ON SYZYGIES OF ABELIAN VARIETIES

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Abstract. In this paper we prove the following result: Let \( X \) be a complex torus and \( M \) a normally generated line bundle on \( X \); then, for every \( p \geq 0 \), the line bundle \( M^p+1 \) satisfies Property \( N_p \) of Green-Lazarsfeld.

1. Introduction

In this paper we prove a new result on syzygies of abelian varieties; precisely, the problem we are considering is the following: let \( X \) be a complex torus, \( L \) a very ample line bundle on \( X \) and \( \varphi_L \) the associated map; we are concerned with the degree of the equations defining \( \varphi_L(X) \), the degree of the syzygies among them and the degree of higher syzygies. In particular, here we examine the case where \( L = M^l \) where \( M \) is a normally generated line bundle.

To review precisely the statements of the known results on syzygies of abelian varieties and to formulate precisely our theorem, we have to recall Green-Lazarsfeld’s definition of Property \( N_p \) (see [Gr1], [G-L], [Gr2], [Laz2], [E-L]): let \( Y \) be a smooth complex projective variety of dimension \( n \) and let \( L \) be a very ample line bundle on \( Y \) defining an embedding \( Y \subset \mathbb{P} = \mathbb{P}(H^0(Y, L)^*) \); set \( S = S(L) = \text{Sym}^*H^0(L) \), the homogeneous coordinate ring of the projective space \( \mathbb{P} \), and consider the graded \( S \)-module \( G = G(L) = \bigoplus_i H^0(Y, L^i) \). Let \( E_* \) be a minimal graded free resolution of \( G \) (that is, an exact sequence with \( E_i \) free \( S \)-modules and such that the matrices of homogenous polynomials giving the maps \( E_i \to E_{i-1} \) has no nonzero constant entries); the line bundle \( L \) satisfies Property \( N_p \) if and only if \( E_0 = S \)

\[
E_i = \bigoplus S(-i-1) \quad \text{for} \ 1 \leq i \leq p.
\]

(Thus \( L \) satisfies Property \( N_0 \) if and only if \( Y \subset \mathbb{P}(H^0(L)^*) \) is projectively normal, that is, \( L \) is normally generated; \( L \) satisfies Property \( N_1 \) if and only if \( L \) satisfies Property \( N_0 \) and the homogeneous ideal \( I \) of \( Y \subset \mathbb{P}(H^0(L)^*) \) is generated by quadrics; \( L \) satisfies Property \( N_2 \) if and only if \( L \) satisfies Property \( N_1 \) and the module of syzygies among quadratic generators \( Q_i \in I \) is spanned by relations of the form \( \sum L_i Q_i = 0 \), where \( L_i \) are linear polynomials; and so on.)

In 1966 Mumford proved that, if \( M \) is an ample line bundle on a complex torus \( X \) and \( l \geq 4 \), then the ideal of \( \varphi_M(X) \) is generated by quadrics ([Mum2]) and in

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Theorem 1 (Mumford-Sekiguchi-Kempf). Let $X$ be a complex torus. If $A$ is an ample line bundle on $X$ we denote $I(A)$ the ideal of $\varphi_A(X)$. Let $M$ be an ample line bundle on $X$;
   a) if $l \geq 4$, the ideal $I(M^l)$ is generated by forms of degree 2,
   b) let $l = 3$, the ideal $I(M^3)$ is generated by forms of degrees 2 and 3,
   c) (Lange-Birkenhake) let $l = 2$; if $M^2$ is normally generated, then the ideal $I(M^2)$ is generated by forms of degrees 2, 3 and 4.

In 1984 Green proved that if $X$ is a Riemann surface of genus $g$ and $L$ is a holomorphic line bundle on $X$ of degree $2g + 1 + p$, then $L$ satisfies Property $N_p$ (see [Gr1] and [Gr2]). Thus, if $M$ is an ample line bundle on an elliptic curve, then $M^{p+3}$ satisfies Property $N_p$ and in [Laz2] Lazarsfeld formulated the following conjecture:

Conjecture 2 (Lazarsfeld). If $M$ is an ample line bundle on a complex torus, then, for every $p \geq 0$, the line bundle $M^{p+3}$ satisfies Property $(N_p)$.

In 1989 Kempf proved a weaker result (see [Ke]):

Theorem 3 (Kempf). Let $M$ be an ample line bundle on a complex torus $X$. If $l \geq 4$, then $M^l$ satisfies Property $N_{\frac{l-3}{2}}$.

In 1993 Ein and Lazarsfeld proved the following theorem (see [E-L]):

Theorem 4 (Ein-Lazarsfeld). Let $Y$ be a smooth complex projective variety of dimension $n$; let $A$ be a very ample line bundle on $Y$, and $B$ a numerically effective line bundle on $Y$; then $K_Y \otimes A^{n+1+p} \otimes B$ satisfies Property $N_p$.

If $(Y, A, B) \neq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n})$ and $p \geq 1$, then $K_Y \otimes A^{n+p} \otimes B$ satisfies Property $N_p$.

Thus, if $M$ is a very ample line bundle on a complex torus of dimension $n$, then $M^{n+p}$ satisfies Property $N_p$.

In this paper, using the ideas of Kempf’s paper [Ke], by a patient adaptation, we prove another theorem on syzygies of abelian varieties:

Theorem 5. If $M$ is a normally generated line bundle on a complex torus $X$, then, for every $p \geq 0$, the line bundle $M^{p+1}$ satisfies Property $N_p$.

Since there is a result on normal generation of primitive line bundles (Lazarsfeld’s theorem on projective normality of $(1, d)$-abelian surfaces; see [Laz1]), Theorem 5 may actually be useful (see Remark 18).

Notation and Definitions. We collect here some notation and standard definitions that we will use throughout the paper.

- $\varphi_L$. If $L$ is a line bundle on a complex manifold $Y$, $\varphi_L$ is the rational map associated to $L$.
- A line bundle $L$ on a complex manifold $Y$ is called **normally generated** if it is very ample and $\varphi_L(Y)$ is projectively normal. We have that $L$ is normally generated if and only if it is ample and the natural maps $S^n H^0(Y, L) \rightarrow H^0(Y, L^n)$ are surjective for all $n \geq 2$ (see [Mum1], p. 38 and [L-B], Chapter 7, §3).
If $X$ is a complex torus of dimension $g$, then

- $t_x$ is the translation on $X$ by the point $x$;
- $\hat{X}$ is the dual complex torus of $X$; it is isomorphic to $Pic^0(X)$;
- $P$ denotes the Poincaré bundle on $X \times \hat{X}$;
- $\phi_L$ is the homomorphism $X \rightarrow \hat{X}$, $x \mapsto t_x^* L \otimes L^{-1}$, where $L$ is a line bundle on $X$;
- $K(L)$ is the kernel of $\phi_L$; it depends only on $H$, the first Chern class of $L$, thus we denote $K(L)$ also by $K(H)$; if $L$ is nondegenerate, then $K(L)$ is a finite group isomorphic to $\left(\mathbb{Z}/d_1 \oplus \ldots \oplus \mathbb{Z}/d_g\right)^2$ with $d_i|d_{i+1}$; we say that $L$ is of type $(d_1, \ldots, d_g)$;
- $W \cdot W'$: if $W$ is a vector subspace of $H^0(X, E)$ and $W'$ is a vector subspace of $H^0(X, E')$ ($E$ and $E'$ line bundles on $X$), $W \cdot W'$ is the image of $W \otimes W'$ under the multiplication map; we often omit $\cdot$.
- $\pi$: if we have a product of tori, we use the notation: $\pi_i$ is the projection on the $i$th factor and $\pi$ is the projection on $\cdot$.

2. Some recalls

First we recall Mumford’s lemma (see [Mum1] or [L-B], Chapter 7, Lemma 3.3) and the following remark and proposition.

**Lemma 6** (Mumford). Let $A$ and $B$ be two ample line bundles on a complex torus $X$. For every nonempty open subset $U$ of $X$, we have

$$\sum_{p \in U} H^0(X, A \otimes P) \cdot H^0(X, B \otimes P^{-1}) = H^0(X, A \otimes B).$$

As Kempf observed in [Ke], Mumford’s lemma can be interpreted in this way: a linear functional $\lambda$ on $H^0(A \otimes B)$ is determined by the family $\{\lambda_p\}_{p \in U}$, where $\lambda_p$ is the linear functional on $H^0(X, A \otimes P) \otimes H^0(X, B \otimes P^{-1})$ given by the composition of the multiplication with $\lambda$.

**Remark 7** (see [Gr2]). Let $V$ be a complex vector space of dimension $r + 1$, let $S = \bigoplus_{q \geq 0} \text{Sym}^q(V)$ and $G = \bigoplus_{q} G_q$ a finitely generated graded $S$-module. Let

$$0 \rightarrow E_n \rightarrow E_{n-1} \rightarrow \ldots \rightarrow E_0 \rightarrow G \rightarrow 0$$

be a minimal free resolution of $G$, that is, an exact sequence with $E_i$ free $S$-modules and such that the matrices of homogeneous polynomials giving the maps $E_i \rightarrow E_{i-1}$ has no nonzero constant entries. Write $E_p = \bigoplus_{q \geq 0} (B_{p,q} \otimes S(-q))$ with $B_{p,q}$ vector spaces on $C$. See $C$ as the $S$-module $S/\bigoplus_{q \geq 1} \text{Sym}^q(V)$. Then

$$\text{Tor}^S_p(G, C)_q \simeq B_{p,q}.$$

**Proposition 8** (Koizumi [Ko]). Let $A$ and $A'$ be two algebraically equivalent ample line bundles on a complex torus $X$. The multiplication map $H^0(X, A^m) \otimes H^0(X, A'^n) \rightarrow H^0(X, A^m \otimes A'^n)$ is surjective for all $m \geq 3$ and $n \geq 2$.

Now we recall some facts, definitions and propositions of Kempf’s paper [Ke].

**Definition 9** (Kempf). For any $A_i$’s ample line bundles on a complex torus $X$, let $K(A_1) = H^0(X, A_1)$ and, for $n > 1$, define $K(A_1, \ldots, A_n)$ inductively by the following exact sequence:

$$0 \rightarrow K(A_1, \ldots, A_n) \rightarrow K(A_1, A_3, \ldots, A_n) \otimes H^0(X, A_2) \rightarrow K(A_1 \otimes A_2, A_3, \ldots, A_n).$$
To follow completely Kempf’s notations, we denote $K(A_1, A_2)$ by $R(A_1, A_2)$ ($= \ker(H^0(A_1) \otimes H^0(A_2) \to H^0(A_1 \otimes A_2))$).

In the sequel $K(A_1, A_3, \ldots, A_n) \cdot H^0(X, A_2)$ will denote the image of the multiplication map $K(A_1, A_3, \ldots, A_n) \otimes H^0(X, A_2) \to K(A_1 \otimes A_2, A_3, \ldots, A_n) (\subset H^0(A_1 \otimes A_2) \otimes H^0(A_3) \otimes \cdots \otimes H^0(A_n))$; we often omit .

**Notation 10** (Kempf). In the remainder of this section, following [Ke], we use the following notation: let $X$ be a complex torus of dimension $g$; fix an ample line bundle $M$ on $X$; $i \in \mathbb{N}$, will denote positive integers and $L_i$ will denote a line bundle algebraically equivalent to $M^i$.

Observe that, if $A$ is a line bundle on $X$, since $H^0((\pi_X^* A \otimes P)|_{X \times \{P\}}) = H^0(A \otimes P)$ is of constant dimension $\forall P \in \mathcal{X}$, then the sheaf $\pi_X^*(\pi_X^* A \otimes P)$ on $\mathcal{X}$ is locally free and its fibre over $P \in \mathcal{X}$ is $H^0(A \otimes P)$, by Grauert’s Theorem (see [Ha] Theorem 12.9 Chapter 3). Analogously the sheaf $\pi_X^*(\pi_X^* A \otimes P^{-1})$ on $\mathcal{X}$ is locally free and its fibre over $P \in \mathcal{X}$ is $H^0(A \otimes P^{-1})$.

Consider the following map:

$$\pi_X^*(\pi_X^* L_1 \otimes P) \otimes \pi_X^*(\pi_X^* L_2 \otimes P^{-1}) \to H^0(X, L_1 \otimes L_2) \otimes \mathcal{O}_X$$

(given by the composition of the maps

$$\pi_X^*(\pi_X^* L_1 \otimes P) \otimes \pi_X^*(\pi_X^* L_2 \otimes P^{-1}) \to \pi_X^*(\pi_X^* L_1 \otimes L_2)$$

and

$$\pi_X^*(\pi_X^* L_1 \otimes L_2) \to H^0(X, L_1 \otimes L_2) \otimes \mathcal{O}_X$$)

This map induces a map:

$$m : H^0(X, L_1 \otimes L_2)^\vee \to H^0(\mathcal{X}, (\pi_X^*(\pi_X^* L_1 \otimes P) \otimes \pi_X^*(\pi_X^* L_2 \otimes P^{-1}))^\vee)$$

**Proposition 11** (Kempf). i) The map

$$m : H^0(X, L_1 \otimes L_2)^\vee \to H^0(\mathcal{X}, (\pi_X^*(\pi_X^* L_1 \otimes P) \otimes \pi_X^*(\pi_X^* L_2 \otimes P^{-1}))^\vee)$$

is an isomorphism.

ii) $H^i(\mathcal{X}, (\pi_X^*(\pi_X^* L_1 \otimes P) \otimes \pi_X^*(\pi_X^* L_2 \otimes P^{-1}))^\vee) = 0$ for $i \geq 1$.

Arguing exactly as in Proposition 4 of [Ke], we have:

**Proposition 12** (Kempf). If $H^0(L_1 \otimes P) \otimes H^0(L_3) \to H^0(L_1 \otimes P \otimes L_3)$ is surjective $\forall P \in Pic^0(X)$ and $H^0(L_1 \otimes L_2) \otimes H^0(L_3) \to H^0(L_1 \otimes L_2 \otimes L_3)$ is surjective, then

$$\sum_{P \in Pic^0(X)} R(L_1 \otimes P, L_3) \cdot H^0(L_2 \otimes P^{-1}) = R(L_1 \otimes L_2, L_3).$$

We reproduce the proof here for later use.

**Proof.** One inclusion is obvious:

$$\sum_{P \in \mathcal{X}} R(L_1 \otimes P, L_3) \cdot H^0(L_2 \otimes P^{-1}) \subset R(L_1 \otimes L_2, L_3).$$

We want to show the other one. It suffices to show that $(\sum_{P \in Pic^0(X)} R(L_1 \otimes P, L_3) \cdot H^0(L_2 \otimes P^{-1}))^\perp$ in $(H^0(L_1 \otimes L_2) \otimes H^0(L_3))^\vee$ is contained in $R(L_1 \otimes L_2, L_3)^\perp$.
For every $P \in \hat{X}$, we have the following commutative diagram:

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
R(L_1 \otimes P, L_3) \otimes H^0(L_2 \otimes P^{-1}) & \rightarrow & R(L_1 \otimes L_2, L_3) \\
\downarrow & & \downarrow \\
H^0(L_1 \otimes P) \otimes H^0(L_3) \otimes H^0(L_2 \otimes P^{-1}) & \rightarrow & H^0(L_1 \otimes L_2) \otimes H^0(L_3) \\
\downarrow & & \downarrow \\
H^0(L_1 \otimes P \otimes L_3) \otimes H^0(L_2 \otimes P^{-1}) & \rightarrow & H^0(L_1 \otimes L_2 \otimes L_3) \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}
\]

The columns are exact by the hypotheses.

To see that $(\sum_{P \in \text{Pic}^0(X)} R(L_1 \otimes P, L_3) \cdot H^0(L_2 \otimes P^{-1}))^\vee$ in $(H^0(L_1 \otimes L_2) \otimes H^0(L_3))^\vee$ is contained in $R(L_1 \otimes L_2, L_3)^\vee$ is equivalent to seeing that any element of $(H^0(L_1 \otimes L_2) \otimes H^0(L_3))^\vee$ inducing a linear form on $H^0(L_1 \otimes L_3 \otimes P) \otimes H^0(L_2 \otimes P^{-1})$ induces a linear form on $H^0(L_1 \otimes L_2 \otimes L_3)$ and this is true by Proposition 11.

**Definition 13** (Kempf). Let $S$ be a graded ring and $G$ a finitely generated graded $S$-module and $k = S/\bigoplus_{n \geq 1} S_n$. Define $d(G) = \min \{j \mid G \text{ is generated by the union of the } G_d's \text{ for } d \leq j\}$ (thus $d(G)$ is the smallest number such that $G \otimes_S k$ is zero in degree $> d(G)$).

1. Define $T^1(G) = \ker(G(1) \otimes_k S_1 \rightarrow G)$.
2. Define $T^0(G) = T^{-1}(T^1(G))$ and $T^0(G) = G$.
3. Define $d^0(G) = d(T^1(G))$.

**Lemma 14** (Kempf). Let $S$ be a graded ring and $G$ a finitely generated graded $S$-module and $k = S/\bigoplus_{n \geq 1} S_n$. If $q > i - j + d(T^1(G))$ for all $0 \leq j \leq i$, then $\text{Tor}^S_i(G, k)$ is zero in degree $q$.

3. Two new lemmas

To prove Theorem 5 we need the following two lemmas.

**Lemma 15.** Let $A$ and $A'$ be two algebraically equivalent normally generated line bundles on a complex torus $X$ and $m, n \in \mathbb{N}$. If $m \geq 2$ and $n \geq 1$, then

$$H^0(A^m) \cdot H^0(A^n) = H^0(A^m \otimes A^n).$$

**Proof.** Observe that the set $U := \{P \in \text{Pic}^0(X) \text{ s.t. } H^0(A \otimes P^{-1})H^0(A') = H^0(A \otimes A' \otimes P^{-1})\}$ is nonempty, since $A$ and $A'$ are normally generated, and open, since, as we have already observed, for any line bundle $L$ on $X$, the vector spaces $H^0(L \otimes P)$ for $P \in \hat{X}$ form a vector bundle on $X$.

Applying twice Mumford’s Lemma, we have

$$H^0(A^m)H^0(A^n) = \sum_{P \in U} H^0(A^{m-1} \otimes P)H^0(A \otimes P^{-1})H^0(A')$$

$$= \sum_{P \in U} H^0(A^{m-1} \otimes P)H^0(A \otimes A' \otimes P^{-1}) = H^0(A^m \otimes A^n)$$

for $m \geq 2$ and $n \geq 1$. 

**Lemma 16.** Let $A, A'$ and $A''$ be three algebraically equivalent normally generated line bundles on a complex torus $X$ and $\alpha, \beta, \gamma \in \mathbb{N}$. If we are in one of the following three cases:
Proof. Let \( \alpha, \beta, \gamma, l \in \mathbb{N} \) and \( \beta, l, \alpha - l \geq 1 \). If
\[
H^0(A^{\alpha - l} \otimes P) \otimes H^0(A^{\beta}) \rightarrow H^0(A^{\alpha - l} \otimes P \otimes A^{\beta})
\]
for all \( P \in Pic^0(X) \), and
\[
H^0(A^\alpha) \otimes H^0(A^{\beta}) \rightarrow H^0(A^\alpha \otimes A^{\beta})
\]
are surjective (we call this condition (a)), then, by Proposition 12, we have
\[
R(A^\alpha, A^{\beta})H^0(A^{\gamma}) = \sum_{P \in Pic^0(X)} R(A^{\alpha - l} \otimes P, A^{\beta})H^0(A^l \otimes P^{-1})H^0(A^{\gamma}).
\]
If
\[
H^0(A^l \otimes P^{-1}) \otimes H^0(A^{\gamma}) \rightarrow H^0(A^l \otimes P^{-1} \otimes A^{\gamma})
\]
for all \( P \in Pic^0(X) \), is surjective (we call this condition (b)), then we have
\[
\sum_{P \in Pic^0(X)} R(A^{\alpha - l} \otimes P, A^{\beta})H^0(A^l \otimes P^{-1})H^0(A^{\gamma}) = \sum_{P \in Pic^0(X)} R(A^{\alpha - l} \otimes P, A^{\beta})H^0(A^l \otimes A^{\gamma} \otimes P^{-1}).
\]
If
\[
H^0(A^{\alpha - l} \otimes P) \otimes H^0(A^{\beta}) \rightarrow H^0(A^{\alpha - l} \otimes P \otimes A^{\beta})
\]
for all \( P \in Pic^0(X) \), and
\[
H^0(A^\alpha \otimes A^{\gamma}) \otimes H^0(A^{\beta}) \rightarrow H^0(A^\alpha \otimes A^{\gamma} \otimes A^{\beta})
\]
are surjective (we call this condition (c)), then, by Proposition 12, we have
\[
\sum_{P \in Pic^0(X)} R(A^{\alpha - l} \otimes P, A^{\beta})H^0(A^l \otimes A^{\gamma} \otimes P^{-1}) = R(A^\alpha \otimes A^{\gamma}, A^{\beta}).
\]
By Lemma 15, if we are in one of the following four cases:
1) \( \alpha \geq 2, \beta \geq 2, \gamma \geq 2, \)
2) \( \alpha \geq 3, \beta = 1, \alpha + \gamma \geq 5, \)
3) \( \alpha \geq 3, \gamma = 1, \alpha + \beta \geq 5, \)
then (a), (b) and (c) hold. Thus we conclude the proof.
4. The proof of Theorem 5 and two remarks

The crucial step to prove Theorem 5 is the following proposition, which is analogous to Theorem 17 in [Ke].

**Proposition 17.** Let $M$ be a normally generated line bundle on a complex torus $X$. We again use Notation 17A that is the $l_i$'s, $i \in \mathbb{N}$, denote positive integers and $L_i$ denotes a line bundle algebraically equivalent to $M_i$.

- Let $m \geq 3$. If $l_1 \geq m - 1, \ldots, l_m \geq m - 1$, then $K(L_1, L_3, \ldots, L_m) \otimes H^0(X, L_2) \rightarrow K(L_1 \otimes L_2, L_3, \ldots, L_m)$ is surjective.

- Let $m \geq 1$. If $l_1 \geq m, \ldots, l_m \geq m$, then $K(L_1, \ldots, L_m)$ form a vector bundle on the appropriate component of $\text{Pic}(X)^m$.

- Let $m \geq 4$. If $l_1 \geq m - 2, l_2 \geq 1$ and $l_3 \geq m - 1, \ldots, l_m \geq m - 1$, then $K(L_1 \otimes L_2, L_3, \ldots, L_m) = \sum_{P \in X} K(L_1 \otimes P, L_3, \ldots, L_m) \cdot H^0(L_2 \otimes P^{-1})$.

- Let $m \geq 2$. Let $l_1 \geq 2m - 1, l_2 \geq 1$, and, if $m \geq 3$, $l_3 \geq m, \ldots, l_m \geq m$ and suppose the family of vector spaces $K(L_1 \otimes P, L_3, \ldots, L_m)$, $P \in \hat{X}$, forms a vector bundle on $\hat{X}$, we call the corresponding sheaf $\mathcal{F}_{m-1}$. Then we have

  i) an isomorphism

  $$K(L_1 \otimes L_2, L_3, \ldots, L_m)^{\vee} = H^0(\hat{X}, (\mathcal{F}_{m-1} \otimes \pi_{\hat{X}}^*(\pi_X^* L_2 \otimes P^{-1}))^{\vee})$$

  ii) $H^i(\hat{X}, (\mathcal{F}_{m-1} \otimes \pi_{\hat{X}}^*(\pi_X^* L_2 \otimes P^{-1}))^{\vee}) = 0$ if $i \geq 1$.

**Proof.** Consider the following four statements depending on the natural number $m$:

- **Statement A($m$):** If $l_1 \geq m - 1, \ldots, l_m \geq m - 1$, then $K(L_1, L_3, \ldots, L_m) \otimes H^0(X, L_2) \rightarrow K(L_1 \otimes L_2, L_3, \ldots, L_m)$ is surjective.

- **Statement B($m$):** If $l_1 \geq m, \ldots, l_m \geq m$, then $K(L_1, \ldots, L_m)$ form a vector bundle on the appropriate component of $\text{Pic}(X)^m$.

- **Statement C($m$):** If $l_1 \geq m - 2, l_2 \geq 1$, and, if $m \geq 3$, $l_3 \geq m - 1, \ldots, l_m \geq m - 1$, then $K(L_1 \otimes L_2, L_3, \ldots, L_m) = \sum_{P \in X} K(L_1 \otimes P, L_3, \ldots, L_m) \cdot H^0(L_2 \otimes P^{-1})$.

- **Statement D($m$):** Let $l_1 \geq 2m - 1, l_2 \geq 1$, and, if $m \geq 3$, $l_3 \geq m, \ldots, l_m \geq m$ and suppose the family of vector spaces $K(L_1 \otimes P, L_3, \ldots, L_m)$, $P \in \hat{X}$, forms a vector bundle on $\hat{X}$, we call the corresponding sheaf $\mathcal{F}_{m-1}$; then we have

  i) an isomorphism

  $$K(L_1 \otimes L_2, L_3, \ldots, L_m)^{\vee} = H^0(\hat{X}, (\mathcal{F}_{m-1} \otimes \pi_{\hat{X}}^*(\pi_X^* L_2 \otimes P^{-1}))^{\vee})$$

  ii) $H^i(\hat{X}, (\mathcal{F}_{m-1} \otimes \pi_{\hat{X}}^*(\pi_X^* L_2 \otimes P^{-1}))^{\vee}) = 0$ if $i \geq 1$.

We know that $A(3)$, $B(1)$, $B(2)$, $D(2)$ are true ($A(3)$ is true by Lemma 16, $B(2)$ by Lemma 15, $D(2)$ by Proposition 11).

We will prove the following four implications:

- $A(m)$. $A(m - 1)$ and $B(m - 2) \Rightarrow B(m - 1)$ for $m \geq 3$.

- $A(m)$. $A(m - 1)$, $B(m - 2)$ and $D(m - 1) \Rightarrow D(m)$ for $m \geq 4$ and $B(1)$ and $D(2) \Rightarrow D(3)$.

- $C(m)$. $D(m)$ and $A(m)$ for $m \geq 3$.

By the second implication also $D(3)$ holds; using the four implications, one can prove by induction that $A(m)$, $B(m - 1)$ and $D(m)$ are true for $m \geq 3$ and conclude.

Thus let us prove the four implications:

- $A(m)$ and $B(m - 2) \Rightarrow B(m - 1)$ for $m \geq 3$: obvious.

- $A(m)$. $A(m - 1)$, $B(m - 2)$ and $D(m - 1) \Rightarrow D(m)$ for $m \geq 4$ and $B(1)$ and $D(2) \Rightarrow D(3)$; it can be proved in an analogous way as Proposition 9 in [Ke]; more precisely:
consider line bundles $L_1, ..., L_m$ with the hypotheses of $D(m)$, thus $l_1 \geq 2m - 1$, $l_2 \geq 1$, and, if $m \geq 3$, $l_3 \geq m$, ..., $l_m \geq m$.

Let $m \geq 4$. Since $l_1, l_3, ..., l_m \geq m - 2$, by $A(m - 1)$ we have the following exact sequence:

$$0 \to K(L_1 \otimes P, L_3, ..., L_m) \otimes H^0(L_2 \otimes P^{-1})$$

$$\to K(L_1 \otimes P, L_4, ..., L_m) \otimes H^0(L_3) \otimes H^0(L_2 \otimes P^{-1})$$

$$\to K(L_1 \otimes L_3 \otimes P, L_4, ..., L_m) \otimes H^0(L_2 \otimes P^{-1}) \to 0;$$

observe that also if $m = 3$ this exact sequence holds, by Proposition $\S$.

The above sequence gives an exact sequence

$$0 \to (\mathcal{F}'_{m-2} \otimes \pi_X^*(\pi_X^* L_2 \otimes \mathcal{P}^{-1}))^\vee$$

$$\to (\mathcal{F}_{m-2} \otimes H^0(L_3) \otimes \pi_X^*(\pi_X^* L_2 \otimes \mathcal{P}^{-1}))^\vee$$

$$\to (\mathcal{F}_{m-1} \otimes \pi_X^*(\pi_X^* L_2 \otimes \mathcal{P}^{-1}))^\vee \to 0,$$

with $\mathcal{F}_{m-2}$ the sheaf corresponding to the bundle whose fibre over $P \in \mathcal{X}$ is $K(L_1 \otimes P, L_4, ..., L_m)$ and $\mathcal{F}'_{m-2}$ the sheaf corresponding to the bundle whose fibre over $P \in \mathcal{X}$ is $K(L_1 \otimes L_3 \otimes P, L_4, ..., L_m)$ (they are bundles by $B(m - 2)$, in fact, the hypotheses of $B(m - 2)$ for them, that is $l_1, l_4, ..., l_m \geq m - 2$ and $l_1 + l_3, l_4, ..., l_m \geq m - 2$, hold).

We take the cohomology sequence associated to the above exact sequence. By ii) of $D(m - 1)$ we have ii) of $D(m)$.

Then we obtain

$$0 \to H^0((\mathcal{F}'_{m-2} \otimes \pi_X^*(\pi_X^* L_2 \otimes \mathcal{P}^{-1}))^\vee)$$

$$\to H^0((\mathcal{F}_{m-2} \otimes H^0(L_3) \otimes \pi_X^*(\pi_X^* L_2 \otimes \mathcal{P}^{-1}))^\vee)$$

$$\to H^0((\mathcal{F}_{m-1} \otimes \pi_X^*(\pi_X^* L_2 \otimes \mathcal{P}^{-1}))^\vee) \to 0,$$

which is equal to

$$0 \to K(L_1 \otimes L_2 \otimes L_3, L_4, ..., L_m)^\vee$$

$$\to (K(L_1 \otimes L_2, L_4, ..., L_m)H^0(L_3))^\vee$$

$$\to H^0((\mathcal{F}_{m-1} \otimes \pi_X^*(\pi_X^* L_2 \otimes \mathcal{P}^{-1}))^\vee) \to 0,$$

by $D(m - 1)$ (we have to verify that $l_1 + l_3 \geq 2(m - 1) - 1$, $l_4, ..., l_m \geq m - 1$, $l_2 \geq 1$ and $l_1 \geq 2(m - 1) - 1$ and that it is actually true).

By $A(m - 1)$ we have

$$0 \to K(L_1 \otimes L_2 \otimes L_3, L_4, ..., L_m)^\vee$$

$$\to (K(L_1 \otimes L_2, L_4, ..., L_m)H^0(L_3))^\vee$$

$$\to K(L_1 \otimes L_2, L_3, L_4, ..., L_m)^\vee \to 0,$$

(to apply $A(m - 1)$ we have to verify that $l_1 + l_2 \geq m - 2$, $l_3, ..., l_m \geq m - 2$ and it is true).

Thus $H^0((\mathcal{F}_{m-1} \otimes \pi_X^*(\pi_X^* L_2 \otimes \mathcal{P}^{-1}))^\vee) = K(L_1 \otimes L_2, L_3, ..., L_m)^\vee$.

• $A(m - 1)$, $B(m - 2)$ and $D(m - 1) \Rightarrow C(m)$ for $m \geq 3$: this implication can be proved in an analogous way as in Proposition $\S$.

• $C(m) \Rightarrow A(m)$ for $m \geq 3$: it can be proved in an analogous way as in Theorem 5 in [Ke]: more precisely, let $l_1, ..., l_m \geq m - 1$; write $L_1 = L'_1 \otimes M$ with $L'_1$
algebraically equivalent to $M^{l_1-1}$. We have

$$K(L_1 \otimes L_2, L_3, ..., L_m) = \sum_{P \in Pic^0(X)} K(L'_1 \otimes P, L_3, ..., L_m)H^0(M \otimes L_2 \otimes P^{-1})$$

if

$$(*_1) \quad l_1 - 1 \geq m - 2, \quad 1 + l_2 \geq 1, \quad l_3, ..., l_m \geq m - 1,$$

by $C(m)$.

We have

$$\sum_{P \in Pic^0(X)} K(L'_1 \otimes P, L_3, ..., L_m)H^0(M \otimes L_2 \otimes P^{-1}) = \sum_{P \in Pic^0(X)} K(L'_1 \otimes P, L_3, ..., L_m)H^0(M \otimes P^{-1})H^0(L_2)$$

if

$$(*_2) \quad l_2 \geq 2,$$

by Lemma 15.

We have

$$\sum_{P \in Pic^0(X)} K(L'_1 \otimes P, L_3, ..., L_m)H^0(M \otimes P^{-1})H^0(L_2) = K(L_1, L_3, ..., L_m)H^0(L_2)$$

if

$$(*_3) \quad l_1 - 1 \geq m - 2, \quad 1 \geq 1, \quad l_3, ..., l_m \geq m - 1$$

by $C(m)$.

$(*_1), (*_2), (*_3)$ are true, thus we conclude the proof of this implication.

Now we are ready to prove Theorem 5.

**Proof of Theorem 5.** For any line bundle $L$ on $X$ we denote $G(L) = \bigoplus_n H^0(L^n)$, a module over the ring $S(L) = SymH^0(L)$.

By Remark 7 we have to prove that $Tor_i^{S(M^{p+1})}(G(M^{p+1}), C)$ is purely of degree $i + 1$ for $1 \leq i \leq p$.

Thus we have to prove that $Tor_i^{S(M^{p+1})}(G(M^{p+1}), C)$ is zero in degree $i + 2$ for $1 \leq i \leq p$.

By Lemma 14 it is sufficient to prove that

$$i + 2 > i - j + d(T^j(G(M^{p+1})))$$

for $0 \leq j \leq i$ and $1 \leq i \leq p$, (we use Definition 13), that is,

$$j + 1 \geq d(T^j(G(M^{p+1})))$$

for $0 \leq j \leq i$ and $1 \leq i \leq p$, that is,

$$j + 1 \geq d(T^j(G(M^{p+1})))$$

for $0 \leq j \leq p$.

Observe that

$$T^j(G(M^{p+1})) = \bigoplus_n K(M^{(p+1)(n-j)}, M^{p+1}, ..., M^{p+1});$$
then using Proposition 17 part a) with \( m - 1 = p + 1 \) we have that, if \( p \geq j \), then \( T^j(G(M^{p+1})) \) is generated by \( K(M^{p+1}, ..., M^{p+1}) \) (where \( M^{p+1} \) repeats \( j + 1 \) times), that is by the part of degree \( n \) with \( n - j = 1 \) that is \( n = j + 1 \); thus \( d(T^j(G(M^{p+1}))) = j + 1 \) and we conclude.

\[ \square \]

**Remark 18.** Let \( X_i \) be a complex torus and \( L_i \) a line bundle on \( X_i \) for \( i = 1, 2 \); one can easily see that, if \( L_i \) satisfies Property \( N_0 \) for \( i = 1, 2 \), then the line bundle \( \pi_i^*L_1 \otimes \pi_i^*L_2 \) on \( X_1 \times X_2 \) satisfies Property \( N_0 \) and if \( L_i \) satisfies Property \( N_1 \) for \( i = 1, 2 \), then the line bundle \( \pi_i^*L_1 \otimes \pi_i^*L_2 \) on \( X_1 \times X_2 \) satisfies Property \( N_1 \).

In [Laz1], Lazarsfeld proved that, if \( X \) is a complex torus of dimension 2, \( L \) is an ample line bundle of type \( (1, d) \) on \( X \), \( |L| \) has no fixed components and \( \varphi_L \) is birational onto its image, then \( \varphi_L(X) \) is projectively normal for \( d \) odd \( \geq 7 \) and \( d \) even \( \geq 14 \).

Thus, for instance, if \( d \in \mathbb{N} \) is even and \( \geq 14 \), one can deduce from Theorem 7 and Lazarsfeld’s Theorem that, if \( (X, c_1(L)) \) is generic in the moduli space of polarized abelian threefolds of type \( (2, 4, 2d) \), the line bundle \( L \) on the complex torus \( X \) satisfies Property \( N_1 \); in fact, one can consider an elliptic curve \( E \) with an ample line bundle \( A \) of type \( (4) \) and an abelian surface \( S \) with a very ample line bundle \( M \) of type \( (1, d) \) satisfying the hypotheses of Lazarsfeld’s Theorem (it exists by Reider’s Theorem, which claims that, if \( M \) is an ample line bundle of type \( (1, d) \) with \( d \geq 5 \) on a complex torus \( X \) of dimension 2, then \( M \) is very ample if and only if there is no elliptic curve \( C \) on \( X \) with \( (C \cdot L) = 2 \); thus generically an ample line bundle of type \( (1, d) \) with \( d \geq 5 \) on a complex torus \( X \) of dimension 2 is very ample; see [Re] or [L-B] Chapter 10, §4); the line bundle \( A \) satisfies Property \( N_1 \) by Theorem 1 and the line bundle \( M^2 \) satisfies Property \( N_1 \) by Lazarsfeld’s Theorem and Theorem 5 thus, considering the product \((E, A) \times (S, M^2)\), we conclude.

More generally, one can prove analogously the following statement: let \( d_i \in \mathbb{N} \) \( i = 1, ..., g, d_i \mid d_{i+1} \), \( 1 < s + 1 \leq t < g, d_1 = ... = d_s = 1, d_{s+1}, ..., d_t \geq 2, d_{t+1}, ..., d_g \in \{ d \in \mathbb{N} \mid d \geq 7 \text{ odd or } d \geq 14 \text{ even} \} \); if \( g - t \geq s \), then, if \( (X, c_1(L)) \) is generic in the moduli space of polarized abelian varieties of type \( (2d_1, ..., 2d_g) \), the line bundle \( L \) on the complex torus \( X \) satisfies Property \( N_1 \).

**Remark 19.** One can conjecture that, if \( M \) is an ample line bundle on a complex torus \( X \) and \( M^s \) satisfies Property \( N_k \), then \( M^{s+p} \) satisfies Property \( N_{k+p} \).

Observe that for \( s = 3 \) and \( k = 0 \) this is Lazarsfeld’s conjecture and for \( s = 1 \) and \( k = 0 \) this is Theorem 5.

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**References**


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