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RELATIVE COMPLETIONS OF LINEAR GROUPS OVER $\mathbb{Z}[t]$ AND $\mathbb{Z}[t, t^{-1}]$

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ABSTRACT. We compute the completion of the groups $SL_n(\mathbb{Z}[t])$ and $SL_n(\mathbb{Z}[t, t^{-1}])$ relative to the obvious homomorphisms to $SL_n(\mathbb{Q})$; this is a generalization of the classical Malcev completion. We also make partial computations of the rational second cohomology of these groups.

The Malcev (or $\mathbb{Q}$-) completion of a group Γ is a prounipotent group $\mathcal{P}$ defined over $\mathbb{Q}$ together with a homomorphism $\varphi : \Gamma \to \mathcal{P}$ satisfying the following universal mapping property: If $\psi : \Gamma \to \mathcal{U}$ is a map of Γ into a prounipotent group, then there is a unique map $\Phi : \mathcal{P} \to \mathcal{U}$ such that $\psi = \Phi\varphi$. If $H_1(\Gamma, \mathbb{Q}) = 0$, then the group $\mathcal{P}$ is trivial and is therefore useless for studying Γ. In particular, the Malcev completions of the groups $SL_n(\mathbb{Z}[t])$ and $SL_n(\mathbb{Z}[t, t^{-1}])$ are trivial when $n \geq 3$ (this follows from the work of Suslin [15]).

Here we consider Deligne's notion of relative completion. Suppose $\rho : \Gamma \to S$ is a representation of Γ in a semisimple linear algebraic group over $\mathbb{Q}$. Suppose that the image of ρ is Zariski dense in S. The completion of Γ relative to ρ is a proalgebraic group $\mathcal{G}$ over $\mathbb{Q}$, which is an extension of S by a prounipotent group $\mathcal{U}$, and homomorphism $\tilde{\rho} : \Gamma \to \mathcal{G}$ which lifts ρ and has Zariski dense image. When S is the trivial group, $\mathcal{G}$ is simply the classical Malcev completion. The relative completion satisfies an obvious universal mapping property. The basic theory of relative completion was developed by R. Hain [6] (and independently by E. Looijenga (unpublished)), and is reviewed in Section 2 below.

In this paper, we consider the completions of the groups $SL_n(\mathbb{Z}[t])$ and $SL_n(\mathbb{Z}[t, t^{-1}])$ relative to the homomorphisms to $SL_n(\mathbb{Q})$ given by setting $t = 0$ (respectively, $t = 1$). There is an obvious candidate for the relative completion, namely the proalgebraic group $SL_n(\mathbb{Q}[[T]])$. The map

$$SL_n(\mathbb{Z}[t]) \to SL_n(\mathbb{Q}[[T]])$$

is the obvious inclusion and the map

$$SL_n(\mathbb{Z}[t, t^{-1}]) \to SL_n(\mathbb{Q}[[T]])$$

is induced by the map $t \mapsto 1 + T$.

Theorem. *For all $n \geq 3$, the group $SL_n(\mathbb{Q}[[T]])$ is the relative completion of both $SL_n(\mathbb{Z}[t])$ and $SL_n(\mathbb{Z}[t, t^{-1}])$.*

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Remark. We expect that the theorem holds for an arbitrary simple group G of sufficiently large rank (large enough to guarantee the vanishing of $H^2(G(\mathbb{Z}), A)$ for nontrivial $G(\mathbb{Q})$-modules A). We have chosen to work with SL_n just to be concrete.

The theorem does not hold for $n = 2$ (see Section 5 below). Our proof breaks down in this case essentially because the $\mathbb{Z}$-Lie algebra $\mathfrak{sl}_2(\mathbb{Z})$ is not perfect.

We use this result to study the cohomology of the groups $SL_n(\mathbb{Z}[t])$ and $SL_n(\mathbb{Z}[t, t^{-1}])$. This is motivated by an attempt to find unstable analogues of the Fundamental Theorem of Algebraic K–theory. Recall that if A is a regular ring, then there are natural isomorphisms $K_\bullet(A[t]) \cong K_\bullet(A)$ and $K_\bullet(A[t, t^{-1}]) \cong K_\bullet(A) \oplus K_{\bullet-1}(A)$. An unstable analogue does exist for infinite fields: If k is an infinite field, then $H_\bullet(SL_n(k[t]), \mathbb{Z}) \cong H_\bullet(SL_n(k), \mathbb{Z})$ for all n [10]. Since $\mathbb{Z}$ is regular, one might guess that an analogous statement holds for n sufficiently large. We note, however, that if such a result holds, we must have $n \geq 3$ since $H_1(SL_2(\mathbb{Z}[t]), \mathbb{Z})$ has infinite rank [4], while $H_1(SL_2(\mathbb{Z}), \mathbb{Z}) \cong \mathbb{Z}/12$.

The basic idea is to use continuous cohomology. Following Hain [5], we define the continuous cohomology of a group π to be

$$H^\bullet_{\mathrm{cts}}(\pi, \mathbb{Q}) = \varinjlim H^\bullet(\pi/\Gamma^r\pi, \mathbb{Q}),$$

where $\Gamma^\bullet\pi$ denotes the lower central series of π. There is a natural map

$$H^\bullet_{\mathrm{cts}}(\pi, \mathbb{Q}) \longrightarrow H^\bullet(\pi, \mathbb{Q})$$

which is injective in degree 2 provided $H_1(\pi, \mathbb{Q})$ is finite dimensional.

Consider the extension

$$1 \longrightarrow K(R) \longrightarrow SL_n(R) \longrightarrow SL_n(\mathbb{Z}) \longrightarrow 1$$

for $R = \mathbb{Z}[t], \mathbb{Z}[t, t^{-1}]$. This yields a spectral sequence for computing the rational cohomology of $SL_n(R)$. In light of the following result, it is reasonable to conjecture that $H^2(SL_n(R), \mathbb{Q}) = 0$ for $n \geq 3$.

Theorem. *If $n \geq 3$, then $H^0(SL_n(\mathbb{Z}), H^2_{\mathrm{cts}}(K(R), \mathbb{Q})) = 0$.*

Of course, one can see that $H^2(SL_n(R), \mathbb{Q}) = 0$ for $n \geq 5$ by using van der Kallen's stability theorem [8] and the Fundamental Theorem of Algebraic K-theory. The above result provides evidence for the vanishing of $H^2(SL_n(R), \mathbb{Q})$ for $n = 3, 4$. We note, however, that $H^2(SL_2(\mathbb{Z}[t]), \mathbb{Q})$ has infinite rank (this is a consequence of results of Grunewald, et al. [4]).

The study of the relative completion of the fundamental group of a complex algebraic variety X is related to the study of variations of mixed Hodge structure over X [6]. Moreover, relative completions were used with great success by R. Hain in his study of mapping class groups $\mathcal{M}_g$ and Torelli groups $\mathcal{T}_g$ [6],[7]. In particular, he was able to provide a presentation of the Malcev Lie algebra of $\mathcal{T}_g$ which in turn gives a partial computation of $H^2(\mathcal{T}_g, \mathbb{Q})$. This also yields a description of the completion $\mathcal{G}_g$ of $\mathcal{M}_g$ with respect to its representation on the first homology of the surface. However, the map $\mathcal{M}_g \to \mathcal{G}_g$ remains a mystery. As far as we know, the results of this paper provide the first concrete descriptions of relative completions and the associated homomorphisms aside from the obvious trivial ones (*e.g.*, $SL_n(\mathbb{Z}) \to SL_n(\mathbb{Q})$, $n \geq 3$).

This paper obviously owes a great deal to the work of Dick Hain and I thank him for suggesting this problem to me. I would also like to thank the referee for many useful comments.

1. Malcev completions

Recall that the Malcev completion of a group Γ is a prounipotent group $\mathcal{M}$ over $\mathbb{Q}$, together with a map $\Gamma \to \mathcal{M}$ which satisfies the obvious universal mapping property. We recall the construction of $\mathcal{M}$ as given by Bousfield [3].

First, suppose that G is a nilpotent group. The Malcev completion of G consists of a group $\widehat{G}$ and a homomorphism $j : G \to \widehat{G}$. It is characterized by the following three properties [13, Appendix A, Cor. 3.8]:

1. $\widehat{G}$ is nilpotent and uniquely divisible.
2. The kernel of j is the torsion subgroup of G.
3. If $x \in \widehat{G}$, then $x^n \in \operatorname{im} j$ for some $n \neq 0$.

Quillen constructs $\widehat{G}$ as the set of grouplike elements of the completed group algebra $\widehat{\mathbb{Q}}G$ (completed with respect to the augmentation ideal).

Now, if G is an arbitrary group, denote by G_r the nilpotent group $G/\Gamma^r G$, where $\Gamma^\bullet G$ is the lower central series of G. Following Bousfield [3], we define the Malcev completion of G to be

$$\mathcal{M} = \varprojlim \widehat{G}_r$$

where $\widehat{G}_r$ is the Malcev completion of G_r. One can easily check that the group $\mathcal{M}$ satisfies the universal mapping property.

2. Relative completions

In this section we review the theory of relative completion. All results in this section are due to R. Hain [6].

Let Γ be a group and $\rho : \Gamma \to S$ a Zariski dense representation of Γ in a semisimple algebraic group S over $\mathbb{Q}$. The completion of Γ relative to ρ may be constructed as follows. Consider all commutative diagrams of the form

$$\begin{tikzcd} 1 \arrow[r] & U \arrow[r] & E \arrow[r] & S \arrow[r] & 1 \\ & & \Gamma \arrow[u, "\tilde{\rho}"] \arrow[ur, "\rho"'] & & \end{tikzcd}$$

where E is a linear algebraic group over $\mathbb{Q}$, U is a unipotent subgroup of E, and the image of $\tilde{\rho}$ is Zariski dense. The collection of all such diagrams forms an inverse system [6, Prop 2.1] and we define the completion of Γ relative to ρ to be

$$\mathcal{G} = \varprojlim E.$$

The group $\mathcal{G}$ satisfies the following universal mapping property. Suppose that $\mathcal{E}$ is a proalgebraic group over $\mathbb{Q}$ such that there is a map $\mathcal{E} \to S$ with prounipotent kernel. If $\varphi : \Gamma \to \mathcal{E}$ is a homomorphism whose composition with $\mathcal{E} \to S$ is ρ, then there is a unique map $\tau : \mathcal{G} \to \mathcal{E}$ such that the following diagram commutes:

$$\begin{tikzcd} & \mathcal{G} \arrow[dr] \arrow[dd, "\tau"'] & \\ \Gamma \arrow[ur] \arrow[dr, "\varphi"'] & & S \\ & \mathcal{E} \arrow[ur] & \end{tikzcd}$$

Denote by L the image of $\rho : \Gamma \to S$ and by T the kernel. Let $\Gamma \to \mathcal{G}$ be the completion of Γ relative to ρ and let $\mathcal{U}$ be the prounipotent radical of $\mathcal{G}$. Consider the commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & T & \to & \Gamma & \to & L & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \mathcal{U} & \to & \mathcal{G} & \to & S & \to & 1. \end{array}$$

Denote by $\mathcal{T}$ the classical Malcev completion of T. The universal mapping property of $\mathcal{T}$ gives a map $\Phi : \mathcal{T} \to \mathcal{U}$ whose composition with $T \to \mathcal{T}$ is the map $T \to \mathcal{U}$. Denote the kernel of Φ by $\mathcal{K}$. We have the following three results.

Proposition 2.1 ([6, Prop. 4.5])**.** *Suppose that $H_1(T, \mathbb{Q})$ is finite dimensional. If the action of L on $H_1(T, \mathbb{Q})$ extends to a rational representation of S, then $\mathcal{K}$ is contained in the center of $\mathcal{T}$.* □

Proposition 2.2 ([6, Prop. 4.6])**.** *Suppose that the natural action of L on $H_1(T, \mathbb{Q})$ extends to a rational representation of S. If $H^1(L, A) = 0$ for all rational representations A of S, then Φ is surjective.* □

Proposition 2.3 ([6, Prop. 4.13])**.** *Suppose $H_1(T, \mathbb{Q})$ is finite dimensional and that $H^1(L, A)$ vanishes for all rational representations A of S. Suppose further that $H^2(L, A) = 0$ for all nontrivial rational representations of S. Then there is a surjective map $H_2(L, \mathbb{Q}) \to \mathcal{K}$.* □

Observe that $H^1(SL_n(\mathbb{Z}), A) = 0$ for $n \geq 3$ by Raghunathan's theorem [14]. Moreover, the second condition that $H^2(SL_n(\mathbb{Z}), A) = 0$ for all nontrivial A holds for $n \geq 9$ [2].[1]

3. The Malcev completion of $K(R)$

Consider the short exact sequences

$$1 \longrightarrow K(R) \longrightarrow SL_n(R) \longrightarrow SL_n(\mathbb{Z}) \longrightarrow 1$$

for $R = \mathbb{Z}[t], \mathbb{Z}[t, t^{-1}]$, and $n \geq 3$. In this section we compute the Malcev completion of $K(R)$.

Denote by $\mathfrak{m}_{\mathbb{Z}[t]}$ (resp. $\mathfrak{m}_{\mathbb{Z}[t,t^{-1}]}$) the ideal (t) (resp. $(t-1)$) of $\mathbb{Z}[t]$ (resp. $\mathbb{Z}[t, t^{-1}]$). For each $l \geq 1$, define a subgroup $K^l(R)$ by

$$K^l(R) = \{X \in K(R) : X \equiv I \mod \mathfrak{m}_R^l\}.$$

One can easily check that $K^\bullet(R)$ is a descending central series; that is,

$$[K^i, K^j] \subseteq K^{i+j}.$$

It follows that for each i, $\Gamma^i K \subseteq K^i$.

For each $i \geq 1$, define homomorphisms ρ_i, σ_i as follows. If $X \in K^i(\mathbb{Z}[t])$, write

$$X = I + t^i X_i + \cdots + t^m X_m$$

where each X_j is a matrix with integer entries. Define

$$\rho_i : K^i(\mathbb{Z}[t]) \to \mathfrak{sl}_n(\mathbb{Z})$$

by $\rho_i(X) = X_i$. Similarly, if $Y \in K^i(\mathbb{Z}[t, t^{-1}])$ we may write

$$Y = I + (t-1)^i Y_i \mod (t-1)^{i+1}$$

[1] The result in [2] only implies vanishing for $n \geq 9$. However, this is easily strengthened to $n \geq 4$; one need only compute a certain constant which depends on the weights of $SL_n(\mathbb{Q})$-modules.

since $(t^{-1}-1)^i \equiv (-1)^i(t-1)^i \mod (t-1)^{i+1}$. Now define

$$\sigma_i : K^i(\mathbb{Z}[t,t^{-1}]) \to \mathfrak{sl}_n(\mathbb{Z})$$

by $\sigma_i(Y) = Y_i$. These maps are well defined since the condition $\det Z = 1$ in $K^i(R)$ forces trace $Z_i = 0$. Moreover, it is easy to see that the maps ρ_i, σ_i are surjective group homomorphisms with kernel K^{i+1}. Thus for each $i \geq 1$, we have

$$K^i(R)/K^{i+1}(R) \cong \mathfrak{sl}_n(\mathbb{Z}).$$

Consider the associated graded $\mathbb{Z}$-Lie algebra

$$\mathrm{Gr}^\bullet K(R) = \bigoplus_{i\geq 1} K^i(R)/K^{i+1}(R).$$

If $n \geq 3$, the Lie algebra $\mathfrak{sl}_n(\mathbb{Z})$ satisfies $\mathfrak{sl}_n(\mathbb{Z}) = [\mathfrak{sl}_n(\mathbb{Z}), \mathfrak{sl}_n(\mathbb{Z})]$. It follows that the graded algebra $\mathrm{Gr}^\bullet K(R)$ is generated by $\mathrm{Gr}^1 K(R)$. The following lemma is easily proved (compare with [13, Appendix A, Prop. 3.5]).

Lemma 3.1. *Let G be a group with filtration $G = G^1 \supseteq G^2 \supseteq \cdots$. Then the associated graded Lie algebra $\mathrm{Gr}^\bullet G$ is generated by $\mathrm{Gr}^1 G$ if and only if $G^r = G^{r+1}\Gamma^r$ for each $r \geq 1$.*

Corollary 3.2. *Suppose $\bigcap G^r = \{1\}$. If $\mathrm{Gr}^\bullet G$ is generated by $\mathrm{Gr}^1 G$, then the completions of G with respect to the filtration $G^\bullet$ and the lower central series $\Gamma^\bullet G$ are isomorphic; that is,*

$$\varprojlim G/G^r \cong \varprojlim G/\Gamma^r G.$$

Proof. Consider the short exact sequence

$$1 \longrightarrow G^r/\Gamma^r \longrightarrow G/\Gamma^r \longrightarrow G/G^r \longrightarrow 1.$$

Since $\mathrm{Gr}^\bullet G$ is generated by $\mathrm{Gr}^1 G$, we have $G^r = G^{r+1}\Gamma^r$ for each r. It follows that the inverse system $\{G^r/\Gamma^r\}$ is surjective. This, in turn, implies that the natural map

$$\varprojlim G/\Gamma^r \longrightarrow \varprojlim G/G^r$$

is surjective. Injectivity follows since the assumption that $\bigcap G^r = \{1\}$ implies that $\varprojlim G^r/\Gamma^r = \{1\}$. □

We now compute the Malcev completions of the groups $K(R)/K^i(R)$. We first provide the following result.

Lemma 3.3. *The completion of $\mathbb{Z}[t,t^{-1}]$ with respect to the ideal $(t-1)$ is the power series ring $\mathbb{Z}[[T]]$. The canonical map $\mathbb{Z}[t,t^{-1}] \to \mathbb{Z}[[T]]$ sends t to $1+T$.*

Proof. This follows easily once we note that in $\mathbb{Z}[t,t^{-1}]/(t-1)^m$, we have $t^{-1} = 1 + (t-1) + \cdots + (t-1)^{m-1}$, so that any polynomial in $\mathbb{Z}[t,t^{-1}]/(t-1)^m$ may be written as a polynomial in nonnegative powers of $(t-1)$. □

Consider the short exact sequence

$$1 \longrightarrow \overline{K} \longrightarrow SL_n(\mathbb{Z}[[T]]) \xrightarrow{T=0} SL_n(\mathbb{Z}) \longrightarrow 1.$$

Corollary 3.4. *The group $\overline{K}$ is the completion of $K(R)$ with respect to the filtration $K^1(R) \supset K^2(R) \supset \cdots$ and with respect to the lower central series of $K(R)$.*

Proof. The first assertion follows from Lemma 3.3 and the second from Corollary 3.2. □

Observe that the group $\overline{K}$ has a filtration given by powers of T (exactly as $K(\mathbb{Z}[t])$ does) and that the successive graded quotients are isomorphic to $\mathfrak{sl}_n(\mathbb{Z})$. Denote the filtration by $\overline{K}^{\bullet}$.

We have an analogous sequence over $\mathbb{Q}$:

$$1 \longrightarrow \mathcal{U} \longrightarrow SL_n(\mathbb{Q}[[T]]) \xrightarrow{T=0} SL_n(\mathbb{Q}) \longrightarrow 1,$$

and the corresponding T-adic filtration $\mathcal{U}^{\bullet}$ in $\mathcal{U}$. In this case, the successive graded quotients are isomorphic to $\mathfrak{sl}_n(\mathbb{Q})$. We can assemble our exact sequences into a commutative diagram

$$\begin{array}{ccccccccc}
1 & \rightarrow & K(R) & \rightarrow & SL_n(R) & \rightarrow & SL_n(\mathbb{Z}) & \rightarrow & 1 \\
 & & \downarrow & & \downarrow & & || & & \\
1 & \rightarrow & \overline{K} & \rightarrow & SL_n(\mathbb{Z}[[T]]) & \stackrel{T=0}{\rightarrow} & SL_n(\mathbb{Z}) & \rightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & \mathcal{U} & \rightarrow & SL_n(\mathbb{Q}[[T]]) & \stackrel{T=0}{\rightarrow} & SL_n(\mathbb{Q}) & \rightarrow & 1.
\end{array}$$

Proposition 3.5. *The map $K(R)/K^r(R) \xrightarrow{j} \mathcal{U}/\mathcal{U}^r$ is the Malcev completion.*

Proof. According to Quillen's criterion (see Section 1) we must check three things. First, the group $\mathcal{U}/\mathcal{U}^r$ is nilpotent and uniquely divisible. Nilpotency is obvious, so suppose

$$Y = I + TY_1 + \cdots + T^{r-1}Y_{r-1}$$

is an element of $\mathcal{U}/\mathcal{U}^r$. For each $n > 0$, we must find a unique $X \in \mathcal{U}/\mathcal{U}^r$ with $X^n = Y$. For an arbitrary $X \in \mathcal{U}/\mathcal{U}^r$, write $X = I + TX_1 + \cdots + T^{r-1}X_{r-1}$, and consider the equation

$$\begin{aligned}
Y &= X^n \\
&= I + TnX_1 + T^2(nX_2 + \binom{n}{2}X_1^2) + \cdots + T^{r-1}(nX_{r-1} + p(\{X_i\}_{i=1}^{r-2}))
\end{aligned}$$

where $p(X_1, \ldots, X_{r-2})$ is a polynomial in the X_i, $i \leq r-2$. Clearly, we can solve this equation inductively for the X_i and find a unique X.

Second, we must show that the kernel of j is the torsion subgroup of $K(R)/K^r(R)$. This is clear since $K(R)/K^r(R)$ is torsion-free (*i.e.,* if some power of $X \in K$ lies in K^r, then $X \in K^r$ already) and the map is injective.

Finally, we must show that if $X \in \mathcal{U}/\mathcal{U}^r$, then $X^m \in \operatorname{im} j$ for some $m \neq 0$. We prove this by induction on r, beginning at $r = 2$. Let $X = I + TX_1$ be an element of $\mathcal{U}/\mathcal{U}^2$. Then there is an $m > 0$ such that mX_1 consists of integer entries. Then $X^m = I + TmX_1$ lies in the image of j. Now suppose the result holds for $r-1$ and consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \rightarrow & K^{r-1}/K^r & \rightarrow & K/K^r & \rightarrow & K/K^{r-1} & \rightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{U}^{r-1}/\mathcal{U}^r & \rightarrow & \mathcal{U}/\mathcal{U}^r & \rightarrow & \mathcal{U}/\mathcal{U}^{r-1} & \rightarrow & 1.
\end{array}$$

Suppose $X \in \mathcal{U}/\mathcal{U}^r$. Denote its image in $\mathcal{U}/\mathcal{U}^{r-1}$ by $\overline{X}$. By the inductive hypothesis, there is an integer $m \neq 0$ with $\overline{X}^m = \overline{Y}$ for some $\overline{Y} \in K/K^{r-1}$. Choose a lift Y of $\overline{Y}$ in K/K^r. Then Y maps to $\overline{X}^m$ in $\mathcal{U}/\mathcal{U}^{r-1}$. But X^m also maps to $\overline{X}^m$ so that $X^mY^{-1} = Z$ for some $Z \in \mathcal{U}^{r-1}/\mathcal{U}^r$. Now, there exists some

$W \in K^{r-1}/K^r (\cong \mathfrak{sl}_n(\mathbb{Z}))$ with $Z^p = W$ for some $p \neq 0$. Since $Y = Z^{-1}X^m$, we have

$$\begin{aligned} Y^p &= (Z^{-1}X^m)^p \\ &= Z^{-p}X^{mp} \qquad (\text{since } \mathcal{U}^{r-1}/\mathcal{U}^r \text{ is central}) \\ &= W^{-1}X^{mp}. \end{aligned}$$

Thus, $X^{mp} = WY^p$ belongs to K/K^r and the induction is complete. □

Theorem 3.6. *The inclusion $K(R) \to \mathcal{U}$ is the Malcev completion.*

Proof. Since $\mathcal{U} = \varprojlim \mathcal{U}/\mathcal{U}^r$ and $\mathcal{U}/\mathcal{U}^r$ is the Malcev completion of $K(R)/K^r(R)$, the theorem will follow immediately if we can show that $K^r(R) = \Gamma^r K(R)$ for each r. This follows from the next two lemmas.

Lemma 3.7 ([3], 13.6)**.** *Let $F^\bullet$ be a central series in a group G such that:*

1. *The natural map $G \to \varprojlim G/F^s$ is an isomorphism.*
2. *F^s/F^{s+1} is torsion-free for $s \geq 1$.*
3. *The Lie product $G/F^2 \otimes F^s/F^{s+1} \to F^{s+1}/F^{s+2}$ is surjective for $s \geq 1$.*
4. *G/F^2 is finitely generated.*

Then $\Gamma^s G = F^s$ for $s \geq 1$. □

Lemma 3.8 ([3], 13.4)**.** *Let G be a group and denote by $\overline{G}$ the completion $\overline{G} = \varprojlim G/\Gamma^r G$. Then the following statements are equivalent:*

1. *The map $\overline{G} \to \overline{\overline{G}}$ is an isomorphism.*
2. *The map $G/\Gamma^r G \to \overline{G}/\Gamma^r \overline{G}$ is an isomorphism for each $r \geq 1$.*

Completion of the proof of Theorem 3.6. Consider the group

$$\overline{K} = \ker(SL_n(\mathbb{Z}[[T]]) \xrightarrow{T=0} SL_n(\mathbb{Z}))$$

with its T-adic filtration $\overline{K}^\bullet$. Note that Lemma 3.7 shows that $\overline{K}^r = \Gamma^r \overline{K}$ for each r: the first two conditions are clear, as is the fourth; the third condition follows since the Lie algebra $\mathfrak{sl}_n(\mathbb{Z})$ is perfect (it is here that we must exclude the case $n = 2$). Since

$$\overline{K} = \varprojlim K(R)/K^r(R) = \varprojlim K(R)/\Gamma^r K(R)$$

(the last equality is Corollary 3.4), and since

$$\begin{aligned} \overline{K} &= \varprojlim \overline{K}/\overline{K}^r \\ &= \varprojlim \overline{K}/\Gamma^r \overline{K} \\ &= \overline{\overline{K}}, \end{aligned}$$

Lemma 3.8 implies that $K(R)/\Gamma^r K(R) \cong \overline{K}/\Gamma^r \overline{K}$ for all r. Consider the commutative diagram

$$\begin{array}{ccc} K(R)/\Gamma^r K(R) & \xrightarrow{\cong} & \overline{K}/\Gamma^r \overline{K} \\ \downarrow & & \downarrow\cong \\ K(R)/K^r(R) & \xrightarrow{\cong} & \overline{K}/\overline{K}^r. \end{array}$$

It follows that $K^r(R) = \Gamma^r K(R)$ and hence $\mathcal{U}$ is the Malcev completion of $K(R)$. □

Remark 3.9. Even if Lemma 3.8 were not available, we could still prove the result as follows. Denote by $\mathcal{M}_r$ the Malcev completion of $K(R)/\Gamma^r K(R)$, and by $\mathcal{M} = \varprojlim \mathcal{M}_r$ the Malcev completion of $K(R)$. Then the map $K(R) \to \mathcal{M}$ factors through $\overline{K}$. Moreover, by the universal property of $\mathcal{M}$, we get a unique map $\mathcal{M} \to \mathcal{U}$ which is easily seen to be an isomorphism since it has an inverse given by the universal property of the Malcev completion $\overline{K} \to \mathcal{U}$.

Corollary 3.10. *If $n \geq 3$, then $H_1(K(R), \mathbb{Z}) \cong H_1(\overline{K}, \mathbb{Z}) \cong \mathfrak{sl}_n(\mathbb{Z})$.*

Proof. The first isomorphism follows from Lemma 3.8 and the second isomorphism from Lemma 3.7. □

4. The relative completion of $SL_n(R)$

We first prove the following result.

Lemma 4.1. *The group $\mathcal{U}$ has trivial center.*

Proof. Let X be a central element of $\mathcal{U}$. For a pair of integers $1 \leq i, j \leq n$, denote by $E_{ij}(a)$ the matrix having i, j-entry equal to a and all other entries 0. By computing the product (in both orders) of X with elementary matrices of the form $I + E_{i,j}(T) \in \mathcal{U}$ for $i \neq j$, we see that X must be a diagonal matrix with all entries equal, say $1 + a_1 T + a_2 T^2 + \cdots$. However, since X must have determinant 1, we see that $a_i = 0$ for all $i \geq 1$. □

Theorem 4.2. *If $n \geq 3$, then the map $SL_n(\mathbb{Z}[t]) \xrightarrow{t \mapsto T} SL_n(\mathbb{Q}[[T]])$ (resp. $SL_n(\mathbb{Z}[t, t^{-1}]) \xrightarrow{t \mapsto 1+T} SL_n(\mathbb{Q}[[T]])$) is the completion with respect to the map $SL_n(\mathbb{Z}[t]) \xrightarrow{t=0} SL_n(\mathbb{Q})$ (resp. $SL_n(\mathbb{Z}[t, t^{-1}]) \xrightarrow{t=1} SL_n(\mathbb{Q})$).*

Proof. The relative completion is a proalgebraic group which is an extension

$$1 \longrightarrow \mathcal{P} \longrightarrow \mathcal{G} \longrightarrow SL_n(\mathbb{Q}) \longrightarrow 1 \tag{4.1}$$

where $\mathcal{P}$ is prounipotent. By the universal property of $\mathcal{U}$, we have a unique map $\Phi : \mathcal{U} \to \mathcal{P}$ induced by the map $K(R) \to \mathcal{P}$. Since $H^1(SL_n(\mathbb{Z}), A) = 0$ for all rational $SL_n(\mathbb{Q})$-modules A [14], we see that Φ is surjective (Proposition 2.2). On the other hand, since $H_1(K(R), \mathbb{Q}) \cong \mathfrak{sl}_n(\mathbb{Q})$ is finite dimensional and the action of $SL_n(\mathbb{Z})$ on $H_1(K(R), \mathbb{Q})$ extends to a rational representation of $SL_n(\mathbb{Q})$, Proposition 2.1 implies that the kernel of Φ is central in $\mathcal{U}$. But by Lemma 4.1, the center of $\mathcal{U}$ is trivial. Thus, Φ is injective and $\mathcal{U} \cong \mathcal{P}$. Since the extension (4.1) is split ([6, Prop. 4.4]), it follows that $\mathcal{G} \cong SL_n(\mathbb{Q}[[T]])$. □

Remark 4.3. An alternate proof of the injectivity of Φ can be obtained via Proposition 2.3 for n sufficiently large. Since $H^2(SL_n(\mathbb{Z}), A)$ vanishes for nontrivial A when $n \geq 9$ [2], Proposition 2.3 asserts that the kernel of Φ is bounded above by $H_2(SL_n(\mathbb{Z}), \mathbb{Q}) = 0$. This can certainly be improved to $n \geq 4$ (but not to $n = 3$ since examples exist where $H^2(SL_3(\mathbb{Z}), A) \neq 0$).

5. The case $n = 2$

The proof of Theorem 4.2 breaks down in the case $n = 2$ for a variety of reasons.

1. The Lie algebra $\mathfrak{sl}_2(\mathbb{Z})$ is not perfect.
2. Raghunathan's theorem on the vanishing of $H^1(SL_n(\mathbb{Z}), A)$ does not apply for $n = 2$.

3. Borel's result for the vanishing of $H^2(SL_n(\mathbb{Z}), A)$ cannot be strengthened to include $n = 2$.

However, one can make the following observations. Denote by $\mathcal{G}(\mathbb{Z})$ the completion of $SL_2(\mathbb{Z})$ relative to its canonical inclusion in $SL_2(\mathbb{Q})$, and by $\mathcal{G}(R)$ the completion of $SL_2(R)$ $(R = \mathbb{Z}[t], \mathbb{Z}[t, t^{-1}])$ relative to the map $SL_2(R) \xrightarrow{\text{mod } \mathfrak{m}_R} SL_2(\mathbb{Q})$. The group $\mathcal{G}(\mathbb{Z})$ is *not* isomorphic to $SL_2(\mathbb{Q})$; in fact, it is an extension of $SL_2(\mathbb{Q})$ by a free prounipotent group with infinite dimensional H_1 (see [7, Rmk. 3.9]).

We have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K(R) & \longrightarrow & SL_2(R) & \overset{\longleftarrow}{\longrightarrow} & SL_2(\mathbb{Z}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathcal{W} & \longrightarrow & \mathcal{G}(R) & \overset{\longleftarrow}{\longrightarrow} & \mathcal{G}(\mathbb{Z}) & \longrightarrow & 1. \end{array}$$

The map $\Phi : \mathcal{G}(R) \to \mathcal{G}(\mathbb{Z})$ is induced by the composition $SL_2(R) \to SL_2(\mathbb{Z}) \to \mathcal{G}(\mathbb{Z})$ and the map $\Psi : \mathcal{G}(\mathbb{Z}) \to \mathcal{G}(R)$ is induced by the composition $SL_2(\mathbb{Z}) \to SL_2(R) \to \mathcal{G}(R)$. Since the composition $SL_2(\mathbb{Z}) \to SL_2(R) \to SL_2(\mathbb{Z})$ is the identity, we see that $\Phi \circ \Psi = \mathrm{id}_{\mathcal{G}(\mathbb{Z})}$.

If $n \geq 3$, then the map $SL_n(\mathbb{Z}) \to SL_n(\mathbb{Q})$ is the relative completion so that the completion of $SL_n(R)$ is an extension of the completion of $SL_n(\mathbb{Z})$ by the Malcev completion of $K(R)$. This leads us to make the following conjecture.

Conjecture 5.1. *The map $K(R) \longrightarrow \mathcal{W}$ is the Malcev completion.*

Note that there is at least some hope for this since $\mathcal{W}$ is properly contained in the kernel of the map $\mathcal{G}(R) \to SL_2(\mathbb{Q})$ so that $\mathcal{W}$ is prounipotent.

6. Cohomology

In this section we provide evidence for the following conjecture.

Conjecture 6.1. *If $n \geq 3$, then $H^2(SL_n(R), \mathbb{Q}) = 0$.*

Note that this conjecture is true for $n \geq 5$ for the following reason. If $n \geq 5$, then by van der Kallen's stability theorem [8], we have

$$H_2(SL_n(R), \mathbb{Z}) \cong H_2(SL(R), \mathbb{Z}) \cong K_2(R).$$

It follows that if $n \geq 5$, then $H_2(SL_n(R), \mathbb{Q}) \cong K_2(R) \otimes \mathbb{Q}$. Since $K_2(\mathbb{Z}[t]) \cong K_2(\mathbb{Z})$ and $K_2(\mathbb{Z}[t, t^{-1}]) \cong K_2(\mathbb{Z}) \oplus K_1(\mathbb{Z})$, we see that $K_2(R) \otimes \mathbb{Q} = 0$.

The tool that we will use is continuous cohomology. We define the continuous cohomology of a group π by

$$H^\bullet_{\mathrm{cts}}(\pi, \mathbb{Q}) = \varinjlim H^\bullet(\pi/\Gamma^r\pi, \mathbb{Q}).$$

The basic properties of continuous cohomology were established by Hain [5]. We note the following facts.

Proposition 6.2 ([5], Thm. 5.1)**.** *The natural map $H^k_{\mathrm{cts}}(\pi, \mathbb{Q}) \to H^k(\pi, \mathbb{Q})$ is an isomorphism for $k = 0, 1$ and is injective for $k = 2$.* □

The map on H^2 need not be surjective in general. A group π is called *pseudo-nilpotent* if the natural map $H^\bullet_{\mathrm{cts}}(\pi, \mathbb{Q}) \to H^\bullet(\pi, \mathbb{Q})$ is an isomorphism. Examples of pseudo-nilpotent groups include the pure braid groups, free groups and the fundamental groups of affine curves over $\mathbb{C}$.

Proposition 6.3 ([5, Thm. 3.7])**.** *Let π be a group with $H_1(\pi,\mathbb{Q})$ finite dimensional. Let $\mathcal{P}$ be the Malcev completion of π and denote by $\mathfrak{p}$ the Lie algebra of $\mathcal{P}$. Then the natural map*

$$H^\bullet_{\mathrm{cts}}(\pi,\mathbb{Q}) \longrightarrow H^\bullet_{\mathrm{cts}}(\mathfrak{p},\mathbb{Q})$$

is an isomorphism.

Thus, if $H_1(\pi,\mathbb{Q})$ is finite dimensional, we can use Lie algebra cohomology to obtain a lower bound on the dimension of $H^2(\pi,\mathbb{Q})$. We will not compute $H^2_{\mathrm{cts}}(K(R),\mathbb{Q})$ explicitly. However, we note the following result.

Proposition 6.4. *If $n \geq 3$, then $\dim H^2_{\mathrm{cts}}(K(R),\mathbb{Q}) \geq (n^2-1)^2/4$.*

Proof. By a result of Lubotzky and Magid [11], if G is a nilpotent group with $b_1 = \dim H_1(G,\mathbb{Q})$ finite, then the second Betti number b_2 satisfies $b_2 \geq b_1^2/4$. In the case of $K(R)/K^r(R)$, since $H_1(K/K^r,\mathbb{Q}) \cong \mathfrak{sl}_n(\mathbb{Q})$, we see that $b_2(K/K^r) \geq (n^2-1)^2/4$ for each r. □

To show that $H^2(SL_n(R),\mathbb{Q})$ vanishes, it would suffice to show the following three things.

1. $H^2(SL_n(\mathbb{Z}),\mathbb{Q}) = 0$.
2. $H^1(SL_n(\mathbb{Z}),H^1(K(R),\mathbb{Q})) = 0$.
3. $H^0(SL_n(\mathbb{Z}),H^2(K(R),\mathbb{Q})) = 0$.

The first statement is clear. The second follows from [14] since $H^1(K(R),\mathbb{Q})$ is the adjoint representation $\mathfrak{sl}_n(\mathbb{Q})$. The third statement is true for $n \geq 5$.

Proposition 6.5. *If $n \geq 5$, then $H^0(SL_n(\mathbb{Z}),H^2(K(R),\mathbb{Q})) = 0$.*

Proof. Consider the Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p(SL_n(\mathbb{Z}),H^q(K(R),\mathbb{Q})) \Longrightarrow H^{p+q}(SL_n(R),\mathbb{Q}).$$

We know that $H^2(SL_n(R),\mathbb{Q}) = 0$ for $n \geq 5$ (see the remarks following Conjecture 5.1). Note also that $H^2(SL_n(\mathbb{Z}),H^1(K(R),\mathbb{Q})) = 0$ and $H^3(SL_n(\mathbb{Z}),\mathbb{Q}) = 0$. It follows that $d_2 : E_2^{0,2} \to E_2^{2,1}$ and $d_3 : E_3^{0,2} \to E_3^{3,0}$ are both the zero map and hence $E_\infty^{0,2} = H^0(SL_n(\mathbb{Z}),H^2(K(R),\mathbb{Q}))$. But this group must vanish since $E_\infty^{1,1}$ and $E_\infty^{2,0}$ do. □

The next result provides evidence for the vanishing of $H^0(SL_n(\mathbb{Z}),H^2(K(R),\mathbb{Q}))$ when $n = 3,4$. We first state the following lemma, which can be proved via direct computation.

Lemma 6.6. *Let $\Gamma_{a_1,\dots,a_{n-1}}$ be the irreducible $SL_n(\mathbb{Q})$-module with highest weight $(a_1+\cdots+a_{n-1})L_1+\cdots+a_{n-1}L_{n-1}$, where $L_1,\dots,L_{n-1}$ are the weights of the fundamental representation. Then we have the following isomorphisms of $SL_n(\mathbb{Q})$-modules:*

1. $\mathfrak{sl}_3(\mathbb{Q})\otimes\mathfrak{sl}_3(\mathbb{Q}) \cong \Gamma_{2,2}\oplus\Gamma_{3,0}\oplus\Gamma_{0,3}\oplus\Gamma_{1,1}\oplus\Gamma_{1,1}\oplus\Gamma_{0,0}$,
2. $\bigwedge^2\mathfrak{sl}_3(\mathbb{Q}) \cong \Gamma_{3,0}\oplus\Gamma_{0,3}\oplus\Gamma_{1,1}$,
3. $\bigwedge^3\mathfrak{sl}_3(\mathbb{Q}) \cong \Gamma_{2,2}\oplus\Gamma_{3,0}\oplus\Gamma_{0,3}\oplus\Gamma_{1,1}\oplus\Gamma_{0,0}$,
4. $\mathfrak{sl}_4(\mathbb{Q})\otimes\mathfrak{sl}_4(\mathbb{Q}) \cong \Gamma_{2,0,2}\oplus\Gamma_{2,1,0}\oplus\Gamma_{0,1,2}\oplus\Gamma_{0,2,0}\oplus\Gamma_{1,0,1}\oplus\Gamma_{1,0,1}\oplus\Gamma_{0,0,0}$,
5. $\bigwedge^2\mathfrak{sl}_4(\mathbb{Q}) \cong \Gamma_{2,1,0}\oplus\Gamma_{0,1,2}\oplus\Gamma_{1,0,1}$,
6. $\bigwedge^3\mathfrak{sl}_4(\mathbb{Q}) \cong \Gamma_{4,0,0}\oplus\Gamma_{0,0,4}\oplus\Gamma_{1,2,1}\oplus\Gamma_{2,0,2}\oplus\Gamma_{2,1,0}\oplus\Gamma_{0,1,2}\oplus\Gamma_{0,2,0}\oplus\Gamma_{1,0,1}\oplus\Gamma_{0,0,0}$.

Theorem 6.7. *If $n \geq 3$, then $H^0(SL_n(\mathbb{Z}), H^2_{\mathrm{cts}}(K(R), \mathbb{Q})) = 0$.*

Proof. We need only consider the cases $n = 3, 4$. It suffices to show that

$$H^0(SL_n(\mathbb{Z}), H^2(K/K^l, \mathbb{Q})) = 0$$

for each l. We use Lie algebra cohomology. Denote by $\mathfrak{u}$ the Lie algebra of $\mathcal{U}$ and consider the T-adic filtration $\mathfrak{u}^\bullet$. The Malcev Lie algebra of K/K^l is the Lie algebra $\mathfrak{u}_l = \mathfrak{u}/\mathfrak{u}^l$. Observe that for each l, the quotient $\mathfrak{u}^{l-1}/\mathfrak{u}^l$ is isomorphic *as an $SL_n(\mathbb{Q})$-module* to the adjoint representation $\mathfrak{sl}_n(\mathbb{Q})$, but as a Lie algebra it is abelian (*i.e.,* to compute the bracket in $\mathfrak{u}^{l-1}/\mathfrak{u}^l$, we lift elements to $\mathfrak{u}^{l-1}$, apply $[\ ,\ ]$, and project back; but the commutator of any two elements in $\mathfrak{u}^{l-1}$ lies in $\mathfrak{u}^l$ and so projects to 0).

We proceed by induction on l, beginning at $l = 2$. The Lie algebra $\mathfrak{u}_2$ is abelian of dimension $n^2 - 1$; as an $SL_n(\mathbb{Q})$-module it is the adjoint representation $\mathfrak{sl}_n(\mathbb{Q})$. Thus $H^2(\mathfrak{u}_2, \mathbb{Q}) \cong \bigwedge^2 \mathfrak{sl}_n(\mathbb{Q})$ as an $SL_n(\mathbb{Q})$-module. By Lemma 6.6, parts 2 and 5, we see that $H^0(SL_n(\mathbb{Z}), H^2(\mathfrak{u}_2, \mathbb{Q})) = 0$. Now, suppose that $l > 2$ and that $H^0(SL_n(\mathbb{Z}), H^2(\mathfrak{u}_{l-1}, \mathbb{Q})) = 0$. Consider the short exact sequence

$$0 \longrightarrow \mathfrak{u}^{l-1}/\mathfrak{u}^l \longrightarrow \mathfrak{u}_l \longrightarrow \mathfrak{u}_{l-1} \longrightarrow 0.$$

The kernel is central in $\mathfrak{u}_l$. Consider the Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p(\mathfrak{u}_{l-1}, H^q(\mathfrak{u}^{l-1}/\mathfrak{u}^l, \mathbb{Q})) \Longrightarrow H^{p+q}(\mathfrak{u}_l, \mathbb{Q}).$$

We have isomorphisms of $SL_n(\mathbb{Q})$-modules:

1. $H^2(\mathfrak{u}_{l-1}, H^0(\mathfrak{u}^{l-1}/\mathfrak{u}^l, \mathbb{Q})) = H^2(\mathfrak{u}_{l-1}, \mathbb{Q})$,
2. $H^1(\mathfrak{u}_{l-1}, H^1(\mathfrak{u}^{l-1}/\mathfrak{u}^l, \mathbb{Q})) \cong \mathfrak{sl}_n(\mathbb{Q}) \otimes \mathfrak{sl}_n(\mathbb{Q})$,
3. $H^0(\mathfrak{u}_{l-1}, H^2(\mathfrak{u}^{l-1}/\mathfrak{u}^l, \mathbb{Q})) \cong \mathrm{Hom}_{\mathbb{Q}}(\bigwedge^2 \mathfrak{sl}_n(\mathbb{Q}), \mathbb{Q})$.

By induction, the $SL_n(\mathbb{Z})$ invariants of the first module are trivial and by Lemma 6.6, parts 2 and 5, so are the invariants of the last group. It follows that

$$H^0(SL_n(\mathbb{Z}), E_\infty^{0,2}) = 0.$$

Also, since

$$H^1(SL_n(\mathbb{Z}), E_\infty^{0,1}) = H^1(SL_n(\mathbb{Z}), \mathfrak{sl}_n(\mathbb{Q})) = 0,$$

the long exact cohomology sequence associated to the extension

$$0 \longrightarrow E_\infty^{0,1} \xrightarrow{d_2} H^2(\mathfrak{u}_{l-1}, \mathbb{Q}) \longrightarrow E_\infty^{2,0} \longrightarrow 0$$

shows that $H^0(SL_n(\mathbb{Z}), E_\infty^{2,0}) = 0$. It remains to show that $H^0(SL_n(\mathbb{Z}), E_\infty^{1,1})$ vanishes.

Note that $E_2^{1,1}$ contains a copy of the trivial representation (parts 1 and 4 of Lemma 6.6). However, the differential (known as *transgression* [1])

$$d_2 : E_2^{1,1} \longrightarrow H^3(\mathfrak{u}_{l-1}, \mathbb{Q})$$

is easily seen to map the trivial representation onto a copy of the trivial representation in the image (this copy arises from the map in cohomology induced by the map $\mathfrak{u}_{l-1} \to \mathfrak{sl}_n(\mathbb{Q})$; use parts 3 and 6 of Lemma 6.6). It follows that $E_\infty^{1,1}$ contains no copies of the trivial representation and hence $H^0(SL_n(\mathbb{Z}), E_\infty^{1,1}) = 0$. Thus

$$H^0(SL_n(\mathbb{Z}), H^2(\mathfrak{u}_l, \mathbb{Q})) = 0$$

and the induction is complete. ∎

One might conjecture that $K(R)$ is pseudo-nilpotent (we do not know if this is the case). If so, it would follow that $H^0(SL_n(\mathbb{Z}), H^2(K(R), \mathbb{Q})) = 0$ and hence $H^2(SL_n(R), \mathbb{Q}) = 0$ for $n \geq 3$.

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