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ON SHIMURA, SHINTANI AND EICHLER-ZAGIER CORRESPONDENCES

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Abstract. In this paper, we set up Shimura and Shintani correspondences between Jacobi forms and modular forms of integral weight for arbitrary level and character, and generalize the Eichler-Zagier isomorphism between Jacobi forms and modular forms of half-integral weight to higher levels. Using this together with the known results, we get a strong multiplicity 1 theorem in certain cases for both Jacobi cusp newforms and half-integral weight cusp newforms. As a consequence, we get, among other results, the explicit Waldspurger theorem.

1. Introduction

The famous work of Atkin and Lehner [1] partitions the space $S_{2k}(N)$ of cusp forms of weight $2k$, level N into eigensubspaces with respect to the Hecke operators $T(n)$, $(n, N) = 1$. In each eigensubspace there exists a normalized Hecke eigenform $g(z)$ of weight $2k$ and level N' (N' is a positive divisor of N), such that the eigensubspace is generated by all the forms $g(dz)$, where d varies over all positive divisors of N/N'. The direct sum of the eigensubspaces when $N' = N$ is known as the *space of newforms* of weight $2k$, level N, denoted by $S_{2k}^{new}(N)$. This theory is known as the theory of newforms of integral weight.

Starting with the remarkable work of Shimura [15], the theory of modular forms of half-integral weight is well developed with an analogous Hecke theory. The space $S_{k+1/2}(\Gamma_0(4N))$ of cusp forms of weight $k + 1/2$ for the group $\Gamma_0(4N)$ is connected with the space $S_{2k}(2N)$ via the Shimura and Shintani correspondences, commuting with the action of Hecke operators. Through the results of Waldspurger [22], [23], the square of the Fourier coefficient of a Hecke eigenform of half-integral weight is proportional to the special value of the twisted L-function associated with the corresponding (via the Shimura correspondence) Hecke eigenform of integral weight at the center of the critical strip. It is therefore natural to set up a parallel theory of newforms for the space $S_{k+1/2}(\Gamma_0(4N))$, which is compatible with the theory of newforms of integral weight. In turn we get the explicit Waldspurger result [6, 8, 11]. Several attempts have been made to set up the theory of newforms; however, there is no complete theory available as in the case of modular forms of integral weight.

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Another development, noted from the earlier work of Jacobi [5] and arising naturally from Siegel modular forms, is the theory of Jacobi forms. Hecke theory and the theory of newforms have been completely studied in the space $J_{k,m}$ of Jacobi forms of weight k and index m for the full Jacobi modular group Γ^J. The analogues of the Shimura and Shintani correspondences between $J_{k,m}$ and $S^-_{2k-2}(m)$, a special subspace of $S_{2k-2}(m)$, are known. In order to complete the Saito–Kurokawa correspondence, Eichler and Zagier [2] established a perfect isomorphism between Jacobi forms and modular forms of half-integral weight in the simplest case when $m = 1$.

Our goal in the present paper is three-fold. First (section 3), we set up Shimura and Shintani correspondences between the spaces $J^{cusp}_{k,m}(M,\chi)$ (the space of Jacobi cusp forms of weight k, index m and level M with character χ) and $S_{2k-2}\left(mM,\chi^2\right)$. To do so, we connect Poincaré series of Jacobi forms and certain holomorphic kernel function of the periods of integral weight cusp forms under the Shimura correspondence. Second (section 4), we extend the Eichler-Zagier perfect isomorphism [2, Theorem 5.4], denoted by $\mathbf{Z}_1$, into a linear map $\mathbf{Z}_m$ commuting with the action of Hecke operators and which maps $J^{cusp}_{k,m}(M,\chi)$ into a special subspace $\mathbf{S}^m_{k-1/2}(mM,\chi)$ (see section 4, (19), for the definition) of $S_{k-1/2}\left(\Gamma_0(4mM),\chi\right)$. By proving that the (n,r)-th Fourier coefficient of a Jacobi newform in $J^{cusp,new}_{k,m}(M,\chi)$ depends only on the discriminant r^2-4mn but not on r modulo $2m$, we obtain that the restriction of $\mathbf{Z}_m$ on newforms is a perfect isomorphism from $J^{cusp,new}_{k,m}(M,\chi)$ onto $\mathbf{S}^{m,new}_{k-1/2}(mM,\chi)$, where m is relatively prime to the conductor of χ (refer to the equation (22) in section 5.1 for the definitions). Now, invoking the known results [7, 12, 18, 19, 20, 21], we have the "strong multiplicity one" theorem for (i) Jacobi cusp newforms of index m and square-free level with real character, and (ii) the newforms in the special space $\mathbf{S}^{m,new}_{k-1/2}(mM,\chi)$, where m is arbitrary, M is square-free, and χ is real. As a consequence (section 5.2), the Waldspurger result is derived for Jacobi cusp newforms. Third (section 5.3), we decompose the space of oldforms both in $J^{cusp}_{k,1}(M)$ and in the space $\mathbf{S}^m_{k-1/2}(m)$, compatible with $\mathbf{Z}_m$. Further, in section 5.2, we obtain the exact eigenvalues for the W-operators on the space $S^+_{k-1/2}\left(\Gamma_0(4mM)\right)$ for the primes p dividing mM.

We feel that the methods used in this paper can also be used to get similar results in the case of skew-holomorphic Jacobi forms. For details in the simplest case (for example, the results of section 3), we refer to the thesis of the first author [10].

2. Preliminaries

Let $k,m,M,N \in \mathbb{N}$, and let $\tau, w \in \mathcal{H}$ (complex upper half-plane), $z \in \mathbb{C}$. We denote by $S_k(N,\psi)$, the space of cusp forms of weight k and level N with character ψ; by $S^-_k(N)$ the subspace of $S_k(N)$ consisting of forms f satisfying $f\big|W_N = (-1)^{k/2+1}f$, where W_N is the Atkin–Lehner involution; by $S_{k+1/2}\left(\Gamma_0(4N),\psi_1\right)$, the space of cusp forms of weight $k+1/2$, for the group $\Gamma_0(4N)$ with character ψ_1 in the sense of Shimura, where $\psi_1 = \left(\frac{4\psi(-1)}{\cdot}\right)\cdot\psi$; by $S^+_{k+1/2}\left(\Gamma_0(4N),\psi_1\right)$, the subspace of $S_{k+1/2}\left(\Gamma_0(4N),\psi_1\right)$, defined by Kohnen, consisting of forms f whose n-th Fourier coefficient vanishes if $\psi(-1)(-1)^k n$ is not a square modulo 4; by $J^{cusp}_{k,m}(N,\psi)$, the space of Jacobi cusp forms of weight k, index m, level N with character ψ. The n-th

Fourier coefficient of a modular form f is denoted as $a(f;n)$ and the $(\frac{r^2-D}{4m}, r)$-th Fourier coefficient of a Jacobi form ϕ is denoted as $a(\phi; D, r)$.

The Hecke operators in $S_{k-1/2}(\Gamma_0(4N), \psi_1)$ (resp. $S^+_{k-1/2}(\Gamma_0(4N), \psi_1)$, N odd) are denoted by $T(p^2)$, $p \nmid 2N$, and $U(p^2), p|2N$ (resp. $T(p^2)$, $p \nmid 2N$, $T^+(4)$ and $U(p^2), p|N$). The Hecke operators in $J^{cusp}_{k,m}(N, \psi)$ are denoted by $T_J(p), p \nmid mN$ and $U_J(p), p|N$. Note that the Hecke operator $U_J(d)$ has already been defined in [13].

We denote by $\langle \cdot, \cdot \rangle$, the Petersson inner product in the spaces of modular forms of both integral and half-integral weights and the space of Jacobi forms. Throughout the paper, p denotes a prime. For $z \in \mathbb{C}$ and $c \in \mathbb{R}$, we put $e_c(z) = e^{2\pi i z/c}$. By $d|n$, we always mean a positive divisor d of n. For a fundamental discriminant D_0, $\left(\frac{D_0}{\cdot}\right)$ denotes the generalized quadratic residue symbol. i_N denotes the index of the congruence subgroup $\Gamma_0(N)$ in $SL_2(\mathbb{Z})$. For an integer D, $D \equiv \ \square \pmod m$ means that D is a square modulo m. For general reference we refer the reader to [1, 2, 15].

3. Correspondence between Jacobi cusp forms and cusp forms of integral weight

Let χ be a primitive Dirichlet character modulo M_1, where $M_1|M$, and let $\epsilon = \chi(-1)$. In [8], W. Kohnen constructed a holomorphic kernel function of the Shimura lift and obtained the adjoint Shintani lift for the spaces $S^+_{k-1/2}(\Gamma_0(4M))$ and $S_{2k-2}(M)$ (M odd). Following this, similar lifts were obtained for the spaces $J^{cusp}_{k,m}$ and $S^-_{2k-2}(m)$ in [3] and for the spaces $J^{cusp}_{k,m}(M, \chi)$ and $S_{2k-2}\left(mM, \chi^2\right)$ (M is odd) in [14]. We also remark here that Kohnen's work [8] has been generalized for the spaces $S_{k-1/2}(\Gamma_0(4N), \psi)$ and $S_{2k-2}\left(2N, \psi^2\right)$ (with some restrictions on the even part of the conductor of ψ (modulo $4N$)) in [11]. In this section, we present a modified way to get these lifts between $J^{cusp}_{k,m}(M, \chi)$ and $S_{2k-2}\left(mM, \chi^2\right)$. The results of this section are straightforward applications of the known results [3, 8, 14], and so we omit the proofs.

We need two basic cusp forms in the respective spaces, which we describe below.

Poincaré series. For a negative discriminant D and an integer r modulo $2m$ with $D \equiv r^2 \pmod{4m}$, we have the (D, r)-th Poincaré series $P_{(D,r)}$ in $J^{cusp}_{k,m}(M, \chi)$ uniquely determined by

$$\langle \phi, P_{(D,r)} \rangle = \alpha_{k,m,D}\, a(\phi; D, r), \tag{1}$$

where

$$\alpha_{k,m,D} = \frac{m^{k-2}\Gamma(k-3/2)}{2\pi^{k-3/2}} |D|^{-k+3/2}, \tag{2}$$

and $\phi \in J^{cusp}_{k,m}(M, \chi)$.

The other cusp form which we need for our purpose is the holomorphic kernel function of the periods of cusp forms in $S_{2k-2}\left(mM, \chi^2\right)$. Let D_0 be a negative fundamental discriminant, and let $r_0 \pmod{2m}$ be an integer such that $D_0 \equiv r_0^2 \pmod{4m}$.

The holomorphic kernel function of the periods of cusp forms. This is the function $f_{k,mM,\chi,M_1^2D_0D,-r_0rM_1,D_0}\big|K \in S_{2k-2}\left(mM,\chi^2\right)$, where

$$f_{k,mM,\chi,M_1^2D_0D,-r_0rM_1,D_0}(w) = \sum_Q \chi(c)\chi_{D_0}(Q)Q(w,1)^{-k+1} \in\ S_{2k-2}\left(mM,\overline{\chi}^2\right).$$

Here the summation varies over all binary quadratic forms $Q = [a,b,c]$ such that $b^2 - 4ac = M_1^2D_0D$, $a \equiv 0 \pmod{mMM_1}$ and $b \equiv -r_0rM_1 \pmod{2m}$, $\chi_{D_0}(Q)$ is the generalized genus character defined in [3] and K is the operator defined by $f(\tau) \mapsto f\big|K(\tau) = \overline{f(-\overline{\tau})}$.

We define

$$S_{2k-2}^m(mM,\chi^2) = \left\{f \in S_{2k-2}(mM,\chi^2)\ :\ f\big|W_m = \epsilon(-1)^{k-1}\overline{\chi}(m)f\right\}. \tag{3}$$

Note that $S_{2k-2}^-(m) = S_{2k-2}^m(m)$.

Proposition 3.1. *Let $(m,M_1) = 1$. Then the function $f_{k,mM,\chi,M_1^2D_0D,-r_0rM_1,D_0}$ belongs to the space $S_{2k-2}^m(mM,\overline{\chi}^2)$.*

A characterization of the function $f_{k,mM,\chi,M_1^2D_0D,-r_0rM_1,D_0}$ is given in the following proposition.

Proposition 3.2. *Let $f \in S_{2k-2}(mM,\chi^2)$ and let*

$$r_{k,mM,\chi,M_1^2D_0D,-r_0rM_1,D_0}(f) = \sum_Q \chi(c)\chi_{D_0}(Q)\int_{C_Q} f(w)\ Q(-\overline{w},1)^{k-2}\ dw, \tag{4}$$

where the sum varies over all quadratic forms $Q = [a,b,c]$ modulo $\Gamma_0(mM)$ such that $b^2 - 4ac = M_1^2D_0D$, $b \equiv -r_0rM_1 \pmod{2m}$ and $a \equiv 0 \pmod{mMM_1}$, and where C_Q is the image in $\Gamma_0(mM)\backslash\mathcal{H}$ of the semicircle $a|w|^2+bu+c = 0$ $(w = u+iv)$ with appropriate orientation (see [3, 14] for details). Then

$$\begin{aligned} i_{mM}^{-1}\pi\binom{2k-4}{k-2}2^{-2k+4}(M_1^2D_0D)^{-k+3/2}&r_{k,mM,\chi,M_1^2D_0D,-r_0rM_1,D_0}(f)\\ &= \langle f, f_{k,mM,\chi,M_1^2D_0D,-r_0rM_1,D_0}\big|K\rangle. \end{aligned} \tag{5}$$

Corollary 3.3. *Let $D \equiv r^2 \pmod{4m}$ be a negative fundamental discriminant such that $(D,mM) = 1$. Assume that $(M_1,m) = 1$. Let $R(\chi,D)$ be the following Gauss type sum:*

$$R(\chi,D) = (DM_1)^{-1/2}\sum_{a(M_1|D|)} \chi(a)\left(\tfrac{D_0}{a}\right)e_{M_1|D|}(a).$$

Then, for a normalized Hecke eigenform f belonging to the space $S_{2k-2}^{m,new}(mM,\chi^2)$ $\left(= S_{2k-2}^m(mM,\chi^2) \cap S_{2k-2}^{new}(mM,\chi^2)\right)$, we have

$$\begin{aligned} &r_{k,mM,\chi,M_1^2D^2,-r^2M_1,D}(f)\\ &\qquad = \frac{-i^k(M_1|D|)^{k-3/2}(k-2)!R(\chi,D)}{2\pi^{k-1}}L(f,\overline{\chi}\left(\tfrac{D}{\cdot}\right),k-1). \end{aligned} \tag{6}$$

The Shimura and Shintani lifts. For a negative fundamental discriminant D_0 and an integer r_0 modulo $2m$ with $D_0 \equiv r_0^2 \pmod{4m}$, we define the Shimura map $\mathcal{S}_{D_0,r_0}$on $J_{k,m}^{cusp}(M,\chi)$ as follows:

$$\phi\big|\mathcal{S}_{D_0,r_0}(w) = \sum_{n\geq 1}\left(\sum_{\substack{d|n\\(d,M_2)=1}} \chi(d)\ d^{k-2}\ a\left(\phi;\frac{n^2}{d^2}D_0,\frac{n}{d}r_0\right)\right)\ e(nw), \tag{7}$$

where $M_2 = M/M_1$.

To prove the mapping property of the Shimura lift $\mathcal{S}_{D_0,r_0}$and to obtain its adjoint Shintani lift $\mathcal{S}^\star_{D_0,r_0}$, we express the image of the Poincaré series in terms of the kernel function of the periods of cusp forms of integral weight in the following main theorem of this section.

Theorem 3.4.

(8)

$$\begin{aligned}P_{(D,r)}\big|\mathcal{S}_{D_0,r_0}(w) = {}& \frac{(-i)^{k-2}2^{k-3}(M_1|D_0|)^{k-3/2}\Gamma(k-3/2)R(\chi,D_0)}{\pi^{k-1/2}\binom{2k-4}{k-2}}\\ &\cdot\sum_{t|M_2}\mu(t)\chi(t)\left(\tfrac{D_0}{t}\right)t^{k-2}f_{k,mM/t,\chi,M_1^2D_0D,-r_0rM_1,D_0}\big|K(tw).\end{aligned}$$

Note. Since all Poincaré series $P_{(D,r)}$, $0 > D \equiv r^2 \pmod{4m}$, $r \pmod{2m}$ generate $J_{k,m}^{cusp}(M,\chi)$, it follows from Theorem 3.4 that $\mathcal{S}_{D_0,r_0}$maps $J_{k,m}^{cusp}(M,\chi)$ to $S_{2k-2}(mM,\chi^2)$. One can directly verify that the map $\mathcal{S}_{D_0,r_0}$commutes with the action of the Hecke operators.

For a negative fundamental discriminant D_0 and an integer r_0 modulo $2m$ with $D_0 \equiv r_0^2 \pmod{4m}$, we define the Shintani lift $\mathcal{S}^\star_{D_0,r_0}$, the adjoint of $\mathcal{S}_{D_0,r_0}$with respect to the Petersson scalar product, as follows.

For $f \in S_{2k-2}(mM,\chi^2)$,

(9)

$$f\big|\mathcal{S}^\star_{D_0,r_0}(\tau,z) = \sum_{\substack{0>D,r\in\mathbb{Z}\\ D\equiv r^2 \pmod{4m}}} \alpha_{k,m,D}^{-1}\langle f, P_{(D,r)}\big|\mathcal{S}_{D_0,r_0}\rangle e\left(\frac{r^2-D}{4m}\tau + rz\right),$$

where $\alpha_{k,m,D}$ is the constant defined in (2). In other words, we have

$$\langle f\big|\mathcal{S}^\star_{D_0,r_0}, P_{(D,r)}\rangle = \langle f, P_{(D,r)}\big|\mathcal{S}_{D_0,r_0}\rangle.$$

From the properties of $\mathcal{S}_{D_0,r_0}$, we conclude that $\mathcal{S}^\star_{D_0,r_0}$maps $S_{2k-2}\left(mM,\chi^2\right)$ to $J_{k,m}^{cusp}(M,\chi)$, and it is the adjoint of $\mathcal{S}_{D_0,r_0}$with respect to the Petersson inner product. Further, it can be easily verified that this commutes with the action of the Hecke operators. Note that the definition of $\mathcal{S}^\star_{D_0,r_0}$is made using (5) and Theorem 3.4. Using (5), we have the explicit Fourier coefficients of $f\big|\ \mathcal{S}^\star_{D_0,r_0}$in the following (see [3, 14])

(10)

$$\begin{aligned}a\left(f\big|\mathcal{S}^\star_{D_0,r_0};D,r\right) = {}&(-i/2m)^{k-2}\,i_{mM}^{-1}M_1^{-k+3/2}\overline{R(\chi,D_0)}\\ &\times\left(\sum_{t|M_2}\mu(t)\overline{\chi}(t)\left(\tfrac{D_0}{t}\right)t^{-k+1}r_{k,mMt,\chi,M_1^2D_0Dt^2,-r_0rM_1,D_0}(f)\right).\end{aligned}$$

We summarize the above discussion in the following theorem.

Theorem 3.5. *$\mathcal{S}_{D_0,r_0}$ maps $J_{k,m}^{cusp}(M,\chi)$ to $S_{2k-2}\left(mM,\chi^2\right)$, and $\mathcal{S}_{D_0,r_0}^{\star}$ is the adjoint of $\mathcal{S}_{D_0,r_0}$ with respect to the Petersson scalar product. Further, the mappings commute with the action of Hecke operators.*

Remark 3.1. When $M=1$, the correspondence given in Theorem 3.5 is actually between the spaces $J_{k,m}^{cusp}$ and $S_{2k-2}^{-}(m)$.

Remark 3.2. In view of Proposition 3.1, whenever $(m,M_1)=1$, the Shimura map $\mathcal{S}_{D_0,r_0}$ maps $J_{k,m}^{cusp}(M,\chi)$ to $S_{2k-2}^{m}(mM,\chi^2)$.

4. A generalization of Eichler–Zagier map

In this section, we define a map $\mathcal{Z}_m$ on $J_{k,m}^{cusp}(M,\chi)$, which is a generalization of the Eichler–Zagier map (see Remark 4.2 below). Using this map as a main tool (Theorem 4.3), we develop, in the subsequent sections, the theory of newforms in the space of Jacobi forms from the already known theory of newforms in the space of cusp forms of half–integral weight, and vice versa. The proofs of the results presented in this section are straightforward calculations, so we omit them.

Definition. Let $\chi_1=\left(\frac{4\epsilon}{\cdot}\right)\chi$. We define the subspaces $S_{k-1/2}^{m}(\Gamma_0(4mM),\chi_1)$ and $S_{k-1/2}^{+,m}(\Gamma_0(4mM),\chi_1)$ as follows:

(11)
$$\begin{aligned}&S_{k-1/2}^{m}(\Gamma_0(4mM),\chi_1)\\ &=\left\{f\in S_{k-1/2}(\Gamma_0(4mM),\chi_1): a(f;n)=0 \ \text{ if } \ \epsilon(-1)^{k-1}n\not\equiv \ \square \pmod m\right\}\end{aligned}$$

and, when mM is odd,

(12)
$$\begin{aligned}S_{k-1/2}^{+,m}(\Gamma_0(4mM),\chi_1) &= S_{k-1/2}^{+}(\Gamma_0(4mM),\chi_1)\cap S_{k-1/2}^{m}(\Gamma_0(4mM),\chi_1)\\ &=\left\{f\in S_{k-1/2}^{+}(\Gamma_0(4mM),\chi_1): a(f;n)=0 \ \text{ if } \ \epsilon(-1)^{k-1}n\not\equiv \ \square \pmod{4m}\right\}.\end{aligned}$$

We also define the operator $\mathcal{A}_4$ as

$$\mathcal{A}_4:\sum_{n\geq 1}a(n)e(n\tau)\longrightarrow \sum_{\substack{n\geq 1\\ \epsilon(-1)^{k-1}n\equiv \ \square \pmod 4}}a(n)e(n\tau). \tag{13}$$

In fact, the operator $\mathcal{A}_4$ is given explicitly in the following (the action is with respect to the usual stroke operation):

$$\mathcal{A}_4=\left[\frac{1}{\alpha}(\xi+\xi')+\frac{1}{2}\right],$$

where

$$\xi=\left(\begin{pmatrix}4&1\\0&4\end{pmatrix},\epsilon^{1/2}e^{i\pi/4}\right),\qquad \xi'=\left(\begin{pmatrix}4&-1\\0&4\end{pmatrix},\epsilon^{1/2}e^{-i\pi/4}\right)$$

and $\alpha=(-1)^{\left[\frac{k-1)}{2}\right]}\epsilon 2\sqrt{2}$. (See [7].)

Let us denote by $\mathcal{P}_m$ the subspace in $S_{k-1/2}(\Gamma_0(4mM),\chi_1)$ (if $2|mM$) or in $S_{k-1/2}^{+}(\Gamma_0(4mM),\chi_1)$ (if $2\nmid mM$) generated by all Poincaré series $P_{|D|}$ (if $2|mM$) or $P_{|D|}^{+}$ (if $2\nmid mM$), where D varies over all discriminants such that $\epsilon(-1)^{k-1}D>0$, $(D,m)=1$, and $D\equiv \ \square \pmod{4m}$. Then we have the following proposition.

Proposition 4.1. *$\mathcal{A}_4$ maps $S_{k-1/2}(\Gamma_0(4mM),\chi_1)$ into $S_{k-1/2}(\Gamma_0(16mM),\chi_1)$, and it is injective on $\mathcal{P}_m$. In other words, the Poincaré series $P_{|D|}$, $D \equiv r^2 \pmod{4m}$, is uniquely determined by its n-th Fourier coefficients whenever $\epsilon(-1)^{k-1}n \equiv \square \pmod 4$.*

We need the following remark at the end of this section for the extension of the map $\mathbf{Z}_m$.

Remark 4.1. Note that for a positive integer d, $P_{|D|}|B(d^2)$ is also determined uniquely by its n-th Fourier coefficients, where $\epsilon(-1)^{k-1}n \equiv \square \pmod 4$; here $B(d)$ ($d \in \mathbb{N}$) is the operator defined on formal Fourier series as follows:

$$B(d) : \sum_{n\geq 1} a(n)e(nz) \longrightarrow \sum_{n\geq 1} a(n)e(ndz).$$

The map $\mathcal{Z}_m$. Define the map $\mathcal{Z}_m$ on $J^{cusp}_{k,m}(M,\chi)$ as follows.

Let $(M_1, m) = 1$. Then for $\phi \in J^{cusp}_{k,m}(M,\chi)$,

$$\phi\big|\mathcal{Z}_m(\tau) = \sum_{D<0}\left(\sum_{\substack{r \pmod{2m} \\ D\equiv r^2 \pmod{4m}}} a(\phi; D, r)\right) e(|D|\tau). \tag{14}$$

Remark 4.2. When $mM = 1$, $\mathcal{Z}_1$ is nothing but the map defined by Eichler and Zagier in [2], which is a canonical map from $J^{cusp}_{k,1}$ onto $S^+_{k-1/2}(\Gamma_0(4))$ (in fact, it acts on noncusp forms also). When $m = 1$ and $2 \nmid M$, $\mathcal{Z}_1$ is the map defined in [13] in connection with the Saito–Kurokawa descent. The present map $\mathcal{Z}_m$, which generalizes the previous maps, is introduced to study the correspondence between $J^{cusp}_{k,m}(M,\chi)$ and $S_{k-1/2}(\Gamma_0(4mM),\chi_1)$, to the extent possible, in general. N.-P. Skoruppa [17] also obtained a similar generalization.

Lemma 4.2. *Let k be even, let χ be an even character and let $f \in \mathcal{P}_m$. If $2 \nmid mM$, then $a(f; |D'|) = 0$ whenever $D' \not\equiv \square \pmod{4m}$; if $2|mM$, then $a(f; |D'|) = 0$ whenever $D' \not\equiv \square \pmod{4m}$ and $D' \equiv \square \pmod 4$.*

In view of the assumptions of the above lemma, from now on we assume that k is even and $\epsilon = 1$.

Denote by $\mathcal{P}_J(m)$ the subspace of $J^{cusp}_{k,m}(M,\chi)$ generated by all Poincaré series $P_{(D,r)}$, where D varies over all negative discriminants such that $(D,m) = 1$ and $D \equiv r^2 \pmod{4m}$.

The main theorem of this section is the following.

Theorem 4.3. *The linear map $\mathcal{Z}_m$ maps $\mathcal{P}_J(m)$ onto $\mathcal{P}_m \big|\mathcal{A}_4$.*

In view of Proposition 4.1, we redefine the map $\mathcal{Z}_m$ on $\mathcal{P}_J(m)$ as follows:

$$\mathbf{Z}_m = \begin{cases} \mathcal{Z}_m & \text{if } 2 \nmid mM, \\ \mathcal{Z}_m\ \mathcal{A}_4^{-1} & \text{if } 2|mM. \end{cases} \tag{15}$$

Using Theorem 4.3, we have established the following.

Theorem 4.4. *$\mathbf{Z}_m$ maps $\mathcal{P}_J(m)$ onto $\mathcal{P}_m$.*

We now state a generalized form of [18, Lemma 3.1], whose proof uses the same techniques.

Proposition 4.5. *Let $0 \neq \phi \in J_{k,m}^{cusp}(M,\chi)$ be such that $a(\phi; D, r) = 0$ for all $0 > D \equiv r^2 \pmod{4m}$ with $(D,m) = 1$. Then*

$$\phi \in \sum_{\substack{d>1 \\ d^2|m}} J_{k,m/d^2}^{cusp}(M,\chi)\big|u_d,$$

where u_d is the operator which maps the function $\phi(\tau,z) \in J_{k,m}(M,\chi)$ to the function $\phi(\tau, dz) \in J_{k,md^2}(M,\chi)$.

The following proposition is a consequence of Proposition 4.5.

Proposition 4.6.

$$J_{k,m}^{cusp}(M,\chi) = \mathcal{P}_J(m) \oplus \sum_{\substack{d>1 \\ d^2|m}} J_{k,m/d^2}^{cusp}(M,\chi)\big|u_d, \tag{16}$$

$$J_{k,m}^{cusp}(M,\chi) = \mathcal{P}_J(m) \qquad (m \text{ square-free}). \tag{17}$$

Lemma 4.7. *Let $\phi \in J_{k,m}^{cusp}(M,\chi)$ such that $\phi \notin \mathcal{P}_J(m)$. Then*

$$\phi\big|u_d\big|\mathcal{Z}_m = \phi\big|\mathcal{Z}_{m/d^2}\big|B(d^2). \tag{18}$$

Note that the mapping property of $\mathbf{Z}_m$ is obtained only on $\mathcal{P}_J(m)$. We extend the mapping property to $J_{k,m}^{cusp}(M,\chi)$ by using i) induction on the square part of m, ii) the commutative property (18), and iii) Remark 4.1.

Definition. Define the space $\mathbf{S}_{k-1/2}^m(mM,\chi)$ as follows:

$$\mathbf{S}_{k-1/2}^m(mM,\chi) = \begin{cases} S_{k-1/2}^m\left(\Gamma_0(4mM),\chi\right) & \text{if } 2|mM, \\ S_{k-1/2}^{+,m}\left(\Gamma_0(4mM),\chi\right) & \text{if } 2\nmid mM. \end{cases} \tag{19}$$

We summarize the above results in the following theorem.

Theorem 4.8. *The linear map $\mathbf{Z}_m$ maps $J_{k,m}^{cusp}(M,\chi)$ to $\mathbf{S}_{k-1/2}^m(mM,\chi)$ and commutes with the action of Hecke operators.*

5. Newforms and Waldspurger's result

Throughout this section we assume that $(m, M_1) = 1$.

5.1. **Newforms.** Let $S_{2k-2}^{new}(mM,\chi^2)$ be the space of newforms in $S_{2k-2}\left(mM,\chi^2\right)$. Let $f_1, f_2, \cdots, f_\nu$ denote the orthogonal basis of normalized Hecke eigenforms in $S_{2k-2}^{new}(mM,\chi^2)$. Define for each i, $1 \leq i \leq \nu$, the following eigensubspaces:

(20)
$$J_{k,m}^{cusp,new}(M,\chi;f_i) = \left\{\phi \in J_{k,m}^{cusp}(M,\chi) \ : \ \phi\big|T_J(p) = a(f_i;p)\phi, p\nmid mM\right\},$$

(21)
$$\mathbf{S}_{k-1/2}^{m,new}(mM,\chi;f_i) = \left\{g \in \mathbf{S}_{k-1/2}^m(mM,\chi) \ : \ g\big|T(p^2) = a(f_i;p)g, p\nmid mM\right\},$$

We define the space of newforms in $J_{k,m}^{cusp}(M,\chi)$ and in $\mathbf{S}_{k-1/2}^m(mM,\chi)$ as follows:

$$\begin{aligned} J_{k,m}^{cusp,new}(M,\chi) &= \bigoplus_{i=1}^{\nu} J_{k,m}^{cusp,new}(M,\chi;f_i), \\ \mathbf{S}_{k-1/2}^{m,new}(mM,\chi) &= \mathcal{P}_m \cap \bigoplus_{i=1}^{\nu} \mathbf{S}_{k-1/2}^{m,new}(mM,\chi;f_i). \end{aligned} \tag{22}$$

As usual we define the space of oldforms to be the orthogonal complement (with respect to Petersson product) of the space of newforms in the respective spaces. (Refer to equations (37), (41) and (52).)

Proposition 5.1. *$\mathbf{Z}_m$ maps $J_{k,m}^{cusp,new}(M,\chi)$ onto $\mathbf{S}_{k-1/2}^{m,new}(mM,\chi)$ and preserves the corresponding eigenspaces.*

Proof. Let $\phi \in J_{k,m}^{cusp,new}(M,\chi)$ be a Hecke eigenform. Using the commutative relation

$$T_J(p)\mathbf{Z}_m = \mathbf{Z}_m T(p^2) \qquad p \nmid mM, \tag{23}$$

we have $\phi\big|\mathbf{Z}_m \in \mathbf{S}_{k-1/2}^{m}(mM,\chi)$, and it is a Hecke eigenform having the same eigenvalues as that of ϕ. So, by definition, $\phi\big|\mathbf{Z}_m \in \mathbf{S}_{k-1/2}^{m,new}(mM,\chi)$. □

For a Jacobi cusp form (resp. an integral or half-integral weight cusp form) ϕ, we write $\phi = \phi^{new} \oplus \phi^{old}$, where ϕ^{new} (resp. ϕ^{old}) belongs to the space of newforms (resp. oldforms).

Theorem 5.2. *$\mathbf{Z}_m$ is injective on $J_{k,m}^{cusp,new}(M,\chi)$. In particular, $P_{(D,r)}^{new} = P_{(D,r')}^{new}$ whenever $r \equiv r' \pmod{2m}$.*

Proof. Let $\phi \in J_{k,m}^{cusp,new}(M,\chi)$ be in the kernel V of $\mathbf{Z}_m$. Suppose that $\phi \neq 0$. In view of (23), we assume that ϕ is a Hecke eigenform. Then there exists a fundamental discriminant $D < 0$, $(D,mM) = 1$ and $D \equiv r^2 \pmod{4m}$, such that $a(\phi;D,r) \neq 0$. This implies that $P_{(D,r)}^{new,V} \neq 0$. (Here, we have put $P_{(D,r)}^{new} = P_{(D,r)}^{new,V} \oplus P_{(D,r)}^{new,V^\perp}$, where $J_{k,m}^{cusp,new}(M,\chi) = V \oplus V^\perp$.) Let $V_k \in S_{2k-2}^{new}(mM,\chi^2)$ be the image of V under all Shimura lifts $\mathcal{S}_{D',r'}$. Now our assumption shows that

$$P_{(D,r)}^{new,V}\big|\mathcal{S}_{D,r} \neq 0.$$

This together with Theorem 3.4 imply that

$$f_{k,mM,\chi,M_1^2D^2,-r^2M_1,D}^{new,V_k} \neq 0,$$

which in turn implies that

$$f_{k,mM,\chi,M_1^2D^2,D}^{new,V_k} \neq 0.$$

(Refer to [11] for the definition of the function.) This implies that $P_{|D|}^{new,V_{k-1/2}} \neq 0$, where $P_{|D|}$ is the $|D|$-th Poincaré series which belongs to $\mathcal{P}_m$ and $V_{k-1/2}$ is the image of V under $\mathbf{Z}_m$. On the other hand, using the relation (whose proof is direct)

$$\mathcal{S}_{D,r}^\star\ \mathbf{Z}_m = \mathcal{S}_D^\star \qquad \text{when restricted onto newforms,}$$

we get

$$0 = P_{(D,r)}^{new,V}\big|\mathbf{Z}_m = \left(P_1^{new,V_k}\big|\mathcal{S}_{D,r}^*\right)\big|\mathbf{Z}_m = P_1^{new,V_k}\big|\mathcal{S}_D^* = P_{|D|}^{new,V_{k-1/2}}. \tag{24}$$

This contradiction proves our claim. □

As a consequence of the above theorem, we have the following.

Corollary 5.3. *Let $\phi \in J_{k,m}^{cusp,new}(M,\chi)$ be a non-zero form. Then the Fourier coefficient $a(\phi;D,r)$ of ϕ depends only on the discriminant D and not on r modulo $2m$, where $D \equiv r^2 \pmod{4m}$.*

Remark 5.1. If $\phi \in J_{k,m}^{cusp,new}(M,\chi)$, then in view of the above corollary, its Fourier coefficient is denoted by $a(\phi; D)$ instead of $a(\phi; D, r)$.

We now state the refinement of $\mathbf{Z}_m$ on newforms as follows.

Theorem 5.4. *The map $\mathbf{Z}_m$, defined by*

(25)
$$\sum_{\substack{0>D,r\in\mathbb{Z}\\ D\equiv r^2 \pmod{4m}}} a\left(\phi; D, r\right) e\left(\frac{r^2-D}{4m}\tau + rz\right) \mapsto \sum_{\substack{0>D\in\mathbb{Z}\\ D\equiv \square \pmod{4m}}} a\left(\phi; D\right) e\left(|D|\tau\right),$$

is a canonical isomorphism between $J_{k,m}^{cusp,new}(M,\chi)$ and $\mathbf{S}_{k-1/2}^{m,new}(mM,\chi)$, which commutes with the action of Hecke operators. It also preserves the Hilbert space structure.

Corollary 5.5. *The strong "multiplicity 1" theorem is valid on $\mathbf{S}_{k-1/2}^{m,new}(mM,\chi)$, where M is square-free and χ is real.*

Proof. If $M=1$, then the result follows using the above theorem and the Skoruppa-Zagier theorem [18, Theorem 5]. Let $M>1$ be square-free. For a form f which belongs to $\mathbf{S}_{k-1/2}^{m,new}(mM,\chi)$ and for a prime p, $p^2|mM$, we have

$$f\big|R_p(\tau) := \sum_{n\geq 1}\left(\tfrac{\epsilon(-1)^{k-1}n}{p}\right) a(f;n)e(n\tau) = f, \tag{26}$$

so that any Hecke eigenform f in this space is an eigenform under the twisting operator R_p with eigenvalue 1. Hence, by the results of Ueda [20, 21], we have the required strong multiplicity one theorem. $\square$

Corollary 5.6. *If M is square-free and χ is real, then we have the strong "multiplicity 1" theorem on $J_{k,m}^{cusp,new}(M,\chi)$.*

Proof. Use Corollary 5.5 in the above theorem. $\square$

Remark 5.2. In the above two cases, the decomposition of oldforms compatible via $\mathbf{Z}_m$ will be discussed in sections 5.3 and 5.4 with some restrictions.

5.2. **Waldspurger's Result.** Let f be one of the basis elements (which belongs to $S_{2k-2}^{new}\left(mM,\chi^2\right)$) described in section 5.1 such that the corresponding eigenspace $J_{k,m}^{cusp,new}(M,\chi;f)$ is nonempty. Let $0>D$ be a fundamental discriminant, and let $r \pmod{2m}$ be such that $D\equiv r^2 \pmod{4m}$. Then $\mathcal{S}_{D,r}$ maps $J_{k,m}^{cusp,new}(M,\chi;f)$ onto $\mathbb{C}f$. If $\{\phi_1,\phi_2,\cdots,\phi_s\}$ constitutes an orthogonal basis of Hecke eigenforms of $J_{k,m}^{cusp,new}(M,\chi;f)$, we have

$$\phi_i\big|\mathcal{S}_{D,r} = a\left(\phi_i; D\right)\ f, \qquad 1\leq i\leq s. \tag{27}$$

Consider

$$\overline{a\left(\phi_i; D\right)}\phi_i\big|\mathcal{S}_{D,r} = \left|a\left(\phi_i; D\right)\right|^2 f.$$

i.e.,

$$\alpha_{k,m,D}^{-1}P_{(D,r)}\big|\mathcal{S}_{D,r} = \sum_{i=1}^{s}\frac{\left|a\left(\phi_i; D\right)\right|^2}{\langle\phi_i,\phi_i\rangle} f \qquad \text{on } \mathbb{C}f. \tag{28}$$

Using Theorem 3.4 and the equations (5) and (6) in (28), we get the following.

Theorem 5.7 (Waldspurger result). *Let f and the ϕ_i's be as above. Then, for a negative fundamental discriminant $D \equiv \square \pmod{4m}$ and $(D, mM) = 1$, we have*

$$\sum_{i=1}^{s} \frac{|a(\phi_i; D)|^2}{\langle \phi_i, \phi_i \rangle} = \frac{(k-2)!|D|^{k-3/2}}{i_{mM} 2^{2k-3} \pi^{k-1} m^{k-2}} \frac{L(f, \overline{\chi}(\frac{D}{\cdot}), k-1)}{\langle f, f \rangle}. \tag{29}$$

Remark 5.3. The above equation (29) is also valid for skew-holomorphic Jacobi newforms (refer to [10] for the simplest case).

5.3. **Decomposition of Jacobi forms.** In this section, using the theory of newforms developed for the space $S^{+}_{k-1/2}(\Gamma_0(4mM))$ when mM is odd and square-free, we obtain the same for the space $J^{cusp,new}_{k,m}(M)$ via the map $\mathbf{Z}_m$. We also decompose the space of oldforms $J^{cusp,old}_{k,m}(M)$, which is compatible with the corresponding decomposition of oldforms in $S^{+,m}_{k-1/2}(\Gamma_0(4mM))$.

First we recall the W-operators defined in $S^{+}_{k-1/2}(\Gamma_0(4mM))$. For $p|mM$, let

$$W(p) = \left(\begin{pmatrix} pa & b \\ 4mMc & p \end{pmatrix}, \ p^{-1/2}(4mMcz + p)^{1/2} \right),$$

where $a, b, c \in \mathbb{Z}$ are such that $b \equiv 1 \pmod{p}$ and $p^2 a - 4mMbc = p$, be the W-operator defined on $S^{+}_{k-1/2}(\Gamma_0(4mM))$. Then the analogous Atkin-Lehner W-operator w_p is defined as follows:

$$w_p = p^{-k/2+3/4} U(p) W(p).$$

Proposition 5.8. *Let $p|mM$ and let $f \in S^{+}_{k-1/2}(\Gamma_0(4mM))$ be an eigenform under w_p. Then*

$$f\big|w_p = \left(\tfrac{-n_0}{p}\right) f,$$

where $n_0 > 0$ is an integer such that $p \nmid n_0$ and $a(f; n_0) \neq 0$.

Proof. The following can be easily verified (see [7]):

$$\begin{aligned} f\big|w_p(\tau) &= \sum_{n \geq 1} \left(\tfrac{(-1)^{k-1} n}{p}\right) a(f; n) e(n\tau) \\ &\quad + \left(\tfrac{-4}{p}\right)^{-k+1/2} p^{-1/2} f\big|W(p) \left(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, p^{-1/4} \right)(\tau). \end{aligned} \tag{30}$$

Now $f\big|w_p = \lambda_p f$ is equivalent to

(31)

$$p^{-k+5/4} f\big|U(p) \left(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, p^{-1/4} \right)(\tau) = \left(\tfrac{-4}{p}\right)^{-k+1/2} f\big|W(p) \left(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, p^{-1/4} \right)(\tau),$$

where we have used the fact that

$$W(p)^2 = \left(\tfrac{-4}{p}\right)^{k-1/2} \quad \text{on } S^{+}_{k-1/2}\left(\Gamma_0(4mM), \left(\tfrac{\cdot}{p}\right)\right).$$

Using this in (30), we see that $f\big|w_p = \lambda_p f$ is equivalent to

$$\lambda_p \sum_{\substack{n \geq 1 \\ p \nmid n}} a(f; n) e(n\tau) = \sum_{n \geq 1} \left(\tfrac{(-1)^{k-1} n}{p}\right) a(f; n) e(n\tau). \tag{32}$$

Since there exists an integer $n_0 > 0$ with $p \nmid n_0$ such that $a(f; n_0) \neq 0$, we get the required result from the above equation. $\square$

A characterization of $S^{+,m}_{k-1/2}(\Gamma_0(4mM))$ is given in the following.

Lemma 5.9. i) *For $p|m$, the W-operator w_p acts as the identity operator on $S^{+,m}_{k-1/2}(\Gamma_0(4mM))$.*

ii) *For $D \equiv r^2 \pmod{4m}$, $D < 0$ and $(D,m) = 1$, the Poincaré series $P^+_{|D|}$ in $S^+_{k-1/2}(\Gamma_0(4mM))$ belongs to $S^{+,m}_{k-1/2}(\Gamma_0(4mM))$.*

Proof. Let p divide m and let f belong to $S^{+,m}_{k-1/2}(\Gamma_0(4mM))$. By the definition of the space $S^{+,m}_{k-1/2}(\Gamma_0(4mM))$, we have $a(f;n) = 0$ whenever $-n \not\equiv \square \pmod{4m}$. i.e., $a(f;n) \neq 0$ whenever $-n \equiv \square \pmod{4p}$ for all $p|m$. Now i) follows from Proposition 5.8. The proof of ii) is contained in the proof of Theorem 4.3. $\square$

Let $S^{+,new}_{k-1/2}(\Gamma_0(4mM))$ be the subspace generated by newforms as carried out by W. Kohnen [7]. Define

$$S^{+,m,new}_{k-1/2}(\Gamma_0(4mM)) = S^{+,m}_{k-1/2}(\Gamma_0(4mM)) \cap S^{+,new}_{k-1/2}(\Gamma_0(4mM)). \tag{33}$$

The following lemma follows from a similar result obtained in $S^{+,new}_{k-1/2}(\Gamma_0(4mM))$.

Lemma 5.10. i) *The space $S^{+,m,new}_{k-1/2}(\Gamma_0(4mM))$ is generated by all Poincaré series $P^+_{|D|}$, where D varies over all negative discriminants such that $D \equiv r^2 \pmod{4m}$ and $(D,m) = 1$.*

ii) *The space $S^{+,m,new}_{k-1/2}(\Gamma_0(4mM))$ has a basis of eigenforms with respect to the Hecke operators $T(p^2)$, $p \nmid mM$, or $U(p^2)$, $p|mM$. Further, these are eigenforms with respect to the W-operators w_p, $p|mM$.*

In order to study the theory of newforms in $J^{cusp}_{k,m}(M)$, we state the following modified decomposition of $S^{+,m}_{k-1/2}(\Gamma_0(4mM))$, which differs (with respect to the old class decomposition) from the decomposition obtained by Kohnen. Before doing that we need the following operators. Set

$$\begin{aligned} L_1 &= I \text{ (Identity)},\\ L_p &= w_p \text{ for } p|m,\\ L_{st} &= L_s L_t \text{ for } s,t|m \text{ with } (s,t) = 1. \end{aligned} \tag{34}$$

Put

$$\begin{aligned} B_\ell &= \sum_{t|\ell} L_t,\\ B_{\ell,d} &= B_\ell U(d^2) \text{ for } \ell|m, d|M. \end{aligned} \tag{35}$$

Using induction on the number of prime factors of m, we have the following.

Theorem 5.11.
(36)
$$S^{+,m}_{k-1/2}(\Gamma_0(4mM)) = S^{+,m,new}_{k-1/2}(\Gamma_0(4mM)) \oplus S^{+,m,old}_{k-1/2}(\Gamma_0(4mM)),$$

where

$$S^{+,m,old}_{k-1/2}(\Gamma_0(4mM)) = \bigoplus_{\substack{\ell|m;r,d|M\\ (\ell,r)\neq(1,M)}} S^{+,m/\ell,new}_{k-1/2}(\Gamma_0(4mr/\ell))\big|B_{\ell,d}. \tag{37}$$

Now we define the corresponding operators in the case of Jacobi forms as follows. The W-operator on $J_{k,m}^{cusp}(M)$ is defined by

$$w_p = \begin{cases} \frac{1}{p} \sum_{\lambda,\mu(p)} \left(I, \left[\frac{\lambda}{p}, \frac{\mu}{p}\right]\right) & \text{if } p|m, \\ \frac{1}{p^2} \sum_{\lambda,\mu,v(p)} \left(\begin{pmatrix} a+cvM/p & v+b/p \\ Mc & p \end{pmatrix}, [\lambda,\mu]\right) & \text{if } p|M, \end{cases} \tag{38}$$

where in the above $a, b, c \in \mathbb{Z}$ are such that $pa - bcM/p = 1$.

The operator $\mathcal{V}_\ell$ from $J_{k,m}^{cusp}(M)$ into $J_{k,m\ell}^{cusp}(M)$, is defined by

(39)

$$\phi\big|\mathcal{V}_\ell(\tau,z) = \sum_{\substack{0>D,r\in\mathbb{Z} \\ D\equiv r^2 \pmod{4m\ell}}} \left(\sum_{d|(\frac{r^2-D}{4m\ell},r,\ell)} d^{k-1} a\left(\phi; \frac{D}{d^2}, \frac{r}{d}\right)\right) e\left(\frac{r^2-D}{4m\ell}\tau + rz\right).$$

A direct verification shows that for $\ell|m$ and $d|M$

$$\mathcal{V}_\ell \; U_J(d) = U_J(d) \; \mathcal{V}_\ell \quad \text{on } J_{k,m}^{cusp}(M). \tag{40}$$

We now define the space of oldforms as

$$J_{k,m}^{cusp,old}(M) = \sum_{\substack{\ell|m; rd|M \\ (\ell,r)\neq(1,M)}} J_{k,m/\ell}^{cusp}(r)\big|\mathcal{V}_\ell \; U_J(d) \tag{41}$$

and let $J_{k,m}^{cusp,new}(M)$ be the orthogonal complement of $J_{k,m}^{cusp,old}(M)$ in $J_{k,m}^{cusp}(M)$ with respect to the Petersson scalar product.

Lemma 5.12. *Let $\phi \in J_{k,m}^{cusp}(M)$. Then, for $p|m$ and also for $p|M$, the operators w_p preserve $J_{k,m}^{cusp}(M)$. For $p|M$, w_p does not depend on the representatives a, b, c. Further, these operators are Hermitian involutions and preserve $J_{k,m}^{cusp,new}(M)$. Finally, the action of w_p for $p|m$ is given by*

$$\phi|w_p(\tau,z) = \sum a\left(\phi; D, r\right) e\left(\frac{{r'}^2 - D'}{4m}\tau + r'z\right), \tag{42}$$

where the sum runs over all $D' < 0$ such that $D \equiv r^2 \pmod{4m}$, $D = D' \equiv {r'}^2 \pmod{4m}$, $D' = {r'}^2 - 4n'm$ and $r, r' \in \mathbb{Z}$ are such that $r' \equiv r \pmod{2m/p}$ and $r' \equiv -r \pmod{2p}$.

Proof. We prove only that $w_p^2 = I$ for $p|M$. The rest of the lemma follows from standard arguments.

Let

$$Tr_{M/p}^{M} : J_{k,m}^{cusp}(M) \longrightarrow J_{k,m}^{cusp}(M/p) \tag{43}$$

be the trace operator adjoint to the inclusion map under the Petersson scalar product. Then we have

$$Tr_{M/p}^{M} - I = \sum_{h(p)} \left(\begin{pmatrix} 1 & h \\ M/p & hM/p+1 \end{pmatrix}, [0,0]\right), \tag{44}$$

where $I = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), [0,0]\right)$. Let $\phi \in J_{k,m}^{cusp}(M/p)$. Then

$$\phi\big|\left(Tr_{M/p}^{M} - I\right) = p\;\phi. \tag{45}$$

Also, for $\psi \in J_{k,m}^{cusp}(M)$,

$$p^{k-1}\psi\big|\left(w_p Tr_{M/p}^M - w_p\right) = p^{k-1}\psi\big|w_p\big|\left(Tr_{M/p}^M - I\right) = p\ \psi\big|U_J(p). \tag{46}$$

Consider

$$\begin{aligned} p^{k-1}\phi\big|w_p^2\big|\left(Tr_{M/p}^M - I\right) &= p\ \phi\big|w_p U_J(p) \quad \text{(using (46))} \\ &= p^k\phi. \end{aligned} \tag{47}$$

The last step follows from the Fourier expansion of w_p on $J_{k,m}^{cusp}(M/p)$ (see Lemma 5.14 below), which is given by

(48)

$$\phi\big|w_p = \sum_{\substack{0>D,r\in\mathbb{Z} \\ D\equiv r^2 \pmod{4m}}} \left(\left(\tfrac{D}{p}\right)a\left(\phi; D, r\right) + p^{k-1}a\left(\phi; \frac{D}{p^2}, \frac{r}{p}\right)\right) e\left(\frac{r^2-D}{4m}\tau + rz\right).$$

Thus,

$$p^{k-1}\left(\phi\big|w_p^2 - \phi\right)\big|\left(Tr_{M/p}^M - I\right) = 0, \tag{49}$$

which implies that

$$\phi\big|w_p^2 = \phi \quad \text{on } J_{k,m}^{cusp}(M/p).$$

Thus, using the definition of $J_{k,m}^{cusp,old}(M)$, we have established that w_p^2 acts as the identity on $J_{k,m}^{cusp,old}(M)$. This completes the proof. $\square$

Proposition 5.13. *Let p divide m. Then w_p is the identity on $J_{k,m}^{cusp,new}(M)$.*

Proof. This follows from Corollary 5.3 and (42). $\square$

We now state a lemma whose proof is simple, and hence we omit the details. Note that when $m = 1$, this has already been stated in [13, Proposition 6 (i)].

Lemma 5.14. *Let $p|M$. Then, we have the following:*

$$\begin{aligned} &U_J(p) + p^{k-1}w_p : J_{k,m}^{cusp}(M) \longrightarrow J_{k,m}^{cusp}(M/p), \\ &U_J(p) + p^{k-1}w_p = \begin{cases} T_J(p) & \text{on } J_{k,m}^{cusp}(M/p), \\ 0 & \text{on } J_{k,m}^{cusp,new}(M). \end{cases} \end{aligned} \tag{50}$$

Note that from the definition of $J_{k,m}^{cusp,old}(M)$, it is clear that $T_J(p)$, $p \nmid mM$, preserves $J_{k,m}^{cusp,old}(M)$ and hence preserves $J_{k,m}^{cusp,new}(M)$. Summarizing the results, we have the following theorem.

Theorem 5.15. *The space $J_{k,m}^{cusp,new}(M)$ has an orthogonal basis of eigenforms with respect to all the Hecke operators $T_J(p)$, $p \nmid mM$, $U_J(p)$, $p|M$, and the W-operators w_p, $p|mM$.*

Definition. We call a Jacobi form $\phi \in J_{k,m}^{cusp,new}(M)$ a *Jacobi newform*, or for short, *newform*, if ϕ is one of the basis elements of Theorem 5.15.

Using the commutative property

$$\mathcal{V}_\ell U_J(d)\mathbf{Z}_m = \mathbf{Z}_{m/\ell}B_{\ell,d} \quad (\ell|m, d|M) \tag{51}$$

on $J_{k,m/\ell}^{cusp}(r)$ with $rd|M$ and $(\ell, r) \neq (1, M)$, together with Theorem 5.4, we get the following theorem.

Theorem 5.16. *The strong "multiplicity 1" theorem is valid on $J_{k,m}^{cusp,new}(M)$. We have the decomposition*

$$J_{k,m}^{cusp}(M) = J_{k,m}^{cusp,new}(M) \bigoplus_{\substack{\ell|m; rd|M \\ (\ell,r)\neq(1,M)}} J_{k,m/\ell}^{cusp,new}(r)\big|\mathcal{V}_\ell\ U_J(d). \tag{52}$$

Remark 5.4. When mM is even and square-free, we have a similar decomposition as in Theorem 5.16. The only changes here are:

i) Replace $S_{k-1/2}^{+,m}(\Gamma_0(4mM))$ by $S_{k-1/2}^{m}(\Gamma_0(4mM))$.

ii) If $2|m$, the operator L_2 defined in (34) is to be replaced by

$$f\big|L_2(\tau) = \sum_{n\geq 1}\left(\left(\tfrac{(-1)^{k-1}n}{2}\right)a(f;n) + 2^{k-1}a(f;n/4)\right)e(n\tau). \tag{53}$$

Remark 5.5. A natural question seems to be the following.

$$\textbf{Is it true that} \quad L_2 = W(4) \textbf{ on } S_{k-1/2}(\Gamma_0(2mM)) \quad (2|mM)? \tag{54}$$

Here $W(4)$ is the W-operator defined in [12].

Remark 5.6. If the answer to the above question is in the affirmative, then the eigenvalues of $W(4)$ are explicitly given by

$$\left(\tfrac{(-1)^{k-1}n_0}{2}\right).$$

More precisely, if f belongs to $S_{k-1/2}(\Gamma_0(2mM))$ $(2|mM)$, and is a Hecke eigenform and an eigenform with respect to $W(4)$, then

$$f\big|W(4) = \left(\tfrac{(-1)^{k-1}n_0}{2}\right)f, \tag{55}$$

where n_0 is the order of f at $i\infty$. This fact is needed to derive the explicit Waldspurger result in [11, Theorem 4].

5.4. **Decomposition of $\mathbf{S}_{k-1/2}^m(m)$.** Let us first consider the following decomposition given by Skoruppa and Zagier in [18].

$$J_{k,m}^{cusp} = J_{k,m}^{cusp,new} \bigoplus_{\substack{\ell d^2|m \\ \ell d^2>1}} J_{k,m/\ell d^2}^{cusp,new}\big|u_d\mathcal{V}_\ell. \tag{56}$$

Our aim is to carry out a similar decomposition in $\mathbf{S}_{k-1/2}^m(m)$, which is compatible via the map $\mathbf{Z}_m$. For this we need certain operators on $\mathbf{S}_{k-1/2}^m(m)$. For $p|m$, define the operator L_p on functions $f \in \mathbf{S}_{k-1/2}^m(m)$ as follows:

(57)
$$L_p : \sum_{n\geq 1} a(f;n)e(n\tau) \mapsto \sum_{\substack{n\geq 1 \\ -n\equiv \square \pmod{4m}}} \left(a(f;n) + p^{k-1}a(f;n/p^2)\right)e(n\tau).$$

For a general square-free divisor t of m, we extend the L operator by

$$L_t = \prod_{p|t} L_p. \tag{58}$$

Put

$$B_\ell = \sum_{t|\ell} L_t \qquad (\ell \text{ is a square-free divisor of } m). \tag{59}$$

Finally, we define

$$B_{\ell,d} = B_\ell B(d^2). \tag{60}$$

The following commutative relations are true on the space of Jacobi newforms:

$$\begin{aligned} u_d \mathbf{Z}_m &= \mathbf{Z}_{m/d^2} B(d^2), \\ \mathcal{V}_\ell \mathbf{Z}_m &= \mathbf{Z}_{m\ell} L_\ell. \end{aligned} \tag{61}$$

Now using the above operators and (56), we have

Theorem 5.17.

$$\mathbf{S}^m_{k-1/2}(m) = \mathbf{S}^{m,new}_{k-1/2}(m) \oplus \mathbf{S}^{m,old}_{k-1/2}(m), \tag{62}$$

where

$$\mathbf{S}^{m,old}_{k-1/2}(m) = \bigoplus_{\substack{\ell d^2 | m \\ \ell d^2 \neq 1}} \mathbf{S}^{m/\ell d^2, new}_{k-1/2}(m/\ell d^2) \big| B_{\ell,d}. \tag{63}$$

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References

1. A. O. L. Atkin and J. Lehner, *Hecke operators on* $\Gamma_0(m)$, Math. Ann. **185** (1970), 134–160. MR **42:**3022
2. M. Eichler and D. Zagier, *The theory of Jacobi forms*, Progress in Mathematics **55**, Birkhaüser, Boston, 1985. MR **86j:**11043
3. B. Gross, W. Kohnen and D. Zagier, *Heegner points and derivatives of L−series II*, Math. Ann. **278** (1987), 497–562. MR **89i:**11069
4. H. Iwaniec, *Fourier coefficients of modular forms of half-integral weight*, Invent. Math. **87** (1987), 385–401. MR **88b:**11024
5. C. G. J. Jacobi, *Fundamenta nova theories functionum ellipticarum*, Königesberg, 1829.
6. W. Kohnen and D. Zagier, *Values of L-series of modular forms at the center of the critical strip*, Invent. Math. **64** (1981), 175–198. MR **83b:**11029
7. W. Kohnen, *Newforms of half-integral weight*, J. Reine Angew Math. **333** (1981), 32–72. MR **84b:**10038
8. W. Kohnen, *Fourier coefficients of modular forms of half-integral weight*, Math. Ann. **271** (1985), 237–268. MR **86i:**11018
9. W. -W. Li, *Newforms and functional equations*, Math. Ann. **212** (1975), 285–315. MR **51:**5498
10. M. Manickam, *Newforms of half-integral weight and some problems on modular forms*, Ph. D. Thesis, Univ. of Madras, 1989.
11. M. Manickam, B. Ramakrishnan and T. C. Vasudevan, *On Shintani correspondence*, Proc. Indian Acad. Sci. (Math. Sci.) **99** (1989), 235–247. MR **90m:**11061
12. M. Manickam, B. Ramakrishnan and T. C. Vasudevan, *On the theory of newforms of half-integral weight*, J. Number Theory **34** (1990), 210–224. MR **91b:**11060
13. M. Manickam, B. Ramakrishnan and T. C. Vasudevan, *On Saito–Kurokawa descent for congruence subgroups*, Manuscripta Math. **81** (1993), 161–182. MR **94j:**11048
14. B. Ramakrishnan, *On Shimura and Shintani correspondences for modular forms and Jacobi forms and some results on Hecke operators*, Ph.D. Thesis, Univ. of Madras 1989.
15. G. Shimura, *On modular forms of half–integral weight*, Ann. Math. **97** (1973), 440–481. MR **48:**10989
16. T. Shintani, *On construction of holomorphic cusp forms of half–integral weight*, Nagoya Math. J. **58** (1975), 83–126. MR **52:**10603

17. N.-P. Skoruppa, *Über den Zussammenhang zwischen Jacobiformen und Modulformen halbganzen Gewichtz*, Bonner Math. Schriften **159** (1995). MR **87a:**11045
18. N.-P. Skoruppa and D. Zagier, *Jacobi forms and a certain space of modular forms*, Invent. Math. **94** (1988), 113–146. MR **89k:**11029
19. M. Ueda, *The decomposition of the spaces of cusp forms of half-integral weight and the trace formula of Hecke operators*, J. Math. Kyoto U. **28** (1988), 505–555. MR **90a:**11054
20. M. Ueda, *The trace formula of the twisting operator on the spaces of cusp forms of half-integral weight and some trace relations*, Japanese J. Math. **17** (1991), 83–135. MR **92g:**11045
21. M. Ueda, *On twisting operators and newforms of half-integral weight*, Nagoya Math. J. **131** (1993), 135–205. MR **95a:**11040
22. J. -L. Waldspurger, *Correspondance de Shimura*, J. Math. Pures Appl. (9) **59** (1980), 1–132. MR **83f:**10029
23. J. -L. Waldspurger, *Sur les coefficients de Fourier des formes modulaires de poids demi-entier*, J. Math. Pures Appl. (9) **60** (1981), 375–484. MR **83h:**10061

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