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SOME PROPERTIES OF PARTITIONS IN TERMS OF CRANK

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ABSTRACT. Let $N(r,m,n)$ (resp. $M(r,m,n)$) denote the number of partitions of n whose ranks (resp. cranks) are congruent to r modulo m. Atkin and Swinnerton-Dyer gave the relations between the numbers $N(r,m,mn+k)$ when $m=5,\ 7$ and $0 \leq r,k < m$. Garvan gave the relations between the numbers $M(r,m,mn+k)$ when $m=5,7,$ and 11, $0 \leq r,k < m$. Here, we show that the methods of Atkin and Swinnerton-Dyer can be extended to prove the relations for the crank.

1. INTRODUCTION

Let $N(m,n)$ and $M(m,n)$ denote the number of partitions of n with rank and crank m, respectively. We change this definition of $M(m,n)$ just a little, setting $M(0,1)=-1$ and $M(-1,1)=1=M(1,1)$, and modify $M(r,m,n)$ accordingly. We shall also suppose that the empty partition of 0 has rank 0.

For convenience, we write N_1 for M and N_3 for N. So, by (12) of [9] and (1.11) of [2] when $k=1$, and by (2.12) of [3] when $k=3$, we have

$$\sum_m \sum_{n\geq 0} N_k(m,n) z^m q^n = \left(\prod_{r=1}^{\infty} \frac{1}{1-q^r}\right)(1-z)\sum_n (-1)^n \frac{q^{n(kn+1)/2}}{1-zq^n}. \tag{1.1}$$

Here, and below, $\sum_n$ denotes a sum over all integers n, while $\sum'_n$ denotes a sum over all non-zero integers n.

For odd positive integers k, defining $N_k(m,n)$ by (1.1), we observe that $N_k(m,n) \geq 0$ for almost all n. If we put $z=1$ in (1.1), we find that

$$\sum_m N_k(m,n) = p(n), \tag{1.2}$$

where $p(n)$ is the number of partitions of n, and, replacing z by z^{-1} in (1.2),

$$N_k(-m,n) = N_k(m,n). \tag{1.3}$$

Thus, one can ask whether there is a "k**-rank**" (1-rank = crank, 3-rank = rank) such that $N_k(m,n)$ counts the number of partitions of n with k-rank m. This question has been answered by Garvan [8]

In [7], Garvan found a number of relations for the crank modulo 5, 7 and 11 analogous to those found by Dyson [5] for the rank modulo 5 and 7. These relations were proved by Garvan using various results for theta functions, including

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Winquist's identity [11]. Dyson's results were proved by Atkin and Swinnerton-Dyer [3] by relating certain mock-theta-like functions with elliptic-theta function identities. In this paper we show how Atkin and Swinnerton-Dyer's method may be extended to prove Garvan's crank relations. In the process we find some new relations for mock-theta-like functions.

Since we use the method of Atkin and Swinnerton-Dyer [3], we adopt their notations: m will take the values 5, 7 and 11, and the variables q and y are always related by the equation

$$y = q^m.$$

We shall regard any power series in q as a polynomial of degree $m-1$ in q whose coefficients are power series in y. Thus, any identity between two power series in q can be regarded, on equating coefficients of powers of q, as equivalent to m identities between power series in y. We write

$$P(z,q) := \prod_{r=1}^{\infty}(1 - zq^{r-1})(1 - z^{-1}q^r), \tag{1.4}$$

so that $P(z,q)$ is a single-valued analytic function of z in any ring-shaped region $0 < z_1 \le |z| \le z_2$, and satisfies

$$P(z^{-1}q,q) = P(z,q),\ P(zq,q) = -z^{-1}P(z,q). \tag{1.5}$$

We also write

$$P(a) := P_m(a) := P(y^a, y^m) = \prod_{r=1}^{\infty}(1 - y^{m(r-1)+a})(1 - y^{mr-a}),$$

$$P(0) := P_m(0) := \prod_{r=1}^{\infty}(1 - y^{mr}),$$

where a is not a multiple of m. It should be noted that $P(0)$ is not the expression that would be obtained by writing 0 instead of a in the definition of $P(a)$. From (1.5), we have

$$P(m-a) = P(a), \quad P(-a) = P(m+a) = -y^{-a}P(a), \tag{1.6}$$

which we shall use without explicit mention below.

2. Preparation

We define

$$M(r) := \sum_{n \ge 0} M(r,m,n)q^n \tag{2.1}$$

and

$$S(a) := \sum_n{}' (-1)^n \frac{q^{n(n+1)/2+an}}{1 - q^{mn}}. \tag{2.2}$$

The power series $S(a)$ differs only from (6.1) of [3] in that 3 is replaced by 1, which appears in the power of q in the numerator of (6.1) of [3]. As in [3], writing $-n$ for n in (2.2), we find that

$$S(a) = -S(m-1-a). \tag{2.3}$$

This gives

$$S(\frac{m-1}{2})=0. \tag{2.4}$$

Replacing z by a primitive m-th root of unity in (1.1), we can see that

$$\begin{aligned} &M(0)=\mathbf{F}\left(S(0)-S(m-1)+1\right), \\ &M(r)=\mathbf{F}\left(S(r)-S(r-1)\right) \qquad (r=1,\ 2,\ 3,\dots,\ m-1), \end{aligned} \tag{2.5}$$

where $\mathbf{F}=\prod_{r=1}^{\infty}(1-q^r)^{-1}$ (see [10] for details). To express the power series $S(a)$ as a polynomial in q of degree $m-1$ in q whose coefficients are power series in y, we define (for any complex number ζ)

$$T(z,\zeta,q):=\sum_n(-1)^n\frac{\zeta^n q^{n(n+1)/2}}{1-zq^n}, \tag{2.6}$$

which is obviously an analytic function of z in every region $0<r_1\le|z|\le r_2$ except for simple poles at point $z=q^n$. Also, let

$$T^*(\zeta,q):={\sum_n}'(-1)^n\frac{\zeta^n q^{n(n+1)/2}}{1-q^n}. \tag{2.7}$$

Following the similar proof of (6.6) of [3], we find

$$\begin{aligned} S(\frac{m-1}{2}-b)=&(-1)^b q^{b(m-b)/2}T(q^{mb},1,q^{m^2}) \\ &+T^*(q^{-mb},q^{m^2})+q^{mb}T(q^{2mb},q^{mb},q^{m^2}) \\ &+\sum_{\substack{a=1\\ a\not\equiv\mp b \bmod (m)}}^{(m-1)/2}(-1)^{a+b}q^{(a-b)(a+b-m)/2}\Big\{q^{ma}T(q^{m(b+a)},q^{ma},q^{m^2}) \\ &\qquad\qquad +T(q^{m(b-a)},q^{-ma},q^{m^2})\Big\}, \end{aligned} \tag{2.8}$$

It follows from Lemma 2 of [9] that

$$\zeta T(z\zeta,\zeta,q)+T(z\zeta^{-1},\zeta^{-1},q)=\frac{P(z;q)P(\zeta^2;q)\mathbf{F}^{-2}}{P(z\zeta^{-1};q)P(z\zeta;q)P(\zeta;q)}, \tag{2.9}$$

and that

$$T(z,1,q)=\frac{\mathbf{F}^{-2}}{P(z;q)}. \tag{2.10}$$

We now write

$$h(z;q):=T^*(z^{-1},q)+zT(z^2,z,q). \tag{2.11}$$

Lemma 1.

(i) $$h(z;q)-h(zq;q)=1,$$

(ii) $$h(z;q)+h(z^{-1};q)=0,$$

(iii) $$h(z;q)+h(z^{-1}q;q)=-1,$$

(iv) $$3h(z;q)-h(z^3;q)=\frac{P^3(z^2;q)\mathbf{F}^{-2}}{P^3(z;q)P(z^3;q)}-\frac{P^3(z^4;q)\mathbf{F}^{-2}}{P^3(z^2;q)P(z^6;q)}.$$

Proof. (i). By Jacobi's triple product identity (Thm.2.8 in) [1] we have

(2.12)
$$-P(z^{-1};q)\mathbf{F}^{-1} = -{\sum_n}' (-1)^n \frac{z^{-n}q^{n(n-1)/2}}{1-q^n} + {\sum_n}' (-1)^n \frac{z^{-n}q^{n(n+1)/2}}{1-q^n} - 1$$

and

(2.13)
$$\begin{aligned} P(z^{-1};q)\mathbf{F}^{-1} &= \sum_n (-1)^n \frac{z^n q^{n(n+1)/2}}{1-z^2q^{n+1}}(1-z^2q^{n+1}) \\ &= \sum_n (-1)^n \frac{z^n q^{n(n+1)/2}}{1-z^2q^{n+1}} - \sum_n (-1)^n \frac{z^{n+2}q^{(n+1)(n+2)/2}}{1-z^2q^{n+1}} \\ &\quad \text{(replace } n \text{ by } n+1) \qquad \text{(replace } n \text{ by } n-1) \\ &= -zq\sum_n (-1)^n \frac{(zq)^n q^{n(n+1)/2}}{1-z^2q^{n+2}} + z\sum_n (-1)^n \frac{z^n q^{n(n+1)/2}}{1-z^2q^n}. \end{aligned}$$

If we add (2.12) and (2.13), we have (i). (ii) is trivial and (iii) is consequence of (i) and (ii).

For (iv), let $f_L(z)$ and $f_R(z)$ denote the left and right sides of (iv). Then (i) shows that $f_L(zq) = f_L(z)$. By (1.5), $f_R(zq) = f_R(z)$. Now, $f_L(z) - f_R(z)$ is free from poles and so, by Lemma 2 of [3], either is free from zeros or is identically zero. By using (1.5), we find that $f_R(z^{-1}) = -f_R(z)$, and by (ii) $h(z^{-1};q) = -h(z;q)$. Therefore $z = 1$ is a zero of $f_L(z) - f_R(z)$. □

Consequently, with the help of (2.9)-(2.11) with $z = q^{ma}$, $\zeta = q^{mb}$, and q^{m^2} for q, we have

(2.14)
$$\begin{aligned} S(\tfrac{m-1}{2} - b) &= h(q^{mb};q^{m^2}) + (-1)^b q^{b(m-b)/2}\frac{P_m^2(0)}{P_m(b)} \\ &\quad + \sum_{\substack{a=1 \\ a\not\equiv\mp b \bmod (m)}}^{(m-1)/2} (-1)^{a+b} q^{(a-b)(a+b-m)/2} \frac{P_m(b)\,P_m(2a)\,P_m^2(0)}{P_m(b-a)\,P_m(b+a)\,P_m(a)}. \end{aligned}$$

We shall need the following, which is Lemma 6 in [3]:

$$\mathbf{F}^{-1} = (-1)^\lambda q^{\lambda(3\lambda+\mu)/2} P_m(0)\left[1 + \sum_{c=1}^{(m-1)/2} (-1)^c q^{c(3c-m)/2}\frac{P_m(2c)}{P_m(c)}\right], \tag{2.15}$$

where $m = 6\lambda + \mu$, λ is a positive integer and $\mu = \mp 1$.

We are now in a position to state and prove the results for crank of ordinary partition in the cases of modulo 5, 7 and 11. For convenience, we shall write

$$R_{ij}(k) := \sum_{n\ge 0} \left(M(i,m,mn+k) - M(j,m,mn+k)\right) y^n, \tag{2.16}$$

and also

$$h(a) := h(y^a;y^m), \tag{2.17}$$

so that Lemma 1, with q replaced by y^m and z by y^a, states that

$$h(a) - h(m+a) = 1, \tag{2.18}$$

$$h(a) + h(m-a) = -1, \tag{2.19}$$

$$3h(a) - h(3a) = \frac{P^3(2a)\,P^2(0)}{P^3(a)\,P(3a)} - \frac{P^3(4a)\,P^2(0)}{P^3(2a)\,P(6a)}. \tag{2.20}$$

3. Some results for cranks modulo 5

Taking $m = 5$ in (2.5), with the help of (2.3) and (2.4), we find that

$$\begin{aligned} M(0) &= \mathbf{F}\,(2S(0)+1)\,, \\ M(1) &= \mathbf{F}\,(S(1)-S(0))\,, \\ M(2) &= -\mathbf{F}\,(S(1))\,. \end{aligned} \tag{3.1}$$

Taking $m = 5$ and $b = 2,\ 1$ in (2.14), we have

$$\begin{aligned} S(0) &= h(2) - q\frac{P^2(0)\,P(2)}{P^2(1)} + q^3\frac{P^2(0)}{P(2)}, \\ S(1) &= h(1) - q^2\frac{P^2(0)}{P(1)} + q^4\frac{P^2(0)\,P(1)}{P^2(2)}. \end{aligned} \tag{3.2}$$

Putting $a = 1$ in (2.18) and (2.20) with $m = 5$, and $a = 2$ in (2.19) and (2.20) with $m = 5$, we also obtain

$$\begin{aligned} h(1) &= -\frac{1}{5} + \frac{1}{5}\left\{\frac{P^2(0)\,P^2(2)}{P^3(1)} + 2y\frac{P^2(0)\,P^2(1)}{P^3(2)}\right\}, \\ h(2) &= -\frac{2}{5} + \frac{1}{5}\left\{2\frac{P^2(0)\,P^2(2)}{P^3(1)} - y\frac{P^2(0)\,P^2(1)}{P^3(2)}\right\}. \end{aligned} \tag{3.3}$$

After all these preparations the following is easily proven.

Theorem 1.

$$R_{01}(0) = \frac{P(2)\,P(0)}{P^2(1)}, \tag{3.4}$$

$$R_{01}(1) = -2\frac{P(0)}{P(1)}, \tag{3.5}$$

$$R_{12}(1) = \frac{P(0)}{P(1)}, \tag{3.6}$$

$$R_{12}(2) = -\frac{P(0)}{P(2)}, \tag{3.7}$$

$$R_{01}(3) = -R_{12}(3) = \frac{P(1)\,P(0)}{P^2(2)} \tag{3.8}$$

and all other functions $R_{b,b+1}(d)$, where $b = 0$ or 1, are zero.

By (3.1), to prove the theorem we only have to show that

$$\begin{aligned} 3S(0) - S(1) + 1 &= \left\{\frac{P(0)\,P(2)}{P^2(1)} - 2q\frac{P(0)}{P(1)} + q^3\frac{P(0)\,P(1)}{P^2(2)}\right\}\mathbf{F}^{-1}, \\ 2S(1) - S(0) &= \left\{q\frac{P(0)}{P(1)} - q^2\frac{P(0)}{P(2)} - q^3\frac{P(0)\,P(1)}{P^2(2)}\right\}\mathbf{F}^{-1}. \end{aligned}$$

Since by (2.15) we have

$$\mathbf{F}^{-1} = P(0)\left\{\frac{P(2)}{P(1)} - q - q^2\frac{P(1)}{P(2)}\right\},$$

these are respectively equivalent to

$$3h(2) - h(1) + 1 = \frac{P^2(0)\,P^2(2)}{P^3(1)} - y\frac{P^2(0)\,P^2(1)}{P^3(2)},$$

$$2h(1) - h(2) = y\frac{P^2(0)\,P^2(1)}{P^3(2)},$$

which are true by (3.3). This proves the theorem.

4. Some results for cranks modulo 7

Here and in the next section we need the following for simplifications, which is Lemma 4 of [3],

$$\begin{aligned} P^2(b)P(c+d)P(c-d) - P^2(c)P(b+d)P(b-d)& \\ + y^{c-d}P^2(d)P(b+c)P(b-c) &= 0, \end{aligned} \tag{4.1}$$

where none of $b,\ c,\ d,\ b \mp c,\ c \mp d,\ b \mp d$ is divisible by m. This gives, for $(b,\ c,\ d) = (3,\ 2,\ 1)$,

$$P(1)P^3(3) - P(3)P^3(2) + yP(2)P^3(1) = 0. \tag{4.2}$$

As in the previous section, taking $m = 7$ in (2.5), with the help of (2.3) and (2.4) we find that

$$\begin{aligned} M(0) &= \mathbf{F}\left(2S(0) + 1\right), \\ M(1) &= \mathbf{F}\left(S(1) - S(0)\right), \\ M(2) &= \mathbf{F}\left(S(2) - S(1)\right), \\ M(3) &= -\mathbf{F}\left(S(2)\right). \end{aligned} \tag{4.3}$$

Taking $m = 7$ and $b = 3,\ 2,$ and 1 in (2.14), we have

$$\begin{aligned} S(0) &= h(3) - q\frac{P^2(0)\,P^2(3)}{P^2(2)\,P(1)} + q^3\frac{P^2(0)}{P(1)} - q^6\frac{P^2(0)}{P(3)}, \\ S(1) &= h(2) - q^2\frac{P^2(0)\,P^2(2)}{P^2(1)\,P(3)} + q^5\frac{P^2(0)}{P(2)} + q^6\frac{P^2(0)}{P(3)}, \\ S(2) &= h(1) - q^3\frac{P^2(0)}{P(1)} + q^5\frac{P^2(0)}{P(2)} - q^{11}\frac{P^2(0)\,P^2(1)}{P^2(3)\,P(2)}. \end{aligned} \tag{4.4}$$

Putting $a = 1,\ 2$ and 3, respectively, in (2.18)-(2.20) with $m = 7$, and using (4.2), we obtain

$$\begin{aligned} h(1) &= -\frac{1}{7} + \frac{1}{7}P^2(0)\left\{\frac{P(3)}{P^2(1)} + 3y\frac{P(1)}{P^2(2)} + 2y\frac{P(2)}{P^2(3)}\right\}, \\ h(2) &= -\frac{2}{7} + \frac{1}{7}P^2(0)\left\{2\frac{P(3)}{P^2(1)} - y\frac{P(1)}{P^2(2)} - 3y\frac{P(2)}{P^2(3)}\right\}, \\ h(3) &= -\frac{3}{7} + \frac{1}{7}P^2(0)\left\{3\frac{P(3)}{P^2(1)} + 2y\frac{P(1)}{P^2(2)} - y\frac{P(2)}{P^2(3)}\right\}. \end{aligned} \tag{4.5}$$

After all these preparations the following is easily proven.

Theorem 2.

$$R_{01}(0) = \frac{P(3)\,P(0)}{P(1)\,P(2)}, \tag{4.6}$$

$$R_{01}(1) = -2\frac{P(0)}{P(1)}, \tag{4.7}$$

$$R_{12}(1) = \frac{P(0)}{P(1)}, \tag{4.8}$$

$$R_{12}(2) = -R_{23}(2) = -\frac{P(2)\,P(0)}{P(1)\,P(3)}, \tag{4.9}$$

$$R_{01}(3) = -R_{23}(3) = \frac{P(0)}{P(2)}, \tag{4.10}$$

$$R_{01}(4) = -R_{12}(4) = \frac{P(0)}{P(3)}, \tag{4.11}$$

$$R_{01}(6) = -R_{12}(6) = R_{23}(6) = -\frac{P(1)\,P(0)}{P(2)\,P(3)}, \tag{4.12}$$

and all other functions $R_{b,b+1}(d)$, *where* $0 \le b \le 2$, *are zero.*

To prove the theorem we only consider the three pairs of values $(i,j) = (0,1)$, $(1,2)$ and $(2,3)$ in (2.16). So, by (4.3), we only have to show that

$$3S(0) - S(1) + 1 = \left\{\frac{P(0)\,P(3)}{P(1)\,P(2)} - 2q\frac{P(0)}{P(1)} + q^3\frac{P(0)}{P(2)} + q^4\frac{P(0)}{P(3)} - q^6\frac{P(0)\,P(1)}{P(2)\,P(3)}\right\}\mathbf{F}^{-1}, \tag{4.13}$$

$$2S(1) - S(2) - S(0) = \left\{q\frac{P(0)}{P(1)} - q^2\frac{P(0)\,P(2)}{P(1)\,P(3)} - q^4\frac{P(0)}{P(3)} + q^6\frac{P(0)\,P(1)}{P(2)\,P(3)}\right\}\mathbf{F}^{-1}, \tag{4.14}$$

$$2S(2) - S(1) = \left\{q^2\frac{P(0)\,P(2)}{P(1)\,P(3)} - q^3\frac{P(0)}{P(2)} - q^6\frac{P(0)\,P(1)}{P(2)\,P(3)}\right\}\mathbf{F}^{-1}. \tag{4.15}$$

Now by (2.15) we have

$$\mathbf{F}^{-1} = P(0)\left\{\frac{P(2)}{P(1)} - q\frac{P(3)}{P(2)} - q^2 + q^5\frac{P(1)}{P(3)}\right\}.$$

Substituting this in each of (4.13)–(4.15) and equating coefficients of powers of q, we have 21 equations to prove. The coefficients of q^0 give us respectively

$$\begin{aligned}
3h(3) - h(2) + 1 &= P^2(0)\left\{y\frac{P(1)}{P^2(2)} + \frac{P(3)}{P^2(1)}\right\},\\
2h(2) - h(1) - h(3) &= -P^2(0)y\left\{\frac{P(2)}{P^2(3)} + \frac{P(1)}{P^2(2)}\right\},\\
2h(1) - h(2) &= -P^2(0)y\left\{\frac{P(2)}{P^2(3)} + \frac{P(1)}{P^2(2)}\right\},
\end{aligned}$$

which are true by (4.5). All the other equations are trivially satisfied except for the coefficients of $q,\ q^2$ and q^4 in (4.13), of q and q^2 in (4.14), and of q^4 in (4.15). The coefficients of $q,\ q^2$ and q^4 are respectively

$$P^2(0)\left\{\frac{P(2)}{P^2(1)}-\frac{P^2(3)}{P(1)\,P^2(2)}-y\frac{P(1)}{P(2)\,P(3)}\right\}=0,$$

$$P^2(0)\left\{y\frac{P(1)}{P^2(3)}-\frac{P^2(2)}{P(3)\,P^2(1)}+\frac{P(3)}{P(1)\,P(2)}\right\}=0,$$

$$P^2(0)\left\{\frac{P(3)}{P^2(2)}+y\frac{P^2(1)}{P(2)\,P^2(3)}-\frac{P(2)}{P(1)\,P(3)}\right\}=0,$$

and each of them reduces to (4.2). This proves the theorem.

5. Some results for cranks modulo 11

As in the previous section, taking $m=11$ in (2.5), with the help of (2.3) and (2.4) we find that

$$\begin{aligned}
M(0)&=\mathbf{F}\left(2S(0)+1\right),\\
M(1)&=\mathbf{F}\left(S(1)-S(0)\right),\\
M(2)&=\mathbf{F}\left(S(2)-S(1)\right),\\
M(3)&=\mathbf{F}\left(S(3)-S(2)\right),\\
M(4)&=\mathbf{F}\left(S(4)-S(3)\right),\\
M(5)&=-\mathbf{F}\left(S(4)\right).
\end{aligned}\tag{5.1}$$

Taking $m=11$ and $b=5,\ 4,\ 3,\ 2$ and 1 respectively in (2.14), we have

(5.2)
$$\begin{aligned}
S(0)=h(5)&-q\tfrac{P^2(0)\,P(3)\,P(5)}{P(1)\,P(2)\,P(4)}+q^3\tfrac{P^2(0)\,P^2(5)}{P(2)\,P^2(3)}-q^4y\tfrac{P^2(0)}{P(5)}\\
&-q^6\tfrac{P^2(0)\,P(5)}{P(2)\,P(3)}+q^{10}\tfrac{P^2(0)\,P(2)}{P(1)\,P(4)},
\end{aligned}$$

$$\begin{aligned}
S(1)=h(4)&-q^2\tfrac{P^2(0)\,P(5)}{P(1)\,P(3)}+q^3y\tfrac{P^2(0)}{P(4)}+q^5\tfrac{P^2(0)\,P^2(4)}{P(5)\,P^2(2)}\\
&-q^9\tfrac{P^2(0)\,P(2)\,P(4)}{P(1)\,P(3)\,P(5)}+q^{10}\tfrac{P^2(0)\,P(4)}{P(2)\,P(5)},
\end{aligned}$$

$$\begin{aligned}
S(2)=h(3)&-qy\tfrac{P^2(0)}{P(3)}-q^3\tfrac{P^2(0)\,P(3)\,P(4)}{P(1)\,P(2)\,P(5)}+q^7\tfrac{P^2(0)\,P(3)}{P(1)\,P(4)}\\
&-q^8y\tfrac{P^2(0)\,P(1)}{P(2)\,P(5)}+q^0\tfrac{P^2(0)\,P^2(3)}{P(1)\,P^2(4)},
\end{aligned}$$

$$\begin{aligned}
S(3)=h(2)&+q^4\tfrac{P^2(0)\,P^2(2)}{P(3)\,P^2(1)}+q^5y^2\tfrac{P^2(0)\,P(1)\,P(2)}{P(3)\,P(4)\,P(5)}-q^6y\tfrac{P^2(0)\,P(3)}{P(4)\,P(5)}\\
&+q^8\tfrac{P^2(0)\,P(2)}{P(1)\,P(3)}+q^9\tfrac{P^2(0)}{P(2)},
\end{aligned}$$

$$\begin{aligned}
S(4)=h(1)&-qy^3\tfrac{P^2(0)\,P^2(1)}{P(4)\,P^2(5)}+q^2y^2\tfrac{P^2(0)\,P(1)}{P(4)\,P(5)}-q^4y\tfrac{P^2(0)\,P(1)\,P(5)}{P(2)\,P(3)\,P(4)}\\
&-q^5\tfrac{P^2(0)}{P(1)}+q^7\tfrac{P^2(0)\,P(4)}{P(2)\,P(3)}.
\end{aligned}$$

Theorem 3.

$$R_{01}(0) = \frac{P(0)}{P(1)}, \tag{5.3}$$

$$-\frac{1}{2}R_{01}(1) = R_{12}(1) = \frac{P(5)\,P(0)}{P(2)\,P(3)}, \tag{5.4}$$

$$R_{12}(2) = -R_{23}(2) = -\frac{P(3)\,P(0)}{P(1)\,P(4)}, \tag{5.5}$$

$$R_{01}(3) = -R_{23}(3) = R_{34}(3) = \frac{P(2)\,P(0)}{P(1)\,P(3)}, \tag{5.6}$$

$$R_{01}(4) = -R_{12}(4) = R_{23}(4) = -R_{34}(4) = R_{45}(4) = \frac{P(0)}{P(2)}, \tag{5.7}$$

$$R_{12}(5) = -R_{23}(5) = R_{34}(5) = -R_{45}(5) = \frac{P(4)\,P(0)}{P(2)\,P(5)}, \tag{5.8}$$

$$R_{01}(7) = -R_{12}(7) = R_{34}(7) = -R_{45}(7) = -\frac{P(0)}{P(3)}, \tag{5.9}$$

$$R_{01}(8) = -R_{12}(8) = R_{23}(8) = -R_{45}(8) = -y\frac{P(1)\,P(0)}{P(4)\,P(5)}, \tag{5.10}$$

$$R_{01}(9) = -R_{34}(9) = R_{45}(9) = -\frac{P(0)}{P(4)}, \tag{5.11}$$

$$R_{23}(10) = -R_{34}(10) = \frac{P(0)}{P(5)}, \tag{5.12}$$

and all other functions $R_{b,b+1}(d)$, where $0 \le b \le 4$, are zero.

Since $R_{ij}(k) = -R_{ji}(k)$ and $R_{is}(k) + R_{sj}(k) = R_{ij}(k)$, to prove the theorem it is sufficient to consider the five pairs of values $(i,j) = (0,5),\ (1,5),\ (2,5),\ (3,5)$ and $(4,5)$, so we have to prove

(5.13)
$$2S(0) + S(4) + 1 = \left\{\frac{P(0)}{P(1)} - q\frac{P(0)\,P(5)}{P(2)\,P(3)} + q^3\frac{P(0)\,P(2)}{P(1)\,P(3)} + q^4\frac{P(0)}{P(2)} - q^9\frac{P(0)}{P(4)}\right\}\mathbf{F}^{-1},$$

(5.14)
$$S(1) + S(4) - S(0) = \left\{q\frac{P(0)\,P(5)}{P(2)\,P(3)} + q^7\frac{P(0)}{P(3)} + q^8 y\frac{P(0)\,P(1)}{P(4)\,P(5)}\right\}\mathbf{F}^{-1},$$

(5.15)
$$S(2) + S(4) - S(1) = \left\{q^2\frac{P(0)\,P(3)}{P(1)\,P(4)} + q^4\frac{P(0)}{P(2)} - q^5\frac{P(0)\,P(4)}{P(2)\,P(5)}\right\}\mathbf{F}^{-1},$$

(5.16)
$$S(3) + S(4) - S(2) = \left\{q^3\frac{P(0)\,P(2)}{P(1)\,P(3)} + q^8 y\frac{P(0)\,P(1)}{P(4)\,P(5)} - q^{10} y\frac{P(0)}{P(5)}\right\}\mathbf{F}^{-1},$$

(5.17)
$$2S(4) - S(3) = \left\{q^4\frac{P(0)}{P(2)} - q^5\frac{P(0)\,P(4)}{P(2)\,P(5)} + q^7\frac{P(0)}{P(3)} + q^8 y\frac{P(0)\,P(1)}{P(4)\,P(5)} - q^9\frac{P(0)}{P(4)}\right\}\mathbf{F}^{-1}.$$

By (2.15) with $m = 11$ ($\lambda = 2$ and $\mu = -1$), we have

$$\mathbf{F}^{-1} = P(0)\left\{\frac{P(4)}{P(2)} - q\frac{P(2)}{P(1)} - q^2\frac{P(5)}{P(3)} - q^4 y\frac{P(1)}{P(5)} + q^5 + q^7\frac{P(3)}{P(4)}\right\}. \tag{5.18}$$

Substituting (5.18) in each of (5.13)-(5.17) and equating the coefficients of powers of q, we have 55 equations to prove. To do this we need the following ten identities, which can be found by taking $(b, c, d) = (5, 4, 1)$, .$(5, 4, 2)$, $(4, 3, 1)$, $(5, 3, 2)$, $(3, 2, 1)$, $(5, 3, 1)$, $(5, 4, 3)$, $(5, 2, 1)$, $(4, 3, 2)$ and $(4, 2, 1)$ in (4.1):

$$(5.19)\qquad \begin{aligned} &P(3)P^3(5) - P(5)P^3(4) + y^3P(2)P^3(1) = 0, &&(a1)\\ &P(2)P^3(5) - P(3)P^3(4) + y^2P(1)P^3(2) = 0, &&(a2)\\ &P(2)P^3(4) - P(5)P^3(3) + y^2P(4)P^3(1) = 0, &&(a3)\\ &P(1)P^3(5) - P(4)P^3(3) + yP(3)P^3(2) = 0, &&(a4)\\ &P(1)P^3(3) - P(4)P^3(2) + yP(5)P^3(1) = 0. &&(a5) \end{aligned}$$

$$(5.20)\qquad \begin{aligned} &P(2)P(4)P^2(5) - P(4)P(5)P^2(3) + y^2P(2)P(3)P^2(1) = 0, &&(b1)\\ &P(1)P(4)P^2(5) - P(2)P(3)P^2(4) + yP(1)P(2)P^2(3) = 0, &&(b2)\\ &P(1)P(3)P^2(5) - P(4)P(5)P^2(2) + yP(3)P(4)P^2(1) = 0, &&(b3)\\ &P(1)P(5)P^2(4) - P(2)P(5)P^2(3) + yP(1)P(4)P^2(2) = 0, &&(b4)\\ &P(1)P(3)P^2(4) - P(3)P(5)P^2(2) + yP(2)P(5)P^2(1) = 0. &&(b5) \end{aligned}$$

Putting $a = 1,\ 2,\ 3,\ 4$ and 5, respectively, in (2.18)–(2.20) with $m = 11$, and using (4.2), we obtain

$$(5.21)\qquad \begin{aligned} 3h(1) - h(3) &= B_1,\\ 3h(2) + h(5) &= B_2 - 1,\\ 3h(3) + h(2) &= B_3 - 1,\\ 3h(4) - h(1) &= B_4 - 1,\\ 3h(5) - h(4) &= B_5 - 1, \end{aligned}$$

where

$$B_i = \frac{P^3(2i)\,P^2(0)}{P^3(i)\,P(3i)} - \frac{P^3(4i)\,P^2(0)}{P^3(2i)\,P(6i)} \qquad (i = 1, 2, \ldots, 5). \tag{5.22}$$

The solution of (5.21) is

$$(5.23)\qquad \begin{aligned} h(1) &= -\frac{1}{11} + \frac{1}{242}(81B_1 - 9B_2 + 27B_3 + B_4 + 3B_5),\\ h(2) &= -\frac{2}{11} + \frac{1}{242}(-3B_1 + 81B_2 - B_3 - 9B_4 - 27B_5),\\ h(3) &= -\frac{3}{11} + \frac{1}{242}(B_1 - 27B_2 + 81B_3 + 3B_4 + 9B_5),\\ h(4) &= -\frac{4}{11} + \frac{1}{242}(27B_1 - 3B_2 + 9B_3 + 81B_4 + B_5),\\ h(5) &= -\frac{5}{11} + \frac{1}{242}(9B_1 - B_2 + 3B_3 + 27B_4 + 81B_5). \end{aligned}$$

We simplify (5.23) by using some results of [4] as follows: Write

$$r=-y^2\frac{P(1)}{P(3)\,P(5)},\quad s=-y\frac{P(2)}{P(1)\,P(5)},\quad t=\frac{P(4)}{P(1)\,P(2)},$$
$$u=y\frac{P(3)}{P(2)\,P(4)},\quad v=y\frac{P(5)}{P(3)\,P(4)}. \tag{5.24}$$

Now, dividing ($b1$)–($b5$) respectively by

$$y^{-1}P(2)\,P(3)\,P(5)\,P^2(4),\qquad P(1)\,P(3)\,P(4)\,P^2(2),\qquad y^{-1}P(1)\,P(4)\,P(5)\,P^2(3),$$
$$P(2)\,P(4)\,P(5)\,P^2(1),\qquad y^{-1}P(1)\,P(2)\,P(3)\,P^2(5)$$

respectively, and dividing ($a1$)–($a5$) by

$$y^{-1}P(2)\,P(4)\,P^3(5),\qquad P(3)\,P(5)\,P^3(2),\qquad y^{-1}P(1)\,P(5)\,P^3(4),$$
$$P(1)\,P(2)\,P^3(3),\qquad P(3)\,P(4)\,P^3(1)$$

respectively, we find that

$$\begin{aligned} B_1 &= (r+u+v-s)\,P^2(0),\\ B_2 &= (t-r-s-v)\,P^2(0),\\ B_3 &= (t+u+s-v)\,P^2(0),\\ B_4 &= (t+r+s-u)\,P^2(0),\\ B_5 &= (u+v+t-r)\,P^2(0). \end{aligned} \tag{5.25}$$

Thus, (5.23) becomes

$$\begin{aligned} h(1) &= -\frac{1}{11}+\frac{1}{11}(4r-2s+t+5u+3v)P^2(0),\\ h(2) &= -\frac{2}{11}+\frac{1}{11}(-3r-4s+2t-u-5v)P^2(0),\\ h(3) &= -\frac{3}{11}+\frac{1}{11}(r+5s+3t+4u-2v)P^2(0),\\ h(4) &= -\frac{4}{11}+\frac{1}{11}(5r+3s+4t-2u+v)P^2(0),\\ h(5) &= -\frac{5}{11}+\frac{1}{11}(-2r+s+5t+3u+4v)P^2(0). \end{aligned} \tag{5.26}$$

Therefore,

(5.27)
$$2h(5)+h(1)+1=\left\{\frac{P(4)}{P(2)\,P(1)}+y\frac{P(3)}{P(2)\,P(4)}+y\frac{P(5)}{P(3)\,P(4)}\right\}P^2(0),$$
$$h(4)+h(1)-h(5)=y^2\frac{P(1)\,P^2(0)}{P(3)\,P(5)},$$
$$h(3)+h(1)-h(4)=y\frac{P(3)\,P^2(0)}{P(2)\,P(4)},$$
$$h(2)+h(1)-h(3)=y\frac{P(2)\,P^2(0)}{P(1)\,P(5)},$$
$$h(1)-h(2)=\left\{\frac{P(5)}{P(3)\,P(4)}-y\frac{P(1)}{P(3)\,P(5)}+y\frac{P(3)}{P(2)\,P(4)}\right\}P^2(0),$$

which are the coefficients of q^0 in each of (5.13)–(5.17). For the other coefficients, we substract the left–hand sides from the right-hand sides in each of (5.13)-(5.17), and see that some coefficients are zero directly, others, by the help of $(b1)$–$(b5)$.

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