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ON SYZYGIES OF ABELIAN VARIETIES

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ABSTRACT. In this paper we prove the following result: Let X be a complex torus and M a normally generated line bundle on X; then, for every $p \geq 0$, the line bundle M^{p+1} satisfies Property N_p of Green-Lazarsfeld.

1. INTRODUCTION

In this paper we prove a new result on syzygies of abelian varieties; precisely, the problem we are considering is the following: let X be a complex torus, L a very ample line bundle on X and φ_L the associated map; we are concerned with the degree of the equations defining $\varphi_L(X)$, the degree of the syzygies among them and the degree of higher syzygies. In particular, here we examine the case where $L = M^l$ where M is a normally generated line bundle.

To review precisely the statements of the known results on syzygies of abelian varieties and to formulate precisely our theorem, we have to recall Green-Lazarsfeld's definition of Property N_p (see [Gr1], [G-L], [Gr2], [Laz2], [E-L]): let Y be a smooth complex projective variety of dimension n and let L be a very ample line bundle on Y defining an embedding $Y \subset \mathbf{P} = \mathbf{P}(H^0(Y, L)^*)$; set $S = S(L) = Sym^* H^0(L)$, the homogeneous coordinate ring of the projective space $\mathbf{P}$, and consider the graded S-module $G = G(L) = \bigoplus_d H^0(Y, L^d)$. Let E_* be a minimal graded free resolution of G (that is, an exact sequence with E_i free S-modules and such that the matrices of homogenous polynomials giving the maps $E_i \longrightarrow E_{i-1}$ has no nonzero constant entries); the line bundle L satisfies Property N_p ($p \in \mathbf{N}$) if and only if

$$E_0 = S,$$
$$E_i = \bigoplus S(-i-1) \quad \text{for } 1 \leq i \leq p.$$

(Thus L satisfies Property N_0 if and only if $Y \subset \mathbf{P}(H^0(L)^*)$ is projectively normal, that is, L is normally generated; L satisfies Property N_1 if and only if L satisfies Property N_0 and the homogeneous ideal I of $Y \subset \mathbf{P}(H^0(L)^*)$ is generated by quadrics; L satisfies Property N_2 if and only if L satisfies Property N_1 and the module of syzygies among quadratic generators $Q_i \in I$ is spanned by relations of the form $\sum L_i Q_i = 0$, where L_i are linear polynomials; and so on.)

In 1966 Mumford proved that, if M is an ample line bundle on a complex torus X and $l \geq 4$, then the ideal of $\varphi_{M^l}(X)$ is generated by quadrics ([Mum2]) and in

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1978 Sekiguchi proved a similiar result for $l = 3$ ([Se]). In 1989 Kempf proved again and generalized these results in [Ke]; precisely the following theorem holds (part *c*) was proved by Lange and Birkenhake using Kempf's proof; see 7.4.1, [L-B]):

Theorem 1 (Mumford-Sekiguchi-Kempf). *Let X be a complex torus. If A is an ample line bundle on X we denote $I(A)$ the ideal of $\varphi_A(X)$. Let M be an ample line bundle on X;*

a) if $l \geq 4$, the ideal $I(M^l)$ is generated by forms of degree 2,

b) let $l = 3$, the ideal $I(M^3)$ is generated by forms of degrees 2 and 3,

c) (Lange-Birkenhake) *let $l = 2$; if M^2 is normally generated, then the ideal $I(M^2)$ is generated by forms of degrees 2, 3 and 4.*

In 1984 Green proved that if X is a Riemann surface of genus g and L is a holomorphic line bundle on X of degree $2g + 1 + p$, then L satisfies Property N_p (see [Gr1] and [Gr2]). Thus, if M is an ample line bundle on an elliptic curve, then M^{p+3} satisfies Property N_p and in [Laz2] Lazarsfeld formulated the following conjecture:

Conjecture 2 (Lazarsfeld). *If M is an ample line bundle on a complex torus, then, for every $p \geq 0$, the line bundle M^{p+3} satisfies Property (N_p).*

In 1989 Kempf proved a weaker result (see [Ke]):

Theorem 3 (Kempf). *Let M be an ample line bundle on a complex torus X. If $l \geq 4$, then M^l satisfies Property $N_{[\frac{l-2}{2}]}$.*

In 1993 Ein and Lazarsfeld proved the following theorem (see [E-L]):

Theorem 4 (Ein-Lazarsfeld). *Let Y be a smooth complex projective variety of dimension n; let A be a very ample line bundle on Y, and B a numerically effective line bundle on Y; then $K_Y \otimes A^{n+1+p} \otimes B$ satisfies Property N_p.*

If $(Y, A, B) \neq (\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1), \mathcal{O}_{\mathbf{P}^n})$ and $p \geq 1$, then $K_Y \otimes A^{n+p} \otimes B$ satisfies Property N_p.

Thus, if M is a very ample line bundle on a complex torus of dimension n, then M^{n+p} satisfies Property N_p.

In this paper, using the ideas of Kempf's paper [Ke], by a patient adaptation, we prove another theorem on syzygies of abelian varieties:

Theorem 5. *If M is a normally generated line bundle on a complex torus X, then, for every $p \geq 0$, the line bundle M^{p+1} satisfies Property N_p.*

Since there is a result on normal generation of primitive line bundles (Lazarsfeld's theorem on projective normality of $(1, d)$-abelian surfaces; see [Laz1]), Theorem 5 may actually be useful (see Remark 18).

Notation and Definitions. We collect here some notation and standard definitions that we will use throughout the paper.

- $\boldsymbol{\varphi_L}$. If L is a line bundle on a complex manifold Y, φ_L is the rational map associated to L.
- A line bundle L on a complex manifold Y is called **normally generated** if it is very ample and $\varphi_L(Y)$ is projectively normal. We have that L is normally generated if and only if it is ample and the natural maps $S^n H^0(Y, L) \longrightarrow H^0(Y, L^n)$ are surjective for all $n \geq 2$ (see [Mum1], p. 38 and [L-B], Chapter 7, §3).

If X is a complex torus of dimension g, then

- $\boldsymbol{t_x}$ is the translation on X by the point x;
- $\hat{\boldsymbol{X}}$ is the dual complex torus of X; it is isomorphic to $Pic^0(X)$;
- $\boldsymbol{\mathcal{P}}$ denotes the Poincaré bundle on $X \times \hat{X}$;
- $\boldsymbol{\phi_L}$ is the homomorphism $X \longrightarrow \hat{X}$, $x \mapsto t_x^* L \otimes \mathrm{L}^{-1}$, where L is a line bundle on X;
- $\boldsymbol{K(L)}$ is the kernel of ϕ_L; it depends only on H, the first Chern class of L, thus we denote $K(L)$ also by $\boldsymbol{K(H)}$; if L is nondegenerate, then $K(L)$ is a finite group isomorphic to $(\mathbf{Z}/d_1 \oplus .... \oplus \mathbf{Z}/d_g)^2$ with $d_i | d_{i+1}$; we say that L is of **type** $(d_1, ..., d_g)$;
- $\boldsymbol{W \cdot W'}$: if W is a vector subspace of $H^0(X, E)$ and W' is a vector subspace of $H^0(X, E')$ (E and E' line bundles on X), $W \cdot W'$ is the image of $W \otimes W'$ under the multiplication map; we often omit $\cdot$.
- $\boldsymbol{\pi}$: if we have a product of tori, we use the notation: π_i is the projection on the ith factor and $\pi_\cdot$ is the projection on $\cdot$.

2. Some recalls

First we recall Mumford's lemma (see [Mum1] or [L-B], Chapter 7, Lemma 3.3) and the following remark and proposition.

Lemma 6 (Mumford)**.** *Let A and B be two ample line bundles on a complex torus X. For every nonempty open subset U of $\hat{X}$, we have*

$$\sum_{P \in U} H^0(X, A \otimes P) \cdot H^0(X, B \otimes P^{-1}) = H^0(X, A \otimes B).$$

As Kempf observed in [Ke], Mumford's lemma can be interpreted in this way: a linear functional λ on $H^0(A \otimes B)$ is determined by the family $\{\lambda_P\}_{P \in U}$, where λ_P is the linear functional on $H^0(X, A \otimes P) \otimes H^0(X, B \otimes P^{-1})$ given by the composition of the multiplication with λ.

Remark 7 (see [Gr2]). Let V be a complex vector space of dimension $r+1$, let $S = \bigoplus_{q \geq 0} Sym^q(V)$ and $G = \bigoplus_q G_q$ a finitely generated graded S-module. Let

$$0 \longrightarrow E_n \longrightarrow E_{n-1} \longrightarrow ... \longrightarrow E_0 \longrightarrow G \longrightarrow 0$$

be a minimal free resolution of G, that is, an exact sequence with E_i free S-modules and such that the matrices of homogenous polynomials giving the maps $E_i \longrightarrow E_{i-1}$ has no nonzero constant entries. Write $E_p = \bigoplus_q (B_{p,q} \otimes S(-q))$ with $B_{p,q}$ vector spaces on $\mathbf{C}$. See $\mathbf{C}$ as the S-module $S / \bigoplus_{q \geq 1} Sym^q(V)$. Then

$$Tor_p^S(G, \mathbf{C})_q \simeq B_{p,q}.$$

Proposition 8 (Koizumi [Ko])**.** *Let A and A' be two algebraically equivalent ample line bundles on a complex torus X. The multiplication map $H^0(X, A^m) \otimes H^0(X, A'^n) \longrightarrow H^0(X, A^m \otimes A'^n)$ is surjective for all $m \geq 3$ and $n \geq 2$.*

Now we recall some facts, definitions and propositions of Kempf's paper [Ke].

Definition 9 (Kempf)**.** For any A_i's ample line bundles on a complex torus X, let $K(A_1) = H^0(X, A_1)$ and, for $n > 1$, define $K(A_1, ..., A_n)$ inductively by the following exact sequence:

$$0 \to K(A_1, ..., A_n) \to K(A_1, A_3, ..., A_n) \otimes H^0(X, A_2) \to K(A_1 \otimes A_2, A_3, ..., A_n).$$

To follow completely Kempf's notations, we denote $K(A_1, A_2)$ by $R(A_1, A_2)$ $(= \ker(H^0(A_1) \otimes H^0(A_2) \longrightarrow H^0(A_1 \otimes A_2)))$.

In the sequel $K(A_1, A_3, ..., A_n) \cdot H^0(X, A_2)$ will denote the image of the multiplication map $K(A_1, A_3, ..., A_n) \otimes H^0(X, A_2) \longrightarrow K(A_1 \otimes A_2, A_3, ..., A_n)$ $(\subset H^0(A_1 \otimes A_2) \otimes H^0(A_3) \otimes ... \otimes H^0(A_n))$; we often omit $\cdot$.

Notation 10 (Kempf). In the remainder of this section, following [Ke], we use the following notation: let X be a complex torus of dimension g; fix an ample line bundle M on X; l_i, $i \in \mathbf{N}$, will denote positive integers and L_i will denote a line bundle algebraically equivalent to M^{l_i}.

Observe that, if A is a line bundle on X, since $H^0((\pi_X^* A \otimes \mathcal{P})|_{X \times \{P\}}) = H^0(A \otimes P)$ is of constant dimension $\forall P \in \hat{X}$, then the sheaf $\pi_{\hat{X}\, *}(\pi_X^* A \otimes \mathcal{P})$ on $\hat{X}$ is locally free and its fibre over $P \in \hat{X}$ is $H^0(A \otimes P)$, by Grauert's Theorem (see [Ha] Theorem 12.9 Chapter 3). Analogously the sheaf $\pi_{\hat{X}\, *}(\pi_X^* A \otimes \mathcal{P}^{-1})$ on $\hat{X}$ is locally free and its fibre over $P \in \hat{X}$ is $H^0(A \otimes P^{-1})$.

Consider the following map:

$$\pi_{\hat{X}\, *}(\pi_X^* L_1 \otimes \mathcal{P}) \otimes \pi_{\hat{X}\, *}(\pi_X^* L_2 \otimes \mathcal{P}^{-1}) \longrightarrow H^0(X, L_1 \otimes L_2) \otimes_{\mathbf{C}} \mathcal{O}_{\hat{X}}$$

(given by the composition of the maps

$$\pi_{\hat{X}\, *}(\pi_X^* L_1 \otimes \mathcal{P}) \otimes \pi_{\hat{X}\, *}(\pi_X^* L_2 \otimes \mathcal{P}^{-1}) \longrightarrow \pi_{\hat{X}\, *}\pi_X^*(L_1 \otimes L_2)$$

and

$$\pi_{\hat{X}\, *}\pi_X^*(L_1 \otimes L_2) \longrightarrow H^0(X, L_1 \otimes L_2) \otimes_{\mathbf{C}} \mathcal{O}_{\hat{X}}.)$$

This map induces a map:

$$m : H^0(X, L_1 \otimes L_2)^\vee \longrightarrow H^0(\hat{X}, (\pi_{\hat{X}\, *}(\pi_X^* L_1 \otimes \mathcal{P}) \otimes \pi_{\hat{X}\, *}(\pi_X^* L_2 \otimes \mathcal{P}^{-1}))^\vee).$$

Proposition 11 (Kempf)**.** *i) The map*

$$m : H^0(X, L_1 \otimes L_2)^\vee \longrightarrow H^0(\hat{X}, (\pi_{\hat{X}\, *}(\pi_X^* L_1 \otimes \mathcal{P}) \otimes \pi_{\hat{X}\, *}(\pi_X^* L_2 \otimes \mathcal{P}^{-1}))^\vee)$$

is an isomorphism.

ii) $H^i(\hat{X}, (\pi_{\hat{X}\, *}(\pi_X^* L_1 \otimes \mathcal{P}) \otimes \pi_{\hat{X}\, *}(\pi_X^* L_2 \otimes \mathcal{P}^{-1}))^\vee) = 0$ *for* $i \geq 1$.

Arguing exactly as in Proposition 4 of [Ke], we have:

Proposition 12 (Kempf)**.** *If* $H^0(L_1 \otimes P) \otimes H^0(L_3) \longrightarrow H^0(L_1 \otimes P \otimes L_3)$ *is surjective* $\forall\, P \in Pic^0(X)$ *and* $H^0(L_1 \otimes L_2) \otimes H^0(L_3) \longrightarrow H^0(L_1 \otimes L_2 \otimes L_3)$ *is surjective, then*

$$\sum_{P \in Pic^0(X)} R(L_1 \otimes P, L_3) \cdot H^0(L_2 \otimes P^{-1}) = R(L_1 \otimes L_2, L_3).$$

We reproduce the proof here for later use.

Proof. One inclusion is obvious:

$$\sum_{P \in \hat{X}} R(L_1 \otimes P, L_3) \cdot H^0(L_2 \otimes P^{-1}) \subset R(L_1 \otimes L_2, L_3).$$

We want to show the other one. It suffices to show that $(\sum_{P \in Pic^0(X)} R(L_1 \otimes P, L_3) \cdot H^0(L_2 \otimes P^{-1}))^\perp$ in $(H^0(L_1 \otimes L_2) H^0(L_3))^\vee$ is contained in $R(L_1 \otimes L_2, L_3)^\perp$.

For every $P \in \hat{X}$, we have the following commutative diagram:

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ R(L_1 \otimes P, L_3) \otimes H^0(L_2 \otimes P^{-1}) & \longrightarrow & R(L_1 \otimes L_2, L_3) \\ \downarrow & & \downarrow \\ H^0(L_1 \otimes P) \otimes H^0(L_3) \otimes H^0(L_2 \otimes P^{-1}) & \longrightarrow & H^0(L_1 \otimes L_2) \otimes H^0(L_3) \\ \downarrow & & \downarrow \\ H^0(L_1 \otimes P \otimes L_3) \otimes H^0(L_2 \otimes P^{-1}) & \longrightarrow & H^0(L_1 \otimes L_2 \otimes L_3) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

The columns are exact by the hypotheses.

To see that $(\sum_{P \in Pic^0(X)} R(L_1 \otimes P, L_3) \cdot H^0(L_2 \otimes P^{-1}))^{\perp}$ in $(H^0(L_1 \otimes L_2) \otimes H^0(L_3))^{\vee}$ is contained in $R(L_1 \otimes L_2, L_3)^{\perp}$ is equivalent to seeing that any element of $(H^0(L_1 \otimes L_2) \otimes H^0(L_3))^{\vee}$ inducing a linear form on $H^0(L_1 \otimes L_3 \otimes P) \otimes H^0(L_2 \otimes P^{-1})$ induces a linear form on $H^0(L_1 \otimes L_2 \otimes L_3)$ and this is true by Proposition 11. □

Definition 13 (Kempf)**.** Let S be a graded ring and G a finitely generated graded S-module and $k = S / \bigoplus_{n \geq 1} S_n$. Define $d(G) = min\ \{j |\ G$ *is generated by the union of the* G_d*'s for* $d \leq j\}$ (thus $d(G)$ is the smallest number such that $G \otimes_S k$ is zero in degree $> d(G)$).

Define $T^1(G) = \ker(G(-1) \otimes_k S_1 \longrightarrow G)$.

Define $T^j(G) = T^{j-1}(T^1(G))$ and $T^0(G) = G$.

Define $d^j(G) = d(T^j(G))$.

Lemma 14 (Kempf)**.** *Let S be a graded ring and G a finitely generated graded S-module and $k = S / \bigoplus_{n \geq 1} S_n$. If $q > i - j + d(T^j(G))$ for all $0 \leq j \leq i$, then $Tor_i^S(G, k)$ is zero in degree q.*

3. Two new lemmas

To prove Theorem 5 we need the following two lemmas.

Lemma 15. *Let A and A' be two algebraically equivalent normally generated line bundles on a complex torus X and $m, n \in \mathbf{N}$. If $m \geq 2$ and $n \geq 1$, then*

$$H^0(A^m) \cdot H^0(A'^n) = H^0(A^m \otimes A'^n).$$

Proof. Observe that the set $U := \{P \in Pic^0(X)\ s.t.\ H^0(A \otimes P^{-1}) H^0(A'^n) = H^0(A \otimes A'^n \otimes P^{-1})\}$ is nonempty, since A and A' are normally generated, and open, since, as we have already observed, for any line bundle L on X, the vector spaces $H^0(L \otimes P)$ $P \in \hat{X}$ form a vector bundle on $\hat{X}$.

Applying twice Mumford's Lemma, we have

$$\begin{aligned} H^0(A^m) H^0(A'^n) &= \sum_{P \in U} H^0(A^{m-1} \otimes P) H^0(A \otimes P^{-1}) H^0(A'^n) \\ &= \sum_{P \in U} H^0(A^{m-1} \otimes P) H^0(A \otimes A'^n \otimes P^{-1}) = H^0(A^m \otimes A'^n) \end{aligned}$$

for $m \geq 2$ and $n \geq 1$. □

Lemma 16. *Let A, A' and A'' be three algebraically equivalent normally generated line bundles on a complex torus X and $\alpha, \beta, \gamma \in \mathbf{N}$. If we are in one of the following three cases:*

1) $\alpha \geq 2,\ \beta \geq 2,\ \gamma \geq 2,$
2) $\alpha \geq 3,\ \beta = 1,\ \alpha + \gamma \geq 5,$
3) $\alpha \geq 3,\ \gamma = 1,\ \alpha + \beta \geq 5,$
then

$$R(A^\alpha, A'^\beta) \cdot H^0(A''^\gamma) = R(A^\alpha \otimes A''^\gamma, A'^\beta).$$

Proof. Let $\alpha, \beta, \gamma, l \in \mathbf{N}$ and $\beta, l, \alpha - l \geq 1$.

If

$$H^0(A^{\alpha-l} \otimes P) \otimes H^0(A'^\beta) \longrightarrow H^0(A^{\alpha-l} \otimes P \otimes A'^\beta)$$

for all $P \in Pic^0(X)$, and

$$H^0(A^\alpha) \otimes H^0(A'^\beta) \longrightarrow H^0(A^\alpha \otimes A'^\beta)$$

are surjective (we call this condition (a)), then, by Proposition 12, we have

$$R(A^\alpha, A'^\beta)H^0(A''^\gamma) = \sum_{P \in Pic^0(X)} R(A^{\alpha-l} \otimes P, A'^\beta)H^0(A^l \otimes P^{-1})H^0(A''^\gamma).$$

If

$$H^0(A^l \otimes P^{-1}) \otimes H^0(A''^\gamma) \longrightarrow H^0(A^l \otimes P^{-1} \otimes A''^\gamma)$$

for all $P \in Pic^0(X)$, is surjective (we call this condition (b)), then we have

$$\sum_{P \in Pic^0(X)} R(A^{\alpha-l} \otimes P, A'^\beta)H^0(A^l \otimes P^{-1})H^0(A''^\gamma)$$
$$= \sum_{P \in Pic^0(X)} R(A^{\alpha-l} \otimes P, A'^\beta)H^0(A^l \otimes A''^\gamma \otimes P^{-1}).$$

If

$$H^0(A^{\alpha-l} \otimes P) \otimes H^0(A'^\beta) \longrightarrow H^0(A^{\alpha-l} \otimes P \otimes A'^\beta)$$

for all $P \in Pic^0(X)$, and

$$H^0(A^\alpha \otimes A''^\gamma) \otimes H^0(A'^\beta) \longrightarrow H^0(A^\alpha \otimes A''^\gamma \otimes A'^\beta)$$

are surjective (we call this condition (c)), then, by Proposition 12, we have

$$\sum_{P \in Pic^0(X)} R(A^{\alpha-l} \otimes P, A'^\beta)H^0(A^l \otimes A''^\gamma \otimes P^{-1}) = R(A^\alpha \otimes A''^\gamma, A'^\beta).$$

By Lemma 15, if we are in one of the following four cases:
1) $\alpha \geq 2,\ \beta \geq 2,\ \gamma \geq 2,\ l = 1,$
2) $\alpha \geq 4,\ \beta = 1,\ \gamma = 1,\ l = 2,$
2') $\alpha \geq 3,\ \beta = 1,\ \gamma \geq 2,\ \alpha + \gamma \geq 5,\ l = 1$
3) $\alpha \geq 3,\ \gamma = 1,\ \alpha + \beta \geq 5,\ l = 2,$
then (a), (b) and (c) hold. Thus we conclude the proof. □

4. The proof of Theorem 5 and two remarks

The crucial step to prove Theorem 5 is the following proposition, which is analogous to Theorem 17 in [Ke].

Proposition 17. *Let M be a normally generated line bundle on a complex torus X. We again use Notation 10, that is the l_i's, $i \in \mathbf{N}$, denote positive integers and L_i denotes a line bundle algebraically equivalent to M^{l_i}.*

a) Let $m \geq 3$. If $l_1 \geq m-1$,..., $l_m \geq m-1$, then $K(L_1, L_3, ..., L_m) \otimes H^0(X, L_2) \longrightarrow K(L_1 \otimes L_2, L_3, ..., L_m)$ is surjective.

b) Let $m \geq 1$. If $l_1 \geq m$,..., $l_m \geq m$, then $K(L_1, ..., L_m)$ form a vector bundle on the appropriate component of $Pic(X)^m$.

c) Let $m \geq 4$. If $l_1 \geq m-2$, $l_2 \geq 1$ and $l_3 \geq m-1$, ..., $l_m \geq m-1$, then $K(L_1 \otimes L_2, L_3, ..., L_m) = \sum_{P \in \hat{X}} K(L_1 \otimes P, L_3, ..., L_m) \cdot H^0(L_2 \otimes P^{-1})$.

d) Let $m \geq 2$. Let $l_1 \geq 2m-1$, $l_2 \geq 1$, and, if $m \geq 3$, $l_3 \geq m$, ..., $l_m \geq m$ and suppose the family of vector spaces $K(L_1 \otimes P, L_3, ..., L_m)$, $P \in \hat{X}$, forms a vector bundle on $\hat{X}$, we call the corresponding sheaf $\mathcal{F}_{m-1}$. Then we have

i) *an isomorphism*

$$K(L_1 \otimes L_2, L_3, ..., L_m)^\vee = H^0(\hat{X}, (\mathcal{F}_{m-1} \otimes \pi_{\hat{X}\,*}(\pi_X^* L_2 \otimes \mathcal{P}^{-1}))^\vee).$$

ii) $H^i(\hat{X}, (\mathcal{F}_{m-1} \otimes \pi_{\hat{X}\,*}(\pi_X^* L_2 \otimes \mathcal{P}^{-1}))^\vee) = 0$ *if* $i \geq 1$.

Proof. Consider the following four statements depending on the natural number m:

• Statement $A(m)$: *If $l_1 \geq m-1$, ..., $l_m \geq m-1$, then $K(L_1, L_3, ..., L_m) \otimes H^0(X, L_2) \longrightarrow K(L_1 \otimes L_2, L_3, ..., L_m)$ is surjective.*

• Statement $B(m)$: *If $l_1 \geq m$,..., $l_m \geq m$, then $K(L_1, ..., L_m)$ form a vector bundle on the appropriate component of $Pic(X)^m$.*

• Statement $C(m)$: *If $l_1 \geq m-2$, $l_2 \geq 1$, and, if $m \geq 3$, $l_3 \geq m-1$, ..., $l_m \geq m-1$, then $K(L_1 \otimes L_2, L_3, ..., L_m) = \sum_{P \in \hat{X}} K(L_1 \otimes P, L_3, ..., L_m) \cdot H^0(L_2 \otimes P^{-1})$.*

• Statement $D(m)$: *Let $l_1 \geq 2m-1$, $l_2 \geq 1$, and, if $m \geq 3$, $l_3 \geq m$, ..., $l_m \geq m$ and suppose the family of vector spaces $K(L_1 \otimes P, L_3, ..., L_m)$, $P \in \hat{X}$, forms a vector bundle on $\hat{X}$, we call the corresponding sheaf $\mathcal{F}_{m-1}$; then we have*

i) *an isomorphism*

$$K(L_1 \otimes L_2, L_3, ..., L_m)^\vee = H^0(\hat{X}, (\mathcal{F}_{m-1} \otimes \pi_{\hat{X}\,*}(\pi_X^* L_2 \otimes \mathcal{P}^{-1}))^\vee),$$

ii) $H^i(\hat{X}, (\mathcal{F}_{m-1} \otimes \pi_{\hat{X}\,*}(\pi_X^* L_2 \otimes \mathcal{P}^{-1}))^\vee) = 0$ *if* $i \geq 1$.

We know that $A(3)$, $B(1)$, $B(2)$, $D(2)$ are true ($A(3)$ is true by Lemma 16, $B(2)$ by Lemma 15, $D(2)$ by Proposition 11).

We will prove the following four implications:

• $A(m-1)$ and $B(m-2) \Rightarrow B(m-1)$ for $m \geq 3$.

• $A(m-1)$, $B(m-2)$ and $D(m-1) \Rightarrow D(m)$ for $m \geq 4$ and $B(1)$ and $D(2) \Rightarrow D(3)$.

• $A(m-1)$, $B(m-2)$ and $D(m-1) \Rightarrow C(m)$ for $m \geq 3$.

• $C(m) \Rightarrow A(m)$ for $m \geq 3$.

By the second implication also $D(3)$ holds; using the four implications, one can prove by induction that $A(m)$, $B(m-1)$ and $D(m)$ are true for $m \geq 3$ and conclude.

Thus let us prove the four implications.

• $A(m-1)$ and $B(m-2) \Rightarrow B(m-1)$ for $m \geq 3$: obvious.

• $A(m-1)$, $B(m-2)$ and $D(m-1) \Rightarrow D(m)$ for $m \geq 4$ and $B(1)$ and $D(2) \Rightarrow D(3)$; it can be proved in an analogous way as Proposition 9 in [Ke]; more precisely:

consider line bundles $L_1, ..., L_m$ with the hypotheses of $D(m)$, thus $l_1 \geq 2m-1$, $l_2 \geq 1$, and, if $m \geq 3$, $l_3 \geq m$, ..., $l_m \geq m$;

Let $m \geq 4$. Since $l_1, l_3, ..., l_m \geq m-2$, by $A(m-1)$ we have the following exact sequence:

$$\begin{aligned} 0 &\longrightarrow K(L_1 \otimes P, L_3, ..., L_m) \otimes H^0(L_2 \otimes P^{-1}) \\ &\longrightarrow K(L_1 \otimes P, L_4, ..., L_m) \otimes H^0(L_3) \otimes H^0(L_2 \otimes P^{-1}) \\ &\longrightarrow K(L_1 \otimes L_3 \otimes P, L_4, ...L_m) \otimes H^0(L_2 \otimes P^{-1}) \longrightarrow 0; \end{aligned}$$

observe that also if $m = 3$ this exact sequence holds, by Proposition 8.

The above sequence gives an exact sequence

$$\begin{aligned} 0 &\longrightarrow (\mathcal{F}'_{m-2} \otimes \pi_{\hat{X}\,*}(\pi_X^* L_2 \otimes \mathcal{P}^{-1}))^\vee \\ &\longrightarrow (\mathcal{F}_{m-2} \otimes H^0(L_3) \otimes \pi_{\hat{X}\,*}(\pi_X^* L_2 \otimes \mathcal{P}^{-1}))^\vee \\ &\longrightarrow (\mathcal{F}_{m-1} \otimes \pi_{\hat{X}\,*}(\pi_X^* L_2 \otimes \mathcal{P}^{-1}))^\vee \longrightarrow 0, \end{aligned}$$

with $\mathcal{F}_{m-2}$ the sheaf corresponding to the bundle whose fibre over $P \in \hat{X}$ is $K(L_1 \otimes P, L_4, ..., L_m)$ and $\mathcal{F}'_{m-2}$ the sheaf corresponding to the bundle whose fibre over $P \in \hat{X}$ is $K(L_1 \otimes L_3 \otimes P, L_4, ..., L_m)$ (they are bundles by $B(m-2)$, in fact, the hypotheses of $B(m-2)$ for them, that is $l_1, l_4, ..., l_m \geq m-2$ and $l_1 + l_3, l_4, ..., l_m \geq m-2$, hold).

We take the cohomology sequence associated to the above exact sequence. By ii) of $D(m-1)$ we have ii) of $D(m)$.

Then we obtain

$$\begin{aligned} 0 &\longrightarrow H^0((\mathcal{F}'_{m-2} \otimes \pi_{\hat{X}\,*}(\pi_X^* L_2 \otimes \mathcal{P}^{-1}))^\vee) \\ &\longrightarrow H^0((\mathcal{F}_{m-2} \otimes H^0(L_3) \otimes \pi_{\hat{X}\,*}(\pi_X^* L_2 \otimes \mathcal{P}^{-1}))^\vee) \\ &\longrightarrow H^0((\mathcal{F}_{m-1} \otimes \pi_{\hat{X}\,*}(\pi_X^* L_2 \otimes \mathcal{P}^{-1}))^\vee) \longrightarrow 0, \end{aligned}$$

which is equal to

$$\begin{aligned} 0 &\longrightarrow K(L_1 \otimes L_2 \otimes L_3, L_4, ..., L_m)^\vee \\ &\longrightarrow (K(L_1 \otimes L_2, L_4, ..., L_m) H^0(L_3))^\vee \\ &\longrightarrow H^0((\mathcal{F}_{m-1} \otimes \pi_{\hat{X}\,*}(\pi_X^* L_2 \otimes \mathcal{P}^{-1}))^\vee) \longrightarrow 0, \end{aligned}$$

by $D(m-1)$ (we have to verify that $l_1 + l_3 \geq 2(m-1) - 1$, $l_4, ..., l_m \geq m-1$, $l_2 \geq 1$ and $l_1 \geq 2(m-1) - 1$ and that it is actually true).

By $A(m-1)$ we have

$$\begin{aligned} 0 &\longrightarrow K(L_1 \otimes L_2 \otimes L_3, L_4, ..., L_m)^\vee \\ &\longrightarrow (K(L_1 \otimes L_2, L_4, ..., L_m) H^0(L_3))^\vee \\ &\longrightarrow K(L_1 \otimes L_2, L_3, L_4, ..., L_m)^\vee \longrightarrow 0, \end{aligned}$$

(to apply $A(m-1)$ we have to verify that $l_1 + l_2 \geq m-2$, $l_3, .., l_m \geq m-2$ and it is true).

Thus $H^0((\mathcal{F}_{m-1} \otimes \pi_{\hat{X}\,*}(\pi_X^* L_2 \otimes \mathcal{P}^{-1}))^\vee) = K(L_1 \otimes L_2, L_3, ..., L_m)^\vee$.

• $A(m-1)$, $B(m-2)$ and $D(m-1) \Rightarrow C(m)$ for $m \geq 3$: this implication can be proved in an analogous way as in Proposition 12.

• $C(m) \Rightarrow A(m)$ for $m \geq 3$: it can be proved in an analogous way as in Theorem 5 in [Ke]; more precisely, let $l_1, ..., l_m \geq m-1$; write $L_1 = L'_1 \otimes M$ with L'_1

algebraically equivalent to M^{l_1-1}. We have

$$K(L_1\otimes L_2, L_3, ..., L_m) = \sum_{P\in Pic^0(X)} K(L_1'\otimes P, L_3, ..., L_m)H^0(M\otimes L_2\otimes P^{-1})$$

if

$$(*_1) \qquad l_1-1\geq m-2,\ \ 1+l_2\geq 1,\ \ l_3,...,l_m\geq m-1,$$

by $C(m)$.

We have

$$\begin{aligned}\sum_{P\in Pic^0(X)} & K(L_1'\otimes P, L_3, ..., L_m)H^0(M\otimes L_2\otimes P^{-1}) \\ &= \sum_{P\in Pic^0(X)} K(L_1'\otimes P, L_3, ..., L_m)H^0(M\otimes P^{-1})H^0(L_2)\end{aligned}$$

if

$$(*_2) \qquad l_2\geq 2,$$

by Lemma 15.

We have

$$\sum_{P\in Pic^0(X)} K(L_1'\otimes P, L_3, ..., L_m)H^0(M\otimes P^{-1})H^0(L_2) = K(L_1, L_3, ..., L_m)H^0(L_2)$$

if

$$(*_3) \qquad l_1-1\geq m-2,\ \ 1\geq 1,\ \ l_3,...,l_m\geq m-1$$

by $C(m)$.

$(*_1), (*_2), (*_3)$ are true, thus we conclude the proof of this implication. □

Now we are ready to prove Theorem 5.

Proof of Theorem 5. For any line bundle L on X we denote $G(L)=\bigoplus_n H^0(L^n)$, a module over the ring $S(L)=SymH^0(L)$.

By Remark 7, we have to prove that $Tor_i^{S(M^{p+1})}(G(M^{p+1}),\mathbf{C})$ is purely of degree $i+1$ for $1\leq i\leq p$.

Thus we have to prove that $Tor_i^{S(M^{p+1})}(G(M^{p+1}),\mathbf{C})$ is zero in degree $\geq i+2$ for $1\leq i\leq p$.

By Lemma 14, it is sufficient to prove that

$$i+2 > i-j+d(T^j(G(M^{p+1})))$$

for $0\leq j\leq i$ and $1\leq i\leq p$, (we use Definition 13), that is,

$$j+1\geq d(T^j(G(M^{p+1})))$$

for $0\leq j\leq i$ and $1\leq i\leq p$, that is,

$$j+1\geq d(T^j(G(M^{p+1})))$$

for $0\leq j\leq p$.

Observe that

$$T^j(G(M^{p+1})) = \bigoplus_n K(M^{(p+1)(n-j)}, \overbrace{M^{p+1}, ..., M^{p+1}}^{J});$$

then using Proposition 17 part a) with $m-1=p+1$ we have that, if $p \geq j$, then $T^j(G(M^{p+1}))$ is generated by $K(M^{p+1},...,M^{p+1})$ (where M^{p+1} repeats $j+1$ times), that is by the part of degree n with $n-j=1$ that is $n=j+1$; thus $d(T^j(G(M^{p+1})))=j+1$ and we conclude. □

Remark 18. Let X_i be a complex torus and L_i a line bundle on X_i for $i=1,2$; one can easily see that, if L_i satisfies Property N_0 for $i=1,2$, then the line bundle $\pi_1^* L_1 \otimes \pi_2^* L_2$ on $X_1 \times X_2$ satisfies Property N_0 and if L_i satisfies Property N_1 for $i=1,2$, then the line bundle $\pi_1^* L_1 \otimes \pi_2^* L_2$ on $X_1 \times X_2$ satisfies Property N_1.

In [Laz1], Lazarsfeld proved that, if X is a complex torus of dimension 2, L is an ample line bundle of type $(1,d)$ on X, $|L|$ has no fixed components and φ_L is birational onto its image, then $\varphi_L(X)$ is projectively normal for d odd ≥ 7 and d even ≥ 14.

Thus, for instance, if $d \in \mathbf{N}$ is even and ≥ 14, one can deduce from Theorem 5 and Lazarsfeld's Theorem that, if $(X, c_1(L))$ is generic in the moduli space of polarized abelian threefolds of type $(2,4,2d)$, the line bundle L on the complex torus X satisfies Property N_1; in fact, one can consider an elliptic curve E with an ample line bundle A of type (4) and an abelian surface S with a very ample line bundle M of type $(1,d)$ satisfying the hypotheses of Lazarsfeld's Theorem (it exists by Reider's Theorem, which claims that, if M is an ample line bundle of type $(1,d)$ with $d \geq 5$ on a complex torus X of dimension 2, then M is very ample if and only if there is no elliptic curve C on X with $(C \cdot L)=2$; thus generically an ample line bundle of type $(1,d)$ with $d \geq 5$ on a complex torus X of dimension 2 is very ample; see [Re] or [L-B] Chapter 10, §4); the line bundle A satisfies Property N_1 by Theorem 1 and the line bundle M^2 satisfies Property N_1 by Lazarsfeld's Theorem and Theorem 5; thus, considering the product $(E,A) \times (S, M^2)$, we conclude.

More generally, one can prove analogously the following statement: let $d_i \in \mathbf{N}$ $i=1,...,g$, $d_i | d_{i+1}$, $1 < s+1 \leq t < g$, $d_1 = ... = d_s = 1$, $d_{s+1},...,d_t \geq 2$, $d_{t+1},...,d_g \in \{d \in \mathbf{N} |\ d \geq 7 \ \mathit{odd\ or}\ d \geq 14\ \mathit{even}\}$; if $g-t \geq s$, then, if $(X, c_1(L))$ is generic in the moduli space of polarized abelian varieties of type $(2d_1,.....,2d_g)$, the line bundle L on the complex torus X satisfies Property N_1.

Remark 19. One can conjecture that, if M is an ample line bundle on a complex torus X and M^s satisfies Property N_k, then M^{s+p} satisfies Property N_{k+p}.

Observe that for $s=3$ and $k=0$ this is Lazarsfeld's conjecture and for $s=1$ and $k=0$ this is Theorem 5.

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