |\sigma_{B,j}| q^{\nu'_j}$$

for all $\nu \in P(l)$, $\nu' \in P(m)$. To complete the proof of Lemma 9.2**(i)**, it suffices to prove that, for generic $\epsilon_0 > \max\big((qc/d)^{\frac{1}{2}}, (qd/c)^{\frac{1}{2}}\big)$,

$$\sup_{k\in\mathbb{N}_0} |\Delta^{KB}_{1,l,m}(\nu, \nu'; \epsilon_k)| \le K c_{B,l} \Delta^B(\rho_B q^\nu, \sigma_B q^{\nu'}; a, b, c, d; t) \prod_{i=1}^{l} \rho_{B,i} q^{\nu_i} \prod_{i=1}^{m} |\sigma_{B,j}| q^{\nu'_j}$$

for all $\nu \in P(l)$ and all $\nu' \in P(m)$, with $K > 0$ independent of ν and ν'. This can be proved by similar arguments as in the little q-Jacobi case (see proof of Lemma 8.1**(i)**). In particular, the estimates for almost all factors of $\Delta^{KB}_{1,l,m}$ can be obtained from one of the three estimates of Lemma 8.2. Only for the factor in the third line of the expression of Ψ_- (9.15) one needs a separate argument to establish the desired estimate. We may assume that this factor is zero unless $\nu \in P_B^{(0)}(l;\epsilon)$ and $\nu' \in P_B^{(1)}(m;\epsilon)$. In view of the limit (9.17), we would like to establish, for generic $\epsilon_0 > \max\big((qc/d)^{\frac{1}{2}}, (qd/c)^{\frac{1}{2}}\big)$, the estimate

$$\sup_{\{k\in\mathbb{N}_0 | \nu\in P_B^{(0)}(l;\epsilon_k) \text{ and } \nu'\in P_B^{(1)}(m;\epsilon_k)\}} \left| \frac{\prod_{i=1}^{l} \big(-\epsilon_k^{-2} q t^{i-1}; q\big)_{\nu_{i-1}}}{\prod_{1\le i<j\le l} \big(\epsilon_k^{-2} t^{i+j-2} q^{\nu_j+\nu_{i-1}+1} c/d; q\big)_{\nu_i-\nu_{i-1}}} \right|$$

$$\le K' \prod_{i=1}^{l} q^{\binom{\nu_{i-1}+1}{2}} t^{(i-1)\nu_{i-1}} \prod_{1\le i<j\le l} \big(t^{i+j-2} q^{\nu_{i-1}+\nu_j+1} c/d\big)^{\nu_{i-1}-\nu_i} q^{-\binom{\nu_i-\nu_{i-1}}{2}}$$

with $K' > 0$ independent of $\nu \in P(l)$ and $\nu' \in P(m)$. This can be done similarly as we have done for the factor N_1^3 (8.19) in the little q-Jacobi case.

The proof of Lemma 9.2**(ii)** is similar to the proof of Lemma 8.1**(iii)**. □

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