RELATIVE COMPLETIONS OF LINEAR GROUPS
OVER \( \mathbb{Z}[t] \) AND \( \mathbb{Z}[t, t^{-1}] \)

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Abstract. We compute the completion of the groups \( SL_n(\mathbb{Z}[t]) \) and \( SL_n(\mathbb{Z}[t, t^{-1}]) \) relative to the obvious homomorphisms to \( SL_n(\mathbb{Q}) \); this is a generalization of the classical Malcev completion. We also make partial computations of the rational second cohomology of these groups.

The Malcev (or \( \mathbb{Q} \)-) completion of a group \( \Gamma \) is a pronipotent group \( P \) defined over \( \mathbb{Q} \) together with a homomorphism \( \varphi : \Gamma \to P \) satisfying the following universal mapping property: If \( \psi : \Gamma \to U \) is a map of \( \Gamma \) into a pronipotent group, then there is a unique map \( \Phi : P \to U \) such that \( \psi = \Phi \varphi \). If \( H_1(\Gamma, \mathbb{Q}) = 0 \), then the group \( P \) is trivial and is therefore useless for studying \( \Gamma \). In particular, the Malcev completions of the groups \( SL_n(\mathbb{Z}[t]) \) and \( SL_n(\mathbb{Z}[t, t^{-1}]) \) are trivial when \( n \geq 3 \) (this follows from the work of Suslin [15]).

Here we consider Deligne's notion of relative completion. Suppose \( \rho : \Gamma \to S \) is a representation of \( \Gamma \) in a semisimple linear algebraic group over \( \mathbb{Q} \). Suppose that the image of \( \rho \) is Zariski dense in \( S \). The completion of \( \Gamma \) relative to \( \rho \) is a proalgebraic group \( G \) over \( \mathbb{Q} \), which is an extension of \( S \) by a pronipotent group \( U \), and homomorphism \( \tilde{\rho} : \Gamma \to G \) which lifts \( \rho \) and has Zariski dense image. When \( S \) is the trivial group, \( G \) is simply the classical Malcev completion. The relative completion satisfies an obvious universal mapping property. The basic theory of relative completion was developed by R. Hain [6] (and independently by E. Looijenga (unpublished)), and is reviewed in Section 2 below.

In this paper, we consider the completions of the groups \( SL_n(\mathbb{Z}[t]) \) and \( SL_n(\mathbb{Z}[t, t^{-1}]) \) relative to the homomorphisms to \( SL_n(\mathbb{Q}) \) given by setting \( t = 0 \) (respectively, \( t = 1 \)). There is an obvious candidate for the relative completion, namely the proalgebraic group \( SL_n(\mathbb{Q}[T]) \). The map
\[
SL_n(\mathbb{Z}[t]) \to SL_n(\mathbb{Q}[T])
\]
is the obvious inclusion and the map
\[
SL_n(\mathbb{Z}[t, t^{-1}]) \to SL_n(\mathbb{Q}[T])
\]
is induced by the map \( t \mapsto 1 + T \).

Theorem. For all \( n \geq 3 \), the group \( SL_n(\mathbb{Q}[T]) \) is the relative completion of both \( SL_n(\mathbb{Z}[t]) \) and \( SL_n(\mathbb{Z}[t, t^{-1}]) \).
Remark. We expect that the theorem holds for an arbitrary simple group $G$ of sufficiently large rank (large enough to guarantee the vanishing of $H^2(G(\mathbb{Z}), A)$ for nontrivial $G(\mathbb{Q})$-modules $A$). We have chosen to work with $SL_n$ just to be concrete.

The theorem does not hold for $n = 2$ (see Section 5 below). Our proof breaks down in this case essentially because the $\mathbb{Z}$-Lie algebra $sl_2(\mathbb{Z})$ is not perfect.

We use this result to study the cohomology of the groups $SL_n(\mathbb{Z}[t])$ and $SL_n(\mathbb{Z}[t, t^{-1}])$. This is motivated by an attempt to find unstable analogues of the Fundamental Theorem of Algebraic $K$-theory. Recall that if $A$ is a regular ring, then there are natural isomorphisms $K_\ast(A[t]) \cong K_\ast(A)$ and $K_\ast(A[t, t^{-1}]) \cong K_\ast(A) \oplus K_{\ast-1}(A)$. An unstable analogue does exist for infinite fields: If $k$ is an infinite field, then $H_\ast(SL_n(k[t]), \mathbb{Z}) \cong H_\ast(SL_n(k), \mathbb{Z})$ for all $n$ \cite{10}. Since $\mathbb{Z}$ is regular, one might guess that an analogous statement holds for $n$ sufficiently large. We note, however, that if such a result holds, we must have $n \geq 3$ since $H_1(SL_2(\mathbb{Z}[t]), \mathbb{Z})$ has infinite rank \cite{4}, while $H_1(SL_2(\mathbb{Z}), \mathbb{Z}) \cong \mathbb{Z}/12$.

The basic idea is to use continuous cohomology. Following Hain \cite{5}, we define the continuous cohomology of a group $\pi$ to be

$$H^\bullet_{cts}(\pi, \mathbb{Q}) = \lim_{\Gamma^\bullet \pi} H^\bullet(\pi, \Gamma^\bullet \pi, \mathbb{Q}),$$

where $\Gamma^\bullet \pi$ denotes the lower central series of $\pi$. There is a natural map

$$H^\bullet_{cts}(\pi, \mathbb{Q}) \rightarrow H^\bullet(\pi, \mathbb{Q})$$

which is injective in degree 2 provided $H_1(\pi, \mathbb{Q})$ is finite dimensional.

Consider the extension

$$1 \rightarrow K(R) \rightarrow SL_n(R) \rightarrow SL_n(\mathbb{Z}) \rightarrow 1$$

for $R = \mathbb{Z}[t], \mathbb{Z}[t, t^{-1}]$. This yields a spectral sequence for computing the rational cohomology of $SL_n(R)$. In light of the following result, it is reasonable to conjecture that $H^2(SL_n(\mathbb{R}), \mathbb{Q}) = 0$ for $n \geq 3$.

**Theorem.** If $n \geq 3$, then $H^0(SL_n(\mathbb{Z}), H^2_{cts}(K(R), \mathbb{Q})) = 0$.

Of course, one can see that $H^2(SL_n(\mathbb{R}), \mathbb{Q}) = 0$ for $n \geq 5$ by using van der Kallen’s stability theorem \cite{8} and the Fundamental Theorem of Algebraic $K$-theory. The above result provides evidence for the vanishing of $H^2(SL_n(\mathbb{R}), \mathbb{Q})$ for $n = 3, 4$. We note, however, that $H^2(SL_2(\mathbb{Z}[t]), \mathbb{Q})$ has infinite rank (this is a consequence of results of Grunewald, et al. \cite{4}).

The study of the relative completion of the fundamental group of a complex algebraic variety $X$ is related to the study of variations of mixed Hodge structure over $X$ \cite{6}. Moreover, relative completions were used with great success by R. Hain in his study of mapping class groups $\mathcal{M}_g$ and Torelli groups $\mathcal{T}_g$ \cite{6,7}. In particular, he was able to provide a presentation of the Malcev Lie algebra of $\mathcal{T}_g$ which in turn gives a partial computation of $H^2(\mathcal{T}_g, \mathbb{Q})$. This also yields a description of the completion $\mathcal{G}_g$ of $\mathcal{M}_g$ with respect to its representation on the first homology of the surface. However, the map $\mathcal{M}_g \rightarrow \mathcal{G}_g$ remains a mystery. As far as we know, the results of this paper provide the first concrete descriptions of relative completions and the associated homomorphisms aside from the obvious trivial ones (e.g., $SL_n(\mathbb{Z}) \rightarrow SL_n(\mathbb{Q})$, $n \geq 3$).

This paper obviously owes a great deal to the work of Dick Hain and I thank him for suggesting this problem to me. I would also like to thank the referee for many useful comments.
1. Malcev completions

Recall that the Malcev completion of a group $\Gamma$ is a prounipotent group $\mathcal{M}$ over $\mathbb{Q}$, together with a map $\Gamma \to \mathcal{M}$ which satisfies the obvious universal mapping property. We recall the construction of $\mathcal{M}$ as given by Bousfield [3].

First, suppose that $G$ is a nilpotent group. The Malcev completion of $G$ consists of a group $\hat{G}$ and a homomorphism $j : G \to \hat{G}$. It is characterized by the following three properties [13, Appendix A, Cor. 3.8]:

1. $\hat{G}$ is nilpotent and uniquely divisible.
2. The kernel of $j$ is the torsion subgroup of $G$.
3. If $x \in \hat{G}$, then $x^n \in \text{im } j$ for some $n \neq 0$.

Quillen constructs $\hat{G}$ as the set of grouplike elements of the completed group algebra $\hat{\mathbb{Q}}G$ (completed with respect to the augmentation ideal).

Now, if $G$ is an arbitrary group, denote by $G_r$ the nilpotent group $G/\Gamma^r G$, where $\Gamma^r G$ is the lower central series of $G$. Following Bousfield [3], we define the Malcev completion of $G$ to be

$$\mathcal{M} = \lim_{\to} \hat{G}_r$$

where $\hat{G}_r$ is the Malcev completion of $G_r$. One can easily check that the group $\mathcal{M}$ satisfies the universal mapping property.

2. Relative completions

In this section we review the theory of relative completion. All results in this section are due to R. Hain [6].

Let $\Gamma$ be a group and $\rho : \Gamma \to S$ a Zariski dense representation of $\Gamma$ in a semisimple algebraic group $S$ over $\mathbb{Q}$. The completion of $\Gamma$ relative to $\rho$ may be constructed as follows. Consider all commutative diagrams of the form

$$
\begin{array}{cccccc}
1 & \longrightarrow & U & \longrightarrow & E & \longrightarrow & S & \longrightarrow & 1 \\
& & \downarrow{\bar{\rho}} & & \downarrow{\rho} & & \\
& & \rho & & \\
\end{array}
$$

where $E$ is a linear algebraic group over $\mathbb{Q}$, $U$ is a unipotent subgroup of $E$, and the image of $\bar{\rho}$ is Zariski dense. The collection of all such diagrams forms an inverse system [6] Prop 2.1] and we define the completion of $\Gamma$ relative to $\rho$ to be

$$G = \lim_{\to} E.$$ 

The group $G$ satisfies the following universal mapping property. Suppose that $E$ is a proalgebraic group over $\mathbb{Q}$ such that there is a map $E \to S$ with prounipotent kernel. If $\varphi : \Gamma \to E$ is a homomorphism whose composition with $E \to S$ is $\rho$, then there is a unique map $\tau : G \to E$ such that the following diagram commutes:

$$
\begin{array}{ccc}
G & \longrightarrow & E \\
\downarrow{\bar{\varphi}} & & \downarrow{\varphi} \\
\Gamma & \longrightarrow & S \\
\end{array}
$$
Denote by $L$ the image of $\rho : \Gamma \to S$ and by $T$ the kernel. Let $\Gamma \to \mathcal{G}$ be the completion of $\Gamma$ relative to $\rho$ and let $U$ be the prounipotent radical of $\mathcal{G}$. Consider the commutative diagram

\[
\begin{array}{ccc}
1 & \to & T & \to & \Gamma & \to & L & \to & 1 \\
& | & & | & & | & & | & \\
1 & \to & U & \to & \mathcal{G} & \to & S & \to & 1.
\end{array}
\]

Denote by $T$ the classical Malcev completion of $T$. The universal mapping property of $T$ gives a map $\Phi : T \to U$ whose composition with $T \to \mathcal{T}$ is the map $T \to U$.

Denote the kernel of $\Phi$ by $K$. We have the following three results.

**Proposition 2.1** ([6 Prop. 4.5]). Suppose that $H_1(T, \mathbb{Q})$ is finite dimensional. If the action of $L$ on $H_1(T, \mathbb{Q})$ extends to a rational representation of $S$, then $K$ is contained in the center of $T$.

**Proposition 2.2** ([6 Prop. 4.6]). Suppose that the natural action of $L$ on $H_1(T, \mathbb{Q})$ extends to a rational representation of $S$. If $H^1(L, A) = 0$ for all rational representations $A$ of $S$, then $\Phi$ is surjective.

**Proposition 2.3** ([6 Prop. 4.13]). Suppose $H_1(T, \mathbb{Q})$ is finite dimensional and that $H^1(L, A)$ vanishes for all rational representations $A$ of $S$. Suppose further that $H^2(L, A) = 0$ for all nontrivial rational representations of $S$. Then there is a surjective map $H_2(L, \mathbb{Q}) \to K$.

Observe that $H^1(SL_n(\mathbb{Z}), A) = 0$ for $n \geq 3$ by Raghunathan’s theorem [14]. Moreover, the second condition that $H^2(SL_n(\mathbb{Z}), A) = 0$ for all nontrivial $A$ holds for $n \geq 9$ [24].

3. The Malcev completion of $K(R)$

Consider the short exact sequences

\[1 \to K(R) \to SL_n(R) \to SL_n(\mathbb{Z}) \to 1\]

for $R = \mathbb{Z}[t], \mathbb{Z}[t, t^{-1}]$, and $n \geq 3$. In this section we compute the Malcev completion of $K(R)$.

Denote by $m_{\mathbb{Z}[t]}$ (resp. $m_{\mathbb{Z}[t, t^{-1}]}$) the ideal $(t)$ (resp. $(t-1)$) of $\mathbb{Z}[t]$ (resp. $\mathbb{Z}[t, t^{-1}]$). For each $l \geq 1$, define a subgroup $K^l(R)$ by

\[K^l(R) = \{ X \in K(R) : X \equiv I \mod m_{\mathbb{Z}[t]}^l \}.
\]

One can easily check that $K^*(R)$ is a descending central series; that is,

\[[K^i, K^j] \subseteq K^{i+j}.
\]

It follows that for each $i$, $\Gamma^i K \subseteq K^i$.

For each $i \geq 1$, define homomorphisms $\rho_i$, $\sigma_i$ as follows. If $X \in K^i(\mathbb{Z}[t])$, write

\[X = I + t^i X_i + \cdots + t^m X_m\]

where each $X_j$ is a matrix with integer entries. Define

\[\rho_i : K^i(\mathbb{Z}[t]) \to gl_n(\mathbb{Z})\]

by $\rho_i(X) = X_i$. Similarly, if $Y \in K^i(\mathbb{Z}[t, t^{-1}])$ we may write

\[Y = I + (t - 1)^i Y_i \mod (t - 1)^{i+1}
\]

\[\text{The result in [24] only implies vanishing for } n \geq 9. \text{ However, this is easily strengthened to } n \geq 4; \text{ one need only compute a certain constant which depends on the weights of } SL_n(\mathbb{Q})\text{-modules.}
since \((t^{-1} - 1)^i \equiv (-1)^i(t - 1)^i \mod (t - 1)^{i+1}\). Now define
\[
\sigma_i : K^i(\mathbb{Z}[t, t^{-1}]) \to \mathfrak{sl}_n(\mathbb{Z})
\]
by \(\sigma_i(Y) = Y_i\). These maps are well defined since the condition \(\det Z = 1\) in \(K^i(R)\) forces trace \(Z_i = 0\). Moreover, it is easy to see that the maps \(\rho_i\), \(\sigma_i\) are surjective group homomorphisms with kernel \(K^{i+1}\). Thus for each \(i \geq 1\), we have
\[
K^i(R) / K^{i+1}(R) \cong \mathfrak{sl}_n(\mathbb{Z}).
\]
Consider the associated graded \(\mathbb{Z}\)-Lie algebra
\[
\text{Gr}^i K(R) = \bigoplus_{i \geq 1} K^i(R) / K^{i+1}(R).
\]
If \(n \geq 3\), the Lie algebra \(\mathfrak{sl}_n(\mathbb{Z})\) satisfies \(\mathfrak{sl}_n(\mathbb{Z}) = [\mathfrak{sl}_n(\mathbb{Z}), \mathfrak{sl}_n(\mathbb{Z})]\). It follows that the graded algebra \(\text{Gr}^i K(R)\) is generated by \(\text{Gr}^1 K(R)\). The following lemma is easily proved (compare with [13 Appendix A, Prop. 3.5]).

**Lemma 3.1.** Let \(G\) be a group with filtration \(G = G^1 \supseteq G^2 \supseteq \cdots\). Then the associated graded Lie algebra \(\text{Gr}^i G\) is generated by \(\text{Gr}^1 G\) if and only if \(G^r = G^{r+1}\Gamma^r\) for each \(r \geq 1\).

**Corollary 3.2.** Suppose \(\bigcap G^r = \{1\}\). If \(\text{Gr}^i G\) is generated by \(\text{Gr}^1 G\), then the completions of \(G\) with respect to the filtration \(G^r\) and the lower central series \(\Gamma^r G\) are isomorphic; that is,
\[
\lim G / G^r \cong \lim G / \Gamma^r G.
\]

**Proof.** Consider the short exact sequence
\[
1 \to G^r / \Gamma^r \to G / \Gamma^r \to G / G^r \to 1.
\]
Since \(\text{Gr}^i G\) is generated by \(\text{Gr}^1 G\), we have \(G^r = G^{r+1}\Gamma^r\) for each \(r\). It follows that the inverse system \(\{G^r / \Gamma^r\}\) is surjective. This, in turn, implies that the natural map
\[
\lim G / \Gamma^r \to \lim G / G^r
\]
is surjective. Injectivity follows since the assumption that \(\bigcap G^r = \{1\}\) implies that \(\lim G / \Gamma^r = \{1\}\).

We now compute the Malcev completions of the groups \(K(R) / K^i(R)\). We first provide the following result.

**Lemma 3.3.** The completion of \(\mathbb{Z}[t, t^{-1}]\) with respect to the ideal \((t - 1)\) is the power series ring \(\mathbb{Z}[[T]]\). The canonical map \(\mathbb{Z}[t, t^{-1}] \to \mathbb{Z}[[T]]\) sends \(t\) to \(1 + T\).

**Proof.** This follows easily once we note that in \(\mathbb{Z}[t, t^{-1}] / (t - 1)^m\), we have \(t^{-1} = 1 + (t - 1) + \cdots + (t - 1)^{m-1}\), so that any polynomial in \(\mathbb{Z}[t, t^{-1}] / (t - 1)^m\) may be written as a polynomial in nonnegative powers of \((t - 1)\).

Consider the short exact sequence
\[
1 \to K \to SL_n(\mathbb{Z}[[T]]) \xrightarrow{T=0} SL_n(\mathbb{Z}) \to 1.
\]

**Corollary 3.4.** The group \(K\) is the completion of \(K(R)\) with respect to the filtration \(K^1(R) \supset K^2(R) \supset \cdots\) and with respect to the lower central series of \(K(R)\).

**Proof.** The first assertion follows from Lemma 3.3 and the second from Corollary 3.2.
Observe that the group $\overline{K}$ has a filtration given by powers of $T$ (exactly as $K(\mathbb{Z}[t])$ does) and that the successive graded quotients are isomorphic to $\mathfrak{sl}_n(\mathbb{Z})$. Denote the filtration by $K^\ast$.

We have an analogous sequence over $\mathbb{Q}$:

$$1 \rightarrow U \rightarrow SL_n(\mathbb{Q}[[T]]) \xrightarrow{T=0} SL_n(\mathbb{Q}) \rightarrow 1,$$

and the corresponding $T$-adic filtration $U^\ast$ in $U$. In this case, the successive graded quotients are isomorphic to $\mathfrak{sl}_n(\mathbb{Q})$. We can assemble our exact sequences into a commutative diagram

$$
\begin{array}{c}
1 \rightarrow K(R) \rightarrow SL_n(R) \rightarrow SL_n(\mathbb{Z}) \rightarrow 1 \\
\downarrow \quad \downarrow \quad \| \\
1 \rightarrow \overline{K} \rightarrow SL_n(\mathbb{Z}[T]) \xrightarrow{T=0} SL_n(\mathbb{Z}) \rightarrow 1 \\
\downarrow \quad \downarrow \\
1 \rightarrow U \rightarrow SL_n(\mathbb{Q}[[T]]) \xrightarrow{T=0} SL_n(\mathbb{Q}) \rightarrow 1.
\end{array}
$$

**Proposition 3.5.** The map $K(R)/K^\ast(R) \xrightarrow{j} U/U^\ast$ is the Malcev completion.

**Proof.** According to Quillen’s criterion (see Section 1) we must check three things. First, the group $U/U^\ast$ is nilpotent and uniquely divisible. Nilpotency is obvious, so suppose

$$Y = I + TY_1 + \cdots + T^{r-1}Y_{r-1}$$

is an element of $U/U^\ast$. For each $n > 0$, we must find a unique $X \in U/U^\ast$ with $X^n = Y$. For an arbitrary $X \in U/U^\ast$, write $X = I + TX_1 + \cdots + T^{r-1}X_{r-1}$, and consider the equation

$$Y = X^n$$

$$= I + TNX_1 + T^2(nX_2 + \binom{n}{2}X_1^2) + \cdots + T^{r-1}(nX_{r-1} + p((X_{i+1}^{r-1}))$$

where $p(X_1, \ldots, X_{r-2})$ is a polynomial in the $X_i$, $i \leq r - 2$. Clearly, we can solve this equation inductively for the $X_i$ and find a unique $X$.

Second, we must show that the kernel of $j$ is the torsion subgroup of $K(R)/K^\ast(R)$. This is clear since $K(R)/K^\ast(R)$ is torsion-free (i.e., if some power of $X \in K$ lies in $K^\ast$, then $X \in K^\ast$ already) and the map is injective.

Finally, we must show that if $X \in U/U^\ast$, then $X^m \in \text{im } j$ for some $m \neq 0$. We prove this by induction on $r$, beginning at $r = 2$. Let $X = I + TX_1$ be an element of $U/U^2$. Then there is an $m > 0$ such that $mX_1$ consists of integer entries. Then $X^m = I + TMX_1$ lies in the image of $j$. Now suppose the result holds for $r - 1$ and consider the commutative diagram

$$
\begin{array}{c}
0 \rightarrow K^{r-1}/K^r \rightarrow K/K^r \rightarrow K/K^{r-1} \rightarrow 1 \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \rightarrow U^{r-1}/U^r \rightarrow U/U^r \rightarrow U/U^{r-1} \rightarrow 1.
\end{array}
$$

Suppose $X \in U/U^\ast$. Denote its image in $U/U^{r-1}$ by $\overline{X}$. By the inductive hypothesis, there is an integer $m \neq 0$ with $\overline{X}^m = \overline{Y}$ for some $Y \in K/K^{r-1}$. Choose a lift $Y$ of $\overline{Y}$ in $K/K^r$. Then $Y$ maps to $\overline{X}^m$ in $U/U^{r-1}$. But $X^m$ also maps to $\overline{X}^m$ so that $X^mY^{-1} = Z$ for some $Z \in U^{r-1}/U^r$. Now, there exists some
$W \in K^{r-1}/K^r(\cong \mathfrak{sl}_n(\mathbb{Z}))$ with $Z^p = W$ for some $p \neq 0$. Since $Y = Z^{-1}X^m$, we have

$$Y^p = (Z^{-1}X^m)^p = Z^{-p}X^{mp} \quad \text{(since } \mathcal{U}^{r-1}/\mathcal{U}^r \text{ is central)}$$

Thus, $X^{mp} = WY^p$ belongs to $K/K^r$ and the induction is complete. \hfill \Box

**Theorem 3.6.** The inclusion $K(R) \to \mathcal{U}$ is the Malcev completion.

*Proof.* Since $\mathcal{U} = \lim \mathcal{U}/\mathcal{U}^r$ and $\mathcal{U}/\mathcal{U}^r$ is the Malcev completion of $K(R)/K^r(R)$, the theorem will follow immediately if we can show that $K^r(R) = \Gamma^r K(R)$ for each $r$. This follows from the next two lemmas.

**Lemma 3.7** ([3], 13.6). Let $F^s$ be a central series in a group $G$ such that:

1. The natural map $G \to \lim G/F^s$ is an isomorphism.
2. $F^s/F^{s+1}$ is torsion-free for $s \geq 1$.
3. The Lie product $G/F^2 \otimes F^s/F^{s+1} \to F^{s+1}/F^{s+2}$ is surjective for $s \geq 1$.
4. $G/F^2$ is finitely generated.

Then $G/F^s$ is $F$ for each $s \geq 1$.

**Lemma 3.8** ([3], 13.4). Let $G$ be a group and denote by $\overline{G}$ the completion $\overline{G} = \lim G/\Gamma^r G$. Then the following statements are equivalent:

1. The map $\overline{G} \to \overline{G}$ is an isomorphism.
2. The map $G/\Gamma^r G \to \overline{G}/\Gamma^r \overline{G}$ is an isomorphism for each $r \geq 1$.

*Completion of the proof of Theorem 3.6* Consider the group $\overline{K} = \ker(SL_n(\mathbb{Z}[T])) \cong SL_n(\mathbb{Z})$ with its $T$-adic filtration $\overline{K}$. Note that Lemma 3.7 shows that $\overline{K}^r = \Gamma^r \overline{K}$ for each $r$: the first two conditions are clear, as is the fourth; the third condition follows since the Lie algebra $\mathfrak{sl}_n(\mathbb{Z})$ is perfect (it is here that we must exclude the case $n = 2$). Since

$$\overline{K} = \lim K(R)/K^r(R) = \lim K(R)/\Gamma^r K(R)$$

(the last equality is Corollary 3.4), and since

$$\overline{K} = \lim \overline{K}/\overline{K}^r = \lim \overline{K}/\Gamma^r \overline{K} = \overline{K},$$

Lemma 3.8 implies that $K(R)/\Gamma^r K(R) \cong \overline{K}/\Gamma^r \overline{K}$ for all $r$. Consider the commutative diagram

$$\begin{array}{ccc}
K(R)/\Gamma^r K(R) & \cong & \overline{K}/\Gamma^r \overline{K} \\
\downarrow & & \downarrow \approx \\
K(R)/K^r(R) & \cong & \overline{K}/K^r.
\end{array}$$

It follows that $K^r(R) = \Gamma^r K(R)$ and hence $\mathcal{U}$ is the Malcev completion of $K(R)$. \hfill \Box
Remark 3.9. Even if Lemma 3.8 were not available, we could still prove the result as follows. Denote by $\mathcal{M}$ the Malcev completion of $K(R)/\Gamma^n K(R)$, and by $\mathcal{M} = \lim_{r \to \infty} \mathcal{M}$, the Malcev completion of $K(R)$. Then the map $K(R) \to \mathcal{M}$ factors through $\mathcal{K}$. Moreover, by the universal property of $\mathcal{K}$, we get a unique map $\mathcal{M} \to U$ which is easily seen to be an isomorphism since it has an inverse given by the universal property of the Malcev completion $\mathcal{K} \to U$.

Corollary 3.10. If $n \geq 3$, then $H_1(K(R), \mathbb{Z}) \cong H_1(\mathcal{K}, \mathbb{Z}) \cong \mathfrak{sl}_n(\mathbb{Z})$.

Proof. The first isomorphism follows from Lemma 3.8 and the second isomorphism from Lemma 3.7.

\[ \square \]

4. The relative completion of $SL_n(R)$

We first prove the following result.

Lemma 4.1. The group $U$ has trivial center.

Proof. Let $X$ be a central element of $U$. For a pair of integers $1 \leq i, j \leq n$, denote by $E_{ij}(a)$ the matrix having $i, j$-entry equal to $a$ and all other entries $0$. By computing the product (in both orders) of $X$ with elementary matrices of the form $I + E_{ij}(T) \in U$ for $i \neq j$, we see that $X$ must be a diagonal matrix with all entries equal, say $1 + a_1 T + a_2 T^2 + \cdots$. However, since $X$ must have determinant 1, we see that $a_i = 0$ for all $i \geq 1$.

Theorem 4.2. If $n \geq 3$, then the map $SL_n(\mathbb{Z}[t]) \xrightarrow{t \to T} SL_n(\mathbb{Q}[\lfloor T\rfloor])$ (resp. $SL_n(\mathbb{Z}[t, t^{-1}]) \xrightarrow{t \to 1 + T} SL_n(\mathbb{Q}[\lfloor T\rfloor])$) is the completion with respect to the map $SL_n(\mathbb{Z}[t]) \xrightarrow{t = 0} SL_n(\mathbb{Q})$ (resp. $SL_n(\mathbb{Z}[t, t^{-1}]) \xrightarrow{t = 1} SL_n(\mathbb{Q})$).

Proof. The relative completion is a proalgebraic group which is an extension

\[ (4.1) \quad 1 \longrightarrow \mathcal{P} \longrightarrow \mathcal{G} \longrightarrow SL_n(\mathbb{Q}) \longrightarrow 1 \]

where $\mathcal{P}$ is prounipotent. By the universal property of $U$, we have a unique map $\Phi : U \to \mathcal{P}$ induced by the map $K(R) \to \mathcal{P}$. Since $H^1(SL_n(\mathbb{Z}), A) = 0$ for all rational $SL_n(\mathbb{Q})$-modules A [13], we see that $\Phi$ is surjective (Proposition 2.2). On the other hand, since $H^1(K(R), \mathbb{Q}) \cong \mathfrak{sl}_n(\mathbb{Q})$ is finite dimensional and the action of $SL_n(\mathbb{Z})$ on $H^1(K(R), \mathbb{Q})$ extends to a rational representation of $SL_n(\mathbb{Q})$, Proposition 2.1 implies that the kernel of $\Phi$ is central in $U$. But by Lemma 4.1 the center of $U$ is trivial. Thus, $\Phi$ is injective and $U \cong \mathcal{P}$. Since the extension (4.1) is split ([6] Prop. 4.4]), it follows that $\mathcal{G} \cong SL_n(\mathbb{Q}[\lfloor T\rfloor])$.

Remark 4.3. An alternate proof of the injectivity of $\Phi$ can be obtained via Proposition 2.3 for $n$ sufficiently large. Since $H^2(SL_n(\mathbb{Z}), A)$ vanishes for nontrivial $A$ when $n \geq 9$ [2], Proposition 2.3 asserts that the kernel of $\Phi$ is bounded above by $H_2(SL_n(\mathbb{Z}), \mathbb{Q}) = 0$. This can certainly be improved to $n \geq 4$ (but not to $n = 3$ since examples exist where $H^2(SL_3(\mathbb{Z}), A) \neq 0$).

5. The case $n = 2$

The proof of Theorem 4.2 breaks down in the case $n = 2$ for a variety of reasons.

1. The Lie algebra $\mathfrak{sl}_2(\mathbb{Z})$ is not perfect.
2. Raghunathan’s theorem on the vanishing of $H^1(SL_n(\mathbb{Z}), A)$ does not apply for $n = 2$. 
3. Borel’s result for the vanishing of $H^2(SL_n(\mathbb{Z}), A)$ cannot be strengthened to include $n = 2$.

However, one can make the following observations. Denote by $G(\mathbb{Z})$ the completion of $SL_2(\mathbb{Z})$ relative to its canonical inclusion in $SL_2(\mathbb{Q})$, and by $G(R)$ the completion of $SL_2(R) (R = \mathbb{Z}[[t]], \mathbb{Z}[t, t^{-1}])$ relative to the map $SL_2(R) \rightarrow SL_2(\mathbb{Q})$. The group $G(\mathbb{Z})$ is not isomorphic to $SL_2(\mathbb{Q})$; in fact, it is an extension of $SL_2(\mathbb{Q})$ by a free pronipotent group with infinite dimensional $H_1$ (see [7, Rmk. 3.9]).

We have a commutative diagram

$$
\begin{array}{cccc}
1 & \rightarrow & K(R) & \rightarrow & SL_2(R) & \rightarrow & SL_2(\mathbb{Z}) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & 1 \\
1 & \rightarrow & \mathcal{V} & \rightarrow & G(R) & \rightarrow & G(\mathbb{Z}) & \rightarrow & 1.
\end{array}
$$

The map $\Phi : G(R) \rightarrow G(\mathbb{Z})$ is induced by the composition $SL_2(R) \rightarrow SL_2(\mathbb{Z}) \rightarrow G(\mathbb{Z})$ and the map $\Psi : G(\mathbb{Z}) \rightarrow G(R)$ is induced by the composition $SL_2(\mathbb{Z}) \rightarrow SL_2(R) \rightarrow SL_2(R)$. Since the composition $SL_2(\mathbb{Z}) \rightarrow SL_2(R) \rightarrow SL_2(\mathbb{Z})$ is the identity, we see that $\Phi \circ \Psi = id_{G(\mathbb{Z})}$.

If $n \geq 3$, then the map $SL_n(\mathbb{Z}) \rightarrow SL_n(\mathbb{Q})$ is the relative completion so that the completion of $SL_n(R)$ is an extension of the completion of $SL_n(\mathbb{Z})$ by the Malcev completion of $K(R)$. This leads us to make the following conjecture.

**Conjecture 5.1.** The map $K(R) \rightarrow \mathcal{V}$ is the Malcev completion.

Note that there is at least some hope for this since $\mathcal{V}$ is properly contained in the kernel of the map $G(R) \rightarrow SL_2(\mathbb{Q})$ so that $\mathcal{V}$ is pronipotent.

### 6. Cohomology

In this section we provide evidence for the following conjecture.

**Conjecture 6.1.** If $n \geq 3$, then $H^2(SL_n(R), \mathbb{Q}) = 0$.

Note that this conjecture is true for $n \geq 5$ for the following reason. If $n \geq 5$, then by van der Kallen’s stability theorem [5], we have

$$H_2(SL_n(R), \mathbb{Z}) \cong H_2(SL(R), \mathbb{Z}) \cong K_2(R).$$

It follows that if $n \geq 5$, then $H_2(SL_n(R), \mathbb{Q}) \cong K_2(R) \otimes \mathbb{Q}$. Since $K_2(\mathbb{Z}[t]) \cong K_2(\mathbb{Z})$ and $K_2(\mathbb{Z}[t, t^{-1}]) \cong K_2(\mathbb{Z}) \oplus K_1(\mathbb{Z})$, we see that $K_2(R) \otimes \mathbb{Q} = 0$.

The tool that we will use is continuous cohomology. We define the continuous cohomology of a group $\pi$ by

$$H^\bullet_{cts}(\pi, \mathbb{Q}) = \lim \ H^\bullet(\pi/\Gamma^r \pi, \mathbb{Q}).$$

The basic properties of continuous cohomology were established by Hain [5]. We note the following facts.

**Proposition 6.2** ([5], Thm. 5.1). The natural map $H^k_{cts}(\pi, \mathbb{Q}) \rightarrow H^k(\pi, \mathbb{Q})$ is an isomorphism for $k = 0, 1$ and is injective for $k = 2$. 

The map on $H^2$ need not be surjective in general. A group $\pi$ is called pseudo-nilpotent if the natural map $H^\bullet_{cts}(\pi, \mathbb{Q}) \rightarrow H^\bullet(\pi, \mathbb{Q})$ is an isomorphism. Examples of pseudo-nilpotent groups include the pure braid groups, free groups and the fundamental groups of affine curves over $\mathbb{C}$. 

Proposition 6.3 ([5 Thm. 3.7]). Let $\pi$ be a group with $H_1(\pi, \mathbb{Q})$ finite dimensional. Let $\mathcal{P}$ be the Malcev completion of $\pi$ and denote by $\mathfrak{p}$ the Lie algebra of $\mathcal{P}$. Then the natural map
\[ H^*_\text{cts}(\pi, \mathbb{Q}) \rightarrow H^*_\text{cts}(\mathfrak{p}, \mathbb{Q}) \]
is an isomorphism.

Thus, if $H_1(\pi, \mathbb{Q})$ is finite dimensional, we can use Lie algebra cohomology to obtain a lower bound on the dimension of $H^2(\pi, \mathbb{Q})$. We will not compute $H^2_{\text{cts}}(K(R), \mathbb{Q})$ explicitly. However, we note the following result.

Proposition 6.4. If $n \geq 3$, then $\dim H^2_{\text{cts}}(K(R), \mathbb{Q}) \geq (n^2 - 1)^2/4$.

Proof. By a result of Lubotzky and Magid [11], if $G$ is a nilpotent group with $b_1 = \dim H_1(G, \mathbb{Q})$ finite, then the second Betti number $b_2$ satisfies $b_2 \geq b_1^2/4$. In the case of $K(R)/K^r(R)$, since $H_1(K/K^r, \mathbb{Q}) \cong \mathfrak{sl}_n(\mathbb{Q})$, we see that $b_2(K/K^r) \geq (n^2 - 1)^2/4$ for each $r$.

To show that $H^2(\text{SL}_n(R), \mathbb{Q})$ vanishes, it would suffice to show the following three things.
1. $H^2(\text{SL}_n(\mathbb{Z}), \mathbb{Q}) = 0$.
2. $H^1(\text{SL}_n(\mathbb{Z}), H^1(K(R), \mathbb{Q})) = 0$.
3. $H^0(\text{SL}_n(\mathbb{Z}), H^2(K(R), \mathbb{Q})) = 0$.
The first statement is clear. The second follows from [13] since $H^1(K(R), \mathbb{Q})$ is the adjoint representation $\mathfrak{sl}_n(\mathbb{Q})$. The third statement is true for $n \geq 5$.

Proposition 6.5. If $n \geq 5$, then $H^0(\text{SL}_n(\mathbb{Z}), H^2(K(R), \mathbb{Q})) = 0$.

Proof. Consider the Hochschild–Serre spectral sequence
\[ E_2^{p,q} = H^p(\text{SL}_n(\mathbb{Z}), H^q(K(R), \mathbb{Q})) \Rightarrow H^{p+q}(\text{SL}_n(R), \mathbb{Q}). \]
We know that $H^2(\text{SL}_n(R), \mathbb{Q}) = 0$ for $n \geq 5$ (see the remarks following Conjecture 5.1). Note also that $H^2(\text{SL}_n(\mathbb{Z}), H^1(K(R), \mathbb{Q})) = 0$ and $H^3(\text{SL}_n(\mathbb{Z}), \mathbb{Q}) = 0$. It follows that $d_2 : E_2^{0,2} \rightarrow E_2^{2,1}$ and $d_3 : E_3^{0,2} \rightarrow E_3^{3,0}$ are both the zero map and hence $E_{\infty}^{0,2} = H^0(\text{SL}_n(\mathbb{Z}), H^2(K(R), \mathbb{Q}))$. But this group must vanish since $E_{\infty}^{1,1}$ and $E_{\infty}^{2,0}$ do.

The next result provides evidence for the vanishing of $H^0(\text{SL}_n(\mathbb{Z}), H^2(K(R), \mathbb{Q}))$ when $n = 3, 4$. We first state the following lemma, which can be proved via direct computation.

Lemma 6.6. Let $\Gamma_{a_1, \ldots, a_{n-1}}$ be the irreducible $\text{SL}_n(\mathbb{Q})$-module with highest weight $(a_1 + \cdots + a_{n-1})L_1 + \cdots + a_{n-1}L_{n-1}$, where $L_1, \ldots, L_{n-1}$ are the weights of the fundamental representation. Then we have the following isomorphisms of $\text{SL}_n(\mathbb{Q})$-modules:
1. $\mathfrak{sl}_3(\mathbb{Q}) \otimes \mathfrak{sl}_3(\mathbb{Q}) \cong \Gamma_{2,2} \oplus \Gamma_{3,0} \oplus \Gamma_{0,3} \oplus \Gamma_{1,1} \oplus \Gamma_{1,1} \oplus \Gamma_{0,0},$
2. $\Lambda^2 \mathfrak{sl}_4(\mathbb{Q}) \cong \Gamma_{3,0} \oplus \Gamma_{0,3} \oplus \Gamma_{1,1},$
3. $\Lambda^3 \mathfrak{sl}_4(\mathbb{Q}) \cong \Gamma_{2,2} \oplus \Gamma_{3,0} \oplus \Gamma_{0,3} \oplus \Gamma_{1,1} \oplus \Gamma_{0,0},$
4. $\mathfrak{sl}_4(\mathbb{Q}) \otimes \mathfrak{sl}_4(\mathbb{Q}) \cong \Gamma_{2,0,2} \oplus \Gamma_{2,1,0} \oplus \Gamma_{0,1,2} \oplus \Gamma_{0,0,2} \oplus \Gamma_{1,0,1} \oplus \Gamma_{0,0,0},$
5. $\Lambda^2 \mathfrak{sl}_4(\mathbb{Q}) \cong \Gamma_{2,1,0} \oplus \Gamma_{0,1,2} \oplus \Gamma_{1,0,1},$
6. $\Lambda^3 \mathfrak{sl}_4(\mathbb{Q}) \cong \Gamma_{4,0,0} \oplus \Gamma_{0,0,4} \oplus \Gamma_{1,2,1} \oplus \Gamma_{2,0,2} \oplus \Gamma_{2,1,0} \oplus \Gamma_{0,1,2} \oplus \Gamma_{0,2,0} \oplus \Gamma_{1,0,1} \oplus \Gamma_{0,0,0}.$
Theorem 6.7. If \( n \geq 3 \), then \( H^0(SL_n(\mathbb{Z}), H^2_{\text{cts}}(K(R), \mathbb{Q})) = 0 \).

Proof. We need only consider the cases \( n = 3, 4 \). It suffices to show that
\[
H^0(SL_n(\mathbb{Z}), H^2(K/K^l, \mathbb{Q})) = 0
\]
for each \( l \). We use Lie algebra cohomology. Denote by \( u \) the Lie algebra of \( U \) and consider the \( T \)-adic filtration \( u^l \). The Malcev Lie algebra of \( K/K^2 \) is the Lie algebra \( u_1 = u/u^2 \). Observe that for each \( l \), the quotient \( u^{l-1}/u^l \) is isomorphic as an \( SL_n(\mathbb{Q}) \)-module to the adjoint representation \( sl_n(\mathbb{Q}) \), but as a Lie algebra it is abelian \( (i.e., \) to compute the bracket in \( u^{l-1}/u^l \), we lift elements to \( u^{l-1} \), apply \([, , \] \), and project back; but the commutator of any two elements in \( u^{l-1} \) lies in \( u^l \) and so projects to 0).

We proceed by induction on \( l \), beginning at \( l = 2 \). The Lie algebra \( u_2 \) is abelian of dimension \( n^2 - 1 \); as an \( SL_n(\mathbb{Q}) \)-module it is the adjoint representation \( sl_n(\mathbb{Q}) \). Thus \( H^2(u_2, \mathbb{Q}) \cong \bigwedge^2 sl_n(\mathbb{Q}) \) as an \( SL_n(\mathbb{Q}) \)-module. By Lemma 6.6 parts 2 and 5, we see that \( H^0(SL_n(\mathbb{Z}), H^2(u_2, \mathbb{Q})) = 0 \). Now, suppose that \( l > 2 \) and that \( H^0(SL_n(\mathbb{Z}), H^2(u_{l-1}, \mathbb{Q})) = 0 \). Consider the short exact sequence
\[
0 \longrightarrow u^{l-1}/u^l \longrightarrow u_l \longrightarrow u_{l-1} \longrightarrow 0.
\]
The kernel is central in \( u_l \). Consider the Hochschild–Serre spectral sequence
\[
E_2^{p,q} = H^p(u_{l-1}, H^q(u^{l-1}/u^l, \mathbb{Q})) \Longrightarrow H^{p+q}(u_l, \mathbb{Q}).
\]
We have isomorphisms of \( SL_n(\mathbb{Q}) \)-modules:
1. \( H^2(u_{l-1}, H^0(u^{l-1}/u^l, \mathbb{Q})) = H^2(u_{l-1}, \mathbb{Q}) \),
2. \( H^1(u_{l-1}, H^1(u^{l-1}/u^l, \mathbb{Q})) \cong sl_n(\mathbb{Q}) \otimes sl_n(\mathbb{Q}) \),
3. \( H^0(u_{l-1}, H^2(u^{l-1}/u^l, \mathbb{Q})) \cong \text{Hom}_\mathbb{Q}(\bigwedge^2 sl_n(\mathbb{Q}), \mathbb{Q}) \).

By induction, the \( SL_n(\mathbb{Z}) \) invariants of the first module are trivial and by Lemma 6.6 parts 2 and 5, so are the invariants of the last group. It follows that
\[
H^0(SL_n(\mathbb{Z}), E_{\infty}^{0,2}) = 0.
\]
Also, since
\[
H^1(SL_n(\mathbb{Z}), E_{\infty}^{0,1}) = H^1(SL_n(\mathbb{Z}), sl_n(\mathbb{Q})) = 0,
\]
the long exact cohomology sequence associated to the extension
\[
0 \longrightarrow E_{\infty}^{0,1} \overset{d_2}{\longrightarrow} H^2(u_{l-1}, \mathbb{Q}) \longrightarrow E_{\infty}^{2,0} \longrightarrow 0
\]
shows that \( H^0(SL_n(\mathbb{Z}), E_{\infty}^{2,0}) = 0 \). It remains to show that \( H^0(SL_n(\mathbb{Z}), E_{\infty}^{1,1}) \) vanishes.

Note that \( E_{\infty}^{1,1} \) contains a copy of the trivial representation (parts 1 and 4 of Lemma 6.6). However, the differential (known as transgression \( [\] \))
\[
d_2 : E_{\infty}^{1,1} \longrightarrow H^3(u_{l-1}, \mathbb{Q})
\]
is easily seen to map the trivial representation onto a copy of the trivial representation in the image (this copy arises from the map in cohomology induced by the map \( u_{l-1} \rightarrow sl_n(\mathbb{Q}) \); use parts 3 and 6 of Lemma 6.6). It follows that \( E_{\infty}^{1,1} \) contains no copies of the trivial representation and hence \( H^0(SL_n(\mathbb{Z}), E_{\infty}^{1,1}) = 0 \). Thus
\[
H^0(SL_n(\mathbb{Z}), H^2(u_l, \mathbb{Q})) = 0
\]
and the induction is complete. \( \square \)
One might conjecture that $K(R)$ is pseudo-nilpotent (we do not know if this is the case). If so, it would follow that $H^0(SL_n(\mathbb{Z}), H^2(K(R), \mathbb{Q})) = 0$ and hence $H^2(SL_n(R), \mathbb{Q}) = 0$ for $n \geq 3$.

References


