TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 352, Number 5, Pages 2331–2367
S 0002-9947(99)02477-0
Article electronically published on December 10, 1999

CONDITIONS FOR THE EXISTENCE OF SBR MEASURES FOR "ALMOST ANOSOV" DIFFEOMORPHISMS

HUYI HU

ABSTRACT. A diffeomorphism f of a compact manifold M is called "almost Anosov" if it is uniformly hyperbolic away from a finite set of points. We show that under some nondegeneracy condition, every almost Anosov diffeomorphism admits an invariant measure μ that has absolutely continuous conditional measures on unstable manifolds. The measure μ is either finite or infinite, and is called SBR measure or infinite SBR measure respectively. Therefore, $\frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^ix}$ tends to either an SBR measure or δ_p for almost every x with respect to Lebesgue measure. (δ_x is the Dirac measure at x.) For each case, we give sufficient conditions by using coefficients of the third order terms in the Taylor expansion of f at p.

0. INTRODUCTION

In this paper we study the existence of SBR measures of two-dimensional diffeomorphisms that are hyperbolic everywhere except at finite points. It is easy to see that our methods extend to the situation in which hyperbolicity fails at only finitely many periodic points.

If $f : M \to M$ is a C^2 Anosov diffeomorphism of a compact connected Riemannian manifold, then a result of Sinai (see e.g. [S]) says that f admits an invariant Borel probability measure μ with the property that μ has absolutely continuous conditional measures on unstable manifolds. With respect to this measure, Lebesgue almost every point is generic. That is, if $\phi : M \to \mathbb{R}$ is a continuous function, then for Lebesgue almost every $x \in M$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \phi(f^i x) \to \int \phi d\mu. \tag{0.1}$$

This result, as well as some other properties of μ, has been extended to Axiom A attractors by Bowen, Ruelle, etc. (See e.g. [B].) In this article we will refer to an invariant measure having absolutely continuous conditional measures on unstable manifolds as a *Sinai-Bowen-Ruelle measure* or an *SBR measure*. Due to the works of Oseledec, Pesin, Ledrappier, Young and others on nonuniformly hyperbolic set, the notion of SBR measure is extended to a more general setting (see [O], [P1] and

Received by the editors November 23, 1997.
1991 *Mathematics Subject Classification.* Primary 58F11, 58F15; Secondary 28D05.
Key words and phrases. Almost Anosov diffeomorphism, SBR measure, infinite SBR measure, local Hölder condition.
The author of this work was supported by NSF under grants DMS-8802593 and DMS-9116391, and by DOE (Office of Scientific Computing).

©2000 American Mathematical Society

[LS]); some properties of SBR measure are obtained (e.g. [L] and [LY]). Furthermore, a few examples of SBR measure outside Axiom A systems are studied (see e.g. [BY], [P2] and [C]).

This paper is motivated by the following question: Does a system $f : M \to M$ admit an SBR measure if hyperbolicity fails at only one fixed point p? We confine our topic to two-dimensional cases. Suppose in this system one of the eigenvalues of the derivative Df_p is larger than 1 and the other is equal to 1. The map is in fact uniformly expanding along unstable directions. It is easy to see that the system admits an SBR measure μ, because the arguments on bounded distortion estimates are standard and the push-forward method for invariant measure works. Consequently, the set of generic points with respect to μ has full Lebesgue measure. If the larger eigenvalue is equal to 1 and the smaller one is less than 1, then the results in [HY] indicate that it does not admit SBR measures, and the limit in the left-hand side in (0.1) is $\phi(p)$ at Lebesgue almost every point. It is also found that this system admits an infinite measure that has absolutely continuous conditional measures on weak unstable manifolds. Here we will refer to this measure as an *infinite SBR measure* (see §1 for precise meaning). In this paper, we investigate the case that the derivative Df_p is identity. Our results show that, under some nondegeneracy conditions, f admits either an SBR measure or an infinite SBR measure (Theorem A), and both cases do occur (Theorem B).

The phenomena in this case are quite different from that mentioned above, and some interesting things happen. For example, the decomposition of the tangent spaces into $T_xM = E^u_x \oplus E^s_x$ is discontinuous at the fixed point p (see Remark 4.3). Consequently, we cannot expect the Hölder condition for E^u_x and the Lipschitz condition for W^s-foliation, which are used for bounded distortion and Lebesgue genericity in [HY]. However, we find out that E^u_x satisfies Hölder condition away from p and the W^s-foliation remains absolutely continuous despite the discontinuity of E^u_x and E^s_x. On the other hand, the failure of Lipschitzness of the W^s-foliation makes both SBR measure and infinite SBR measure possible. Notice that whether the measure is finite or infinite depends on whether the area of the sets $P_n = \{x \in P : f^i x \in P, i = 0, 1, \cdots, n\}$ can be summed up or not for any rectangle P containing p in its interior. For a normal Anosov system or a system considered in [HY], because of the Lipschitzness of W^s-foliation this solely depends on how fast a point x in $W^u(p)\backslash\{p\}$ approachs p under backwards iteration. But in our case it depends on how much a stable manifold $W^s(x)$ bends to $W^s(p)$ for x near p.

The proofs of the theorems involve detailed analysis of the dynamics near the "hyperbolic type" indifferent fixed point, which hasn't been done before. We find out a similarity, i.e., the behavior of f^{n^2} at $\frac{x}{n}$ is about the same as f at x but in a smaller scale. We prove the existence and differentiability of stable and unstable manifolds for the fixed point p. We use eigenspaces as coordinate systems to study properties of the splittings $E^u_x \oplus E^s_x$, and use the "local Hölder condition" away from p to avoid the problem caused by discontinuity of the splittings at p; the latter is crucial for the estimates of bounded distortion.

SBR measures on the Hénon attractors, whose existence is found by Benedicks and L.-S. Young ([BY]), are notable examples for non-uniformly hyperbolic systems. The Hénon attractors involve nontrivial interchanges between stable and unstable directions. There exist some areas in which the maps contract very severely along the directions corresponding to positive Lyapunov exponents. However, the orbits jump out immediately and then stay outside the areas for a while so that expansion can be recovered. In our case, the stable and unstable directions are bounded away from each other and the map is always expanding in unstable directions. But near the fixed point p the expansion is very weak and the orbits spend a very long time there. This causes non-uniform hyperbolicity. The speed of expansion may or may not be able to keep exponentially fast eventually, depending on the local behaviors of the maps (see Remark 1.5 and Theorem B).

Acknowledgment. Part of this work was done while I was in the Department of Mathematics at the University of Arizona. It is my pleasure to thank my advisor, Professor Lai-Sang Young, for introducing me to this topic and for giving valuable advice. I also wish to think Professor Marek Rychlik for his helpful conversation.

1. Definitions and statement of results

Let M be a C^∞ two-dimensional compact Riemannian manifold without boundary and let m denote the Riemannian measure on M. Let $f \in \mathrm{Diff}^4(M)$ be the set of C^4 diffeomorphisms.

Definition 1. $f \in \mathrm{Diff}^4(M)$ is said to be an *almost Anosov diffeomorphism*, if there exist two continuous families of cones $x \to \mathcal{C}^u_x, \mathcal{C}^s_x$ such that, except for a finite set S,

i) $Df_x\mathcal{C}^u_x \subseteq \mathcal{C}^u_{fx}$ and $Df_x\mathcal{C}^s_x \supseteq \mathcal{C}^s_{fx}$,
ii) $|Df_xv| > |v|\ \forall v \in \mathcal{C}^u_x$ and $|Df_xv| < |v|\ \forall v \in \mathcal{C}^s_x$.

Remark 1.1. We may assume S is an invariant set. Moreover, by considering f^n instead of f we may also assume $fp = p\ \forall p \in S$.

Remark 1.2. By the continuity, it is easy to see that, if $p \in S$, then

i) $Df_p\mathcal{C}^u_p \subseteq \mathcal{C}^u_p$ and $Df_p\mathcal{C}^s_p \supseteq \mathcal{C}^s_p$,
ii) $|Df_pv| \geq |v|\ \forall v \in \mathcal{C}^u_p$ and $|Df_pv| \leq |v|\ \forall v \in \mathcal{C}^s_p$.

For $\Gamma \subset M$ and $r > 0$, we denote $B(\Gamma, r) = \{y \in M :\ d(y, \Gamma) \leq r\}$.

The ratio of $|Df_xv|$ to $|v|$ may tend to 1 if $x \to S$ and $v \in \mathcal{C}^u_x$ or $\mathcal{C}^s_x$. We need some condition to control the speed of the ratio.

Definition 2. An almost Anosov diffeomorphism f is said to be *nondegenerate (up to the third order)*, if there exist constants $r_0 > 0$ and $\kappa^u, \kappa^s > 0$ such that for all $x \in B(S, r_0)$,

$$\begin{aligned} |Df_xv| &\geq (1 + \kappa^u d(x,S)^2)|v| \qquad \forall v \in \mathcal{C}^u_x, \\ |Df_xv| &\leq (1 - \kappa^s d(x,S)^2)|v| \qquad \forall v \in \mathcal{C}^s_x. \end{aligned} \tag{1.1}$$

Remark 1.3. If f is an almost Anosov diffeomorphism, then for any constant $r > 0$, there exist constants $0 < K^s < 1 < K^u$, depending on r, such that for all $x \notin B(S, r)$,

$$\begin{aligned} |Df_xv| &\geq K^u|v| \qquad \forall v \in \mathcal{C}^u_x, \\ |Df_xv| &\leq K^s|v| \qquad \forall v \in \mathcal{C}^s_x. \end{aligned} \tag{1.2}$$

The following paragraphs concerning the definition of SBR measures can be seen in [LS]. We state the following for the convenience of the reader.

Let ξ be a measurable partition of a measure space X, and let ν be a probability measure on X. Then there is a family of probability measures $\{\nu^\xi_x : x \in X\}$ with

$\nu_x^\xi\big(\xi(x)\big) = 1$, such that for every measurable set $B \subset X$, $x \to \nu_x^\xi(B)$ is measurable and

$$\nu B = \int_X \nu_x^\xi(B) d\nu(x).$$

The family $\{\nu_x^\xi\}$ is called a *canonical system of conditional measures* for ν and ξ. (For reference, see e.g. [R].)

Suppose now that $f : (M, \mu) \to (M, \mu)$ has positive Lyapunov exponents almost everywhere. Then for μ-a.e. x, the unstable manifold $W^u(x)$ exists and is an immersed submanifold of M (see [P1]). A measurable partition ξ of M is said to be *subordinate to unstable manifolds* if for μ-a.e. x, $\xi(x) \subset W^u(x)$ and contains an open neighborhood of x in $W^u(x)$. Let m_x^u denote the Riemannian measure induced on $W^u(x)$. We say that μ has *absolutely continuous conditional measures on unstable manifolds* if for every measurable partition ξ that is subordinate to unstable manifolds, μ_x^ξ is absolutely continuous with respect to m_x^u (written $\mu_x^\xi << m_x^u$) for μ-a.e. $x \in M$. (For more details, see [LS].)

Definition 3. An f-invariant Borel probability measure μ on M is called an *SBR measure* for $f : M \to M$ if

i) (f, μ) has positive Lyapunov exponents almost everywhere;
ii) μ has absolutely continuous conditional measures on unstable manifolds.

Let Γ be a subset of M. We denote by f_Γ the first return map on Γ, i.e., $f_\Gamma x = f^{\tau(x)}(x)$ $\forall x \in \Gamma$, where $\tau(x)$ is the smallest positive integer such that $f^{\tau(x)}(x) \in \Gamma$. We also denote by μ_Γ the normalization of $\mu|_\Gamma$ as $\mu\Gamma < \infty$.

Note that the notion of absolutely continuous conditional measures on unstable manifolds makes sense if even f is a piecewise diffeomorphism.

Definition 4. An f-invariant Borel measure μ on M is called an *infinite SBR measure*, if $\mu M = \infty$ and for any open set $U \supset S$,

i) $\mu(M \backslash U) < \infty$,
ii) $(f_{M\backslash U}, \mu_{M\backslash U})$ has positive Lyapunov exponents almost everywhere, and $\mu_{M\backslash U}$ has absolutely continuous conditional measures on unstable manifolds of f.

Remark 1.4. In this paper, the term "SBR measure" without any qualifications will always be reserved for probability measures.

Remember our assumption that M is a C^∞ two-dimensional compact Riemannian manifold, m is the Riemannian measure on M and $f \in \mathrm{Diff}^4(M)$.

Theorem A. *Every topologically transitive nondegenerate almost Anosov diffeomorphism f on M has either an SBR measure or an infinite SBR measure.*

The nondegeneracy condition in the theorem is only for technical reasons. The author believes that the result would remain true under some weaker conditions.

Corollary. *Let f be as in Theorem A.*

i) *If f admits an SBR measure μ, then for any continuous function $\phi : M \to \mathbb{R}$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i x) = \int \phi d\mu, \qquad m\text{-a.e. } x \in M.$$

ii) *If f admits an infinite SBR measure μ, then for any open neighborhood U of S,*

$$\lim_{n\to\infty} \frac{1}{n} \#\{k : f^k x \in U,\ 0 \le k \le n-1\} = 1, \qquad m\text{-a.e. } x \in M.$$

Remark 1.5. In case i) Lebesgue almost every point has a positive Lyapunov exponent because of the definition of the SBR measure. On the contrary, it is evident by the corollary that in case ii) there is no positive Lyapunov exponent at Lebesgue almost every point in M.

In this paper we only consider the case that S contains a single point, i.e., $S = \{p\}$. It is not difficult for the reader to adjust the proof to the case that S contains more than one point.

If $S = \{p\}$, then part ii) of the above corollary is equivalent to the following.

ii′) *If f admits an infinite SBR measure, then for any continuous function $\phi : M \to \mathbb{R}$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i x) = \phi(p), \qquad m\text{-a.e. } x \in M.$$

By Remark 1.3, we see that Df_p has two eigenvectors. As we mentioned in §0, it is known that Theorem A holds if at least one of eigenvalues of Df_p is not equal to 1. So to prove Theorem A, we only need consider the case that $Df_p = \mathrm{id}$.

For further analysis we need the following technical assumptions. Denote

$$W^u_\epsilon(p) = \{y \in M : d(f^{-n}y, p) \le \epsilon \ \ \forall n \ge 0\},$$
$$W^s_\epsilon(p) = \{y \in M : d(f^n\ y,\ p) \le \epsilon \ \ \forall n \ge 0\}.$$

We will prove in §4 that both are differentiable curves. To state and prove Theorem B we give the following.

Assumption A. $W^u_\epsilon(p)$ and $W^s_\epsilon(p)$ are C^4 curves.

Note that $Df_p = \mathrm{id}$. We will show $D^2 f_p = 0$ (Proposition 2.1). Thus we can take a suitable coordinate system such that in some neighborhood of p, f can be expressed as

$$f(x,y) = \Big(\ x\big(1 + \phi(x,y)\big),\ y\big(1 - \psi(x,y)\big)\ \Big), \tag{1.3}$$

where $(x,y) \in \mathbb{R}^2$ and

$$\phi(x,y) = a_0 x^2 + a_1 xy + a_2 y^2 + O(|(x,y)|^3),$$
$$\psi(x,y) = b_0 x^2 + b_1 xy + b_2 y^2 + O(|(x,y)|^3). \tag{1.4}$$

In this circumstance we have the following.

Theorem B. *Let f be a topologically transitive nondegenerate almost Anosov diffeomorphism on M with $S = \{p\}$. Assume $Df_p = \mathrm{id}$, and $W^u_\epsilon(p)$ and $W^s_\epsilon(p)$ are C^4 curves. With the notation above,*

(I) *if $\alpha a_2 > 2b_2$ for some $0 < \alpha < 1$, $a_1 = 0 = b_1$ and $a_0 b_2 - a_2 b_0 > 0$, then f admits an SBR measure;*

(II) *if $2a_2 < \alpha b_2$ for some $0 < \alpha < 1$ and $a_1 b_1 \ne 0$, then f admits an infinite SBR measure.*

The systems satisfying the conditions in Theorem B exist. For example, we can take a torus T^2 and constants $0 < r_0 << r_1 << 1 = \operatorname{diam} T^2$ and then construct a dynamical system in such a way that f is a hyperbolic toral automorphism outside the r_1-neighborhood and has the form (1.3) and (1.4) within the r_0-neighborhood of the origin O.

Remark 1.6. The nondegeneracy conditions (1.1) guarantee that $a_0, a_2, b_0, b_2 > 0$. Hence, the conditions in Theorem B(I) imply that $\phi > \psi$ in some small neighborhood of p, while the conditions in Theorem B(II) imply that in some quadrants, $\phi < \psi$ near the y-axis.

Recall the cases discussed in §0 that $Df_p \neq \mathrm{id}$ but one of the eigenvalues is equal to 1. It seems that in such systems the existence of SBR measures depends on whether expansion is "stronger" than contraction. The results in Theorem B is consistent with this observation, because we can think that expansion is "stronger" if $\phi > \psi$ and is "weaker" if $\phi < \psi$.

PART 1: Proof of Theorem A

In this part we always assume that $B(p, r_0)$ is in the Euclidean plane $\mathbb{R}^2$, where r_0 is as in Definition 2. Take a coordinate system in the plane such that the origin is the fixed point p. Thus, we can write $|x| = d(x, p)$. Also, $\forall x \in B(p, r_0)$, we identify $T_x M$ with the same Euclidean plane. Let $\Theta(x, y)$ denote the angle from x to y counterclockwise in $\mathbb{R}^2$.

Assumption A is not going to be used in this part.

2. Dynamics near the fixed point

We first prove in Proposition 2.1 that the second derivative $D^2 f_p$ is 0 and the third derivative $D^3 f_p$ is not 0. Then we show that the action of f^{n^2} at $\frac{x}{n}$ is similar to the action of f at x, on a smaller scale. This can be seen in Propositions 2.6 and 2.8, where the former deals with norms and the latter deals with angles.

Proposition 2.1. *$D^2 f_p = 0$ and $D^3 f_p(a, a, \cdot) \neq 0$ $\forall a \in \mathbb{R}^2$. So,*

$$Df_x = \mathrm{id} + \frac{1}{2} D^3 f_p(x, x, \cdot) + R_F(x),$$

and

$$fx = x + \frac{1}{6} D^3 f_p(x, x, x) + R_f(x),$$

where $R_F(x)$ and $R_f(x)$ are remainders with $\|R_F(x)\| = O(|x|^3)$ and $|R_f(x)| = O(|x|^4)$.

Proof. Since $Df_p = \mathrm{id}$, by Taylor expansion

$$Df_x = \mathrm{id} + D^2 f_p(x, \cdot) + \frac{1}{2} D^3 f_p(x, x, \cdot) + R_F(x). \tag{2.1}$$

Let $\langle \cdot, \cdot \rangle$ denote the inner product. We have

$$|Df_x v|^2 = |v|^2 + 2\langle D^2 f_p(x, v),\ v\rangle + O(|x|^2)|v|^2.$$

By (1.1) we know that for any sufficiently small x and $v \in \mathcal{C}_x^s$, $\langle D^2 f_p(x, v),\ v\rangle \leq 0$.

Note that $x \to \mathcal{C}_x^s$ is continuous. There exists a small $0 < r < r_0$ such that $\bigcap_{x \in B(p,r)} \mathcal{C}_x^s$ has nonempty interior. Thus, we get $\langle D^2 f_p(x, v),\ v\rangle \leq 0$ for all $x \in B(p,r)$ and $v \in \bigcap_{x\in B(p,r)} \mathcal{C}^s_x$. Since $\langle D^2 f_p(\cdot,\cdot),\ \cdot\rangle$ is trilinear, this implies $\langle D^2 f_p(x,v),\ v\rangle = 0\ \forall x, v \in \mathbb{R}^2$. Now it is easy to conclude that $D^2 f_p = 0$.

The inequality $D^3 f_p(a,a,\cdot) \neq 0$ follows from (1.1), (2.1) and the fact $D^2 f_p = 0$. ☐

Remark 2.2. Similarly, $D^2 f_p^{-1} = 0$ and $D^3 f_p^{-1}(x,x,\cdot) \neq 0\ \forall x \in \mathbb{R}^2$.

The next two corollaries follow directly from the above facts.

Corollary 2.3. *There exists a constant $\eta > 0$ such that $\forall x \in B(p,r_0)$,*

$$\|Df_x^{\pm 1} - \mathrm{id}\,\| \le 3\eta|x|^2 \qquad \textit{and} \qquad |f^{\pm 1}x - x| \le \eta|x|^3.$$

Corollary 2.4. *For all $x, y \in B(p,r_0)$ with $d(x,y) \le |x|^2$,*

$$d(f^{\pm 1}x, f^{\pm 1}y) \le (1 + 4\eta|f^{\pm 1}x|^2)d(x,y).$$

Lemma 2.5. *Let $x \in B(p,r_0)$. If $|x| \ge \big(3\eta k\big)^{-\frac{1}{2}}$ for some $k > 1$, then $|fx| \ge \big(3\eta(k+1)\big)^{-\frac{1}{2}}$.*

Proof. By Corollary 2.3,

$$|fx| \ge |x|\big(1 - \eta|x|^2\big) \ge \sqrt{\frac{1}{3k\eta}}\Big(1 - \eta\cdot\frac{1}{3k\eta}\Big) \ge \sqrt{\frac{1}{3(k+1)\eta}}\ .$$

☐

Proposition 2.6. *For any $\epsilon > 0$, there exists $0 < r_\rho = r_\rho(\epsilon) \le r_0$ such that $\forall x \in B(p,r_\rho)$, $t \in (0,1]$, $j = 1, \cdots, [\frac{2}{t^2}]$,*

$$(1-\epsilon)|tx| \le \big|f^j(tx)\big| \le (1+\epsilon)|tx|.$$

Proof. Take $0 < r_\rho \le r_0$ such that $e^{2\eta(1+\epsilon)^2 r_\rho^2} \le (1+\epsilon)$. Let $x \in B(p,r_\rho)$ arbitrarily.

Suppose $\forall i = 1, \cdots, j-1$, $\big|f^i(tx)\big| \le (1+\epsilon)|tx|$. By Corollary 2.3,

$$\begin{aligned}\big|f^j(tx)\big| &\le \big|f^{j-1}(tx)\big|\big(1 + \eta\big|f^{j-1}(tx)\big|^2\big) \le t|x|\big(1 + \eta(1+\epsilon)^2 t^2|x|^2\big)^j \\ &\le t|x|\big(1 + \eta(1+\epsilon)^2 t^2|x|^2\big)^{\frac{2}{t^2}} \le t|x|e^{2\eta(1+\epsilon)^2|x|^2} \le (1+\epsilon)t|x|.\end{aligned}$$

Therefore, the second inequality in the lemma follows from induction.

The first inequality can be obtained similarly. ☐

For any $0 \neq x \in B(p,r_0)$, we denote $e_x = x/|x|$ or, equivalently, $x = |x|e_x$.

Lemma 2.7. *For any e_x in the unit circle, uniformly*

$$\lim_{|t|\to 0} \frac{\Theta\big(te_x, f(te_x)\big)}{|t|^2} = \frac{1}{6}\big|D^3 f_p(e_x,e_x,e_x)\big| \sin\Theta(e_x, D^3 f_p(e_x,e_x,e_x)).$$

Proof. Denote $A(x) = \dfrac{1}{6}D^3 f_p(x,x,x)$. The result follows from the facts $|R_f(x)| = o(|x|^3)$ as $x \to 0$, $A(x) = |x|^3 A(e_x)$ and

$$\tan\Theta(x, fx) = \frac{\big|A(x)\big|\sin\Theta\big(x, A(x)\big) + \big|R_f(x)\big|\sin\Theta\big(x, R_f(x)\big)}{|x| + \big|A(x)\big|\cos\Theta\big(x, A(x)\big) + \big|R_f(x)\big|\cos\Theta\big(x, R_f(x)\big)}.$$

☐

Proposition 2.8. *For any $\epsilon > 0$ there exists a constant $0 < r_\theta = r_\theta(\epsilon) \le r_\rho$ such that $\forall r \in (0, r_\theta)$, if $x, y \in B(p, r)$ with $|\Theta(x, y)| \le |\Theta(x, fx)|$ and $|y| = t|x|$, $t \in (0, 1]$, then*

$$|\Theta(y, f^j y)| \le |\Theta(x, fx)| + \epsilon|x|^2 \quad \forall\ 0 \le j \le \frac{1}{t^2}, \tag{2.2}$$

$$|\Theta(y, f^j y)| \ge |\Theta(x, fx)| - \epsilon|x|^2 \quad \forall\ \frac{1}{t^2} \le j \le \frac{2}{t^2}. \tag{2.3}$$

Moreover, if $D^3 f_p(e_x, e_x, e_x)$ and e_x are not collinear, then we can choose $\bar r_\theta = \bar r_\theta(\epsilon, e_x)$ in such a way that $\forall x \in B(p, \bar r_\theta)$,

$$|\Theta(y, f^j y)| \ge |\Theta(x, fx)| \quad \forall\ \frac{1}{(1-\epsilon)t^2} \le j \le \frac{2}{t^2}. \tag{2.4}$$

Proof. Denote $A(z) = \frac{1}{6} D^3 f_p(z, z, z)$.

Note that $A(e_x) \sin \Theta(e_x, A(e_x))$ is continuous on the unit circle. By Lemma 2.7, there exist $r'_\theta > 0$ and $\theta_0 > 0$ such that $\forall x, z \in B(p, r'_\theta)$ with $\Theta(x, z) \le \theta_0$,

$$\frac{|z|^2}{|x|^2}|\Theta(x, fx)| - \frac{\epsilon}{4}|z|^2 \le |\Theta(z, fz)| \le \frac{|z|^2}{|x|^2}|\Theta(x, fx)| + \frac{\epsilon}{4}|z|^2. \tag{2.5}$$

Also, there exists $\beta \in (0, \frac{1}{2}]$ such that $\forall x \in B(p, r'_\theta)$,

$$|\Theta(x, fx)| \le \frac{\epsilon}{3\beta}|x|^2. \tag{2.6}$$

By Proposition 2.6, there exists $r''_\theta > 0$ such that $\forall x \in B(p, r''_\theta)$, $t \in (0, 1]$,

$$(1-\beta)|tz|^2 \le |f^j(tz)|^2 \le (1+\beta)|tz|^2 \qquad j = 0, 1, \cdots, [\frac{2}{t^2}]. \tag{2.7}$$

Take

$$r_\theta = \min\{r'_\theta,\ r''_\theta,\ \frac{\beta}{2\epsilon}\theta_0\}. \tag{2.8}$$

Now let $x, y \in B(p, r_\theta)$ with $|\Theta(x, y)| < |\Theta(x, fx)|$ and $|y| = t|x|$, $t \in (0, 1]$. Suppose for $i = 0, 1, \cdots, j$, $|\Theta(x, f^i y)| < \theta_0$, where $0 \le j \le \frac{2}{t^2}$. By (2.5), (2.7) and (2.6),

$$\begin{aligned} |\Theta(f^j y, f^{j+1} y)| \le& \frac{|f^j y|^2}{|x|^2}\big(|\Theta(x, fx)| + \frac{\epsilon}{4}|x|^2\big) \\ \le& (1+\beta)t^2\big(|\Theta(x, fx)| + \frac{\epsilon}{4}|x|^2\big) \le t^2|\Theta(x, fx)| + \epsilon t^2 |x|^2. \end{aligned} \tag{2.9}$$

Similarly,

$$|\Theta(f^j y, f^{j+1} y)| \ge t^2|\Theta(x, fx)| - \epsilon t^2|x|^2. \tag{2.10}$$

By (2.9),

$$\begin{aligned} |\Theta(x, f^{j+1} y)| \le |\Theta(x, y)| + |\Theta(y, f^{j+1} y)| \le& |\Theta(x, y)| + 2|\Theta(x, fx)| + 2\epsilon|x|^2 \\ \le& 3|\Theta(x, fx)| + 2\epsilon|x|^2. \end{aligned}$$

Using (2.6) and (2.8) we get

$$|\Theta(x, f^{j+1} y)| \le \big(\frac{\epsilon}{\beta} + 2\epsilon\big) \cdot \frac{\beta}{2\epsilon}\theta_0 \le \theta_0.$$

Therefore (2.2) and (2.3) can be obtained by induction.

If $D^3f_p(e_x,e_x,e_x)$ and e_x are not collinear, then $A(e_x)\sin\Theta(e_x,A(e_x))\neq 0$. We can use $\epsilon' = \dfrac{\epsilon}{2}|A(e_x)||\sin\Theta(e_x,A(e_x))|$ instead of ϵ, and therefore obtain $\bar r_\theta = \bar r_\theta(\epsilon', e_x) > 0$ such that if $x,y\in B(p,\bar r_\theta)$ with $|\Theta(x,y)|\le|\Theta(x,fx)|$ and $|y| = t|x|$ for some $t\in(0,1]$, then $\epsilon'\le\epsilon\dfrac{|\Theta(x,fx)|}{|x|^2}$ and (2.10) holds for ϵ'. That is,

$$|\Theta(f^{j-1}y,f^jy)|\ge t^2|\Theta(x,fx)|-\epsilon' t^2|x|^2\ge(1-\epsilon)t^2|\Theta(x,fx)|,\quad j=0,1,\cdots,[\frac{1}{t^2}].$$

Thus (2.4) follows. □

3. Eigenvectors of $D^3f_p(x,x,\cdot)$ near the fixed point

We prove in Proposition 3.2 that $\mathrm{id}+\frac{1}{2}D^3f_p(x,x,\cdot)$, the approximation of Df_x up to the second order of x, has two eigenspaces Ξ^u_x and Ξ^s_x. This fact will be used to construct coordinate systems in §5. Usually, the angle between x and Ξ^u_x is not zero. However, there exists a unique line E^+ on which every point x and its corresponding Ξ^u_x are collinear (Lemma 3.6). We will see in §4 that E^+ is in fact the tangent line, denoted by E^u_p, of "weak" unstable manifold $W^u(p)$ at p.

Lemma 3.1. *There exist constants $0<\tilde r\le r_0$, $0<\tilde\kappa^u\le\kappa^u$ and $0<\tilde\kappa^s\le\kappa^s$, cones $\tilde{\mathcal C}^u$ and $\tilde{\mathcal C}^s$ such that $\forall x\in B(p,\tilde r)$, $\tilde{\mathcal C}^u\supseteq\mathcal C^u_x$, $\tilde{\mathcal C}^s\supseteq\mathcal C^s_x$ and*

$$|Df_xv|\ge(1+\tilde\kappa^u|x|^2)|v|\qquad\forall v\in\tilde{\mathcal C}^u_x,$$

$$|Df_xv|\le(1-\tilde\kappa^s|x|^2)|v|\qquad\forall v\in\tilde{\mathcal C}^u_x.$$

Proof. Denote $A_x=\frac{1}{2}D^3f_p(x,x,\cdot)$. Also, denote

$$\mathcal C_x(\beta)=\{v\in\mathbb R^2:\ \langle v,A_xv\rangle\ge\beta|x|^2|v|^2\}.$$

By Proposition 2.1, $Df_xv = v+A_xv+R_F(x)v$. Hence, $|Df_xv|^2=|v|^2+2\langle A_xv,v\rangle+o(|x|^2)|v|^2$. So if $|x|$ is small enough, then

$$\mathcal C^u_x\subseteq\mathcal C_x(\frac{4}{5}\kappa^u)\subset\mathcal C_x(\frac{3}{5}\kappa^u)\subseteq\big\{v\in T_xM:|Df_xv|\ge(1+\frac{1}{2}\kappa^u|x|^2)|v|\big\}. \tag{3.1}$$

Note that $\mathcal C^u_{tx}(\beta)$ is independent of t provided $t\neq0$. By the continuity of $\mathcal C^u_x$, we get that for any e_x in the unit circle, $\mathcal C^u_p\subseteq\mathcal C_{e_x}(\frac{4}{5}\kappa^u)$ and therefore $\mathcal C^u_p\subseteq\bigcap_{e_x\in S^1}\mathcal C_{e_x}(\frac{4}{5}\kappa^u)$. Put $\tilde{\mathcal C}^u=\bigcap_{e_x\in S^1}\mathcal C_{e_x}(\frac{3}{5}\kappa^u)$. It is easy to see by (3.1) that $\tilde{\mathcal C}^u$ is strictly larger than $\mathcal C^u_p$. Again, by the continuity of $\mathcal C^u_x$, there exists $\tilde r>0$ such that $B(p,\tilde r)\subseteq\{x:\mathcal C^u_x\subseteq\tilde{\mathcal C}^u\}$. Now for any $x\in B(p,\tilde r)$ and $v\in\tilde{\mathcal C}^u$, we have $v\in\tilde{\mathcal C}_x(\frac{3}{5}\kappa^u)$. Therefore by (3.1), $|Df_xv|\ge(1+\frac{1}{2}\kappa^u|x|^2)|v|$. This finishes the proof. □

Proposition 3.2. *For any $a\in\mathbb R^2\backslash\{0\}$, $\mathrm{id}+A_a$ has an eigenvector in $\tilde{\mathcal C}^u$, where A_a denotes $\frac{1}{2}D^3f_p(a,a,\cdot)$.*

Proof. Suppose there is $a\in\mathbb R^2$ with $|a|=1$ such that $\mathrm{id}+A_a$ has no eigenvector in $\tilde{\mathcal C}^u$. We may assume $\Theta(v,v+A_av)>0$ $\forall v\in\tilde{\mathcal C}^u$. Thus there exists $\theta_0>0$ such that for all unit vectors $v\in\tilde{\mathcal C}^u$, and unit vectors a' and v' with $|\Theta(a,a')|\le\theta_0$ and $|\Theta(v,v')|\le\theta_0$,

$$|A_{a'}v'|\sin\Theta(v',A_{a'}v')>\frac{1}{2}|A_av|\sin\Theta(v,A_av). \tag{3.2}$$

Take $x = ta$ for some $t \in (0, \frac{1}{2})$ such that $\Theta(x, fx) + \frac{1}{2}|x|^2 \le \theta_0$.

Let $\bar{v}_z$ be a unit vector in the boundary of $\mathcal{C}^u_z$ satisfying $\Theta(v, \bar{v}_z) \ge 0\ \forall v \in \mathcal{C}^u_z$. By continuity we can take $n > 0$ such that for any $z_1, z_2 \in B(p, \frac{1}{n})$,

$$\Theta(\bar{v}_{z_1}, \bar{v}_{z_1}) < \min_{v \in \tilde{\mathcal{C}}^u} \big\{\theta_0,\ \frac{1}{32}|A_x v| \sin\Theta(v, A_x v)\big\}. \tag{3.3}$$

We may assume that n is large enough such that

$$\Theta(v, Df_z v) > \frac{1}{2}|A_z v| \sin\Theta(v, A_z v) \qquad \forall v \in \tilde{\mathcal{C}}^u, \tag{3.4}$$

provided $z \in B(p, \frac{1}{n})$ with $|\Theta(x, z)| \le \theta_0$. This is possible because

$$\tan\Theta(v, Df_z v) = \frac{|A_z v| \sin\Theta(v, A_z v) + |R_F(z)| \sin\Theta(v, R_F(z)v)}{|v| + |A_z v| \cos\Theta(v, A_z v) + |R_F(z)| \cos\Theta(v, R_F(z)v)}\ .$$

Put $y = \dfrac{x}{n}$. Denote $y_i = f^i y$ and $\bar{v}^{(i)} = \dfrac{Df^i_y \bar{v}_y}{|Df^i_y \bar{v}_y|}$. We may also assume that Proposition 2.6 and Proposition 2.8 can be applied with $\epsilon = \frac{1}{2}$. Therefore $\Theta(x, y_i) \le \Theta(x, fx) + \dfrac{1}{2}|x|^2 \le \theta_0$ and $\dfrac{|x|}{2n} \le |y_i| \le \dfrac{3|x|}{2n} \le \dfrac{1}{n}\ \forall i = 0, 1, \cdots, n^2$.

Note that $Df^i_y \mathcal{C}^u_y \subseteq \mathcal{C}^u_{y_i}$. We have that $\forall i = 0, 1, \cdots, n^2$, $\bar{v}^{(i)} \in \mathcal{C}^u_{y_i} \subseteq \tilde{\mathcal{C}}^u$ and

$$\Theta(\bar{v}_{y_0}, \bar{v}_{y_i}) = \Theta(\bar{v}_{y_0}, Df^i_x \bar{v}_{y_0}) + \Theta(\bar{v}^{(i)}, \bar{v}_{y_i}) \ge \Theta(\bar{v}_{y_0}, Df^i_x \bar{v}_{y_0}). \tag{3.5}$$

By (3.4) and (3.2) we get

$$\begin{aligned}
\Theta(\bar{v}_{y_0}, Df^{n^2}_x \bar{v}_{y_0}) &= \sum_{i=0}^{n^2-1} \Theta(\bar{v}^{(i)}, Df_{y_i} \bar{v}^{(i)}) \ge \sum_{i=0}^{n^2-1} \frac{1}{2}|A_{y_i} \bar{v}^{(i)}| \sin\Theta(\bar{v}^{(i)}, A_{y_i} \bar{v}^{(i)}) \\
&\ge n^2 \cdot \frac{1}{2} \cdot |\frac{x}{2n}|^2 \cdot \frac{1}{2}|A_{e_x} \bar{v}_x| \sin\Theta(\bar{v}_x, A_x \bar{v}_x) \ge \frac{1}{16}|A_x \bar{v}_x| \sin\Theta(\bar{v}_x, A_x \bar{v}_x).
\end{aligned}$$

It contradicts (3.5) and (3.3). ∎

We denote by ξ^u_a and ξ^s_a the unit eigenvectors of $\mathrm{id} + A_a$ in $\tilde{\mathcal{C}}^u$ and $\tilde{\mathcal{C}}^s$, respectively, and by $1 + \lambda^u_a |a|^a$ and $1 - \lambda^s_a |a|^2$ the corresponding eigenvalues. Also, denote by Ξ^u_a and Ξ^s_a the subspaces generated by ξ^u_a and ξ^s_a. Since $A_{ta} = t^2 A_a$, we know that

$$\xi^u_{ta} = \xi^u_a, \quad \xi^s_{ta} = \xi^s_a, \quad \text{and} \quad \lambda^u_{ta} = \lambda^u_a, \quad \lambda^u_{ta} = \lambda^u_a. \tag{3.6}$$

By Lemma 3.1 and Corollary 2.3, $\tilde{\kappa}^u \le \lambda^u_a \le 3\eta$ and $\tilde{\kappa}^s \le \lambda^s_a \le 3\eta$.

Lemma 3.3. *For any $a_1, \cdots, a_n \in \mathbb{R}^2 \backslash \{0\}$, $\prod_{i=1}^{n} (\mathrm{id} + A_{a_i})$ has an eigenvector in $\tilde{\mathcal{C}}^u$, where $A_a = \frac{1}{2} D^3 f_p(a, a, \cdot)$.*

Proof. By the proof of Proposition 3.2, we have $\xi^u_x \in \mathrm{int}\, \tilde{\mathcal{C}}^u$. (Otherwise we can shrink $\tilde{\mathcal{C}}^u$ a little bit and the arguments still work.) This implies $(\mathrm{id} + A_x)\tilde{\mathcal{C}}^u \subset \tilde{\mathcal{C}}^u$. So $\prod_{i=0}^{n} (\mathrm{id} + A_{a_i})\tilde{\mathcal{C}}^u \subset \tilde{\mathcal{C}}^u$. Then the result follows. ∎

Lemma 3.4. *If r is small enough, then $\forall x \in B(p, r)$, $Df_x \tilde{\mathcal{C}}^u \subset \tilde{\mathcal{C}}^u$ and $Df_x \tilde{\mathcal{C}}^s \supset \tilde{\mathcal{C}}^s$.*

Proof. This is because $(\mathrm{id} + A_x)\tilde{\mathcal{C}}^u \subset \tilde{\mathcal{C}}^u$ and $\|Df_x - (\mathrm{id} + A_x)\| = O(|x|^3)$. ∎

Remark 3.5. Df_x has an eigenvector in $\tilde{\mathcal{C}}^u$ and an eigenvector in $\tilde{\mathcal{C}}^s$ if $|x|$ is small.

Lemma 3.6. *For $a, b \in \mathbb{R}^2 \backslash \{0\}$, denote the matrix of identity $\Xi_a^u \oplus \Xi_a^s \to \Xi_b^u \oplus \Xi_b^s$ by*

$$\begin{pmatrix} 1 + h_{11}(a,b) & h_{12}(a,b) \\ h_{21}(a,b) & 1 + h_{22}(a,b) \end{pmatrix}.$$

Then $\forall i, j = 1, 2$, $|h_{ij}(a', b) - h_{ij}(a'', b)| \leq C|\Theta(a', a'')|$, $|h_{ij}(a, b') - h_{ij}(a, b'')| \leq C|\Theta(b', b'')|$, and $|h_{ij}(a, b)| \leq C|\Theta(a, b)|$, where C is a generic constant depending only on f.

Proof. Note that $\forall a \neq 0$, A_a has one positive eigenvalue and one negative eigenvalue. So it is clear that the maps $a \to \xi_a^u, \xi_a^u$ are smooth on $\mathbb{R}^2 \backslash \{0\}$. By (3.6) the result follows. $\square$

Lemma 3.7. *There exists a unique subspace $E^+ \subseteq \mathbb{R}^2$ such that $\forall a \in E^+ \backslash \{0\}$, a is an eigenvector of A_a with positive eigenvalue, where $A_a = \frac{1}{2} D^3 f_p(a, a, \cdot)$*

Proof. The existence follows from the continuity of the map $a \to \Theta(a, A_a a)$.

Now we suppose that there exist x, y with $x, y \in \tilde{\mathcal{C}}^u$ and $\Theta(x, y) > 0$ such that $A_x x = \lambda_x^u |x|^2 x$ and $A_y y = \lambda_y^u |y|^2 y$.

First, we assume $\lambda_x^u = \lambda_y^u$. Take such x and y with $x - y \in \tilde{\mathcal{C}}^s$ and $|x| \neq |y|$. By the law of cosines, Lemma 2.7 and the fact that $|f(tx)| = t|x| + \dfrac{1}{3}\lambda_x^u t^3 |x|^3 + O(|t|^4)$, we obtain that

$$\begin{aligned} &\frac{3}{t^4}\big(|f(ty) - f(tx)|^2 - |ty - tx|^2\big) \\ =&2\lambda_x^u |x|^4 + 2\lambda_y^u |y|^4 - 2|x||y|(\lambda_x^u |x|^2 + \bar{\lambda}_y^u |y|^2)\cos\Theta(y, x) + O(t) \end{aligned}$$

as $t \to 0^+$. Since

$$\lambda_x^u |x|^4 + \lambda_y^u |y|^4 - |x|\,|y|(\lambda_x^u |x|^2 + \lambda_y^u |y|^2) = \big(\lambda_x^u |x|^3 - \lambda_y^u |y|^3\big)\big(|x| - |y|\big).$$

The right-hand side is positive if t is small. So $|f(ty) - f(tx)| > |ty - tx|$. But the fact $y - x \in \tilde{\mathcal{C}}^s$ implies $|f(ty) - f(tx)| < |ty - tx|$, a contradiction.

Next, we assume $\lambda_x^u \neq \lambda_y^u$. In this case, for any x and y, $f(tx) - f(ty)$ is not parallel to $x - y$. Hence we can find x and y such that $x - y \notin \tilde{\mathcal{C}}^s$ but $f(tx) - f(ty) \in \tilde{\mathcal{C}}^s$. Therefore, there is a point tz between tx and ty such that Df_{tz} maps a vector outside the cone $\tilde{\mathcal{C}}^s$ into $\tilde{\mathcal{C}}^s$. This contradicts Lemma 3.4 which says that $Df_{tz}\tilde{\mathcal{C}}^s \supset \tilde{\mathcal{C}}^s$ if t is small. $\square$

4. Unstable manifolds on M

In this section we prove the existence of invariant decomposition of the tangent bundle into $TM = E^u \oplus E^s$ (Proposition 4.2) and the existence of “weak” unstable manifolds $W^u(x) := \{y \in M : \lim_{n\to\infty} d(f^{-n}x, f^{-n}y) = 0\}$ (Proposition 4.4). E_x^u is tangent to $W^u(x)$ for all $x \in M$ and is continuous everywhere else except the fixed point p. Most of the arguments are routine except for proving that E_p^u is tangent to $W^u(p)$ (Lemma 4.5).

For convenience we will refer to $W^u(x)$ and $W_\epsilon^u(x) = \{y \in M : d(f^{-n}x, f^{-n}y) \leq \epsilon \ \forall n \geq 0\}$ as the “unstable manifold” and “local unstable manifold” at x, even though points on the manifolds may not be contracted exponentially in backwards time.

Lemma 4.1. *Let $x \in M\backslash\{p\}$. Then $|Df_x^n v| \to \infty$ $\forall v \in \mathcal{C}_x^u$ and $|Df_x^{-n} v| \to \infty$ $\forall v \in \mathcal{C}_x^s$, as $n \to \infty$.*

Proof. If $f^n x \notin B(p, r_0)$ for infinite number of n's, then by (1.2) the result is clear.

If there exists $N > 0$ such that $\forall n > N$, $f^n x \in B(p, r_0)$, then by Lemma 2.5,

$$\left|Df_x^n v\right| \geq \left|Df_x^N v\right| \prod_{i=N+1}^{n-1} \left(1 + \kappa^u |f^i x|^2\right) \geq \left|Df_x^N v\right| \prod_{i=N+1}^{n-1} \left(1 + \kappa^u \frac{1}{3(k+i)\eta}\right)$$

for some $k > 1$. Thus $\left|Df_x^n v\right| \to \infty$. $\square$

Proposition 4.2. *There exists an invariant decomposition of the tangent bundle into $TM = E^u \oplus E^s$ such that $\forall x \in M$, $E_x^u \subset \mathcal{C}_x^u$, $E_x^s \subset \mathcal{C}_x^s$, and $Df_x E_x^u = E_{fx}^u$, $Df_x E_x^s = E_{fx}^s$. Except for the fixed point p, the decomposition is continuous.*

Proof. E_p^u and E_p^s have been defined in the above section.

For $x \in M\backslash\{p\}$, let $\mathcal{E}_x^u = \bigcap_{n=0}^{\infty} Df_{f^{-n}x}^n \mathcal{C}_{f^{-n}x}^u$. Clearly, $\mathcal{E}_x^u \subset \mathcal{C}_x^u$ and $Df_x \mathcal{E}_x^u = \mathcal{E}_{fx}^u$.

We show that $\mathcal{E}_x^u$ is a one-dimensional subspace in $T_x M$. In fact, if there are two independent vectors in $\mathcal{E}_x^u$, then we can choose $v, v' \in \mathcal{E}_x^u$ such that $0 \neq v - v' \in \mathcal{C}_x^s$. By Lemma 4.1 $|Df_x^{-n}(v - v')| \to \infty$ as $n \to \infty$. On the other hand, since $v, v' \in Df_{f^{-n}x}^n \mathcal{C}_{f^{-n}x}^u$, $|Df_x^{-n}(v - v')| \leq |Df_x^{-n} v'| + |Df_x^{-n} v| \leq |v| + |v'|$ $\forall n > 0$. This is a contradiction.

Put $E_x^u = \mathcal{E}_x^u$ $\forall x \in M\backslash\{p\}$.

Now we prove the continuity of E_x^u. Suppose for some $x_0 \in M\backslash\{p\}$, there exists a sequence $\{x_i\}$ such that $\lim_{i\to\infty} x_i = x_0$ and $\lim_{i\to\infty} E_{x_i}^u = E'_{x_0} \neq E_{x_0}^u$. Take $v \in E'_{x_0}$ with $|v| = 1$. Let v^u and v^s denote the projection of v in $E_{x_0}^u$ and $E_{x_0}^s$ respectively. Since $E'_{x_0} \neq E_{x_0}^u$, $v^s \neq 0$. By Lemma 4.1 $|Df_{x_0}^{-n} v^s| \to \infty$ as $n \to \infty$. Hence, we can find an $n > 0$ such that $|Df_{x_0}^{-n} v| \geq 2$. On the other hand, if we take $v_i \in E_{x_i}^u$ $\forall i > 0$ such that $\lim_{i\to\infty} v_i = v$, then by the continuity of Df_x^{-n}, we have $|Df_{x_0}^{-n} v| = \lim_{i\to\infty} |Df_{x_i}^{-n} v_i| \leq 1$, a contradiction. $\square$

Remark 4.3. The decomposition into $T_x M = E_x^u \oplus E_x^s$ is not continuous at p. This can be seen by using a similar method as in the proof of Proposition 3.2 for vectors in unstable subspaces instead of those in the boundaries of unstable cones. We leave the details to the reader.

Proposition 4.4. *For any $x \in M$, $W_\epsilon^u(x)$ is a curve tangent to E^u.*

Proof. Let $x \in M\backslash\{p\}$. Construct a continuous vector field v_y in a suitable neighborhood of x such that $v_y \in E_y^u$. It is easy to see that the integral curve of the vector field that passes throuth x is contained in $W^u(x)$. To prove that $W^u(x)$ is also contained in the integral curve passing through x, it is enough to show that any piece of stable curve γ^s intersects $W_\epsilon^u(x)$ at most one point.

In fact, if $\gamma^s \cap W_\epsilon^u(x) \supset \{y, z\}$ with $y \neq z$, then by Lemma 4.1, we have $\lim_{n\to\infty} d(f^{-n}y, f^{-n}x) = 0$ and $\lim_{n\to\infty} d(f^{-n}z, f^{-n}x) = 0$. So $\lim_{n\to\infty} d(f^{-n}y, f^{-n}z) = 0$. This contradicts the fact that Df^{-n} is expanding along the tangent lines of γ^s.

Now we consider the case $x = p$. Let Ω be the set of points in $B(p, \epsilon)$ that can be jointed from p by a curve tangent to vectors in $\tilde{\mathcal{C}}^u$. Note that $f^n(\Omega) \cap B(p, \epsilon)$ is decreasing as $n \to \infty$. It is easy to check $W_\epsilon^u(p) = \bigcap_{i=0}^{\infty} \left(f^i \Omega \cap B(p, \epsilon)\right)$. The above argument on uniqueness shows that $W^u_\epsilon(p)$ is a curve. The differentiability of $W^u_\epsilon(p)$ at p is proved in Lemma 4.5. □

Lemma 4.5. *$W^u_\epsilon(p)$ is tangent to E^u_p.*

Proof. Let $\gamma : [-1,1] \to W^u_\epsilon(p)$ be the parameter expression with $\gamma(0) = p$. For $x_1, x_2 \in W^u_\epsilon(p)$, we say $x_1 > x_2$, if $x_1 = \gamma(s_1)$, $x_2 = \gamma(s_2)$ and $s_1 > s_2$.

First, we prove that the one side limit $\lim_{s\to 0^+} \dfrac{\gamma(s)}{|\gamma(s)|}$ exists. Suppose there are two sequences $x'_n = \gamma(s'_n)$, $x''_n = \gamma(s''_n)$, $s'_n, s''_n \to 0$ as $n \to \infty$, and two unit vectors e', e'' such that $\dfrac{x'_n}{|x'_n|} = e'$ and $\dfrac{x''_n}{|x''_n|} = e''$ $\forall n > 0$. Without loss of generality we assume $\Theta(e', e'') \geq 0$ and $x'_1 > x''_1 > x'_2 > x''_2 > \cdots$. Let $e = \dfrac{\beta' e' + \beta'' e''}{|\beta' e' + \beta'' e''|}$ for some $\beta', \beta'' > 0$. Take $y_n, z_n \in W^u_\epsilon(p)$ such that

$$y_n = \sup\{y < x'_n : \ y = |y|e\} \qquad \text{and} \qquad z_n = \sup\{z < x''_n : \ z = |z|e\}.$$

By Corollary 2.3, $|fy_n - y_n| \leq \eta |y_n|^3$. Thus, if n is large enough, then $y_n < fy_n < x'_n$, and therefore $\Theta(fy_n, y_n)$ and $\Theta(x'_n, y_n)$ have the same sign, i.e., $\Theta(fy_n, y_n) > 0$. Similarly, we have $\Theta(fz_n, z_n) < 0$. Thus, by Lemma 2.7 we get $\Theta\big(e, D^3 f_p(e,e,e)\big) = 0$. Since β' and β'' are arbitrary, by Lemma 3.7 we must have $e' = e''$. So the limit $e^u_+ := \lim_{s\to 0^+} \dfrac{\gamma(s)}{|\gamma(s)|}$ exists.

Now we prove $e^u_+ \in E^u_p$. Suppose it is not true. We may assume $\Theta(e^u_+, A_{e^u_+} e^u_+) > 0$. By Lemma 2.7 $\lim_{t\to 0^+} \dfrac{\Theta\big(te^u_+, f(te^u_+)\big)}{t^2} > 2\beta$ for some $\beta > 0$. Hence, we can find $t_0 > 0$, $\theta_0 > 0$ such that for any $z \in \Gamma := \{y : |y| \leq t_0, |\Theta(y, e^u_+)| \leq \theta_0\}$, $\Theta(z, fz) > \beta |z|^2$. Also, we can find $s_0 > 0$ such that the piece of unstable curve $\{\gamma(s) : 0 \leq s \leq s_0\}$ is contained in Γ. Take any point x in this curve. We have $\Theta(f^{-n}x, x) = \sum_{i=1}^{n} \Theta(f^{-i}x, f^{-i+1}x) \geq \beta \sum_{i=1}^{n} |f^{-i}x|^2$. By Lemma 2.5, $\sum_{i=1}^{n} |f^{-i}x|^2$ is unbounded. This contradicts the fact that $f^{-i}x \in \Gamma$ $\forall i \geq 0$.

Similarly, the limit $e^u_- := \lim_{s\to 0^-} \dfrac{\gamma(s)}{|\gamma(s)|}$ exists and satisfies $e^u_- \in E^u_p$. □

Though E^u_x and E^s_x are not continuous at p, they are contained in $\mathcal{C}^u_x$ and $\mathcal{C}^s_x$. Therefore, we know that f has a local product structure, i.e, there exist constants $\epsilon > 0$, $\delta > 0$, such that $\forall x, y \in M$ with $d(x,y) \leq \delta$, $[x,y] := W^u_\epsilon(x) \cap W^u_\epsilon(y)$ contains exactly one point.

Lemma 4.6. *Let $x \in B(p, r_0)$ and y be in the W^u-segment connecting x and $[x, fx]$. Then*

$$d(x,y) \leq C d(x, fx) \qquad \textit{and} \qquad d(x,y) \leq \eta C |x|^3,$$

for some generic constant, which is allowed to depend only on f.

Proof. Use the fact that $\mathcal{C}^u_x$ and $\mathcal{C}^s_x$ are bounded away from each other to get the first inequality. The second one then follows from Corollary 2.3. □

5. Coordinate systems

The purpose of this section is to choose a suitable coordinate system at each tangent space, under which we can prove that at most points in M, Df_x contract angles between vectors in $\tilde{\mathcal{C}}^u$. This is important for the proof of the local Hölder condition in next section.

Let $\tilde{E}^u_x \oplus \tilde{E}^s_x$ be any coordinate system such that $\tilde{E}^u_x \subset \mathcal{C}^u_x$ and $\tilde{E}^s_x \subset \mathcal{C}^s_x$ $\forall x \in M$. Under the coordinate systems, for each $x \in M$ there exists a correspondence π_x from the unit circle in the tangent space T_xM to $\mathbb{R} \cup \{\infty\}$ such that if $\pi_x e = \sigma$, then e and $(1, \sigma)$ are collinear. The correspondence is one-to-one if we identify $-e$ with e.

For simplicity of notations, we say $\sigma_x \in \mathcal{C}^u_x$ if $\pi_x^{-1}\sigma_x \in \mathcal{C}^u_x$, and $\sigma \in \mathcal{C}^u$ if $\sigma_x \in \mathcal{C}^u_x$ $\forall x \in M$. In particular, we denote by σ^u the unique function that satisfies $\sigma \in E^u$.

Suppose $\big(f_{ij}(x)\big)$ is the matrix expression of $Df_x : \tilde{E}^u_x \oplus \tilde{E}^s_x \to \tilde{E}^u_{fx} \oplus \tilde{E}^s_{fx}$. With respect to the coordinate systems, Df_x induces a map F_x on $\mathbb{R}$ given by

$$F_x\sigma = \big(f_{21}(x) + f_{22}(x)\sigma\big)\big(f_{11}(x) + f_{12}(x)\sigma\big)^{-1} \qquad \forall \sigma \in \mathbb{R}. \tag{5.1}$$

For any σ, we define $\mathcal{F}\sigma = F \circ \sigma \circ f^{-1}$, i.e., $(\mathcal{F}\sigma)(x) = F_{f^{-1}x}\sigma(f^{-1}x)$. Clearly, $\mathcal{F}\sigma \in \mathcal{C}^u$ if $\sigma \in \mathcal{C}^u$ and $\mathcal{F}\sigma \in E^u$ if $\sigma \in E^u$.

Lemma 5.1. *For any $\sigma \in \mathcal{C}^u$, $\lim_{n\to\infty} (\mathcal{F}^n\sigma)(x) = \sigma^u(x)$ $\forall x \in M\backslash\{p\}$.*

Proof. Note that $(\mathcal{F}^n\sigma)(x) \in Df^n_{f^{-n}x}\mathcal{C}^u_{f^{-n}x}$. By the proof of Proposition 4.2,

$$\lim_{n\to\infty} (\mathcal{F}^n\sigma)(x) \in \bigcap_{n=0}^{\infty} Df^n_{f^{-n}x}\mathcal{C}^u_{f^{-n}x} = E^u_x.$$

□

Consider a particular coordinate system $\Xi^u_x \oplus \Xi^s_x$ on T_xM $\forall x \in B(p, r_0)$, where Ξ^u_x and Ξ^s_x are eigenspaces of $A_x = \frac{1}{2}D^3 f_p(x, x, \cdot)$.

Lemma 5.2. *Suppose $r_0 > 0$ is small sufficiently. Then with respect to the decomposition $TM_x = \Xi^u_x \oplus \Xi^s_x$ $\forall x \in B(p, r_0)$, the induced map F_x of Df_x satisfies the following.*

i) *There exists a constant $C_0 > 0$ such that $\forall \sigma \in \tilde{\mathcal{C}}^u$,*

$$\big|F_x\sigma - F_y\sigma\big| \le C_0|x|^{-1}d(x, y) \qquad \forall x \in B(p, r_0),\ y \in B(x, |x|^3). \tag{5.2}$$

ii) *There exist constants $\kappa', \eta' > 0$, sectors $\mathcal{S}^u_{r_0} = \{y \in B(p, r_0) :\ |\Theta(y, E^u_p)| \le \theta^u\}$ and $\mathcal{S}^s_{r_0} = \{y \in B(p, r_0) :\ |\Theta(y, E^s_p)| \le \theta^s\}$ for some $\theta^u, \theta^s > 0$ such that $\forall \sigma_1, \sigma_2 \in \tilde{\mathcal{C}}^u_x$,*

$$\big|F_x(\sigma_1 - \sigma_2)\big| \le (1 - \kappa'|x|^2)\big|\sigma_1 - \sigma_2\big| \qquad \forall x \in \mathcal{S}^u_{r_0} \cup \mathcal{S}^s_{r_0}, \tag{5.3}$$

$$\big|F_x(\sigma_1 - \sigma_2)\big| \le (1 + \eta'|x|^2)\big|\sigma_1 - \sigma_2\big| \qquad \forall x \in B(p, r_0)\backslash\big(\mathcal{S}^u_{r_0} \cup \mathcal{S}^s_{r_0}\big). \tag{5.4}$$

Proof. Note that $Df_x\xi^u_x = (1 + \lambda^u_x|x|^2)\xi^u_x + R_F(x)\xi^u_x$ and $Df_x\xi^s_x = (1 - \lambda^s_x|x|^2)\xi^s_x + R_F(x)\xi^s_x$. The matrix of $Df_x : \Xi^u_x \oplus \Xi^s_x \to \Xi^u_{fx} \oplus \Xi^s_{fx}$ can be expressed as

$$\begin{aligned}\begin{pmatrix} 1 + g_{11}(x) & g_{12}(x) \\ g_{21}(x) & 1 + g_{22}(x) \end{pmatrix} &= \begin{pmatrix} 1 + h_{11}(x, fx) & h_{12}(x, fx) \\ h_{21}(x, fx) & 1 + h_{22}(x, fx) \end{pmatrix} \\ &\quad\times \begin{pmatrix} 1 + \lambda^u_x|x|^2 + O(|x|^2) & O(|x|^2) \\ O(|x|^2) & 1 - \lambda^s_x|x|^2 + O(|x|^2) \end{pmatrix},\end{aligned} \tag{5.5}$$

where h_{ij} are as in Lemma 3.6.

By (5.1) we get

$$F_x\sigma = \big(g_{21}(x) + \sigma + g_{22}(x)\sigma\big)\big(1 + g_{11}(x) + g_{12}(x)\sigma\big)^{-1}, \tag{5.6}$$

$$\begin{aligned}&\big|F_x(\sigma_1 - \sigma_2)\big|\\ &\quad= \frac{1 + g_{11}(x) + g_{22}(x) + g_{11}(x)g_{22}(x) - g_{12}(x)g_{21}(x)}{\big(1 + g_{11}(x) + g_{12}(x)\sigma\big)^2}|\sigma_1 - \sigma_2|,\end{aligned} \tag{5.7}$$

where σ in (5.7) is between σ_1 and σ_2.

By Lemma 3.6, $|h_{ij}(x, fx) - h_{ij}(y, fy)| \le C|\Theta(x, y)| + C|\Theta(fx, fy)| \le 3C|\Theta(x, y)|$. It is easy to see by (3.6) and the linearity of A_x that $|\lambda^u_x - \lambda^u_y| \le 3C|\Theta(x, y)|$, if C is large enough. Note that $\sin|\Theta(x, y)| \le |x|^{-1}d(x, y)$ and $\big||x|^2 - |y|^2\big| \le 3|x|d(x, y)$. Therefore, (5.2) follows from (5.5) and (5.6).

By Lemma 2.7, $|\Theta(x, fx)| = O(|x|^2)$. So (5.4) follows from (5.5) and (5.7).

Moreover, if $x \in E^u_p$ or E^s_p, then $\Theta(x, fx) = o(|x|^2)$. Hence $h_{ij}(x, fx) = o(|x|^2)$ as $x \to 0$. So $g_{11}(x) = \lambda^u_x + o(|x|^2)$, $g_{22}(x) = -\lambda^s_x + o(|x|^2)$, and $g_{12}(x), g_{21}(x) = o(|x|^2)$. Thus, (5.3) holds for these x. Then we use continuity. $\square$

Let $\beta > 0$ small. For $x \in B(p, r_0)$, take $k = k(x)$ such that

$$1 + 2\beta \le \prod_{i=0}^{k-1}\big(1 + |f^i x|^2\big) \le 1 + 4\beta.$$

Considering Proposition 2.6, we have that there exist $0 < c_1 \le c_2 \le 1$ such that

$$c_1 \le k|x|^2 \le c_2. \tag{5.8}$$

Lemma 5.3. *There exists $C'_\beta > 0$ such that*

$$\|Df^k_x - Df^k_y\| \le kC'_\beta|x|d(x, y) \qquad \forall x \in B(p, r_0),\ y \in B(x, |x|^3).$$

Proof. We have

$$\|Df^k_x - Df^k_y\| \le \sum_{j=0}^{k-1}\|Df^{k-j-1}_{f^{j+1}x}\|\|Df_{f^j x} - Df_{f^j y}\|\|Df^j_y\|.$$

By Proposition 2.1, $\|Df_{f^j x} - Df_{f^j y}\| \le C'|f^j x|d(f^j x, f^j y)$ for some $C' > 0$. Also, by Corollary 2.3, $\|Df^j_x\| \le \prod_{i=0}^{j-1}(1+3\eta|f^i x|^2) \le \prod_{i=0}^{j-1}(1+|f^i x|^2)^{3\eta}$. Since the difference among x, $|f^i x|$ and $|f^i y|$ are of higher order, we have

$$\|Df^k_x - Df^k_y\| \le kC'(1 + 4\beta)^{3\eta}|x|d(x, y).$$

$\square$

Suppose $\big(f^{(k)}_{ij}(x)\big)$ is the matrix expression of $Df^k_x : \Xi^u_{x,k} \oplus \Xi^s_{x,k} \to \Xi^u_{f^k x,k} \oplus \Xi^s_{f^k x,k}$, where $\Xi^u_{x,k}$ and $\Xi^s_{x,k}$ are eigenspaces of $\prod_{i=0}^{k-1}\big(\mathrm{id} + \frac{1}{2}D^3 f_p(f^i x, f^i x, \cdot)\big)$. It induces a map on $\mathbb{R}$ by

$$F^{(k)}_x\sigma = \big(f^{(k)}_{21}(x) + f^{(k)}_{22}(x)\sigma\big)\big(f^{(k)}_{11}(x) + f^{(k)}_{12}(x)\sigma\big)^{-1} \qquad \forall\sigma \in \mathbb{R}. \tag{5.9}$$

Lemma 5.4. *The map $F^{(k)}_x$ satisfies the following.*

i) *There exists a constant* $C_\beta > 0$ *such that* $\forall \sigma \in \tilde{\mathcal{C}}^u$,

$$\big|F_x^{(k)}\sigma - F_y^{(k)}\sigma\big| \le C_\beta |x|^{-1} d(x,y) \qquad \forall x \in B(p, r_0),\ y \in B(x, |x|^3). \tag{5.10}$$

ii) *There exist constants* $1 > C_\kappa > 0$, $C_\eta > 0$ *such that* $\forall \sigma_1, \sigma_2 \in \tilde{\mathcal{C}}^u_x$,

$$\big|F_x^{(k)}(\sigma_1 - \sigma_2)\big| \le C_\kappa \big|\sigma_1 - \sigma_2\big| \qquad \forall x \in \mathcal{S}^u_{r_0} \cup \mathcal{S}^s_{r_0}, \tag{5.11}$$

$$\big|F_x^{(k)}(\sigma_1 - \sigma_2)\big| \le C_\eta \big|\sigma_1 - \sigma_2\big| \qquad \forall x \in B(p, r_0) \backslash \big(\mathcal{S}^u_{r_0} \cup \mathcal{S}^s_{r_0}\big). \tag{5.12}$$

Proof. The matrix $\big(f_{ij}^{(k)}(x)\big)$ can be written as

$$\begin{aligned} &\begin{pmatrix} f_{11}^{(k)}(x) & f_{12}^{(k)}(x) \\ f_{21}^{(k)}(x) & f_{22}^{(k)}(x) \end{pmatrix} \\ &\qquad = \begin{pmatrix} 1 + h_{11}^{(k)}(x, f^k x) & h_{12}^{(k)}(x, f^k x) \\ h_{21}^{(k)}(x, f^k x) & 1 + h_{22}^{(k)}(x, f^k x) \end{pmatrix} \begin{pmatrix} \bar{f}_{11}^{(k)}(x) & \bar{f}_{12}^{(k)}(x) \\ \bar{f}_{21}^{(k)}(x) & \bar{f}_{22}^{(k)}(x) \end{pmatrix}, \end{aligned} \tag{5.13}$$

where $\big(\bar{f}_{ij}^{(k)}(x)\big)$ is the matrix of Df_x^k with respect to the coordinate system $\Xi^u_{x,k} \oplus \Xi^s_{x,k}$, and $\big(\delta_{ij} + h_{ij}^{(k)}(x)\big)$ is the matrix of identity $\Xi^u_{x,k} \oplus \Xi^s_{x,k} \to \Xi^u_{f^k x,k} \oplus \Xi^s_{f^k x,k}$.

By (5.9),

$$\big|F_x^{(k)}(\sigma_1 - \sigma_2)\big| = \frac{f_{11}^{(k)}(x) f_{22}^{(k)}(x) - f_{12}^{(k)}(x) f_{21}^{(k)}(x)}{\big(f_{11}^{(k)}(x) + f_{12}^{(k)}(x)\sigma\big)^2} |\sigma_1 - \sigma_2|, \tag{5.14}$$

where σ is between σ_1 and σ_2.

Similarly, we have $|h_{ij}^{(k)}(x, f^k x) - h_{ij}^{(k)}(y, f^k y)| \le C|\Theta(x,y)| + C|\Theta(f^k x, f^k y)| \le 3C|\Theta(x,y)| \le 4C|x|^{-1} d(x,y)$. By (5.8) and Lemma 5.3, $|\bar{f}_{21}^{(k)}(x) - \bar{f}_{21}^{(k)}(y)| \le C'_\beta k|x| d(x,y) \le C'_\beta |x|^{-1} d(x,y)$. Hence (5.10) follows from (5.13), (5.8) and (5.9).

Since $\bar{f}_{ij}^{(k)}(x)$ and $h_{ij}^{(k)}(x, f^k x)$ are bounded, (5.12) follows from (5.13) and (5.14).

Now it remains to prove (5.11). Note that $\forall v \in \Xi^u_{f^k x,k}$ with $|v| = 1$,

$$\Big|\big(\prod_{i=0}^{k-1} (\mathrm{id} + A_{f^i x})\big)\ v\Big| \ge \prod_{i=0}^{k-1} (1 + \kappa^u |f^i x|) \ge (1+\beta)^{\kappa^u},$$

where $A_x = \frac{1}{2} D^3 f_p(x, x, \cdot)$. Also note that by Proposition 2.1 and (5.8),

$$\Big\|Df_x^k - \prod_{i=0}^{k-1} \big(\mathrm{id} + A_{f^i x}\big)\Big\| = kO(|x|^3) = O(|x|).$$

So given $\epsilon > 0$, we can take r_0 such that $\bar{f}_{11}^{(k)}(x) \ge (1+\beta)^{\kappa^u} - \epsilon$, and $|\bar{f}_{12}^{(k)}(x)| \le \epsilon$ $\forall x \in B(p, r_0)$. For the same reason, we have $|\bar{f}_{21}^{(k)}(x)| \le \epsilon$, $\bar{f}_{22}^{(k)}(x) \le (1-\beta)^{\kappa^s} + \epsilon$.

Since $x \in \mathcal{S}^u_{r_0} \cup \mathcal{S}^s_{r_0}$, $fx, \cdots, f^k x$ are close to E^u_p or E^s_p, and the eigenvectors are close to Ξ^u_x and Ξ^s_x. Hence $|h_{ij}^{(k)}(x, f^k x)| \le \epsilon$. Now we obtain that the matrics $\big(f_{ij}^{(k)}(x)\big)$ satisfies the following condition:

$$\begin{aligned} f_{11}^{(k)}(x) &\ge (1+\beta)^{\kappa^u} - c\epsilon, & |f_{12}^{(k)}(x)| &\le c\epsilon, \\ |f_{21}^{(k)}(x)| &\le c\epsilon, & f_{22}^{(k)}(x) &\le (1-\beta)^{\kappa^s} + c\epsilon, \end{aligned}$$

where c is a constant independent of ϵ. So we can take ϵ small enough such that (5.14) implies (5.11). ∎

For $r \in (0, r_0]$, denote $\mathcal{S}_r^u = B(p,r) \cap \mathcal{S}_{r_0}^u$, $\mathcal{S}_r^s = B(p,r) \cap \mathcal{S}_{r_0}^s$ and $\mathcal{S}_r^c = B(p,r)\backslash\big(\mathcal{S}_r^u \cup \mathcal{S}_r^s\big)$.

Let $\Lambda \subset M$. For $x \in \Lambda$, denote by $\tau_\Lambda^\pm(x)$ the maximal nonnegative integers such that $\forall k = 0, 1, \cdots, \tau_\Lambda^\pm(x)$, $f^{\pm k}x \in \Lambda$.

Lemma 5.5. *Let $\beta \geq 0$. For each $x \in \mathcal{S}_{r_0}^c$, take $-\tau_{\mathcal{S}_{r_0}^c}^-(x) = n_0 < n_1 < \cdots < n_m = \tau_{\mathcal{S}_{r_0}^c}^+(x)$ such that $1 + 2\beta \leq \prod_{i=n_{j-1}}^{n_j - 1}\big(1 + |f^i x|^2\big) \leq 1 + 4\beta$ $\forall j = 1, \cdots, m$. Then m is bounded for all $x \in \mathcal{S}_{r_0}^c$.*

Proof. Denote $\tau^\pm(x) = \tau_{\mathcal{S}_{r_0}^c}^\pm(x)$. Fix an $z \in \mathcal{S}_{r_0}^c$ such that $f^{-\tau^-(z)-1}(z) \in \mathcal{S}_{r_0}^s$ and $f^{\tau^+(z)+1}(z) \in \mathcal{S}_{r_0}^u$. We may assume $e^{4|f^j z|^2} \leq 1 + 4\beta$ $\forall -\tau^-(z) - 1 \leq j \leq \tau^+(z)$.

We only need to consider the case that

$$|f^i x| \leq \frac{1}{2}\min\{|f^j z| : -\tau^-(z) - 1 \leq j \leq \tau^+(z)\},\ -\tau^-(x) - 1 \leq i \leq \tau^+(x).$$

Otherwise, the result is clear.

For $-\tau^-(z) - 1 \leq j \leq \tau^+(z)$, let $S_j \subset B(p, r_0)$ be the smaller sector bounded by rays from the origin to $f^j z$ and $f^{j+1}z$. Let i_j be the smallest integer k such that $f^k x \in S_j$. Thus, $|f^{i_j}x| = t_j |f^j z|$ for some $0 < t_j \leq \frac{1}{2}$. By Proposition 2.8, there are at most $2t_j^{-2}$ successive $f^i x$'s in S_j. By Proposition 2.6, each of them satisfies $|f^i x| \leq \sqrt{2}\ t_j |f^j z|$. Thus,

$$\prod_{f^i x \in S_j}(1 + |f^i x|^2) \leq \big(1 + 2t_j^2|f^j z|^2\big)^{[\frac{2}{t_j^2}]} \leq e^{4|f^j z|^2} \leq 1 + 4\beta.$$

It means that $m = m(x) \leq \tau^+(z) + \tau^-(z) + 1$. $\square$

Lemma 5.6. *There exists $D > 0$ such that $\forall x \in \mathcal{S}_{r_0}^c$,*

$$\prod_{i=-\tau^-(x)}^{\tau^+(x)}(1 + |f^{-i}x|^2) < D,$$

where $\tau^\pm(x) = \tau_{\mathcal{S}_{r_0}^c}^\pm(x)$.

Proof. This is a direct corollary of Lemma 5.5. $\square$

Now we construct the coordinate systems.

Choose $0 < \hat{r} < r_0$ such that for any $x \in \mathcal{S}_{r_0}^s \backslash \mathcal{S}_{\hat{r}}^s$ with $f^{\tau^+(x)+1}(x) \in \mathcal{S}_{\hat{r}}^s$, then

$$\prod_{i=-\tau^-(x)+1}^{\tau^+(x)-1}(1 - \frac{1}{2}\kappa'|f^i x|^2) \leq \frac{1}{D^{2\eta'}},$$

where $\tau^\pm(x) = \tau_{\mathcal{S}_{r_0}^s \backslash \mathcal{S}_{\hat{r}}^s}^\pm(x)$ and κ' and η' are as in Lemma 5.2.ii).

Denote

$$\Gamma = \big\{y \in B(p, r_0) : \exists -\tau_{B(p,r_0)}^-(y) \leq i \leq \tau_{B(p,r_0)}^+(y) \text{ s.t. } f^i y \in \mathcal{S}_{\hat{r}}^s\big\}.$$

Choose $0 < r^* < \hat{r}$ such that $B(p, r^*)\backslash W_{r_0}^u(p) \subset \Gamma$. In other words, for any $x \in B(p, r^*)\backslash W_{r_0}^u(p)$, if we iterate backwards, then the orbit of x goes to $\mathcal{S}_{\hat{r}}^s$ before it leaves $B(p, r_0)$. We also assume that r^* is small enough such that $\forall x \notin B(x, r_0)$,

$\hat{\mathcal{C}}^u_x := \bigcup_{y\in B(x,r^{*3})} \pi_x^{-1}\pi_y\mathcal{C}^u_y$ and $\hat{\mathcal{C}}^s_x := \bigcup_{y\in B(x,r^{*3})} \pi_x^{-1}\pi_y\mathcal{C}^s_y$ are still bounded away from eath other.

For $x \in \mathcal{S}^s_{r_0} \cup \mathcal{S}^c_{r^*/2} \cup \mathcal{S}^u_{r_0}$, let $\tilde{E}^u_x \oplus \tilde{E}^s_x$ coincide with $\Xi^u_x \oplus \Xi^s_x$. Then we extend these coordinate systems to $M\backslash\{p\}$ in such a way that $\tilde{E}^u_x \oplus \tilde{E}^s_x$ remains C^1 and with respect to these coordinate systems, $\zeta_x \le 1 + \eta'|x|^2$ for $x \in \mathcal{S}^c_{\hat{r}}$ and $\zeta_x < 1$ for $x \notin B(p, r^*)$, where

$$\zeta_x = \min\{\zeta : \big|F_x\sigma_1 - F_x\sigma_2\big| \le \zeta|\sigma_1 - \sigma_2| \ \forall\sigma_1, \sigma_2 \in \hat{\mathcal{C}}^u_x\}. \tag{5.15}$$

Since $\tilde{E}^u_x \oplus \tilde{E}^s_x$ are C^1, there exists a constant $C_0 > 0$ such that $\forall x \in M\backslash B(p, r^*)$, $y \in B(x, r^{*3})$,

$$\big|F_x\sigma - F_y\sigma\big| \le \frac{C_0}{r^*} d(x, y) \qquad \forall\sigma \in \tilde{\mathcal{C}}^u_x. \tag{5.16}$$

We may regard this C_0 the same as that in (5.2).

6. Local Hölder condition

In this section we prove that for any $x \in M\backslash\{p\}$, there exists a neighborhood in which Hölder condition holds with a constant depending on x. We call this property *local Hölder condition.*

Proposition 6.1. *There exist constants $H > 0$, $\theta > 0$ and $r^* > 0$, such that for all $x \in M\backslash\{p\}$,*

$$d(E^u_x, E^u_y) \le \frac{H}{\rho_x^{3\theta}} d(x, y)^\theta \qquad \forall y \in B(x, \rho_x^3),$$

where $\rho_x = \min\{|x|, r^*\}$.

We will prove it by showing the following result.

Lemma 6.2. *Let $\{\tilde{E}^u_x \oplus \tilde{E}^s_x : x \in M\}$ be constructed as in the last section. There exist constants $H > 0$, $\theta > 0$ and $r^* > 0$ such that if $x \in M\backslash\{p\}$, $y \in B(x, \rho_x^3)$, where $\rho_x = \min\{|x|, r^*\}$, then*

$$\big|\sigma^u(x) - \sigma^u(y)\big| \le \frac{H}{\rho_x^{3\theta}} d(x, y)^\theta.$$

Before we prove the lemma, we give the following fact.

Lemma 6.3. *Lemma 6.2 implies Proposition 6.1.*

Proof. Recall that π_y is a map from the unit circle of T_yM to $\mathbb{R}\cup\{\infty\}$. By choosing a suitable branch we may think π_y^{-1} is well defined. Since $\tilde{\mathcal{C}}^u$ and $\tilde{\mathcal{C}}^s$ are bounded away from each other, restricted to $\tilde{\mathcal{C}}^u$, both π_y and π_y^{-1} are Lipschitz with a Lipschitz constant L $\forall y \in M$. Denote $e^u(y) = \pi_y^{-1}\sigma^u(y)$.

First, consider the case that $x \in B(p, r^*)$. We have

$$\begin{aligned} d\big(e^u(x), e^u(y)\big) &\le L\big|\pi_x e^u(x) - \pi_x e^u(y)\big| \\ &\le L\Big(\big|\pi_x e^u(x) - \pi_y e^u(y)\big| + \big|\pi_y e^u(y) - \pi_x e^u(y)\big|\Big). \end{aligned}$$

Denote $\sigma' = \pi_x e^u(y)$. We know $\pi_y e^u(y) - \pi_x e^u(y) = \pi_y \pi_x^{-1}\sigma' - \sigma'$. By Lemma 3.6 and the definition of π_x,

$$\begin{aligned}\left|\pi_y\pi_x^{-1}\sigma' - \sigma'\right| &= \left|\frac{h_{21}(x,y)+h_{22}(x,y)\sigma'}{1+h_{11}(x,y)+h_{12}(x,y)\sigma'} - \sigma'\right| \\ =&\left|\frac{h_{21}(x,y)+h_{22}(x,y)\sigma' - h_{11}(x,y)\sigma' - h_{12}(x,y)\sigma'^2}{1+h_{11}(x,y)+h_{12}(x,y)\sigma'}\right| \le C'\left|\Theta(x,y)\right|\end{aligned}$$

for some $C' > 0$. Since $d(x,y) \le \rho_x^3 = |x|^3$, we have

$$\left|\Theta(x,y)\right| \le \frac{d(x,y)}{|x|} = \frac{d(x,y)^{1-\theta}}{|x|^{1-3\theta}} \cdot \frac{d(x,y)^\theta}{|x|^{3\theta}} \le |x|^2 \cdot \frac{d(x,y)^\theta}{\rho_x^{3\theta}}.$$

By Lemma 6.2,

$$\left|\pi_x e^u(x) - \pi_y e^u(y)\right| = \left|\sigma^u(x) - \sigma^u(y)\right| \le \frac{H}{\rho_x^{3\theta}} d(x,y)^\theta.$$

Thus,

$$d\big(e^u(x), e^u(y)\big) \le L(H + C'|x|^2)\frac{d(x,y)^\theta}{\rho_x^{3\theta}}.$$

We get the inequality in Proposition 6.1 for $x \in B(p, r^*)$ with $L(H + C' {r^*}^2)$ instead of H.

Similarly, since the coordinate systems is C^1 on $M \backslash B(p, r^*)$, we can find a constant $C'' > 0$ such that

$$\left|\pi_y\pi_x^{-1}\sigma' - \sigma'\right| \le C'' d(x,y) \le C'' {r^*}^{3\theta} \frac{d(x,y)^\theta}{\rho_x^{3\theta}}.$$

The rest of the arguments are the same as above. □

Proof of Lemma 6.2. Take $\theta > 0$ such that for $x \in B(p, r_0)$,

(6.1a) $\quad (1 - \frac{3}{4}\kappa'|x|^2)(1+\eta|x|^2)^{3\theta}(1+4\eta|x|^2)^\theta \le 1 - \frac{1}{2}\kappa'|x|^2 \quad \forall x \in B(p, r_0),$

(6.1b) $\quad (1 + \frac{3}{2}\eta'|x|^2)(1+\eta|x|^2)^{3\theta}(1+4\eta|x|^2)^\theta \le 1 + 2\eta'|x|^2 \quad \forall x \in B(p, r_0),$

(6.1c) $\quad \frac{1+\zeta_x}{2}\left\|Df^{-1}|_{fB(x,\rho_x^3)}\right\|^\theta \le 1 \quad \forall x \notin B(p, r^*),$

where κ' and η' are as in Lemma 5.2.ii), and ζ_x is as in (5.15).

Take

$$H \ge \max\Big\{\frac{4C_0}{\kappa'}, \frac{2C_0}{\eta'}, \frac{2C_0 {r^*}^2}{1-\zeta_x},\ 2^{1+\theta}B\|Df^{-1}\|^\theta\Big\},$$

where C_0 is as in (5.2) and (5.16) and B is the bound of all $\sigma \in \tilde{\mathcal{C}}^u$ with respect to coordinate systems $\tilde{E}_x^u \oplus \tilde{E}_x^s$.

Define a function $s : M \to [1, D^{2\eta'}]$ by putting

$$s_x = \begin{cases} \max\Big\{1,\ D^{2\eta'} \prod_{i=1}^{\tau_B^-(x)} \big(1 - \frac{\kappa'}{2}|f^{-i}x|^2\big)\ \Big\} & x \in \mathcal{S}_{r_0}^s \backslash \mathcal{S}_{\hat{r}}^s; \\ 1, & x \in \mathcal{S}_{\hat{r}}^s; \\ \prod_{i=1}^{\tau_{\mathcal{S}}^-(x)} \big(1 + 2\eta'|f^{-i}x|^2\big), & x \in \mathcal{S}_{\hat{r}}^c \cap \Gamma; \\ D^{2\eta'}, & \text{otherwise,} \end{cases}$$

where $\tau_B^-(x) = \tau_{B(p,r_0)}^-(x)$ and $\tau_{\mathcal{S}}^-(x) = \tau_{\mathcal{S}_{\hat{r}}^c \cap \Gamma}^-(x)$.

Let

$$\mathcal{H} = \Big\{\sigma \in \mathcal{C}^u :\ |\sigma^u(x) - \sigma^u(y)| \le \frac{s_x H}{\rho_x^{3\theta}} d(x,y)^\theta \quad \forall x \in M, y \in B(x, \rho_x^3)\ \Big\}.$$

$\mathcal{H}$ is closed in the sense that for a sequence $\{\sigma_n\}_{n=1}^\infty \subset \mathcal{H}$ if $\lim_{n\to\infty} \sigma_n = \sigma$ pointwise, then $\sigma \in \mathcal{H}$. In fact, $\forall x \in M$ $\forall \beta > 0$, we can find $N > 0$ such that for all $n > N$

$$|\sigma_n(x) - \sigma(x)| \le \beta.$$

Thus $\forall y \in B(x, \rho_x^3)$,

$$|\sigma_n(y) - \sigma(x)| \le |\sigma_n(y) - \sigma_n(x)| + |\sigma_n(x) - \sigma(x)| \le \frac{s_x H}{\rho_x^{3\theta}} d(x,y)^\theta + \beta.$$

Since $\lim_{n\to\infty} \sigma_n(y) = \sigma(y)$ and β are arbitrary, $|\sigma(y) - \sigma(x)| \le \dfrac{s_x H}{\rho_x^3} d(x,y)^\theta$. Hence $\sigma \in \mathcal{H}$.

Let $\sigma \in \mathcal{H}$. Take $x \in M$ and $y \in B(x, \rho_x^3)$. We will show that

$$\big|(\mathcal{F}\sigma)(fx) - (\mathcal{F}\sigma)(fy)\big| \le \frac{s_{fx} H}{\rho_{fx}^{3\theta}} d(fx, fy)^\theta. \tag{6.2}$$

If it is true, then we have $\mathcal{F}\mathcal{H} \subset \mathcal{H}$. So by Lemma 5.1 and the closeness of $\mathcal{H}$, for any $\sigma \in \mathcal{H}$, $\sigma^u = \lim_{n\to\infty} \mathcal{F}^n \sigma \in \mathcal{H}$, which will complete the proof.

We point out here that it is enough to consider the case $y \in B(x, \rho_x^3)$. Because if $d(x,y) > \rho_x^3$, then

$$d(fx, fy) \ge \frac{d(x,y)}{\|Df^{-1}\|} > \frac{\rho_x^3}{\|Df^{-1}\|} \ge \frac{\rho_{fx}^3}{2\|Df^{-1}\|}.$$

So by the choice of H we have $\dfrac{H}{\rho_{fx}^{3\theta}} d(fx, fy)^\theta > \dfrac{H}{(2\|Df^{-1}\|)^\theta} \ge 2B$, which means that (6.2) is always true.

Now suppose $x \in M \backslash \{p\}$ and $y \in B(x, \rho_x^3)$. We have

$$\begin{aligned} \big|\big(\mathcal{F}\sigma\big)(fx) - \big(\mathcal{F}\sigma\big)(fy)\big| &= \big|F_x\sigma(x) - F_y\sigma(y)\big| \\ \le &\big|F_x\sigma(y) - F_y\sigma(y)\big| + \big|F_x\sigma(x) - F_x\sigma(y)\big|. \end{aligned} \tag{6.3}$$

We consider the following cases.

Case I: $x \in \mathcal{S}_{r_0}^s \cup \mathcal{S}_{r_0}^u$.

By (6.3) and Lemma 5.2,

$$\begin{aligned}&\big|\big(\mathcal{F}\sigma\big)(fx)-\big(\mathcal{F}\sigma\big)(fy)\big|\leq\frac{C_0d(x,y)}{|x|}+(1-\kappa'|x|^2)|\sigma(x)-\sigma(y)|\\ \leq&\frac{C_0d(x,y)}{|x|}+(1-\kappa'|x|^2)\frac{s_xH}{\rho_x^{3\theta}}d(x,y)^\theta\\ =&\Big(\frac{C_0d(x,y)^{1-\theta}\rho_x^{3\theta}}{|x|}+(1-\kappa'|x|^2)s_xH\Big)\Big(\frac{\rho_{fx}}{\rho_x}\Big)^{3\theta}\Big(\frac{d(x,y)}{d(fx,fy)}\Big)^\theta\frac{1}{\rho_{fx}^{3\theta}}d(fx,fy)^\theta.\end{aligned}$$

Since $\rho_x\leq|x|$, $d(x,y)\leq|x|^3$ and $s_x\kappa'H\geq 4C_0$, we have

$$\begin{aligned}&\frac{C_0d(x,y)^{1-\theta}\rho_x^{3\theta}}{|x|}+(1-\kappa'|x|^2)s_xH\\ &\qquad\leq C_0|x|^2+(1-\kappa'|x|^2)s_xH\leq(1-\frac{3}{4}\kappa'|x|^2)s_xH.\end{aligned}$$

By the definition of s_x and ρ_x, Corollary 2.3 and 2.4,

$$\Big(\frac{\rho_{fx}}{\rho_x}\Big)^{3\theta}\leq\big(1+\eta|x|^2\big)^{3\theta},\ \Big(\frac{d(x,y)}{d(fx,fy)}\Big)^\theta\leq\big(1+4\eta|x|^2\big)^\theta,\ (1-\frac{\kappa'}{2}|x|^2)s_x\leq s_{fx}.$$

Therefore, (6.2) follows from (6.1a).

Case II: $x\in\mathcal{S}_{\hat r}^c\cap\Gamma$.

Similarly,

$$\begin{aligned}&\big|\big(\mathcal{F}\sigma\big)(fx)-\big(\mathcal{F}\sigma\big)(fy)\big|\leq\frac{C_0d(x,y)}{|x|}+(1+\eta'|x|^2)|\sigma(x)-\sigma(y)|\\ \leq&\frac{C_0d(x,y)}{|x|}+(1+\eta'|x|^2)\frac{s_xH}{\rho_x^{3\theta}}d(x,y)^\theta\\ =&\Big(\frac{C_0d(x,y)^{1-\theta}\rho_x^{3\theta}}{|x|}+(1+\eta'|x|^2)s_xH\Big)\Big(\frac{\rho_{fx}}{\rho_x}\Big)^{3\theta}\Big(\frac{d(x,y)}{d(fx,fy)}\Big)^\theta\frac{1}{\rho_{fx}^{3\theta}}d(fx,fy)^\theta.\end{aligned}$$

We get

$$\begin{aligned}&\frac{C_0d(x,y)^{1-\theta}\rho_x^{3\theta}}{|x|}+(1+\eta'|x|^2)s_xH\leq C_0|x|^2+(1+\eta'|x|^2)s_xH\\ &\qquad\leq(1+\frac{3}{2}\eta'|x|^2)s_xH.\end{aligned}$$

Also,

$$\Big(\frac{\rho_{fx}}{\rho_x}\Big)^{3\theta}\leq\big(1+\eta|x|^2\big)^{3\theta},\ \Big(\frac{d(x,y)}{d(fx,fy)}\Big)^\theta\leq\big(1+4\eta|x|^2\big)^\theta,\ (1+2\eta'|x|^2)s_x\leq s_{fx}.$$

Therefore, (6.2) follows from (6.1b).

Case III: $x\in M\backslash\big(\mathcal{S}_{r_0}^s\cup\Gamma\cup\mathcal{S}_{r_0}^u\big)$.

This is the complement of the above two cases. In this case $\rho_x=\rho_{fx}=r^*$ and $s_x=s_{fx}=D^{2\eta'}$. By (6.3), (5.16) and (5.15),

$$\begin{aligned}&\big|\big(\mathcal{F}\sigma\big)(fx)-\big(\mathcal{F}\sigma\big)(fy)\big|\leq\frac{C_0d(x,y)}{r^*}+\zeta_x|\sigma(x)-\sigma(y)|\\ \leq&\frac{C_0d(x,y)}{r^*}+\zeta_x\frac{s_xH}{r^{*3\theta}}d(x,y)^\theta\\ =&\Big(C_0d(x,y)^{1-\theta}r^{*3\theta-1}+\zeta_xs_xH\Big)\Big(\frac{d(x,y)}{d(fx,fy)}\Big)^\theta\cdot\frac{1}{r^{*3\theta}}d(fx,fy)^\theta.\end{aligned}$$

By the choice of H,

$$C_0 d(x,y)^{1-\theta} r^{*3\theta-1} + \zeta_x s_x H \le C_0 r^{*2} + \zeta_x s_x H \le \frac{1+\zeta_x}{2} s_x H = \frac{1+\zeta_x}{2} s_{fx} H.$$

Since $\dfrac{d(x,y)}{d(fx,fy)} \le \big\|Df^{-1}|_{fB(x,\rho_x^3)}\big\|$, by (6.1c) we get (6.2). ◻

7. Distortion estimates and proof of Theorem A

The main work in this section is to prove bounded distortion estimates for small unstable curves away from p (Proposition 7.5). This result makes it possible to prove Theorem A through standard arguments. The absolute continuity of stable foliation is also proved (implied in Proposition 7.7) and is then applied to the proof of the corollary. In the end of the section we give a criterion distinguishing between the two cases in Theorem A (Proposition 7.8), which will be used in Part 2.

We fix a rectangle of the form $P = [W_r^u(p), W_r^s(p)]$, where $r \in (0, r^*]$, such that $fP \cup P \cup f^{-1}P \subset B(p, r^*)$.

Lemma 7.1. *There exist constants $H^+ > 0$ and $\theta > 0$ such that if $\gamma \subset fP\backslash P$ is a W^u-segment with $f^{-i}\gamma \subset P$ for $i = 1, \cdots, n-1$, then for any $x, y \in \gamma$,*

$$d\big(E^u(f^{-n}x), E^u(f^{-n}y)\big) \le H^+ \frac{d(f^{-n}x, f^{-n}y)^\theta}{r^{3\theta}}.$$

Proof. We assume $\theta < \frac{1}{2}$. Take $H > 0$ satisfying Lemma 6.1 and

$$H \ge \frac{r^{3\theta} C_\beta}{1 - C_\kappa},$$

where C_β and C_κ are as in Lemma 5.4.

Suppose N is the minimal positive integer such that $f^{-N}\gamma \not\subset P$. By the same argument in the proof of Lemma 6.3, it is enough to show that for any $x, y \in f^{-N}\gamma$, $n = 1, \cdots, N$,

$$\big|\sigma^u(f^n x) - \sigma^u(f^n y)\big| \le H^+ \frac{d(f^n x, f^n y)^\theta}{r^{3\theta}}.$$

Let $1 \le n \le N$. Fix $\beta \ge 0$ such that $\|Df_z\| \le 1 + \beta \ \forall\ z \in B(p, r^*)$. Take $0 = n_0 < n_1 < \cdots < n_l < n_{l+1} = n$ such that $1 + 2\beta \le \prod_{i=n_j}^{n_{j+1}-1} \big(1 + |f^i x|^2\big) \le 1 + 4\beta$ $\forall j = 0, \cdots, l$. Then put $k_j = n_{j+1} - n_j$, $j = 0, \cdots, l$.

For $j = 0, \cdots, l$, define $s_j = \max\{1, \bar{s}_j\}$, and

$$\bar{s}_j = \begin{cases} 1, & f^{n_j}x \in \mathcal{S}^s_{r^*}; \\ s_{j-1} C_\eta + \frac{r^{3\theta} C_\beta}{H}, & f^{n_j}x \in \mathcal{S}^c_{r^*}; \\ s_{j-1}, & f^{n_j}x \in \mathcal{S}^u_{r^*}, \end{cases}$$

where C_η and C_β are is as in Lemma 5.4. By Lemma 5.5, there are at most m different n_j with $f^{n_j}x \in \mathcal{S}^c_{r^*}$. So there are at most $m+1$ different s_j, and therefore s_j are bounded.

The result of the lemma follows with $H^+ = \max_j \{s_j H\}$ if we prove that for all $0 \le j \le l+1$,

$$\big|\sigma^u(f^{n_j}x) - \sigma^u(f^{n_j}y)\big| \le s_j H \frac{d(f^{n_j}x, f^{n_j}y)^\theta}{r^{3\theta}}. \tag{7.1}$$

By Lemma 6.2, (7.1) is true for $j = 0$. Suppose it is true for some $0 \le j \le l$. Recall $F^{k_j}_{f^{n_j}x}\sigma^u(f^{n_j}x) = \sigma^u(f^{n_{j+1}}x)$. We have

$$\begin{aligned}&\big|\sigma^u(f^{n_{j+1}}x) - \sigma^u(f^{n_{j+1}}y)\big| = \big|F^{k_j}_{f^{n_j}x}\sigma^u(f^{n_j}x) - F^{k_j}_{f^{n_j}y}\sigma^u(f^{n_j}y)\big|\\ \le&\big|F^{k_j}_{f^{n_j}x}\sigma^u(f^{n_j}y) - F^{k_j}_{f^{n_j}y}\sigma^u(f^{n_j}y)\big| + \big|F^{k_j}_{f^{n_j}x}\sigma^u(f^{n_j}x) - F^{k_j}_{f^{n_j}x}\sigma^u(f^{n_j}y)\big|.\end{aligned}$$

By Lemma 4.6, $d(f^ix, f^iy) \le \eta C|f^ix|^3$. So

$$|f^{n_j}x|^{-1} \cdot d(f^{n_j}x, f^{n_j}y)^{1-\theta} \le \eta^{1-\theta}C^{1-\theta}|f^{n_j}x|^{2-3\theta)} \le 1,$$

for all $f^{n_j}x \in B(p, r^*)$ if r^* is small enough.

Hence by Lemma 5.4.i), we get

$$\begin{aligned}&\big|F^{k_j}_{f^{n_j}x}\sigma^u(f^{n_j}y) - F^{k_j}_{f^{n_j}y}\sigma^u(f^{n_j}y)\big|\\ &\qquad\le C_\beta|f^{n_j}x|^{-1} \cdot d(f^{n_j}x, f^{n_j}y) \le C_\beta d(f^{n_j}x, f^{n_j}y)^\theta.\end{aligned}$$

To prove (7.1) for $j+1$, we need use Lemma 5.4.ii). Consider two cases.

Case I: $f^jx \in \mathcal{S}^s_{r^*} \cup \mathcal{S}^u_{r^*}$. Note that in this case, $s_j = s_{j+1}$. By the choice of H,

$$\big|\sigma^u(f^{n_{j+1}}x) - \sigma^u(f^{n_{j+1}}y)\big| \le C_\beta d(f^{n_j}x, f^{n_j}y)^\theta + s_jC_\kappa\frac{H}{r^{3\theta}}d(f^{n_j}x, f^{n_j}y)^\theta$$

$$\le\big(C_\beta + s_jC_\kappa\frac{H}{r^{3\theta}}\big)d(f^{n_j}x, f^{n_j}y)^\theta \le s_{j+1}H\frac{d(f^{n_{j+1}}x, f^{n_{j+1}}y)^\theta}{r^{3\theta}}. \tag{7.2}$$

Case II: $f^jx \in \mathcal{S}^c_{r^*}$. By the definition of s_j, we have

$$\big|\sigma^u(f^{n_{j+1}}x) - \sigma^u(f^{n_{j+1}}y)\big| \le C_\beta d(f^{n_j}x, f^{n_j}y)^\theta + s_jC_\eta\frac{H}{r^{3\theta}}d(f^{n_j}x, f^{n_j}y)^\theta$$

$$\le\big(C_\beta + s_jC_\eta\frac{H}{r^{3\theta}}\big)d(f^{n_j}x, f^{n_j}y)^\theta \le s_{j+1}H\frac{d(f^{n_{j+1}}x, f^{n_{j+1}}y)^\theta}{r^{3\theta}}. \tag{7.3}$$

Hence, we get (7.1) for $j+1$ instead of j. This completes the proof. □

Lemma 7.2. *There exist constants $H^- > 0$ and $\theta > 0$ such that if $\gamma \subset f^{-1}P\backslash P$ is a W^s-segment with $f^i\gamma \subset P$ for $i = 1, \cdots, n-1$, then for any $x, y \in \gamma$,*

$$d\big(E^u(f^nx), E^u(f^ny)\big) \le \frac{H^-}{r^{3\theta}}d(f^nx, f^ny)^\theta.$$

Proof. The idea is the same as that in the Lemma 7.1. However, since $d(f^{n_j}x, f^{n_j}y)$ is decreasing, the fact $d(f^{n_j}x, f^{n_j}y)^\theta \le d(f^{n_{j+1}}x, f^{n_{j+1}}y)^\theta$ cannot be used. We need some adjustment to obtain the last inequalities in (7.2) and (7.3).

By Corollary 2.3, we have

$$\frac{d(f^{n_j}x, f^{n_j}y)}{d(f^{n_{j+1}}x, f^{n_{j+1}}y)} \le \prod_{i=n_j}^{n_{j+1}-1}\big(1 + \eta|f^ix|^2\big) \le \prod_{i=n_j}^{n_{j+1}-1}\big(1 + |f^ix|^2\big)^\eta \le \big(1 + 4\beta\big)^\eta.$$

So if we take θ small enough such that $\big(1 + 4\beta\big)^{\eta\theta}C_\kappa \le 1$, and then take $H \ge \dfrac{r^{3\theta}\big(1 + 4\beta\big)^{\eta\theta}C_\beta}{1 - \big(1 + 4\beta\big)^{\eta\theta}C_\kappa}$, and set $s_j = \big(s_{j-1}C_\eta + \dfrac{r^{3\theta}C_\beta}{H}\big) \cdot \big(1 + 4\beta\big)^{\eta\theta}$ for $f^{n_j}x \in \mathcal{S}^c_{r^*}$, then the last step of (7.2) and (7.3) can go through. □

Lemma 7.3. *There exists constant $\overline{D} > 0$ such that if $\gamma \subset fP\backslash P$ is a W^u-segment with $f^{-i}\gamma \subset P$ for $i = 1, \cdots, n-1$, then $\forall x, y \in \gamma$,*

$$\sum_{i=0}^{n} d(f^{-i}x, f^{-i}y) \le \overline{D}.$$

Proof. By Lemma 4.6,

$$\sum_{i=0}^{n} d(f^{-i}x, f^{-i}y) \le C \sum_{i=0}^{n} d(f^{-i}x, f^{-i+1}x).$$

Choose $0 \le n^u \le n^s \le n$ such that $f^{-i}x \in \mathcal{S}^u_{r^*}$ if $0 \le i < n^u$, $f^{-i}x \in \mathcal{S}^s_{r^*}$ if $n^s < i \le n$, and $f^i x \in \mathcal{S}^c_{r^*}$ if $n^u \le i \le n^s$.

Note that if $z \in E^u_p$ with $|z|$ small, then $z \in \mathcal{S}^u_{r^*}$ and therefore $d(z, fz)$ is close to $|fz| - |z|$. So we may assume that the sectors $\mathcal{S}^u$ were chosen in such a way that for all $z \in \mathcal{S}^u_{r^*}$, $d(z, fz) = |fz - z| \le K(|fz| - |z|)$ for some $K > 0$. Similarly, if $z \in \mathcal{S}^s_{r^*}$, then $d(z, fz) = |fz - z| \le K(|z| - |fz|)$. Thus,

$$\sum_{i=0}^{n^u-1} d(f^{-i}x, f^{-i+1}x) \le K \sum_{i=0}^{n^u-1} \big(|f^{-i+1}x| - |f^{-i}x|\big) \le K|fx| \le Kr^*.$$

$$\sum_{i=n^s+1}^{n} d(f^{-i}x, f^{-i+1}x) \le K \sum_{i=n^s+1}^{n} \big(|f^{-i}x| - |f^{-i+1}x|\big) \le K|f^{-n}x| \le Kr^*.$$

By Corollary 2.3 and Lemma 5.6,

$$\sum_{i=n^u}^{n^s} d(f^{-i}x, f^{-i+1}x) \le \eta \sum_{i=n^u}^{n^s} |f^{-i}x|^3 \le 2\eta r^* \log\Big(\prod_{i=n^u}^{n^s} (1 + |f^{-i}x|^2)\Big) \le 2\eta r^* \log D.$$

So the result follows by putting $\overline{D} \ge Cr^*(2\eta \log D + 2K)$. $\square$

For $y \in W^u(x)$, let $d^u(x, y)$ denote the distance between x and y measured along $W^u(x)$, and for $z \in W^s(x)$, let $d^s(x, z)$ be defined in an analogous way.

Lemma 7.4. *There exists constant $I > 0$ such that if $\gamma \subset fP\backslash P$ is a W^u-segment with $f^{-i}\gamma \subset P$ for $i = 1, \cdots, n-1$, then $\forall x, y \in \gamma$,*

$$\log \frac{\big|Df_y^{-n}|_{E^u_y}\big|}{\big|Df_x^{-n}|_{E^u_x}\big|} \le I d^u(x, y)^\theta.$$

Proof. Denote $x_i = f^{-i}x$, $y_i = f^{-i}y$ and $\gamma_i = f^{-i}\gamma$.

Choose $\beta \ge 0$ such that $\|Df_z\| \le 1 + \beta \ \forall\, z \in B(p, r^*)$. Take $0 = n_0 < n_1 < \cdots < n_l = n$ such that $1 + 2\beta \le \prod_{i=n_{j-1}+1}^{n_j} \big(1 + |x_i|^2\big) \le 1 + 4\beta \ \forall j = 1, \cdots, l$. Then put $k_j = n_j - n_{j-1}$, $j = 1, \cdots, l$.

Let $e(x) = e^u(x)$ be a unit vector in E^u_x. We have

$$\frac{\big|Df_y^{-n}|_{E^u_y}\big|}{\big|Df_x^{-n}|_{E^u_x}\big|} = \frac{|Df^n_{x_n}e(x_n)|}{|Df^n_{x_n}e(y_n)|}\cdot\frac{|Df^n_{x_n}e(y_n)|}{|Df^n_{y_n}e(y_n)|}$$

$$= \prod_{j=1}^{l}\Big(1+\frac{|Df^{k_j}_{x_{n_j}}e(x_{n_j}) - Df^{k_j}_{x_{n_j}}e(y_{n_j})|}{|Df^{k_j}_{y_{n_j}}e(y_{n_j})|}\Big)\cdot\prod_{i=1}^{n}\Big(1+\frac{|Df_{x_i}e(y_i)-Df_{y_i}e(y_i)|}{|Df_{y_i}e(y_i)|}\Big)$$

$$\le \prod_{j=1}^{l}\Big(1+\|Df^{k_j}_{x_{n_j}}\|\cdot|e(x_{n_j})-e(y_{n_j})|\Big)\cdot\prod_{i=1}^{n}\Big(1+\|Df\|d(x_i,y_i)\Big).$$

By Corollary 2.3, $\|Df_{x_j}\| \le 1+3\eta|x_j|^2$. The choice of n_i implies that $\forall 1\le j\le l$,

$$\|Df^{k_j}_{x_{n_j}}\| \le \prod_{i=n_{j-1}+1}^{n_j}(1+3\eta|x_i|^2) \le \prod_{i=n_{j-1}+1}^{n_j}(1+|x_i|^2)^{3\eta} \le (1+4\beta)^{3\eta}.$$

So by Lemma 7.1,

$$\log\frac{\big|Df_y^{-n}|_{E^u_y}\big|}{\big|Df_x^{-n}|_{E^u_x}\big|} \le (1+4\beta)^{3\eta}\frac{H^+}{r^{3\theta}}\sum_{j=1}^{l} d(x_{n_j},y_{n_j})^\theta + \|Df\|\sum_{i=1}^{n} d(x_i,y_i). \tag{7.4}$$

For any $z_i\in\gamma_i$, $|z_i|^2 > \frac{|x_i|^2}{2}$. Hence for all $1\le j\le l$,

$$\big|Df^{k_j}_{z_{n_j}}|_{E^u_{z_{n_j}}}\big| \ge \prod_{i=n_{j-1}+1}^{n_j}(1+\kappa^u|z_i|^2) \ge \prod_{i=n_{j-1}+1}^{n_j}(1+|x_i|^2)^{\frac{\kappa^u}{2}} \ge (1+2\beta)^{\frac{\kappa^u}{2}}.$$

This means that $d^u(x_{n_j},y_{n_j})$ and therefore $d(x_{n_j},y_{n_j})$ decrease exponentially as j increases. Thus the first sum in (7.4) is bounded. By Lemma 7.4, the second sum is also bounded. Now it is easy to conclude that

$$\log\frac{\big|Df_y^{-n}|_{E^u_y}\big|}{\big|Df_x^{-n}|_{E^u_x}\big|} \le I_0$$

for some $I_0>0$ independent of the choice of γ, x and y.

Take $\bar\gamma\supset\gamma$ in such a way that $\bar\gamma\subset fP\backslash P$ with maximal length. The arguments above tell us that

$$d^u(x_i,y_i) \le e^{2I_0}\frac{d^u(\bar x_i,\bar y_i)}{d^u(\bar x,\bar y)}d^u(x,y),$$

where $\bar x_i$ and $\bar y_i$ are the extreme points of $\bar\gamma_i$. Note that $d(x,y)$ and $d^u(x,y)$ are equivalent for any $y\in W^u_{r_0}(x)$ and that $d^u(\bar x,\bar y)$ is bounded away from 0 for any given P. Using (7.4) again we get the result. □

Proposition 7.5. *There exist constants $\delta>0$ and $J_u>1$, depending on P, such that if γ is a W^u-segment with $l(\gamma)\le\delta$ and $\gamma\cap P=\emptyset$, then $\forall x,y\in\gamma$ and $n>0$,*

$$J_u^{-1} \le \frac{\big|Df_y^{-n}|_{E^u_y}\big|}{\big|Df_x^{-n}|_{E^u_x}\big|} \le J_u.$$

Proof. Use Lemma 7.4 and the fact that $Df_x|_{E^u_x}$ is uniformly expanding outside P. See the proof in [HY], Proposition 3.1 for more details. □

Lemma 7.6. *There exist constants $\delta > 0$ and $J_s > 1$, depending on P, such that if γ is a W^s-segment with* $\operatorname{diam}(\gamma) \le \delta$ *and $\gamma \cap P = \emptyset$, then $\forall x, z \in \gamma$ and $n > 0$,*

$$J_s^{-1} \le \frac{\left|Df_z^n|_{E_z^u}\right|}{\left|Df_x^n|_{E_x^u}\right|} \le J_s.$$

Proof. By using Lemma 7.2, we can prove a result parallel to Lemma 7.4. The rest is the same as in the proof of the previous proposition. □

Proof of Theorem A. Take a rectangle P as in [HY], Lemma 5.1. Define the first return map $g : M \backslash P \to M \backslash P$. Using the same proof as in [HY], Lemma 5.2, we know that there exists a g-invariant Borel probability measure $\bar{\mu}$ with the property that $\bar{\mu}$ has absolutely continuous conditional measures on the unstable manifolds of f. By pushing forward, we can extend $\bar{\mu}$ to an f-invariant measure on M. That is, let

$$\mu = \sum_{i=0}^{\infty} f_*^i(\bar{\mu}|_{Q_i}), \tag{7.5}$$

where $f_*\bar{\mu} = \bar{\mu} \circ f^{-1}$, $Q_0 = M \backslash P$ and $Q_i = \{x \in M \backslash P : fx, \cdots, f^i x \in P\}$ $\forall i \ge 1$. It is easy to see that μ also has absolutely continuous conditional measures on the unstable manifolds of f. So if the series in (7.5) converges, then μM is finite, and after normalization we get an SBR measure of f. Otherwise, μ is a σ-finite measure on M and therefore is an infinite SBR measure of f. □

Let Σ_1 and Σ_2 be two W^u-leaves, and let $\iota : \Sigma_1 \to \Sigma_2$ be a continuous map defined by sliding along the W^s-leaves, i.e., for $x \in \Sigma_1$, $\iota(x) \in \Sigma_2 \cap W^s(x)$.

Proposition 7.7. *The W^s-foliation is locally Lipschitz away from p. More precisely, for any rectangle $P = [W_r^u(p), W_r^s(p)]$, there exist constants $L > 0$ and $\epsilon > 0$ such that $\forall x \in M \backslash P$ with $[W_\epsilon^u(x), W_\epsilon^s(x)] \cap P = \emptyset$, and $\forall z \in W_\epsilon^s(x)$, $\iota : W_\epsilon^u(x) \to W^u(z)$ is Lipschitz with Lipschitz constant L.*

Proof. Let γ be an arbitrarily short segment in $W_\epsilon^u(x)$. We will prove $l(\iota\gamma) \le L \cdot l(\gamma)$ for some $L > 0$ independent of γ, where $l(\gamma)$ denote the length of γ.

Denote by γ^s the stable curve connecting $x_1 \in \gamma$ and $\iota x \in W_\epsilon^s(x_1)$. Take $n > 0$ such that $l(f^n\gamma^s) \le \min\{l(f^n\gamma), l(f^n(\iota\gamma))\}$. Without loss generality we assume the lengths of $f^n\gamma \cap P$ and $f^n(\iota\gamma) \cap P$ are 0 and $l(f^n\gamma), l(f^n\gamma^s) \le \delta$, where δ is as in Proposition 7.5 and Lemma 7.6. This is possible because, otherwise, we can take a shorter γ and larger n.

By the continuity of E_x^u and E_x^s, there is an $L' > 0$ such that $l\big(f^n(\iota\gamma)\big) \le L' \cdot l\big(f^n(\gamma)\big)$. The distorsion estimates in Proposition 7.5 and Lemma 7.6 imply that

$$l(\iota\gamma) \le L' J_s J_u \cdot l(\gamma).$$

□

Proof of the Corollary of Theorem A. The Lipschitzness of the W^s-foliation is given by Proposition 7.7. This enables us to follow the proof of Lemma 5.3 and Theorem B in [HY] to get the desired results. □

The following facts will be used in the proof of Theorem B in the next part.

Take $p^+, p^- \in W^u_{r^*}(p)$ in such a way that p^+ and p^- are in different components of $W^u_{r^*}(p)\backslash\{p\}$, and take $q^+, q^- \in W^s_{r^*}(p)$ similarly. We have $[q, f^{-i}p^\pm] \in W^u_{r^*}(q)$ and $\lim_{n\to\infty}[q, f^{-i}p^\pm] = q$ for $q = q^+$ or q^-. Put

$$\text{(7.6)} \quad \Delta^{++} = \sum_{n=1}^{\infty} d^u(q^+, [q^+, f^{-n}p^+]) \quad \text{and} \quad \Delta = \Delta^{++} + \Delta^{+-} + \Delta^{-+} + \Delta^{--},$$

where Δ^{+-}, Δ^{-+} and Δ^{--} are understood in a similar way as Δ^{++}. It is clear whether $\Delta < \infty$ is independent of the choice of $p^\pm$. Since W^s-foliation is Lipschitz away from p, whether $\Delta < \infty$ is also independent of the choice of $q^\pm$.

Proposition 7.8. *If $\Delta < \infty$, then f admits an SBR measure. Otherwise, f admits an infinite SBR measure.*

Proof. We may assume $p^\pm$ and $q^\pm$ are in the boundary of P. Denote by P^{++} the quarter of P bounded by $W^u_{r^*}(p)$, $W^s_{r^*}(p)$, $W^u_{r^*}(q^+)$ and $W^s_{r^*}(p^+)$. It is enough to show that $\mu P^{++} < \infty$ if and only if $\Delta^{++} < \infty$.

Put $Q_i^{++} = \{x \in Q : \ fx, \cdots, f^i x \in P\}$. By (7.5) $\mu P^{++} = \sum_{i=1}^{\infty} \mu Q_i^{++}$. Denote $q_i = [q^+, f^{-i}p^+]$. Clearly, Q_i^{++} is a rectangle whose unstable direction is bounded by $W^s_{r^*}(q^+)$ and $W^s_{r^*}(q_i)$. The distortion estimates imply that the densities of the conditional measures of μ on unstable manifolds are bounded away from 0 and ∞ (see e.g. the proof of [HY], Lemma 5.2). So, μQ_i^{++} is proportional to $d(q^+, q_i)$, i.e. there exists $J_u \geq 1$ such that ${J_u}^{-1} d(q^+, q_i) \leq \mu Q_i^{++} \leq J_u d(q^+, q_i)$ for all $i \geq 1$. Now the result becomes clear. □

PART 2: Proof of Theorem B

In this part we assume that Assumption A holds. Therefore f can be expressed as (1.3) and (1.4) in some neighborhood $B(p, r)$ of the fixed point p.

Take $z = (x, y) \in B(p, r)$ and denote $\bar z = (\bar x, \bar y) = [z, fz]$. To determine whether the series in (7.6) converges, we should estimate the order of $\bar x - x$ as $x \to 0$. Since both $\bar z$ and fz are in $W^s_r(fz)$, we need to know the slope of $W^s_r(fz)$. Let v^s_z be a real number or ∞ such that $(v^s_z, 1)$ is a tangent vector of $W^s_r(z)$. Most of the work in this part is estimating v^s_z for z near the $y-$axis.

8. Preliminaries

We assume that v^s_z has the form $-\rho\frac{x}{y}$, where $\rho = \rho(x, y)$. Results in Lemma 8.3 and Proposition 8.4 imply that $\rho(x, y) = \frac{a_2}{b_2} + \hat\rho(y) + R_\rho(x, y)$, where $\hat\rho(y)$ is determined by equations (8.6) and (8.7) in Proposition 8.4, and $R_\rho(x, y) \to 0$ as $x \to 0$. These facts allow us to estimate the bounds of $\rho(x, y)$ in §9 and §10.

By (1.3) we know that for $z = (x, y) \in B(p, r)$,

$$\text{(8.1)} \qquad Df_z = \begin{pmatrix} 1 + \phi(x, y) + x\phi_x(x, y) & x\phi_y(x, y) \\ -y\psi(x, y) & 1 - \psi(x, y) - y\psi_y(x, y) \end{pmatrix}$$

where ϕ_x denotes the partial derivative of ϕ with respect to x, etc. The nondegeneracy condition (1.1) implies that on $B(p, r)$, $\phi(x, y) + x\phi_x(x, y) \geq \kappa^u|(x, y)|^2$ and $\psi(x, y) + y\psi_y(x, y) \geq \kappa^s|(x, y)|^2$. In other words, for all $z = (x, y) \in B(p, r)$,

$$\text{(8.2)} \qquad \begin{aligned} 3a_0x^2 + 2a_1xy + a_2y^2 &\geq \kappa^u(x^2 + y^2), \\ b_0x^2 + 2b_1xy + 3b_2y^2 &\geq \kappa^s(x^2 + y^2). \end{aligned}$$

Therefore

$$a_0, a_2, b_0, b_2 > 0. \tag{8.3}$$

In this part we always assume that $z = (x, y) \in B(p, r)$. We also assume that r is small enough such that all higher order terms can be controlled by corresponding lower order terms. We will denote $z_n = (x_n, y_n) = f^n z$, $\phi_n = \phi(x_n, y_n)$, $\psi_n = \psi(x_n, y_n)$, etc. Without loss generality we only consider the case that z is in the first quarter. So we have $x_n, y_n \geq 0$ except for the opposite statement. Also we may assume

$$-1 \leq v_z^s \leq 1 \qquad \forall z \in B(p, r). \tag{8.4}$$

Otherwise, we can rescale the x-axis and y-axis by constant factors. This does not change the conditions of Theorem B because by (1.3) and (1.4) the ratio of a_i/b_i and the signs of a_i and b_i, $i = 0, 1, 2$, remain the same.

Lemma 8.1. *Let $\{t_n\}$ be a sequence of positive numbers, and let $C > 0$ and $\alpha > 0$.*

i) *If $\forall n \geq 1$, $t_{n-1} \geq t_n + Ct_n^{1+\alpha}$, then there exist $D > 0$, $k_0 \geq 1$ such that $t_n \leq D(n - k_0)^{-\frac{1}{\alpha}}$ for all large n. Therefore $\sum_{n=1}^{\infty} t_n$ converges as $\alpha < 1$.*

ii) *If $\forall n \geq 1$, $t_{n-1} \leq t_n + Ct^{1+\alpha}$, then there exist $D > 0$, $k_0 \geq 1$ such that $t_n \geq D(n + k_0)^{-\frac{1}{\alpha}}$ for all large n. Therefore $\sum_{n=1}^{\infty} t_n$ diverges as $\alpha \geq 1$.*

Proof. i) Take $D > 0$ such that $\alpha CD^\alpha \geq 2$. Suppose $t_n \geq Dk^{-\frac{1}{\alpha}}$ for some integer $k > 1$. Then

$$t_{n-1} \geq Dk^{-\frac{1}{\alpha}}\big(1 + C \cdot (Dk^{-\frac{1}{\alpha}})^\alpha\big) = Dk^{-\frac{1}{\alpha}}\big(1 + CD^\alpha k^{-1}\big).$$

It is easy to see that if k is large enough , then $(1 - k^{-1})(1 + CD^\alpha k^{-1})^\alpha > 1$ or $k^{-\frac{1}{\alpha}}\big(1 + CD^\alpha k^{-1}\big) \geq (k-1)^{-\frac{1}{\alpha}}$. So $t_{n-1} \geq D(k-1)^{-\frac{1}{\alpha}}$.

The arguments show that if $t_{n-1} \leq D(n - k_0 - 1)^{-\frac{1}{\alpha}}$ for some $k_0 \geq 1$, then $t_n \leq D(n - k_0)^{-\frac{1}{\alpha}}$.

ii) The proof is similar. □

Lemma 8.2. *Let $z = (0, y)$ and $z_n = (0, y_n) = f^n z$. Then $\sum_{n=0}^{\infty} y_n^3 \leq \frac{2}{b_2} y_0$ and the convergence is uniform for all $y_0 \in [0, r]$.*

Proof. Since $y_{n+1} = y_n - y_n\psi(0, y_n)$, (1.3) and (1.4) imply $y_n - y_{n+1} = y_n\psi(0, y_n) \geq \frac{b_2}{2} y_n^3$. So

$$\sum_{n=0}^{\infty} y_n^3 \leq \frac{2}{b_2} \sum_{n=0}^{\infty} \big(y_n - y_{n+1}\big) = \frac{2}{b_2}\ y.$$

The uniformity is clear. □

Denote $z_1 = fz$ and $\rho_1 = \rho(z_1)$. Note that $\rho_1 = \rho\big(x(1+\phi), y(1-\psi)\big)$ is also a function of $z = (x, y)$. Let

$$\begin{aligned} \Delta_\rho(x, y) =& (\rho - \rho_1)(1+\phi)(1-\psi) + \rho_1 y(1+\phi)\psi_y - y(1-\psi)\phi_y \\ & - \rho_1\rho x(1+\phi)\psi_x + \rho x(1-\psi)\phi_x. \end{aligned} \tag{8.5}$$

Lemma 8.3. *If $v_z^s \leq -\rho(z)\dfrac{x}{y}$ and $0 \leq \Delta_\rho(x, y)$, then $v_{z_1}^s \leq -\rho(z_1)\dfrac{x_1}{y_1}$. The result also holds if all "$\leq$" are replaced by "$\geq$".*

Proof. Since the map $v \to \dfrac{(1+c_{11})v + c_{12}}{c_{21}v + (1-c_{12})}$ is increasing as c_{ij} small, by (8.1),

$$v^s_{z_1} \le \frac{-(1+\phi+x\phi_x)\cdot \rho x + x\phi_y \cdot y}{-y\psi_x \cdot \rho x + (-\psi - y\psi_y)\cdot y}.$$

To get the result, use the fact that $0 \le \Delta_\rho(x,y)$ is equivalent to

$$\frac{-(1+\phi+x\phi_x)\cdot \rho x + x\phi_y \cdot y}{-y\psi_x \cdot \rho x + (1-\psi - y\psi_y)\cdot y} \le -\rho_1 \frac{x(1+\phi)}{y(1-\psi)}.$$

□

The next proposition plays a key role for the proof of Theorem B.

Proposition 8.4. *There exist a Lipschitz function $\hat\rho$ on $[0,r]$ with $\hat\rho(0)=0$ satisfying the following two equations:*

$$\begin{aligned}\Delta_{\frac{a_2}{b_2}+\hat\rho}(0,y) =& \big(\hat\rho(y) - \hat\rho(y_1^{(0)})\big)(1+\phi)(1-\psi)\\ &+\big(\frac{a_2}{b_2} + \hat\rho(y_1^{(0)})\big)y(1+\phi)\psi_y - y(1-\psi)\phi_y = 0\end{aligned}$$

and

$$b_2 \log(1+\phi) + a_2\log(1-\psi) - b_2\int_{y_1^{(0)}}^{y} \frac{\hat\rho(t)}{t}dt = 0, \tag{8.7}$$

where $\phi=\phi(0,y)$, $\psi=\psi(0,y)$ and $y_1^{(0)} = y(1-\psi(0,y))$.

Proof. Denote $z=(0,y)$, $z_n = f^n z = (0,y_n)$, $\phi_n = \phi(0,y_n)$, etc. So $y_1^{(0)}$ can be written as y_1. Define $\hat\rho(0)=0$ and for $y\ne 0$ define

(8.8)
$$\hat\rho(y) = \sum_{n=0}^{\infty} \frac{b_2 y_n(1-\psi_n)\phi_y(0,y_n) - a_2 y_n(1+\phi_n)\psi_y(0,y_n)}{b_2(1+\phi_n)(1-\psi_n)} \prod_{k=0}^{n-1}\Big(1-\frac{y_k\psi_y(0,y_k)}{1-\psi_k}\Big).$$

By (1.4) $b_2 y_n\phi_y(0,y_n) - a_2 y_n\psi_y(0,y_n) = O(y_n^3)$. The product in each term in (8.8) is less than 1 because $y\psi_y > 0$. So by Lemma 8.2,

$$|\hat\rho(y)| \le L_{\hat\rho}\, y \tag{8.9}$$

for some $L_{\hat\rho} > 0$.

It is easy to check

$$\hat\rho(y_1) = \sum_{n=1}^{\infty} \frac{b_2 y_n(1-\psi_n)\phi_y(0,y_n) - a_2 y_n(1+\phi_n)\psi_y(0,y_n)}{b_2(1+\phi_n)(1-\psi_n)} \prod_{k=1}^{n-1}\Big(1-\frac{y_k\psi_y(0,y_k)}{1-\psi_k}\Big).$$

Therefore,

$$\hat\rho(y) = \Big(1-\frac{y\psi_y(0,y)}{1-\psi}\Big)\hat\rho(y_1) + \frac{b_2 y(1-\psi)\phi_y(0,y) - a_2 y(1+\phi)\psi_y(0,y)}{b_2(1+\phi)(1-\psi)}.$$

Multiplying by $(1+\phi)(1-\psi)$, we get (8.6).

Note $y_1 = y(1-\psi)$. Differentiating the left-hand side of (8.7) with respect to y, we get

$$\frac{b_2\phi_y}{1+\phi} - \frac{a_2\psi_y}{1-\psi} + b_2\frac{\hat\rho(y)}{y} - b_2\frac{\hat\rho(y_1)}{y(1-\psi)}(1-\psi-y\psi_y).$$

By (8.6) it is equal to 0. This means that the left-hand side of (8.7) is a constant. Since $\hat\rho(y), \phi(0,y), \psi(0,y) \to 0$ as $y \to 0$, this constant must be zero.

Now we prove Lipschitzness of $\hat\rho$. We will prove that the derivative of $\hat\rho$ is bounded by a constant for $y \neq 0$. This combined with (8.9) implies Lipschitzness of $\hat\rho$ on $[0,r]$.

By (1.4) we can write

$$\frac{b_2 y_n(1-\psi_n)\phi_y(0,y_n) - a_2 y_n(1+\phi_n)\psi_y(0,y_n)}{b_2(1+\phi_n)(1-\psi_n)} = C' y_n^3 + o(y_n^3)$$

and

$$1 - \frac{y_k\psi_y(0,y_k)}{1-\psi_k} = (1-\psi_k)^2 \frac{1-\psi_k - y_k\psi_y(0,y_k)}{(1-\psi_k)^3} = (1-\psi_k)^2\big(1 + C'' y_k^3 + o(y_k^3)\big)$$

for some constants C' and C''. Clearly, $\prod_{k=0}^{n-1}(1-\psi_k) = \frac{y_n}{y}$. Hence we can write (8.8) as

$$\begin{aligned}\hat\rho(y) &= \sum_{n=0}^{\infty}\big[C' y_n^3 + o(y_n^3)\big]\prod_{k=0}^{n-1}(1-\psi_k)^2\big(1 + C'' y_k^3 + o(y_k^3)\big)\\ &= \sum_{n=0}^{\infty}\frac{C' y_n^5 + o(y_n^5)}{y^2}\prod_{k=0}^{n-1}\big(1 + C'' y_k^3 + o(y_k^3)\big).\end{aligned}$$

Since $\dfrac{dy_{k+1}}{dy_k} = 1 - \psi_k - y_k\psi_y(0,y_k) \le 1 - \psi_k = \dfrac{y_{k+1}}{y_k}$, we get $\dfrac{dy_k}{dy} \le \dfrac{y_k}{y} \le 1$. Thus,

$$\Big|\frac{d}{dy}\frac{C' y_n^5 + o(y_n^5)}{y^2}\Big| \le C_1\frac{y_n^5}{y^3} \le C_1\frac{y_n^3}{y}$$

for some $C_1 > 0$. By Lemma 8.2, $\prod_{k=0}^{n-1}\big(1 + C'' y_k^3 + o(y_k^3)\big) \le C_2$. Hence,

$$\begin{aligned}&\Big|\frac{d}{dy}\prod_{k=0}^{n-1}\big(1 + C'' y_k^3 + o(y_k^3)\big)\Big|\\ =&\Big|\sum_{k=0}^{n-1}\frac{3C'' y_k^2 + o(y_k^2)}{1 + C'' y_k^3 + o(y_k^3)}\cdot\frac{y_k}{y}\cdot\prod_{k=0}^{n-1}\big(1 + C'' y_k^3 + o(y_k^3)\big)\Big|\\ \le&\sum_{k=0}^{n-1}4|C''|y_k^2\cdot\frac{y_k}{y}\cdot C_2 \le \frac{4|C''|C_2}{y}\sum_{k=0}^{n-1}y_k^3 \le C_3.\end{aligned}$$

From these inequalities we get

$$\Big|\sum_{n=0}^{\infty}\frac{d}{dy}\Big[\frac{C' y_n^5 + o(y_n^5)}{y^2}\prod_{k=0}^{n-1}\big(1 + C'' y_k^3 + o(y_k^3)\big)\Big]\Big| \le \sum_{n=0}^{\infty}\big(C_1\frac{y_n^3}{y}\cdot C_2 + 2C' y_n^3\cdot C_3\big).$$

Now it is easy to see by Lemma 8.2 that the convergence is uniform. Thus, we know that $\big|\frac{d\hat\rho(y)}{dy}\big|$ is bounded by $\frac{2}{b_2}(C_1C_2 + 2C'C_3 r)$ for all $y \in (0,r]$. This proves the result. $\square$

9. Existence of SBR measures

In this section we prove the first part of Theorem B. We estimate the upper bounds of $|v_z^s|$ in Lemma 9.1 and then show that the general terms of the series in (7.6) decrease at a rate faster than $n^{-\frac{1}{\alpha}}$ (implied in Lemma 9.2).

Lemma 9.1. *Suppose $\alpha a_2 > 2b_2$, $a_1 = 0 = b_1$ and $a_0 b_2 - a_2 b_0 > 0$. Then for any point $q = (0, y_q)$ with $y_q > 0$ small, there exists $\epsilon > 0$ such that $\forall z_0 = (x_0, y_0) \in W_\epsilon^u(q)$ with $x_0 > 0$,*

$$v_{z_0}^s \geq -\Big(\frac{a_2}{b_2} + \hat{\rho}(y_0)\Big)(1 - x^\alpha)\frac{x_0}{y_0}. \tag{9.1}$$

Proof. For each $z_0 = (x_0, y_0) \in W_r^u(q)$, $z_n = (x_n, y_n) = f^n z_0$, we denote

$$\rho_i = \rho(z_i) = \Big(\frac{a_2}{b_2} + \hat{\rho}(y)\Big)\big(1 - x_i^\alpha\big) \qquad \forall i \geq 0. \tag{9.2}$$

We show that for $y_q > 0$ small, there exists $\epsilon > 0$ such that $\forall z_0 = (x_0, y_0) \in W_\epsilon^u(q)$ with $x_0 > 0$,

$$v_{z_i}^s \leq -\rho_i \frac{x_i}{y_i} \quad \text{and} \quad \rho_i x_i \leq y_i$$

imply

$$v_{z_{i+1}}^s \leq -\rho_{i+1}\frac{x_{i+1}}{y_{i+1}}.$$

This proves the result of the lemma. In fact, if (9.1) is not true, then for some large n, $\rho_n x_n \geq y_n$ and therefore $v_{z_n}^s \leq -\rho_n \cdot \dfrac{x_n}{y_n} < -1$, contradicting to (8.4).

By Lemma 8.3, it is enough to show for such $z_i = (x_i, y_i)$,

$$\begin{aligned} \Delta_{\rho_i}(x_i, y_i) = {} & (\rho_i - \rho_{i+1})(1 + \phi_i)(1 - \psi_i) \\ & + \rho_{i+1} y_i (1 + \phi_i)\psi_y(x_i, y_i) - y_i(1 - \psi_i)\phi_y(x_i, y_i) \\ & - \rho_i \rho_{i+1} x_i (1 + \phi_i)\psi_x(x_i, y_i) + \rho_i x_i (1 - \psi_i)\phi_x(x_i, y_i) \geq 0. \end{aligned} \tag{9.3}$$

Note that $x_{i+1}^\alpha - x_i^\alpha = x_i^\alpha(1 + \phi_i)^\alpha - x_i^\alpha = \alpha x_i^\alpha\big(\phi_i + O(\phi_i^2)\big) = \alpha a_2 x_i^\alpha y_i^2 + x_i^\alpha O(x_i^2 + y_i^3)$. By (9.2) we have

$$\begin{aligned} \rho_i - \rho_{i+1} = & \big(\hat{\rho}(y_i) - \hat{\rho}(y_{i+1})\big)(1 - x_{i+1}^\alpha) + \Big(\frac{a_2}{b_2} + \hat{\rho}(y_i)\Big)(x_{i+1}^\alpha - x_i^\alpha) \\ = & \hat{\rho}(y_i) - \hat{\rho}(y_{i+1}) + \frac{a_2}{b_2}\alpha a_2 x_i^\alpha y_i^2 + x_i^\alpha O(x_i^2 + y_i^3). \end{aligned}$$

Since $a_1 = 0 = b_1$, by (1.4) we see that $\phi_y(x, y) = \phi_y(0, y) + xO(x+y)$, $\psi_y(x, y) = \psi_y(0, y) + xO(x + y) = 2b_2 y + O(x^2 + y^2)$. So

$$\begin{aligned} \rho_{i+1} y_i \psi_y(x_i, y_i) = & \Big(\frac{a_2}{b_2} + \hat{\rho}(y_{i+1})\Big) y_i \big(\psi_y(0, y_i) + x_i O(x_i + y_i)\big) \\ & - \frac{a_2}{b_2} x_i^\alpha \big(2b_2 y_i^2 + y_i O(x_i^2 + y_i^2)\big), \end{aligned}$$

and

$$y_i \phi_y(x_i, y_i) = y_i\big(\phi_y(0, y_i) + x_i O(x_i + y_i)\big).$$

By Lemma 8.4,

$$\begin{aligned}&\big(\hat\rho(y_i)-\hat\rho(y_{i+1}^{(0)})\big)(1+\phi_i)(1-\psi_i)\\&+\big(\frac{a_2}{b_2}+\hat\rho(y_{i+1}^{(0)})\big)y_i(1+\phi_i)\psi_y(x_i,y_i)-y_i(1-\psi_i)\phi_y(x_i,y_i)=0,\end{aligned}$$

where $y_{i+1}^{(0)}=y_i(1-\psi(0,y_i))$. Note that $y_{i+1}=y_i(1-\psi(x_i,y_i))$. So both $y_{i+1}-y_{i+1}^{(0)}$ and $\hat\rho(y_{i+1})-\hat\rho(y_{i+1}^{(0)})$ have order $O(x_i^2y_i)$. Hence

$$\begin{aligned}&(\rho_i-\rho_{i+1})(1+\phi_i)(1-\psi_i)+\rho_{i+1}y_i(1+\phi_i)\psi_y(x_i,y_i)-y_i(1-\psi_i)\phi_y(x_i,y_i)\\=&\frac{a_2}{b_2}(\alpha a_2-2b_2)x_i^\alpha\big(y_i^2+O(x_i^2+y_i^3)\big)+x_iy_iO(x_i+y_i).\end{aligned}$$

Also, since $\psi_x(x,y)=2a_0x+O(x^2+y^2)$ and $\psi_x(x,y)=2b_0x+O(x^2+y^2)$,

$$-\rho_i\rho_{i+1}x_i\psi_x(x_i,y_i)+\rho_ix_i\phi_x(x_i,y_i)=2\rho_i(-\rho_{i+1}b_0+a_0)x_i^2+x_iO(x_i^2+y_i^2).$$

By (9.2), we have

$$-\rho_{i+1}b_0+a_0=\frac{a_0b_2-a_2b_0}{b_2}+\frac{a_2}{b_2}x_{i+1}^\alpha+O(y_{i+1}).$$

So

$$\begin{aligned}&-\rho_i\rho_{i+1}x_i\psi_x(x_i,y_i)+\rho_ix_i\phi_x(x_i,y_i)\\&\quad=2\rho_i\Big(\frac{a_0b_2-a_2b_0}{b_2}+\frac{a_2}{b_2}x_{i+1}^\alpha\Big)x_i^2+x_iO(x_i^2+y_i^2).\end{aligned}$$

Therefore, (9.3) can be written as

$$\begin{aligned}\Delta_{\rho_i}(x_i,y_i)=&\frac{a_2}{b_2}(\alpha a_2-2b_2)x_i^\alpha\big(y_i^2+O(x_i^2+y_i^3)\big)\\&+2\rho_i\Big(\frac{a_0b_2-a_2b_0}{b_2}+\frac{a_2}{b_2}x_{i+1}^\alpha\Big)x_i^2+x_iO(x_i^2+y_i^2).\end{aligned}$$

Now the assumptions $a_2>4b_2$ and $a_0b_2-a_2b_0>0$ imply $\Delta_\rho(x_i,y_i)\ge 0$ if y_0 and x_0 are small. $\square$

Lemma 9.2. *Let $z_0=(x_0,y_0)$ with $x_0,y_0>0$. If for all $z=(x,y)$ in the stable curve that joins $\bar z_0$ and z_1,*

$$v_z^s\ge-\big(\frac{a_2}{b_2}+\hat\rho(y)\big)\big(1-x^\alpha\big)\frac{x}{y}, \tag{9.4}$$

then

$$\bar x_0\ge x_0+Cx_0^{1+\alpha} \tag{9.5}$$

for some constant $C=C(y_q)>0$.

Proof. We regard $W_r^s(x)$ as the graph of a function $x=x(y)$. Then (9.6) gives

$$\frac{dx}{dy}\ge-\big(\frac{a_2}{b_2}+\hat\rho(y)\big)(1-x^\alpha)\frac{x}{y},$$

i.e.,

$$\frac{dx}{x(1-x^\alpha)}+\big(\frac{a_2}{b_2}+\hat\rho(y)\big)\frac{dy}{y}\ge 0.$$

Integrating it from $z_1 = (x_1, y_1)$ to $\bar{z}_0 = (\bar{x}_0, \bar{y}_0)$, we get

$$\log \frac{\bar{x}_0}{x_1} - \frac{1}{\alpha} \log \frac{1-\bar{x}_0^\alpha}{1-x_1^\alpha} + \frac{a_2}{b_2} \log \frac{\bar{y}_0}{y_1} + \int_{y_1}^{\bar{y}_0} \frac{\hat{\rho}(y)}{y} dy \geq 0.$$

From now on we omit subscript 0 in the rest of proof. This inequality implies

$$\bar{x} \geq x_1 \Big(\frac{1-\bar{x}^\alpha}{1-x_1^\alpha}\Big)^{\frac{1}{\alpha}} \Big(\frac{y_1}{\bar{y}}\Big)^{\frac{a_2}{b_2}} \exp\Big[-\int_{y_1}^{\bar{y}} \frac{\hat{\rho}(y)}{y} dy\Big].$$

Since $x_1 = x(1+\phi)$ and $y_1 = y(1-\psi)$, we can write

$$\begin{aligned}\frac{\bar{x}}{x} \geq& (1+\phi)(1-\psi)^{\frac{a_2}{b_2}} \Big(\frac{1-\bar{x}^\alpha}{1-x_1^\alpha}\Big)^{\frac{1}{\alpha}} \Big(\frac{y}{\bar{y}}\Big)^{\frac{a_2}{b_2}} \exp\Big[-\int_{y_1}^{\bar{y}} \frac{\hat{\rho}(y)}{y} dy\Big] \\ =& \exp \frac{1}{b_2}\Big\{b_2 \log(1+\phi) + a_2 \log(1-\psi) - b_2 \int_{y_1}^{y} \frac{\hat{\rho}(y)}{y} dy\Big\} \\ &\cdot \Big(\frac{1-\bar{x}^\alpha}{1-x_1^\alpha}\Big)^{\frac{1}{\alpha}} \Big(\frac{y}{\bar{y}}\Big)^{\frac{a_2}{b_2}} \exp \int_{\bar{y}}^{y} \frac{\hat{\rho}(y)}{y} dy.\end{aligned}$$

The first factor on the right-side is of the form $1 + O(x)$, because by Lemma 8.4 and (8.9),

$$b_2 \log(1+\phi) + a_2 \log(1-\psi) - b_2 \int_{y_1}^{y} \frac{\hat{\rho}(y)}{y} dy = b_2 \int_{y_1^{(0)}}^{y_1} \frac{\hat{\rho}(y)}{y} dy = O(x),$$

where $y_1^{(0)} = y\big(1-\psi(0,y)\big)$.

Note that $x_1 = x + x\phi$. We have

$$\frac{1-\bar{x}^\alpha}{1-x_1^\alpha} = 1 + \frac{x^\alpha(1+\phi)^\alpha - \bar{x}^\alpha}{1-x_1^\alpha} = 1 + \frac{x^\alpha + \alpha x^\alpha \phi - \bar{x}^\alpha + O(\phi^2)}{1-x_1^\alpha}.$$

Without loss of generality, we may assume $\bar{x}^\alpha - x^\alpha \leq \frac{1}{2} x^\alpha \phi$; otherwise, we have $\frac{\bar{x}}{x} > \big(1 + \frac{1}{2}\phi\big)^{\frac{1}{\alpha}} \geq 1 + Cx^\alpha$ for some $C = O(y^2) > 0$ and therefore (9.5) is true. Thus, we get

$$\Big(\frac{1-\bar{x}^\alpha}{1-x_1^\alpha}\Big)^{\frac{1}{\alpha}} \geq \Big(1 + \frac{\alpha x^\alpha \phi + O(\phi^2)}{2(1-x_1^\alpha)}\Big)^{\frac{1}{\alpha}} \geq 1 + \frac{1}{4} x^\alpha \phi.$$

Since both $\bar{z} = (\bar{x}, \bar{y})$ and $z = (x, y)$ are in the same local unstable manifold, $|\bar{y} - y| \leq N(\bar{x} - x) \leq N(x_1 - x) = Nx\phi$ for some $N > 0$. So,

$$\Big|\Big(\frac{y}{\bar{y}}\Big)^{\frac{a_2}{b_2}}\Big| \leq \Big(1 + \frac{Nx\phi}{\bar{y}}\Big)^{\frac{a_2}{b_2}} = 1 + O(x).$$

By (8.9) we have

$$\exp \int_{\bar{y}}^{y} \frac{\hat{\rho}(y)}{y} dy \leq \exp\big(L_{\hat{\rho}} |y - \bar{y}|\big) \leq \exp\big(L_{\hat{\rho}} N x \phi\big) = 1 + O(x).$$

Summarizing these results, we get that if x is small enough, then

$$\frac{\bar{x}}{x} \geq 1 + \frac{1}{4} x^\alpha \phi + O(x) \geq 1 + \frac{a_2 y^2}{8} x^\alpha.$$

□

Proof of Theorem B(I). By Lemmas 9.1, 9.2 and 8.1.i) we know that the sum Δ^{++} in (7.6) converges. Similary, we have Δ^{-+}, Δ^{--}, $\Delta^{+-} \le \infty$. Thus the result follows from Proposition 7.8. $\square$

Remark 9.3. By Lemma 9.2 and 8.1.i), we can see that under the assumptions of the theorem, the rate of convergence of the series in (7.6) is faster than $n^{-\frac{1}{\alpha}}$.

10. Existence of infinite SBR measures

We prove the second part of Theorem B. We get the lower bounds of $|v^s_z|$ in Lemma 10.2 and then prove in Lemma 10.3 that there is at least one series in (7.6) whose general terms decrease at a rate slower than $n^{-\frac{1}{2}}$.

Lemma 10.1. *Suppose $2a_2 < \alpha b_2$. Then for any constants $K, N > 0$, point $q = (0, y_q)$ with $y_q > 0$ small, there exists $\epsilon > 0$ such that $\forall z_0 = (x_0, y_0) \in W^u_\epsilon(q)$ with $x_0 > 0$, the inequalities*

$$x_0 y_0 \prod_{j=0}^{n} \Big(1 - \frac{7}{8} y_i \psi_y(0, y_i)\Big)^{-1} \ge N \quad \textit{and} \quad K x_n \le y_n$$

hold simultaneously for some $n = n(z_0) > 0$.

Proof. We may assume that K is large enough such that if $Kx \le y$, then

$$1 - \frac{7}{8} y \psi_y(0, y) \le 1 - \frac{13}{8} b_2 y^2 \le (1 - \psi)^{\frac{3}{2}}$$

and

$$(1 + \phi)^2 (1 - \psi)^{\alpha_+} \le 1$$

for some $\alpha < \alpha_+ < 1$. Hence for any $z_0 = (x_0, y_0)$ with $Kx_0 < y_0$, we have $x_1^2 y_1^{\alpha_+} \le x_1^2 (1 + \phi_1)^2 y_1^{\alpha_+} (1 - \psi_1)^{\alpha_+} \le x_0^2 y_0^{\alpha_+}$. Let $n = n(z_0)$ be the largest integer such that $K x_n \le y_n$. Thus $K x_{n+1} > y_{n+1}$. So we have

$$x_0^2 y_0^{\alpha_+} \ge x_{n+1}^2 y_{n+1}^{\alpha_+} \ge K^{-2} y_{n+1}^{2+\alpha_+}$$

and therefore

$$\frac{x_0 y_0}{\prod\limits_{j=0}^{n} \big(1 - \frac{7}{8} y_i \psi_y(0, y_i)\big)} \ge \frac{y_0^{\frac{3}{2}} \cdot x_0 y_0}{y_0^{\frac{3}{2}} \cdot \prod\limits_{j=0}^{n} (1 - \psi_j)^{\frac{3}{2}}} \ge \frac{x_0 y_0^{\frac{5}{2}}}{y_{n+1}^{\frac{3}{2}}} \ge \frac{y_0^{(5-\alpha_+)/2}}{K y_{n+1}^{(1-\alpha_+)/2}}.$$

Since $y_{n+1} \to 0$ as $z_0 = (x_0, y_0) \to q = (0, y_q)$, we get the result. $\square$

Lemma 10.2. *Suppose $2a_2 < \alpha b_2$ and $b_1 < 0 < a_1$. Then for any point $q = (0, y_q)$ with $y_q > 0$, there exists $\epsilon > 0$ such that $\forall z_0 = (x_0, y_0) \in W^u_\epsilon(q)$ with $x_0 > 0$,*

$$v^s_{z_0} \le -\Big(\frac{a_2}{b_2} + \hat\rho(y_0)\Big) \frac{x_0}{y_0}. \tag{10.1}$$

Proof. For each $z_0 = (x_0, y_0) \in W^u_r(q)$, define $c_0 = 0$ and

$$c_i = \frac{a_2}{b_2} (a_1 - \frac{a_2}{b_2} b_1) x_0 y_0 \prod_{j=1}^{i-1} \Big(1 - \frac{7}{8} y_i \psi_y(0, y_i)\Big)^{-1} \qquad \forall i > 0. \tag{10.2}$$

We have

$$c_{i+1} - c_i = c_{i+1} \cdot \frac{7}{8} y_i \psi_y(0, y_i) \qquad \forall i > 0. \tag{10.3}$$

Put

$$\rho_i = \max\Big\{ -\frac{1}{3},\ \frac{a_2}{b_2} + \hat\rho(y_i) - c_i \Big\} \qquad \forall i \ge 0. \tag{10.4}$$

We prove that for each small $y_q > 0$, these exists $\epsilon > 0$ such that $\forall z_0 = (x_0, y_0) \in W^u_\epsilon(q)$, $q = (0, y_q)$, with $x_0 > 0$,

$$v^s_{z_i} \ge -\rho_i \frac{x_i}{y_i} \tag{10.5}$$

implies

$$v^s_{z_{i+1}} \ge -\rho_{i+1} \frac{x_{i+1}}{y_{i+1}}. \tag{10..6}$$

This will prove the result of the lemma. In fact, if (10.1) is not true, then for some large n, $\dfrac{x_n}{y_n} > 3$ and $v^s_{z_n} \ge \dfrac{1}{3} \cdot \dfrac{x_n}{y_n} > 1$, contradicting to (8.4).

By Lemma 8.3, to prove that (10.5) implies (10.6), it is enough to show that

$$\begin{aligned}\Delta_{\rho_i}(x_i, y_i) = {} & (\rho_i - \rho_{i+1})(1+\phi_i)(1-\psi_i) \\ & + \rho_{i+1} y_i (1+\phi_i)\psi_y(x_i,y_i) - y_i(1-\psi_i)\phi_y(x_i,y_i) \\ & - \rho_i\rho_{i+1} x_i (1+\phi_i)\psi_x(x_i,y_i) + \rho_i x_i (1-\psi_i)\phi_x(x_i,y_i) \le 0.\end{aligned}$$

By (10.3) and (10.4),

$$\rho_i - \rho_{i+1} = \begin{cases} \hat\rho(y_0) - \hat\rho(y_1) + c_1, & \text{if } i = 0; \\ \hat\rho(y_i) - \hat\rho(y_{i+1}) + \frac{7}{8} c_{i+1} y_i \psi_y(0, y_i) - d_{i+1}, & \text{if } i > 0 \ \&\ \rho_i > -\frac{1}{3}; \\ 0, & \text{if } i > 0 \ \&\ \rho_i = -\frac{1}{3}, \end{cases}$$

where $d_{i+1} = \rho(y_{i+1}) - \big(\frac{a_2}{b_2} + \hat\rho(y_{i+1}) - c_{i+1}\big) \ge 0$, and $d_{i+1} = 0$ except for the case $\rho_i > -\frac{1}{3}$ and $\frac{a_2}{b_2} - \hat\rho(y_{i+1}) + c_{i+1} < \frac{1}{3}$.

If $\rho_{i+1} > -\frac{1}{3}$, we write

$$\begin{aligned}\rho_{i+1} y_i \psi_y(x_i,y_i) - y_i\phi_y(x_i,y_i) = {} & \big(\frac{a_2}{b_2} + \hat\rho(y_{i+1})\big) y_i \psi_y(0,y_i) - y_i \phi_y(0,y_i) \\ & -(c_{i+1} - d_{i+1}) y_i \psi_y(0,y_i) + (\rho_{i+1} b_1 - a_1) x_i y_i + x_i y_i O(x_i + y_i);\end{aligned}$$

otherwise,

$$\begin{aligned}& \rho_{i+1} y_i \psi_y(x_i,y_i) - y_i \phi_y(x_i,y_i) \\ = {} & 2\big(\rho_{i+1} b_2 - a_2\big) y_i^2 + \big(\rho_{i+1} b_1 - a_1\big) x_i y_i + x_i y_i O(x_i + y_i).\end{aligned}$$

Also,

$$\begin{aligned}& -\rho_i \rho_{i+1} x_i \psi_x(x_i,y_i) + \rho_i x_i \phi_x(x_i,y_i) \\ = {} & -\rho_i(\rho_{i+1} b_1 - a_1) x_i y_i - 2\rho_i(\rho_{i+1} b_0 - a_0) x_i^2 + x_i y_i O(x_i + y_i).\end{aligned}$$

Similarly, as in the proof of Lemma 9.1, we use Lemma 8.4 to get

$$\begin{aligned}\Delta_{\rho_i}(x_i,y_i) = {} & D_i(x_i,y_i) + (1-\rho_i)(\rho_{i+1} b_1 - a_1) x_i y_i \\ & - 2\rho_i(\rho_{i+1} b_0 - a_0) x_i^2 + x_i y_i O(x_i + y_i),\end{aligned}$$

where

$$D_i(x_i,y_i) \le \begin{cases} \frac{a_2}{b_2}(a_1 - \frac{a_2}{b_2} b_1) x_0 y_0 & \text{if } i = 0; \\ -\frac{1}{8} c_{i+1} y_i \psi_y(0,y_i) & \text{if } i > 0 \ \&\ \rho_i > -\frac{1}{3}; \\ 2(\rho_{i+1} b_2 - a_2) y_i^2 & \text{if } i > 0 \ \&\ \rho_i = -\frac{1}{3}. \end{cases}$$

Recall $2a_2 < \alpha b_2$ and $b_1 < 0 < a_1$.

Since $\rho_0 = \frac{a_2}{b_2} + O(y_0) \le \frac{\alpha}{2} + O(y_0)$ and $\rho_1 = \rho_0 + O(y_0^2)$, we have $\rho_1 b_1 - a_1 = \frac{a_2}{b_2} b_1 - a_1 + O(y_0) \le 0$ and $1 - \rho_0 - \frac{a_2}{b_2} > 0$ if y_0 is small. So if $Kx_0 \le y_0$ for some large K, then $\Delta_{\rho_0}(x_0, y_0) \le 0$.

For the case $i > 0$ and $\rho_i > -\frac{1}{3}$, by Lemma 10.1, we can choose ϵ small enough such that $Kx_i \le y_i$ for some large K. Also, $y_i \psi_y(0, y_i) = 2b_2 y_i^2 + O(y_i^3)$. So if $\rho_{i+1} \ge 0$, then both y_i^2 and $x_i y_i$ terms has negative sign, and if $\rho_{i+1} \le 0$, then $c_{i+1} > \frac{a_2}{2b_2}$ and therefore the first term can control other terms.

For the case $\rho_i = -\frac{1}{3}$, we can see that $\Delta_{\rho_i}(x_i, y_i)$ is equal to

$$\begin{aligned}
&2(-\frac{1}{3}b_2 - a_2)y_i^2 + \frac{4}{3}(-\frac{1}{3}b_1 - a_1)x_i y_i + \frac{2}{3}(-\frac{1}{3}b_0 - a_0)x_i^2 + x_i y_i O(x_i + y_i) \\
=& -\frac{2}{3}(3a_2 y_i^2 + 2a_1 x_i y_i + a_0 x_i^2) - \frac{2}{9}(3b_2 y_i^2 + 2b_1 x_i y_i + b_0 x_i^2) + x_i y_i O(x_i + y_i).
\end{aligned}$$

By (8.2) it is less than 0. This proves the lemma. ☐

Lemma 10.3. *Let $z_0 = (x_0, y_0)$ with $x_0, y_0 > 0$. If for all $z = (x, y)$ in the stable curve that joins $\bar{z}_0$ and z_1,*

$$v_{z_0}^s \le -\Big(\frac{a_2}{b_2} + \hat{\rho}(y_0)\Big)\frac{x_0}{y_0},$$

then

$$\bar{x}_0 \le x_0 + Cx_0^2$$

for some constant $C = C(y_q) > 0$, where $\bar{x}$ satisfies $\bar{z} = (\bar{x}, \bar{y}) = [z, fz]$.

Proof. Using a similar way as in the proof of Lemma 9.3, we can get

$$\begin{aligned}
\frac{\bar{x}_0}{x_0} \le & \exp\frac{1}{b_2}\Big[b_2 \log(1+\phi_0) + a_2 \log(1-\psi_0) - b_2 \int_{y_1}^{y_0} \frac{\hat{\rho}(y)}{y} dy\Big]\Big(\frac{y_0}{\bar{y}_0}\Big)^{\frac{a_2}{b_2}} \\
& \cdot \exp\int_{\bar{y}_0}^{y_0} \frac{\hat{\rho}(y)}{y} dy \le 1 + O(x_0).
\end{aligned}$$

Therefore the result follows. ☐

Proof of Theorem B(II). First we assume that $b_1 < 0 < a_1$. By Lemma 10.3 and 8.1.ii), $\Delta^{++} = \infty$, where Δ^{++} is defined in (7.6). Therefore by Proposition 7.8 f has an infinite SBR measure.

By (1.3) and (1.4), the signs of a_1 and b_1 change under the transformation $x \to -x$ and $y \to -y$ respectively. So under the assumption of Theorem B (II), we can always find a suitable coordinate system such that $b_1 < 0 < a_1$. ☐

References

[B] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lecture Notes in Math. **470** Springer, New York, 1975. MR **56:**1364

[BY] M. Benedicks and L.-S. Young, *Sinai-Bowen-Ruelle measure for certain Hénon maps*, Invent. Math. **112** (1993), 541-576. MR **94e:**58074

[C] M. Carvalho, *Sinai-Ruelle-Bowen measures for N-dimensional derived from Anosov diffeomorphisms*, Ergodic Theory Dynamical Systems **13** (1993), 21-44. MR **94h:**58102

[HY] H. Huyi and L.-S. Young, *Nonexistence of SBR measures for some diffeomorphisms that are "almost Anosov"*, Ergodic Theory Dynamical Systems **15** (1995), 67–76. MR **95j:**58096

[L] F. Ledrappier, *Propriétés ergodiques des mesures de Sinaï*, Inst. Hautes. Études Sci. Publ. Math. **59** (1984), 163-188. MR **86f:**58092

[LS] F. Ledrappier and J.-M. Strelcyn, *A proof of the estimation from below in Pesin's entropy formula*, Ergodic Theory Dynamical Systems **2** (1982), 203-219. MR **85f:**58070

[LY] F. Ledrappier and L.-S. Young, *The metric entropy of diffeomorphisms.* I, Ann. of Math. **122** (1985), 509-574. MR **87i:**58101a

[O] V. I. Oseledec, *A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems*, Trans. Moscow Math. Soc. **19** (1968), 197-221.

[P1] Ya. B. Pesin, *Families of invariant manifolds corresponding to non-zero characteristics exponents*, Math. USSR-Izv. **10** (1978), 1261-1305

[P2] ———, *Dynamical systems with generalized hyperbolic attractors: hyperbolic, ergodic and topological properties*, Ergodic Theory Dynamical Systems **12** (1992), 123-151. MR **93b:**58095

[R] V. A. Rohlin, *Lectures on the theory of entropy of transformations with invariant measures*, Russian Math. Surveys **22** (1967), 1-54.

[S] Ya. G. Sinai, *Gibbs measure in ergodic theory*, Uspehi Mat. Nauk **27** (1972), 21-64. MR **53:**3265

Department of Mathematics, University of Maryland, College Park, Maryland 20742

Current address: Department of Mathematics, Penn State University, University Park, Pennsylvania 16801

E-mail address: `hu@math.psu.edu`