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UNIVERSAL FORMULAE FOR SU(n) CASSON INVARIANTS OF KNOTS

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Abstract. An SU(n) Casson invariant of a knot is an integer which can be thought of as an algebraic-topological count of the number of characters of SU(n) representations of the knot group which take a longitude into a given conjugacy class. For fibered knots, these invariants can be characterized as Lefschetz numbers which, for generic conjugacy classes, can be computed using a recursive algorithm of Atiyah and Bott, as adapted by Frohman. Using a new idea to solve the Atiyah-Bott recursion (as simplified by Zagier), we derive universal formulae which explicitly compute the invariants for all n. Our technique is based on our discovery that the generating functions associated to the relevant Lefschetz numbers (and polynomials) satisfy certain integral equations.

Introduction

A knot in a closed oriented 3-manifold is said to fibered if its complement fibers over the circle. For a fibered knot K and $\alpha \in \mathrm{SU}(n)$, the SU($n$) Casson invariant of K, denoted by $\lambda_{n,\alpha}(K)$, is an integer which can be thought of as an algebraic-topological count of the number of characters of SU(n) representations of the knot group which take a longitude into the conjugacy class of α. (For a more detailed description of these invariants, see [8, 2] in case K is fibered and [9, 10] in the general case.) For generic $\alpha \in \mathrm{SU}(n)$, including all generators of the center of SU(n), there exist homogeneous polynomials $p_{n,\alpha}(x_0, x_2, \dots, x_{2n-2})$ (depending only on the conjugacy class of α) of degree $n-1$ such that, for any fibered knot K,

$$\lambda_{n,\alpha}(K) = p_{n,\alpha}\left(\Delta_K(1), \Delta_K^{(2)}(1), \dots, \Delta_K^{(2n-2)}(1)\right),$$

where $\Delta_K^{(2j)}(1)$ is the $2j$-th derivative at $t=1$ of $\Delta_K(t)$, the (balanced) Alexander polynomial of K. In this paper, we present an explicit calculation of $p_{n,\omega}$ for all n, where ω is $e^{2\pi i/n}$ times the $n \times n$ identity matrix. For generic $\alpha \in \mathrm{SU}(n)$, $p_{n,\alpha}$ is determined from $p_{n,\omega}$ via the "wall-crossing" formulae of [2]. Thus computing $p_{n,\omega}$ is an important step in the general calculation of $\lambda_{n,\alpha}(K)$.

Given any knot K, there is a unique polynomial called its *Conway polynomial* and denoted by $\nabla_K(z)$, such that $\nabla_K(t^{1/2} - t^{-1/2}) = \Delta_K(t)$. Since the Conway

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and Alexander polynomials carry equivalent information, there exist polynomials $q_{n,\alpha}(y_0, y_2, \dots, y_{2n-2})$ such that

$$\lambda_{n,\alpha}(K) = q_{n,\alpha}(C_0, C_2, \dots, C_{2n-2})$$

for any fibered knot K with Conway polynomial $\nabla_K(z) = \sum_{i\geq 0} C_{2i} z^{2i}$. (It is well-known that for knots, $\nabla_K(z)$ is a polynomial in z^2.) The formula for $q_{n,\alpha}$ is independent of K, so these polynomials give universal formulae for the invariants $\lambda_{n,\alpha}$.

For $\alpha = \omega$ and $n \leq 5$, a remarkable cancellation occurs, revealing that $q_{n,\omega}$ is not only homogeneous but also *weighted homogeneous* of weighted degree $2n-2$, where y_{2i} has weighted degree $2i$. In Conjecture 1.9 we assert that this is true for all n provided $\alpha = \omega$. Consequently, in computations of $q_{n,\omega}$ one can drop all terms of lower order, which has the effect of making general computations of the universal formulae possible.

For $\alpha \in \mathrm{SU}(n)$, let m_α be the Euler characteristic of its conjugacy class. (Note that m_α is a positive integer for all $\alpha \in \mathrm{SU}(n)$.) The wall-crossing formulae of [2] imply that for generic $\alpha, \beta \in \mathrm{SU}(n)$, $m_\alpha\, q_{n,\alpha} - m_\beta\, q_{n,\beta}$ has weighted degree strictly less than $2n-2$. Thus the weighted homogeneous part of $\frac{1}{m_\alpha} q_{n,\alpha}$ of highest weighted degree, denoted by $\nu_n(y_0, y_2, \dots, y_{2n-2})$, is independent of (generic) $\alpha \in \mathrm{SU}(n)$. Even in the absence of Conjecture 1.9, our results give a complete computation of those terms in the universal polynomials which are invariant under wall-crossing.

Using Zagier's summation formula [15] for solving the Atiyah-Bott recursion [1], we are able to express each coefficient of ν_n as an explicit sum of rational numbers over the set of all compositions of n (i.e., ordered partitions of n). Since there are 2^{n-1} compositions of n, direct evaluation of these sums becomes impractical, even for relatively modest values of n. To overcome this difficulty, a new method for evaluating the sums is required.

Our technique is based on our discovery that certain generating functions $\Phi(s,t)$ associated to these sums satisfy integral equations of the form

$$\Phi(s,t) + \int_0^1 \frac{\gamma(xs)}{x} \Phi(tx,t)dx \;=\; f(s,t),$$

where $\gamma(s) = \sum_{n=1}^{\infty} s^n/b_n$ for $b_n = 4^n n!(n-1)!$ and $f(s,t)$ is a given formal power series in s and t.

Indeed, assembling the solutions to the Atiyah-Bott recursion into a generating function $\widetilde{\Phi}(s,t)$, we prove that $\widetilde{\Phi}(s,t)$ satisfies the contour integral equation

$$\widetilde{\Phi}(s,t) + \oint \rho(sy)\widetilde{\Phi}(ty,t)\eta(x,y)dy \;=\; \rho(s),$$

where $\rho(s)$ and $\eta(x,y)$ are certain given power series and the contour integral is taken over the unit circle in $\mathbb{C}$.

The contour integral equation yields a new, highly efficient method for computing the Lefschetz polynomials of certain self-maps of $\mathcal{M}_{n,1}$, the moduli space of rank n degree 1 bundles over a Riemann surface. These maps are induced by orientation preserving homeomorphisms of the surface with an open disk deleted. In particular, this applies to the identity map of $\mathcal{M}_{n,1}$ and thus provides a rapid method to generate the Poincaré polynomials of $\mathcal{M}_{n,1}$.

Applying these techniques to integral equations of the first type, we are able to determine ν_n for moderate values of n. Appendix B includes a table of ν_n for

$2 \leq n \leq 10$ (the computation is due to Casson for $n = 2$, [5], and to Frohman for $n = 3$, [8]).

Stronger results are obtained whenever one can solve the integral equations in closed form, which we have done in a number of instances. For example, the simplest terms appearing in ν_n are the two monomials $A_n y_0^{n-2} y_{2n-2}$ and $B_n y_2^{n-1}$, where

$$(\dagger) \qquad A_n = 4\sum_{k=1}^{n}(-1)^{k+1} \sum_{n_1+\cdots+n_k=n} \frac{\sum_{i=1}^{k}\sum_{j=1}^{n_i}(2j-1)^{2n-2}}{\prod_{i=1}^{k} b_{n_i} \prod_{j=1}^{k-1}(n_j+n_{j+1})},$$

$$(\ddagger) \qquad B_n = 4\sum_{k=1}^{n}(-1)^{k+1} \sum_{n_1+\cdots+n_k=n} \frac{\prod_{i=1}^{k}\prod_{j=1}^{n_i}(2j-1)^{2}}{\prod_{i=1}^{k} b_{n_i} \prod_{j=1}^{k-1}(n_j+n_{j+1})}.$$

In the above equations, $b_{n_j} = 4^{n_j} n_j!(n_j-1)!$ and the interior sums are over all compositions of n into k parts. Solving the relevant integral equations allows us to evaluate these sums for all n, and we prove that $A_n = \binom{2n-2}{n-1}\frac{1}{n}$, the Catalan number, and that $B_n = 1$ (see Theorems 2.18 and 2.5).

More generally, we express each coefficient of ν_n as an explicit linear combination of sums of the form

$$(*) \qquad 4\sum_{k=1}^{n}(-1)^{k+1} \sum_{n_1+\cdots+n_k=n} \frac{\prod_{\ell=1}^{d}\left[\sum_{i=1}^{k}\sum_{j=1}^{n_i}(2j-1)^{2\lambda_\ell}\right]}{\prod_{i=1}^{k} b_{n_i} \prod_{j=1}^{k-1}(n_j+n_{j+1})},$$

where λ_ℓ is a positive integer for $\ell = 1, \ldots, d$ such that $\lambda_1 + \cdots + \lambda_d < n$. Although these sums are apparently quite complicated, their evaluation is achieved by a remarkably simple formula, and we conjecture that the expression in $(*)$ equals 0 for $\lambda_1 + \cdots + \lambda_d < n-1$ and equals $\binom{2\lambda_1}{\lambda_1}\cdots\binom{2\lambda_d}{\lambda_d} n^{d-2}$ for $\lambda_1 + \cdots + \lambda_d = n-1$ (see Conjecture 1.16 and, for a simpler formulation, Conjecture 2.19).

Two corollaries of Theorems 2.18 and 2.5 are stated in section 3. The first strengthens an earlier result of Frohman (Theorem 1.7 in [8]) on the existence of irreducible SU(n) representations of fibered knot groups. The second presents a simple formula for $\lambda_{n,\omega}(K)$ in terms of Casson's SU(2) invariant $\lambda_{2,\omega}(K)$ for knots K in 3-manifolds N with first Betti number $b_1(N) > 0$. These results do not depend on the previously stated conjectures.

This paper is organized into three sections and two appendices. In the first section, we describe our universal formulae and show how their computation can be reduced to the evaluation of sums of type $(*)$. In the second section, we present our integral equation technique for analyzing such sums and also explain how contour integral equations yield a new algorithm for computing the relevant Lefschetz polynomials. In the third section, we present two corollaries. Proofs of certain combinatorial identities used in §2 are given in Appendix A. The polynomials ν_n for $2 \leq n \leq 10$ are tabulated in Appendix B.

1. Universal Formulae

This section presents universal formulae determining the SU(n) Casson invariants $\lambda_{n,\alpha}(K)$ in terms of the Alexander (or Conway) polynomial of K for all fibered knots K. After recalling the relevant definitions in §1.1 and the earlier results of Frohman concerning the existence of polynomials $p_{n,\alpha}$ in §1.2, we provide new direct computations of p_{n,ω^d} for n, d relatively prime and $n \leq 5$. Next in §1.3 we describe

a change of variables leading to new polynomials $q_{n,\alpha}$, which, though equivalent to $p_{n,\alpha}$, are given by much simpler formulae, at least for $\alpha = \omega$ (cf. Conjecture 1.9).

In §§1.4 & 1.5, we reduce the computation of $q_{n,\omega}$ to the evaluation of the sums $(*)$ from the introduction. This involves computing the weighted homogeneous part of p_{n,ω^d} (which is independent of d provided $(n,d)=1$), and deducing the weighted homogeneous part ν_n of q_{n,ω^d}. The last part of §1.5 describes the coefficients of ν_n as linear combinations of the sums $(*)$, and §2 presents general methods to evaluate such sums. For the purposes of §1, Conjecture 1.16 provides a complete solution determining ν_n for arbitrary n. Conjecture 1.9 then asserts that $q_{n,\omega} = \nu_n$, allowing one to recover $p_{n,\omega}$ from just its weighted homogeneous part. Finally, we mention that one could employ the wall-crossing formulae of [2] to determine $p_{n,\alpha}$ for all generic $\alpha \in \mathrm{SU}(n)$, yielding universal formulae for $\lambda_{n,\alpha}(K)$ for all fibered knots K.

1.1. **Basic definitions.** In this subsection we present a precise definition of the $\mathrm{SU}(n)$ Casson invariants for fibered knots K in closed 3-manifolds N. Before doing that, we introduce the notation for fibered knots and the classical knot invariants given by the Alexander and Conway polynomials.

Suppose K is a knot in a closed oriented 3-manifold N.

Definition 1.1. A knot $K \subset N$ is said to be *fibered* if there is an open tubular neighborhood, $\tau(K)$, of K such that $N \setminus \tau(K)$ is homeomorphic to the mapping torus of an orientation preserving homeomorphism $\varphi : F \longrightarrow F$, where F is a compact connected oriented surface with one boundary component, i.e.,

$$N \setminus \tau(K) \cong F \times [0,1]/(x,0) \sim (\varphi(x),1).$$

The mapping torus structure of $N \setminus \tau(K)$ is essentially unique; if $N \setminus \tau(K)$ is also homeomorphic to the mapping torus of an orientation preserving homeomorphism $\varphi : F' \longrightarrow F'$, then there is an orientation preserving homeomorphism $h : F \longrightarrow F'$ and an isotopy between φ and $h^{-1} \circ \varphi' \circ h$ (for details, see chapter 5 of [4]). We refer to φ as the *monodromy map* of the fibered knot K.

Definition 1.2. (i) The balanced *Alexander polynomial* of a fibered knot $K \subset N$ is given by $\Delta_K(t) \equiv t^{-g} \det(\mathrm{id} - t\varphi_*)$, where $\varphi : F \longrightarrow F$ is the monodromy map of K, g is the genus of F, and $\varphi_* : H_1(F;\mathbb{Z}) \longrightarrow H_1(F;\mathbb{Z})$ is the induced map in homology.

(ii) The Conway polynomial of the fibered knot $K \subset N$ is the unique polynomial $\nabla_K(z)$ such that $\nabla_K(t^{1/2} - t^{-1/2}) = \Delta_K(t)$.

For $N = S^3$, both (i) and (ii) coincide with the common definitions which are usually given in terms of the Alexander module for (i) and in terms of skein theory for (ii). In any case, $\nabla_K(z)$ is in fact a polynomial in z^2 for knots K.

Now, we are almost ready to define the $\mathrm{SU}(n)$ Casson knot invariants for fibered knots. These are given as Lefschetz numbers on representation varieties. We review Lefschetz polynomials and Lefschetz numbers first, and then describe the relevant representation varieties.

Definition 1.3. Given a space Y with the homotopy type of a finite CW complex and a map $f : Y \longrightarrow Y$, the *Lefschetz polynomial* of the pair (Y,f) is defined by

$$\sum_{j\geq 0} (-1)^j \,\mathrm{trace}\,(H_j(f) : H_j(Y;\mathbb{Q}) \longrightarrow H_j(Y;\mathbb{Q}))\, t^j,$$

and the *Lefschetz number* of (Y, f) is the integer obtained by evaluating this polynomial at $t = 1$.

Let F be a compact connected oriented surface of genus g with one boundary component. Choose a basepoint $b \in \partial F$. The fundamental group $\pi_1(F, b)$ is a free group on $2g$ generators. Let $\delta \in \pi_1(F, b)$ be the element determined by ∂F and its orientation. For $\alpha \in \mathrm{SU}(n)$, define $\widetilde{R}_{n,\alpha}$ to be the set of homomorphisms $\rho : \pi_1(F, b) \longrightarrow \mathrm{SU}(n)$ such that $\rho(\delta)$ is conjugate to α. The set $\widetilde{R}_{n,\alpha}$ can be viewed as a real algebraic subset of $\mathrm{SU}(n)^{2g}$, and thus acquires a topology. Define $R_{n,\alpha}$ to be the quotient of $\widetilde{R}_{n,\alpha}$ by the conjugation action of $\mathrm{SU}(n)$. An orientation preserving homeomorphism $h : (F, \partial F) \longrightarrow (F, \partial F)$ and a choice of a path from the basepoint b to $h(b)$ determines an automorphism $h_\# : \pi_1(F, b) \longrightarrow \pi_1(F, b)$. Precomposition with $h_\#$ induces a map $\widetilde{h}^* : \widetilde{R}_{n,\alpha} \longrightarrow \widetilde{R}_{n,\alpha}$, which in turn induces a map $h^* : R_{n,\alpha} \longrightarrow R_{n,\alpha}$, and h^* depends only the homotopy class of h as a map of pairs. In particular, this process defines an action of the *mapping class group*, $\pi_0(\mathrm{Homeo}^+(F))$, on $R_{n,\alpha}$, where $\mathrm{Homeo}^+(F)$ is the topological group of orientation preserving self-homeomorphisms of F.

Definition 1.4. Suppose that $K \subset N$ is a fibered knot with surface F and monodromy map $\varphi : F \longrightarrow F$.

(i) Denote by $L_{n,\alpha}(t; K)$ the Lefschetz polynomial of the pair $(R_{n,\alpha}, \varphi^*)$.
(ii) Define the $\mathrm{SU}(n)$ *Casson invariant* of K by setting $\lambda_{n,\alpha}(K) \equiv L_{n,\alpha}(1; K)$, i.e., $\lambda_{n,\alpha}(K)$ equals the Lefschetz number of the pair $(R_{n,\alpha}, \varphi^*)$.

The Lefschetz number of a map can be thought of as an algebraic-topological count of the number of fixed points of the map; here, fixed points of φ^* correspond to characters of $\mathrm{SU}(n)$ representations of the knot group of K which take a longitude into the conjugacy class of α.

The following is a restatement of Proposition 1.2 of [8].

Proposition 1.5. *If $H_1(N; \mathbb{Z})$ is finite, then $\Delta_K(1) = |H_1(N; \mathbb{Z})|$. Otherwise, $\Delta_K(1) = 0$.*

Consequently, we shall be primarily interested in the case when K is a knot in a rational homology sphere N.

1.2. **Universal polynomials.** This subsection gives a brief description of some results of Frohman and serves as the starting point of our inquiry. Suppose $K \subset N$ is a fibered knot of genus g in a rational homology 3-sphere N. For positive integers d, n satisfying $d < n$ and $(d, n) = 1$, Theorem 3.14 of [8] establishes the existence of homogeneous polynomials $p_{n,\omega^d}(x_0, x_2, \ldots, x_{2n-2})$ of degree $n - 1$ such that $\lambda_{n,\omega^d}(K)$ is obtained by replacing x_{2j} in p_{n,ω^d} by $\Delta_K^{(2j)}(1)$, the value of the $2j$-th derivative of the balanced Alexander polynomial of K at $t = 1$. (Recall that $\omega = e^{2\pi i/n}$ times the $n \times n$ identity matrix. In [8], p_{n,ω^d} is denoted by $p(n, d)$.)

Frohman showed in [8] that the Lefschetz polynomials $L_{n,\omega^d}(K; t)$ satisfy a modified version of the Atiyah-Bott recursion [1]. He also gave an algorithm, based on solving this recursion, for determining the polynomials p_{n,ω^d} (see equation (2.13) in §2.2). By explicitly solving the recursion for $n = 2$ and $n = 3$, he computed $p_{2,\omega}$ and $p_{3,\omega}$.

Zagier subsequently found a general method for inverting the Atiyah-Bott recursion and derived a summation formula for its solution (Theorem 2 of [15], see also

Theorem 2.2 in §2.2). Adapting Zagier's result to Frohman's modification of the Atiyah-Bott recursion yields the following formula for the Lefschetz polynomials $L_{n,\omega^d}(t;K)$.

Before presenting this formula, we need to introduce some notation. For $n>0$, set $c(t)=t^g\Delta_K(t)$, the unbalanced Alexander polynomial, and

$$Q_n=\frac{t^{n^2(1-g)}c(t)c(t^3)\cdots c(t^{2n-1})}{(1-t^2)^2\cdots(1-t^{2n-2})^2(1-t^{2n})}. \tag{1.1}$$

For any $a\in\mathbb{R}$, let $\langle a\rangle=[a]-a+1$ be the unique $b\in(0,1]$ such that $a+b\in\mathbb{Z}$. Define also

$$M(n_1,\dots,n_k;s)=\sum_{i=1}^{k-1}(n_i+n_{i+1})\langle(n_1+\cdots+n_i)s\rangle. \tag{1.2}$$

The following proposition is proved by combining Proposition 2.1 of [8] with Theorem 2 of [15].

Proposition 1.6. *If K is a fibered knot of genus g and $(n,d)=1$, then the Lefschetz polynomial $L_{n,\omega^d}(t;K)$ equals*

$$\frac{t^{n^2(g-1)}(1-t^2)}{c(t)}\sum_{k=1}^{n}\sum_{n_1+\cdots+n_k=n}\frac{(-1)^{k-1}t^{2M(n_1,\dots,n_k;d/n)}Q_{n_1}\cdots Q_{n_k}}{(1-t^{2n_1+2n_2})\cdots(1-t^{2n_{k-1}+2n_k})},$$

where the interior sum is over all compositions of n into k parts.

Note that $\lambda_{n,\omega^d}(K)=L_{n,\omega^d}(1;K)$ but that the rational expressions in the summation formula for $L_{n,\omega^d}(t;K)$ have poles of order $2n-2$ at $t=1$. Thus, to determine the polynomials p_{n,ω^d}, one must apply L'Hôpital's rule $2n-2$ times. Table 1 summarizes the known results. The cases $n=2$ and $n=3$ are due to Casson and Frohman, respectively; the other cases are new. Since $p_{n,\omega^d}=p_{n,\omega^{n-d}}$ (because of 'duality'), these are complete results for $n\le 5$.

TABLE 1. The polynomials p_{n,ω^d} for $2\le n\le 5$

$p_{2,\omega}$	$\frac{1}{2}x_2$
$p_{3,\omega}$	$\frac{1}{12}x_0x_4-x_0x_2+\frac{1}{4}x_2^2$
$p_{4,\omega}$	$\frac{1}{144}x_0^2x_6-\frac{5}{8}x_0^2x_4+5x_0^2x_2+\frac{7}{48}x_0x_2x_4-\frac{7}{4}x_0x_2^2+\frac{1}{8}x_2^3$
$p_{5,\omega}$	$\frac{1}{2880}x_0^3x_8-\frac{7}{60}x_0^3x_6+\frac{91}{12}x_0^3x_4-56x_0^3x_2+\frac{11}{576}x_0^2x_4^2+\frac{13}{720}x_0^2x_2x_6$ $-\frac{25}{12}x_0^2x_2x_4+\frac{63}{4}x_0^2x_2^2+\frac{1}{6}x_0x_2^2x_4-2x_0x_2^3+\frac{1}{16}x_2^4,$
p_{5,ω^2}	$\frac{1}{2880}x_0^3x_8-\frac{41}{360}x_0^3x_6+\frac{22}{3}x_0^3x_4-54x_0^3x_2+\frac{11}{576}x_0^2x_4^2+\frac{13}{720}x_0^2x_2x_6$ $-\frac{49}{24}x_0^2x_2x_4+\frac{61}{4}x_0^2x_2^2+\frac{1}{6}x_0x_2^2x_4-2x_0x_2^3+\frac{1}{16}x_2^4.$

1.3. **Conway coordinates.** In this subsection, we make a change of variables which corresponds to replacing the Alexander polynomial $\Delta_K(t)$ by the Conway polynomial $\nabla_K(z)$. By its very definition, $\Delta_K(t)$ determines $\nabla_K(z)$ and vice versa. This is made precise in the following result.

Proposition 1.7. *Suppose that a knot K in a closed 3-manifold N has Alexander polynomial $\Delta_K(t) = a_0 + \sum_{j=1}^{g} a_j(t^j + t^{-j})$. Then its Conway polynomial $\nabla_K(z)$ equals*

$$a_0 + \sum_{j=1}^{g} 2a_j + \sum_{k=1}^{g}\sum_{j=k}^{g} \binom{j+k-1}{j-k} \frac{ja_j z^{2k}}{k}.$$

Proof. We first solve for the polynomials P_n such that $P_n(z) = t^n + t^{-n}$, where $z = t^{1/2} - t^{-1/2}$. It is straightforward to see that these satisfy the recursion

$$\begin{aligned} P_0(z) &= 2, \qquad P_1(z) = z^2 + 2, \\ P_{n+1}(z) &= (z^2+2)P_n(z) - P_{n-1}(z), \quad n = 1, 2, \dots . \end{aligned}$$

Using the formula

$$\begin{aligned} &\frac{n+1}{k}\binom{n+k}{n-k+1} \\ &\quad = \frac{n}{k-1}\binom{n+k-2}{n-k+1} + \frac{2n}{k}\binom{n+k-1}{n-k} - \frac{n-1}{k}\binom{n+k-2}{n-k-1}, \end{aligned}$$

one can verify directly that

$$P_n(z) = 2 + \sum_{k=1}^{n} \frac{n}{k}\binom{n+k-1}{n-k} z^{2k}$$

by showing that, defined this way, P_n satisfy the recursion. Inserting this explicit formula for $P_n(z)$ into $\nabla_K(z) = \Delta_K(t) = a_0 + \sum_{j=1}^{g} a_j P_j(z)$ and interchanging the order of summation, we obtain the statement of the proposition. □

Define variables $y_0, y_2, \dots, y_{2n-2}$ so that y_{2k} represents the $2k$-th coefficient of the Conway polynomial, i.e., so that $\nabla_K^{(2k)}(0) = (2k)!y_{2k}$. Using the inequality

$$x_{2j} = \Delta_K^{(2j)}(1) = \frac{d^{2j}}{dt^{2j}} \nabla_K(t^{1/2} - t^{-1/2}),$$

together with Di Bruno's formula for the Bell polynomials (§2.8 in [14]), it follows that

$$x_{2j} = \sum_{k=0}^{\infty} \sum \frac{(2j)!(2k)!y_{2k}}{\ell_1! \cdots \ell_{2j}!} \left(\frac{z_1}{1!}\right)^{\ell_1} \cdots \left(\frac{z_{2j}}{(2j)!}\right)^{\ell_{2j}},$$

where

$$z_i \equiv \frac{d^i}{dt^i}(t^{1/2} - t^{-1/2})\bigg|_{t=1} = (-1)^{n-1}\frac{n}{2^{n-1}} \prod_{i=1}^{n-1}(2i-1),$$

and where the interior sum is over all $\ell_1, \dots, \ell_{2j} \geq 0$ such that $\ell_1 + \cdots + \ell_{2j} = k$ and $\ell_1 + 2\ell_2 + \cdots + 2j\ell_{2j} = 2j$ (as multisets, these are just partitions $\{1^{\ell_1}, \dots, (2j)^{\ell_{2j}}\}$ of $2j$ into $2k$ parts; see §5.1 for an explanation of this notation).

By making the above change of coordinates to rewrite the polynomials $p_{n,\alpha}$ in terms of $y_0, y_2, \dots, y_{2n-2}$ and calling the result $q_{n,\alpha}$, we now see that the following result is a direct consequence of Theorem 3.14 of [8], as extended in [3].

Corollary 1.8. *Suppose $\alpha \in \mathrm{SU}(n)$ is generic. There exist homogeneous polynomials $q_{n,\alpha}(y_0, y_2, \dots, y_{2n-2})$ of degree $n-1$ such that, for any fibered knot K with Conway polynomial $\nabla_K(z) = \sum_{i\geq 0} C_{2i} z^{2i}$,*

$$\lambda_{n,\alpha}(K) = q_{n,\alpha}(C_0, C_2, \dots, C_{2n-2}).$$

Table 2 is obtained by applying Proposition 1.7 to our previous computations of p_{n,ω^d}.

TABLE 2. The polynomials q_{n,ω^d} for $2 \leq n \leq 5$

$q_{2,\omega}$	y_2
$q_{3,\omega}$	$2y_0y_4 + y_2^2$
$q_{4,\omega}$	$5y_0^2y_6 + 7y_0y_2y_4 + y_2^3$
$q_{5,\omega}$	$14y_0^3y_8 + 26y_0^2y_2y_6 + 11y_0^2y_4^2 + 16y_0y_2^2y_4 + y_2^4$
q_{5,ω^2}	$14y_0^3y_8 + 26y_0^2y_2y_6 + 11y_0^2y_4^2 + 16y_0y_2^2y_4 + y_2^4$ $+2y_0^2y_2y_4 + 2y_0^3y_6$

Notice that this change of variables yields strikingly simpler formulae than before. Notably, if we assign to y_{2i} the weighted degree $2i$, then $q_{n,\omega}$ is seen to be weighted homogeneous for $2 \leq n \leq 5$. We believe this is true for all n.

Conjecture 1.9. *$q_{n,\omega}(y_0, y_2, \dots, y_{2n-2})$ is weighted homogeneous for all $n > 1$.*

It is not generally true that $q_{n,\alpha}$ is weighted homogeneous for $\alpha \in \mathrm{SU}(n)$.

In all our computations, the coefficients of q_{n,ω^d} are integers. Moreover, the coefficients of $q_{n,\omega}$ are all *positive* integers. We ask if these statements are true in general.

Question 1.10.
1. Is it true that $q_{n,\alpha} \in \mathbb{Z}[y_0, y_2, \dots, y_{2n-2}]$ for all n and all generic $\alpha \in \mathrm{SU}(n)$?
2. Are the coefficients of $q_{n,\omega}$ always positive integers?

Remark 1.11. Viewing $q_{n,\alpha}$ as a map from $\mathbb{Q}^n$ to $\mathbb{Q}$, it follows that $q_{n,\alpha}$ takes the integer lattice $\mathbb{Z}^n$ to $\mathbb{Z}$. This is easily demonstrated by interpreting $q_{n,\omega}(k_1, \dots, k_n)$ as the Lefschetz number of the fibered knot K with Conway polynomial $\nabla_K(z) = \sum_n k_n z^{2n}$.

1.4. **The weighted homogeneous part of p_{n,ω^d} and enumerative sums.** In the next subsection, we shall assume Conjecture 1.9 and derive a formula for $q_{n,\omega}$ in terms of certain sums. In this subsection, we determine a formula for the weighted homogeneous part of p_{n,ω^d} and observe that this is vastly simpler to compute than all of p_{n,ω^d}. The general idea is to apply L'Hôpital's rule $2n-2$ times to the summation formula for $\widetilde{L}_{n,\omega^d}(t;K)$, which is the balanced polynomial associated to $L_{n,\omega^d}(t;K)$ of Proposition 1.6. Exploiting the fact that we only care about the terms in p_{n,ω^d} of highest weighted degree, we are able to perform what would otherwise be an unreasonably complicated calculation.

Sample calculations of $p_{n,\omega}$ for $n = 2$ and $n = 3$ can be found in [8], and the material here assumes some familiarity with those techniques. We only explain

those features relevant to the more general calculation. It was observed in [8] that the formula for the balanced Lefschetz polynomial

$$\widetilde{L}_{n,\omega^d}(t;K) \equiv t^{-\frac{1}{2}\dim(R_{n,\omega^d})} L_{n,\omega^d}(t;K)$$

is nearly identical to the formula given in Proposition 1.6, except that it is independent of the genus g. Specifically, letting $\tilde{c}(t)$ denote the balanced Alexander polynomial of K (so $\tilde{c}(t) = t^{-g}c(t)$), it follows from (1.1) that

$$Q_n = \frac{t^{n^2}\tilde{c}(t)\tilde{c}(t^3)\cdots\tilde{c}(t^{2n-1})}{(1-t^2)^2\cdots(1-t^{2n-2})^2(1-t^{2n})}.$$

Since $\dim R_{n,\omega^d} = 2(n^2-1)(g-1)$, we see from Proposition 1.6 that

$$\widetilde{L}_{n,\omega^d}(t;K) = \frac{(1-t^2)}{\tilde{c}(t)} \sum_{k=1}^{n} \sum_{n_1+\cdots+n_k=n} \frac{(-1)^{k-1}t^{2M(n_1,\ldots,n_k;d/n)-1}Q_{n_1}\cdots Q_{n_k}}{(1-t^{2n_1+2n_2})\cdots(1-t^{2n_{k-1}+2n_k})},$$

where $M(n_1,\ldots,n_k;d/n)$ is as defined in equation (1.2).

Now for some general comments. Suppose that I is an index set and that

$$\widetilde{L}(t) = \sum_{i\in I} \frac{p_i(t)}{q_i(t)},$$

where $\widetilde{L}(t)$ and the $p_i(t), i \in I$, are Laurent polynomials which are analytic at t_0, and where $q_i(t),\ i \in I$, are polynomials with zeroes of order m at t_0. Writing $q_i(t) = (t-t_0)^m \overline{q}_i(t)$ and noting that $\overline{q}_i(t_0) \neq 0$, we observe that

$$\widetilde{L}(t_0) = \left(\frac{d^m}{dt^m}\bigg|_{t=t_0} \sum_{i\in I} p_i(t) \prod_{\substack{j\in I\\ j\neq i}} \overline{q}_j(t) \right) \Big/ m! \prod_{i\in I} \overline{q}_i(t_0). \tag{1.3}$$

We are interested in using this to evaluate $\widetilde{L}_{n,\omega^d}(t;K)$ at $t=1$, so let

$$I = \{\underline{n} = (n_1,\ldots,n_k) \mid k \geq 1,\ n_1+\cdots+n_k = n\}$$

be the set of all compositions of n, and define

$$p_{\underline{n}}(t) = (-1)^{k-1}t^{2M(\underline{n};d/n)-1}(\tilde{c}(t))^{-1} \prod_{i=1}^{k}\prod_{j=1}^{n_i} \tilde{c}(t^{2j-1}), \tag{1.4}$$

$$q_{\underline{n}}(t) = \frac{\left[\prod_{i=1}^{k}\prod_{j=1}^{n_i}(1-t^{2j})^2\right]\left[\prod_{i=1}^{k}(1-t^{2n_i+2n_{i+1}})\right]}{(1-t^2)\prod_{i=1}^{k-1}(1-t^{2n_i})}. \tag{1.5}$$

Notice that $p_{\underline{n}}(t)$ is indeed a Laurent polynomial and is analytic at $t=1$, and that $q_{\underline{n}}(t) \in \mathbb{Z}[t]$ has a zero of order $2n-2$ at $t=1$. Writing $q_{\underline{n}}(t) = (t-1)^{2n-2}\overline{q}_{\underline{n}}(t)$, it is straightforward to verify that

$$\overline{q}_{\underline{n}}(1) = 2^{2n-2}\prod_{i=1}^{k} n_i!(n_i-1)! \prod_{j=1}^{k-1}(n_j+n_{j+1}). \tag{1.6}$$

To determine $\widetilde{L}_{n,\omega^d}(1;K)$ using (1.3), we compute

$$\left(\frac{d^{2n-2}}{dt^{2n-2}}\bigg|_{t=1} \sum_{\underline{n}\in I} p_{\underline{n}}(t) \prod_{\substack{\underline{m}\in I\\ \underline{m}\neq\underline{n}}} \overline{q}_{\underline{m}}(t) \right) \Big/ (2n-2)! \prod_{\underline{n}\in I} q_{\underline{n}}(1). \tag{1.7}$$

For a fixed knot K, this allows one to determine $\lambda_{n,\omega^d}(K)$. However, as Frohman observes (see p. 137 of [8]), since the formula obtained is independent of the genus g of the knot, one actually obtains a formula for $\lambda_{n,\omega^d}(K)$ in terms of the derivatives of $\tilde{c}(t)$ at $t=1$ which is universal in the sense that it does not depend on K.

Although it is possible to compute (1.7) for certain restricted values of n, the general computation appears intractable. For example, in the table of §1.2, the results of computer calculations of p_{n,ω^d} are listed. As n increases, the complexity of this computation grows exponentially, making it impossible to directly determine p_{n,ω^d} in general.

On the other hand, we are only interested in the weighted homogeneous part of p_{n,ω^d} of highest weighted degree; thus we can ignore all terms of (1.7) involving fewer than $2n-2$ derivatives of $p_{\underline{n}}(t)$. This means that, for our purposes, the expression in (1.7) simplifies to give

$$\sum_{\underline{n}\in I}\left(\frac{d^{2n-2}}{dt^{2n-2}}\Big|_{t=1} p_{\underline{n}}(t)\right)\Big/\,(2n-2)!\;\overline{q}_{\underline{n}}(1). \tag{1.8}$$

We are not claiming that (1.8) equals $\widetilde{L}_{n,\omega^d}(1;K)$, just that (1.8) can be used to find the weighted homogeneous part of p_{n,ω^d} of highest weighted degree. The expression in (1.8) is still quite cumbersome to calculate, but notice that there is a further simplification. To make this precise, we introduce the following definition.

Definition 1.12. Given a Laurent polynomial $\tilde{c}(t)$, suppose

$$p(t)=\prod_{i=1}^{k}\tilde{c}(u_i(t))\,f_i(t),$$

where $u_i(t), f_i(t)$ for $i=1,\dots,k$ are polynomials. Then

$$\frac{d^m}{dt^m}p(t)=\sum_{\ell_1+\ldots+\ell_k\le m}\;\prod_{i=1}^{k}\tilde{c}^{(\ell_i)}(u_i(t))\,g_{i,\ell}(t), \tag{1.9}$$

where the sum is over all $\ell=(\ell_1,\dots,\ell_k)\in\mathbb{N}^k$ and where $g_{i,\ell}(t)$ are polynomials depending on $\ell_i, f_i(t), u_i(t)$, $i=1,\dots,k$, and their derivatives. For each term $\prod_{i=1}^{k}\tilde{c}^{(\ell_i)}(u_i(t))g_{i,\ell}(t)$ of (1.9), define its *$\tilde{c}$-order* to be $\sum_{i=1}^{k}\ell_i$.

Notice that for terms in (1.9) of $\tilde{c}$-order m, we have $g_{i,\ell}(t)=\binom{m}{\ell_1,\dots,\ell_k}f_i(t)$.

We now introduce the index set $B_{\underline{n}}$, defined for $\underline{n}=(n_1,\dots,n_k)$ a composition of n by

$$B_{\underline{n}}=\{(i,j)\mid 1\le i\le k,\ 1\le j\le n_i,\ (i,j)\ne(1,1)\}.$$

($B_{\underline{n}}$ is just the Ferrers board associated to $\underline{n}$ with the block at $(1,1)$ removed.) Thus, we can rewrite (1.4) as

$$p_{\underline{n}}(t)=(-1)^{k-1}t^{2M(\underline{n};d/n)-1}\prod_{(i,j)\in B_{\underline{n}}}\tilde{c}(t^{2j-1}). \tag{1.10}$$

Now use the basic derivative formula for products:

$$\frac{d^m}{dt^m}\prod_{i=1}^{k}h_i(t)=\sum\binom{m}{\ell_1,\dots,\ell_k}\prod_{i=1}^{k}h_i^{(\ell_i)}(t),$$

where the sum is over all solutions to $\ell_1+\cdots+\ell_k=m$ in nonnegative integers $\ell_1,\dots,\ell_k$. Taking $m=2n-2$, applying the above to (1.10), and noting that we can

drop any term of $\tilde{c}$-order less than $2n-2$, again because we only need the weighted homogeneous part of p_{n,ω^d} of highest weighted degree, it is apparent that

$$\begin{aligned}\tfrac{d^{2n-2}}{dt^{2n-2}}p_{\underline{n}}(t) &= (-1)^{k-1}t^{2M(\underline{n};d/n)-1}\sum_{\ell}\binom{2n-2}{(\ell_{ij})}\prod_{(i,j)\in B_{\underline{n}}}(2j-1)^{\ell_{ij}}\ \tilde{c}^{(\ell_{ij})}(t^{2j-1})\\ &\quad + (\text{terms of } \tilde{c}\text{-order less than } 2n-2),\end{aligned}\tag{1.11}$$

where the sum is over all $\ell_{ij}\in\mathbb{N}$ satisfying $\sum_{(i,j)\in B_{\underline{n}}}\ell_{ij}=2n-2$ and where $\binom{2n-2}{(\ell_{ij})}$ indicates a multinomial coefficient.

By Proposition 1.4 of [8], since $\tilde{c}(t)=\tilde{c}(t^{-1})$, it follows that any odd order derivative $\tilde{c}^{(2\ell+1)}(1)$ can be written as a linear combination of the even order derivatives $\tilde{c}(1),\tilde{c}''(1),\dots,\tilde{c}^{(2\ell)}(1)$. Keeping track of only those terms of $\tilde{c}$-order $2n-2$, we may thus ignore any term from (1.11) where ℓ_{ij} is odd for some $(i,j)\in B_{\underline{n}}$.

Evaluating (1.11) at $t=1$ and dropping these irrelevant terms, we are left with

$$\begin{aligned}\tfrac{d^{2n-2}}{dt^{2n-2}}p_{\underline{n}}(t)\Big|_{t=1} &= (-1)^{k-1}\sum_{\lambda}\binom{2n-2}{(2\lambda_{ij})}\prod_{(i,j)\in B_{\underline{n}}}(2j-1)^{2\lambda_{ij}}\ \tilde{c}^{(2\lambda_{ij})}(1)\\ &\quad + (\text{terms of } \tilde{c}\text{-order less than } 2n-2)\\ &\quad + (\text{terms of odd order derivatives of } \tilde{c}(t) \text{ at } t=1),\end{aligned}\tag{1.12}$$

where $\lambda_{ij}\in\mathbb{N}$ satisfy $\sum_{(i,j)\in B_{\underline{n}}}\lambda_{ij}=n-1$.

Finally, we obtain a formula for the weighted homogeneous part of p_{n,ω^d} by replacing $\tilde{c}^{(2e)}(1)$ by x_{2e} in (1.12), dividing by (1.6) and summing over all $\underline{n}$. Setting $b_\ell=4^\ell\ell!(\ell-1)!$, we deduce that the weighted homogeneous part of p_{n,ω^d} equals

$$4\sum_{k=1}^{n}(-1)^{k+1}\sum_{n_1+\cdots+n_k=n}\frac{\sum_\lambda\binom{2n-2}{(2\lambda_{ij})}\prod_{(i,j)\in B_{\underline{n}}}(2j-1)^{2\lambda_{ij}}x_{2\lambda_{ij}}}{(2n-2)!\prod_{i=1}^{k}b_{n_i}\prod_{j=1}^{k-1}(n_j+n_{j+1})}.\tag{1.13}$$

The monomial terms of (1.13) are indexed by partitions of $n-1$ determined by (λ_{ij}). Writing $\lambda=(\lambda_1,\dots,\lambda_d)$ for the partition obtained from (λ_{ij}) by removing all occurrences of 0 and setting $x_{2\lambda}=x_0^{2n-2d-2}x_{2\lambda_1}\cdots x_{2\lambda_d}$, we observe from (1.13) that

$$p_{n,\omega^d}=\sum_{\lambda}\frac{C_\lambda x_{2\lambda}}{i_1!\cdots i_{n-1}!}+(\text{terms of lower weighted degree}),\tag{1.14}$$

where the sum is over partitions $(\lambda_1,\dots,\lambda_d)$ of $n-1$, which we have written as $\{1^{i_1},\dots,(n-1)^{i_{n-1}}\}$ in multiset notation (see the beginning of §1.5 for an explanation of this notation), and where C_λ is given in Definition 1.13.

If d is a positive integer, denote elements $((i_1,j_1),\dots,(i_d,j_d))\in B_{\underline{n}}^d$, the d-fold Cartesian product of $B_{\underline{n}}$, by $(\underline{i},\underline{j})$, where $\underline{i}=(i_1,\dots,i_d)$ and $\underline{j}=(j_1,\dots,j_d)$. Let

$$\triangle_{\underline{n}}^d=\big\{(\underline{i},\underline{j})\in B_{\underline{n}}^d\mid (i_k,j_k)=(i_\ell,j_\ell)\text{ for some } k\neq\ell\big\}$$

be the large diagonal subset of $B_{\underline{n}}^d$ and set ${}^0B_{\underline{n}}^d=B_{\underline{n}}^d\setminus\triangle_{\underline{n}}^d$. With respect to the obvious S_d action, $\triangle_{\underline{n}}^d$ is an invariant subset.

Definition 1.13. Suppose that $\lambda=(\lambda_1,\dots,\lambda_d)$ is a partition of $n-1$. Set $b_\ell=4^\ell\ell!(\ell-1)!$ and define

$$C_\lambda=\frac{4}{(2\lambda_1)!\cdots(2\lambda_d)!}\sum_{k=1}^{n}(-1)^{k+1}\sum_{\underline{n}=(n_1,\dots,n_k)}\frac{a_\lambda(\underline{n})}{\prod_{i=1}^{k}b_{n_i}\prod_{j=1}^{k-1}(n_j+n_{j+1})},\tag{1.15}$$

where the interior sum is over all compositions $\underline{n} = (n_1, \dots, n_k)$ of n into k parts and

$$a_\lambda(\underline{n}) = \sum_{(\underline{i},\underline{j}) \in {}^0B^d_{\underline{n}}} (2j_1 - 1)^{2\lambda_1} \cdots (2j_d - 1)^{2\lambda_d}.$$

1.5. **The weighted homogeneous part of q_{n,ω^d}.** Before proceeding, we introduce some useful notation. Given a partition $\underline{n} = (n_1, \dots, n_k)$ of n, we may write this as the multiset $\{1^{\ell_1}, \dots, n^{\ell_n}\}$, where

(i) ℓ_i is the number of times i occurs in $(n_1, \dots, n_k)$,
(ii) i^ℓ means $i, \dots, i$, repeated ℓ times (in particular $i^0 = \emptyset$), and
(iii) $\ell_1 + 2\ell_2 + \cdots + n\ell_n = n$.

Suppose that $\alpha \in \mathrm{SU}(n)$ has eigenvalues $\lambda_1, \dots, \lambda_k$ of multiplicities $n_1, \dots, n_k$. It is elementary to see that the conjugacy class of α in $\mathrm{SU}(n)$ has Euler characteristic given by the multinomial coefficient $\binom{n}{n_1,\dots,n_k}$. Define $m_\alpha = \binom{n}{n_1,\dots,n_k}$. It follows from the results of [2, 3] that the weighted homogeneous part of $q_{n,\alpha}/m_\alpha$, denoted by $\nu_n(y_0, y_2, \dots, y_{2n-2})$, is independent of generic $\alpha \in \mathrm{SU}(n)$. Note that $m_{\omega^d} = 1$, and that ω^d is generic if and only if $(n, d) = 1$.

In this subsection, we derive a formula for each coefficient of $\nu_n(y_0, y_2, \dots, y_{2n-2})$, the weighted homogeneous part of q_{n,ω^d}, as a linear combination of the sums $p(\lambda_1, \dots, \lambda_d; n)$ appearing in Definition 1.15. The computation of ν_n is vastly simpler than that of $q_{n,\omega}$. By the general results of §1.3, it follows that

$$x_{2j} = (2j)!y_{2j} + (\text{terms of lower weighted degree}).$$

Thus, if $\lambda = (\lambda_1, \dots, \lambda_d)$ is a partition of $n-1$ and if we denote by $y_{2\lambda}$ the monomial $y_0^{2n-2d-2} y_{2\lambda_1} \cdots y_{2\lambda_d}$, then ν_n is obtained from (1.14) by replacing $x_{2\lambda}$ by

$$(2\lambda_1)! \cdots (2\lambda_d)! y_{2\lambda}.$$

Remark 1.14. As invariants of knots, ν_n and p_{n,ω^d} do not appear to bear any relationship to one another. In fact, considering how many terms were dropped from $\widetilde{L}_{n,\omega^d}(1; K)$ in our calculations of the weighted homogeneous part of p_{n,ω^d} (not to mention that $(2i)!y_{2i} \neq x_{2i}$), Conjecture 1.9 seems to be quite miraculous.

Writing

$$\nu_n(y_0, y_2, \dots, y_{2n-2}) = \sum_\lambda \frac{c_\lambda y_{2\lambda}}{i_1! \cdots i_{n-1}!},$$

we see from (1.15) that

$$c_\lambda = 4 \sum_{k=1}^{n} (-1)^{k+1} \sum_{\underline{n}=(n_1,\dots,n_k)} \frac{a_\lambda(\underline{n})}{\prod_{i=1}^k b_{n_i} \prod_{j=1}^{k-1} (n_j + n_{j+1})}. \tag{1.16}$$

where $a_\lambda(\underline{n})$ is given in Definition 1.13 and $b_\ell = 4^\ell \ell!(\ell - 1)!$ as before.

Examples. We relate (1.16) to the summation formulae (†) and (‡) for A_n and B_n from the introduction.

(i) Suppose $d = 1$ and $\lambda = (n - 1)$. Then for any composition $\underline{n} = (n_1, \dots, n_k)$ of n we have $\triangle^1_{\underline{n}} = \emptyset$, and we see from Definition 1.13 that

$$a_\lambda(\underline{n}) = -1 + \sum_{i=1}^{k} \sum_{j=1}^{n_i} (2j - 1)^{2n-2}.$$

Thus, using (1.16), the coefficient of $y_0^{n-2}y_{2n-2}$ in ν_n is given by

$$c_{(n-1)} = 4\sum_{k=1}^{n}(-1)^{k+1}\sum_{n_1+\cdots+n_k=n}\frac{-1+\sum_{i=1}^{k}\sum_{j=1}^{n_i}(2j-1)^{2n-2}}{\prod_{i=1}^{k}b_{n_i}\prod_{j=1}^{k-1}(n_j+n_{j+1})}.$$

It will follow from Theorem 2.18 that the above sum equals A_n as defined in (†) in the introduction, and moreover that this sum evaluates to give $\frac{1}{n}\binom{2n-2}{n-1}$, the $(n-1)$-st Catalan number.

(ii) Now suppose $d = n-1$ and $\lambda = (1,\dots,1)$. Because $B_{\underline{n}}$ is a finite set with $n-1$ elements and because the set ${}^0B^k_{\underline{n}}$ consists of k-tuples of *distinct* elements of $B_{\underline{n}}$, it follows that, up to the action of the symmetry group S_{n-1}, every element of ${}^0B^{n-1}_{\underline{n}}$ is equivalent to the $(n-1)$-tuple obtained by listing each element of $B_{\underline{n}}$ once. Thus,

$$\begin{aligned} a_{(1,\dots,1)} &= \sum_{(\underline{i},\underline{j})\in{}^0B^{n-1}_{\underline{n}}}(2j_1-1)^2\cdots(2j_{n-1}-1)^2 \\ &= (n-1)!\prod_{i=1}^{k}\prod_{j=1}^{n_i}(2j-1)^2, \end{aligned}$$

and it follows that the coefficient of y_2^{n-1} in ν_n is given by

$$\frac{c_{(1,\dots,1)}}{(n-1)!} = 4\sum_{k=1}^{n}(-1)^{k+1}\sum_{n_1+\cdots+n_k=n}\frac{\prod_{i=1}^{k}\prod_{j=1}^{n_i}(2j-1)^2}{\prod_{i=1}^{k}b_{n_i}\prod_{j=1}^{k-1}(n_j+n_{j+1})}.$$

This equals the sum for B_n as defined in (‡) of the introduction, and in Theorem 2.5 we will evaluate this sum and prove it equals 1.

The evaluation of both sums in (i) and (ii) above involve the method of integral equations. These methods apply more generally to the sums in the following definition, which are what one gets for (1.16) by replacing ${}^0B^d_{\underline{n}}$ by the larger set $B^d_{\underline{n}}$ in Definition 1.13.

Definition 1.15. For any λ and for $\underline{n}$ a partition of n, define

$$\widetilde{a}_\lambda(\underline{n}) = \sum_{(\underline{i},\underline{j})\in B^d_{\underline{n}}}(2j_1-1)^{2\lambda_1}\cdots(2j_d-1)^{2\lambda_d}. \tag{1.17}$$

Define

$$p(\lambda_1,\dots,\lambda_d;n) = 4\sum_{k=1}^{n}(-1)^{k+1}\sum_{\underline{n}=(n_1,\dots,n_k)}\frac{\widetilde{a}_\lambda(\underline{n})}{\prod_{i=1}^{k}b_{n_i}\prod_{j=1}^{k-1}(n_j+n_{j+1})}. \tag{1.18}$$

Notice that λ is not assumed to be a partition of $n-1$ in the above definition. The following conjecture gives a formula for the evaluation of these sums assuming that $\lambda_1+\cdots+\lambda_d \le n-1$. The conjecture has been extensively verified, and will be proved in the case $d=1$ in the next section (see Theorem 2.18).

Conjecture 1.16.

$$p(\lambda_1,\dots,\lambda_d;n) = \begin{cases} \binom{2\lambda_1}{\lambda_1}\cdots\binom{2\lambda_d}{\lambda_d}n^{d-2} & \text{if } \lambda_1+\cdots+\lambda_d = n-1, \\ 0 & \text{if } \lambda_1+\cdots+\lambda_d < n-1. \end{cases}$$

We shall describe the coefficients c_λ of ν_n as linear combinations of $p(\lambda';n)$. Our description is independent of Conjecture 1.16. For ease of notation, simply write $p(\lambda)$ for $p(\lambda;n)$ whenever $\lambda_1+\cdots+\lambda_d=n-1$ (i.e., whenever λ is a partition of $n-1$).

In the special case where $d=1$, then of course $\triangle^d_{\underline{n}}=\emptyset$ and $p(\lambda)=c_\lambda$. However, for $d>1$ one must correct by subtracting the contribution to (1.18) made by those terms in the diagonal $\triangle^d_{\underline{n}}$ in (1.17).

The argument proceeds by utilizing the principle of inclusion-exclusion as follows. For an arbitrary set X, the d-fold product X^d is "stratified" with respect to the group action S_d, as we now explain. Given a partition $\underline{d}=(d_1,\dots,d_r)$ of d, we say that $(x_1,\dots,x_d)\in X^d$ has type $\underline{d}$, if, up to reordering, $x_1=\cdots=x_{d_1}$, and $x_{d_1+1}=\cdots=x_{d_1+d_2}$, and so on. Define the subsets $\Omega^{\underline{d}}\subset X^d$ by

$$\Omega^{\underline{d}}=\{(x_1,\dots,x_d)\text{ has type }\underline{d}\}.$$

For example, $\Omega^{\{1^d\}}=X^d$ and $\Omega^{\{2,1^{d-2}\}}=\triangle^d$, the large diagonal subset of X^d. Clearly these sets are nested, in fact, $\Omega^{\underline{d}}\subset\Omega^{\underline{e}}$ if and only if $\underline{e}$ is a refinement of $\underline{d}$. We will write $\underline{e}<\underline{d}$ whenever $\underline{e}$ is a proper refinement of $\underline{d}$. Let

$${}^0\Omega^{\underline{d}}=\Omega^{\underline{d}}\setminus\bigcup_{\underline{e}>\underline{d}}\Omega^{\underline{e}}$$

be the set of "pure" $\underline{d}$ type.

Let $H_{\underline{d}}$ be the stabilizer of $\Omega^{\underline{d}}$ with respect to the S_d action. Writing the partition $\underline{d}$ as a multiset $\{1^{\ell_1},\dots,d^{\ell_d}\}$, one sees that $H_{\underline{d}}$ sits in the short exact sequence

$$(S_1)^{\ell_1}\times\cdots\times(S_d)^{\ell_d}\longrightarrow H_{\underline{d}}\longrightarrow S_{\ell_1}\times\cdots\times S_{\ell_d}$$

and is, in fact, a direct sum of wreath products of $(S_i)^{\ell_i}$ and S_{ℓ_i} (it being understood that $(S_i)^\ell=\{e\}$, the trivial group, whenever $\ell=0$). The order of $H_{\underline{d}}$ is therefore

$$|H_{\underline{d}}|=(1!)^{\ell_1}\cdots(d!)^{\ell_d}\ell_1!\cdots\ell_d!$$

and its index is given by the following quotient:

$$h_{\underline{d}}\equiv\frac{d!}{|H_{\underline{d}}|}=\binom{d}{d_1,\dots,d_r}\frac{1}{\ell_1!\cdots\ell_d!}. \tag{1.19}$$

Now suppose $X=B_{\underline{n}}$. Using this notation, stratify the large diagonal subset $\triangle^d_{\underline{n}}$ of our d-fold Ferrers board $B^d_{\underline{n}}$ into the sets denoted by $\Omega^{\underline{d}}_{\underline{n}}$ and ${}^0\Omega^{\underline{d}}_{\underline{n}}$. The strategy is to correct our initial guess for c_λ by successively adding and subtracting terms associated with these subsets.

For example, notice what happens to the summand of (1.17) for a typical term $(\underline{i},\underline{j})\in B^d_{\underline{n}}$ in the diagonal. Taking $(i_1,j_1)=(i_2,j_2)$, then the summand is the one that arises in the power sum with λ replaced by $\lambda'=(\lambda_1+\lambda_2,\lambda_3,\dots,\lambda_d)$. More generally if $(i_k,j_k)=(i_\ell,j_\ell)$, then λ is replaced by λ', where

$$\lambda'=(\lambda_k+\lambda_\ell,\lambda_1,\dots,\widehat{\lambda_k},\dots,\widehat{\lambda_\ell},\dots,\lambda_d).$$

The first approximation is to subtract off these associated power sums, and it will prove convenient to do this in an S_d-equivariant way.

Definition 1.17. For $\underline{d}=(d_1,\dots,d_r)$ a partition of d and $\sigma\in S_d$, define

$$\Lambda_k=\lambda_{\sigma(d_1+\cdots+d_{k-1}+1)}+\cdots+\lambda_{\sigma(d_1+\cdots+d_k)}$$

for $k = 1, \dots, r$ (we suppress the dependence of Λ on $\underline{d}$ and σ). Set

$$p_{\underline{d}}(\lambda_1, \dots, \lambda_d) = \sum_{[\sigma] \in S_d/H_{\underline{d}}} p(\Lambda_1, \dots, \Lambda_r),$$

where $[\sigma]$ denotes the coset of $H_{\underline{d}}$ in S_d represented by σ.

Returning to the example where $(i_k, j_k) = (i_\ell, j_\ell)$ for some $k \neq \ell$, the relevant partition is $\underline{d} = (2, 1, \dots, 1)$ and the first approximation is to subtract $p_{\underline{d}}(\lambda_1, \dots, \lambda_d)$. This successfully subtracts all terms in the sum (1.17) coming from ${}^0\Omega^{\underline{d}}_{\underline{n}}$; however, it over-subtracts terms in $\Omega^{\underline{e}}_{\underline{n}}$ whenever $\underline{e} < \underline{d}$, because $\Omega^{\underline{e}}_{\underline{n}}$ has a different stabilizer group.

For a specific example, suppose $d = 3, \underline{d} = (2, 1)$ and $\underline{e} = (3)$. Because $h_{\underline{d}} = 3$, $\Omega^{\underline{d}}_{\underline{n}}$ has three components intersecting in $\Omega^{\underline{e}}_{\underline{n}}$. Subtracting $p_{\underline{d}}$ from $p(\lambda_1, \lambda_2, \lambda_3)$ over-subtracts $p_{\underline{e}}$ by a factor of 3, and to compensate we need to adjust by adding back $2p_{\underline{e}}$.

We continue this procedure, using the principle of inclusion-exclusion, which explains the occurrence of the numbers $\alpha_{\underline{d}}$ in Proposition 1.18. We work for the moment with extended partitions, which are non-increasing sequences of positive integers (p_i) such that $\lim_{i \to \infty} p_i = 1$. One may think of these as partitions of ∞, two examples being (1^∞) and $(2, 1^\infty)$ in multiset notation. There is a partial ordering on these sequences given by refinement, and we write $(p_i) < (p'_i)$ if (p_i) is a proper refinement of (p'_i). For example, $(\ell, 1^\infty) < (\ell', 1^\infty)$ if and only if $\ell < \ell'$. For any such $p = (p_i)$, there is a chain $(1^\infty) = p' < p'' < \cdots < p^{(\ell)} = p$, and we define the *level* of p, denoted $\ell(p)$, to be the maximal such ℓ. For example, $(\ell, 1^\infty)$ and $(2^\ell, 1^\infty)$ both have level ℓ.

We now define integer-valued functions on the set $\{p = (p_i)\}$ of extended partitions by setting

$$a_0(p) = \begin{cases} 1 & \text{if } p = (1^\infty), \\ 0 & \text{if otherwise,} \end{cases}$$

and defining $a_{j+1}(p)$ in terms of $a_j(p)$ by

$$a_{j+1}(p) = \begin{cases} a_j(p) & \text{if } \ell(p) \leq j, \\ a_j(p) - \displaystyle\sum_{\substack{p' < p \\ \ell(p') = j}} m_{p',p} a_j(p') & \text{if } \ell(p) > j. \end{cases}$$

Here, $m_{p',p}$ is the multiplicity with which p occurs in p'. Because putting a string of ones at the end of $\underline{d}$ and $\underline{e}$ has no effect on the numbers $m_{\underline{e},\underline{d}}$ of Definition 1.20, we may use that definition to define $m_{p',p}$ here.

Clearly, all terms of level ℓ are determined after ℓ iterations, and we claim that $a_j(p) \longrightarrow \alpha_p$ as $j \to \infty$. This is the content of the following proposition.

Proposition 1.18. *Suppose λ is a partition of $n-1$ into d parts. For any partition $\underline{d} = (d_1, \dots, d_r)$ of d, define $\alpha_{\underline{d}} = (d_1 - 1)! \cdots (d_r - 1)!(-1)^{d+r}$. Then*

$$c_\lambda = \sum_{\underline{d}} \alpha_{\underline{d}}\, p_{\underline{d}}(\lambda_1, \dots, \lambda_d), \tag{1.20}$$

the sum being over all partitions of d.

The proof of the proposition rests on some well-known properties of Stirling numbers of the first kind, which is illustrated in the following lemma.

Lemma 1.19. *Suppose $\underline{d}$ is a partition of d and $h_{\underline{d}}$ is given by (1.19). If $d > 1$, then $\sum_{\underline{d}} \alpha_{\underline{d}}\, h_{\underline{d}} = 0$, the sum being taken over all partitions of d.*

Proof. The standard results which this proof uses can be found in many introductory texts on combinatorics, e.g. §§3.2 and 3.3 of [14].

Write the partition $\underline{d}$ as the multiset $\{1^{\ell_1}, \dots, d^{\ell_d}\}$ and notice that

$$(-1)^{d+r} \alpha_{\underline{d}}\, h_{\underline{d}} = \frac{d!}{1^{\ell_1} \cdots d^{\ell_d} \ell_1! \cdots \ell_d!}$$

equals the number of cycles in S_d consisting of ℓ_1 1-cycles, ℓ_2 2-cycles, etc. It is a well-known fact that the function defined by

$$f_d(t) \quad = \quad \sum_{r=1}^{d} \left(\sum_{\underline{d}=(d_1,\dots,d_r)} \alpha_{\underline{d}}\, h_{\underline{d}} \right) t^r$$

is the generating function for the Stirling numbers $s(d,r)$ of the first kind (here, the inside sum is over partitions of d into r parts). Thus $f_d(t) = t(t-1)\cdots(t-d+1)$ and $f_d(1) = 0$ for $d > 1$, which completes the proof. □

The proof of the proposition follows from the lemma, as we now explain.

Proof. We must show that, after performing the sum on the right-hand side of (1.20), terms in the diagonal $\triangle^d_{\underline{n}}$ do not contribute, i.e., the overall coefficient of each and every summand term associated to the stratum ${}^0\Omega^{\underline{d}}_{\underline{n}}$ is zero whenever $\underline{d} \neq (1, \dots, 1)$. A given stratum $\Omega^{\underline{e}}_{\underline{n}}$ contributes terms of $\underline{d}$ type if and only if $\underline{e} \leq \underline{d}$, where we recall that $\underline{e} \leq \underline{d}$ if $\underline{e}$ is a refinement of $\underline{d}$. Hence we claim that

$$\sum_{\underline{e} \leq \underline{d}} \alpha_{\underline{e}}\, m_{\underline{e},\underline{d}} = 0. \tag{1.21}$$

Here, $m_{\underline{e},\underline{d}}$ is the multiplicity of ${}^0\Omega^{\underline{d}}_{\underline{n}}$ in $\Omega^{\underline{e}}_{\underline{n}}$, and roughly it counts the number of times a term of pure type $\underline{d}$ appears in $p_{\underline{e}}$. In order to define $m_{\underline{e},\underline{d}}$ precisely, we shall need to make a subtle distinction between *refinements* and *sub-partitions* of $\underline{d}$. We denote the former by $\underline{e}$ and the latter by $\underline{\underline{d}}$, where $\underline{\underline{d}} = (\underline{d}_1, \dots, \underline{d}_r)$ is a sub-partition of $\underline{d} = (d_1, \dots, d_r)$ (i.e., $\underline{d}_i$ is a partition of d_i for each $i = 1, \dots, r$).

The difference between refinements and sub-partitions is nicely illustrated by the following example. Let $\underline{d} = (4,2)$, $\underline{e} = (2,2,1,1)$, and consider $\underline{\underline{d}} = ((2,2),(1,1))$ and $\underline{\underline{d}}' = ((2,1,1),(2))$. Both $\underline{\underline{d}}$ and $\underline{\underline{d}}'$ are sub-partitions of $\underline{d}$ with underlying refinement $\underline{e}$, but they are not equal.

Write $\underline{\underline{d}} \sim \underline{e}$ if $\underline{\underline{d}}$ is a sub-partition whose underlying partition equals $\underline{e}$.

Definition 1.20. (i) Define $h_{\underline{\underline{d}}} = h_{\underline{d}_1} \cdots h_{\underline{d}_r}$ if $\underline{\underline{d}} = (\underline{d}_1, \dots, \underline{d}_r)$ is a sub-partition of $\underline{d}$, where $h_{\underline{d}}$ is defined in (1.19).

(ii) If $\underline{e} \leq \underline{d}$, define $m_{\underline{e},\underline{d}} = \sum_{\underline{\underline{d}} \sim \underline{e}} h_{\underline{\underline{d}}}$, the sum being over distinct ways of writing $\underline{e}$ as a sub-partition $\underline{\underline{d}}$.

For example, taking as before $\underline{d} = (4,2)$, $\underline{e} = (2,2,1,1)$, $\underline{\underline{d}} = ((2,2),(1,1))$ and $\underline{\underline{d}}' = ((2,1,1),(2))$, then $h_{\underline{\underline{d}}} = \binom{4}{2}\frac{1}{2} = 3$ and $h_{\underline{\underline{d}}'} = \binom{4}{2} = 6$; hence $m_{\underline{e},\underline{d}} = 3+6 = 9$.

Since $\alpha_{\underline{e}} = \alpha_{\underline{d}_1} \cdots \alpha_{\underline{d}_r}$ whenever $\underline{e} \sim \underline{\underline{d}} = (\underline{d}_1, \dots, \underline{d}_r)$, we obtain

$$\begin{aligned} \sum_{\underline{e} \le \underline{d}} \alpha_{\underline{e}} m_{\underline{e},\underline{d}} &= \sum_{\underline{e} \le \underline{d}} \sum_{\underline{\underline{d}} \sim \underline{e}} \alpha_{\underline{d}_1} \cdots \alpha_{\underline{d}_r} h_{\underline{\underline{d}}} \\ &= \left(\sum_{\underline{d}_1} \alpha_{\underline{d}_1} h_{\underline{d}_1} \right) \cdots \left(\sum_{\underline{d}_r} \alpha_{\underline{d}_r} h_{\underline{d}_r} \right) = 0. \end{aligned}$$

The second set of sums is over all partitions $\underline{d}_i$ of d_i, and the last step follows from the lemma. This proves the claim (1.21), and the proposition now follows. □

Computations of ν_n for $2 \le n \le 10$ can be found in Appendix B. To rigorously perform these calculations, we use the techniques of the next section to verify Conjecture 1.16 in the relevant cases.

2. Integral Equations

In §§2.1 we introduce, in a general context, integral equation techniques for analyzing sums of the form

$$\sum_{k=1}^{n} (-1)^{k-1} \sum_{n_1 + \cdots + n_k = n} \frac{a_{n_1} \cdots a_{n_k}}{\prod_{j=1}^{k-1} (n_j + n_{j+1})}.$$

These ideas are extended in §2.2 to contour integral equations and then applied to obtain an efficient new recursive algorithm for computing the Lefschetz polynomials $L_{n,\omega^d}(t; K)$. Strong results are obtained when the relevant integral equations can be solved in closed form. By finding such a closed form solution, we show in §2.3 that the sum B_n (see (‡) in the introduction) evaluates to 1 (Theorem 2.5). Our approach to Conjecture 1.16 requires that we obtain explicit solutions to a certain class of integral equations (2.25); this is accomplished in §2.4. In §2.6 these solutions, together with some facts (see §2.5) concerning sums of powers of odd numbers, are the main ingredients of the proof of Conjecture 1.16 in the case $d = 1$ (Theorem 2.18). In particular, we prove that the sum A_n (see (†) in the introduction) evaluates to the Catalan number $\frac{1}{n}\binom{2n-2}{n-1}$.

Our development of the integral equations in §§2.1 and 2.2 is slightly more general than we need. There, we consider an algebra $\mathcal{A}$ which need not be commutative. It should be noted that a commutative algebra would suffice for the applications in the sequel.

2.1. **Generating functions and integral equations.** Let $\mathcal{A}$ be a (not necessarily commutative) algebra over a field of characteristic 0 and let $\mathcal{A}[Y_1, \dots, Y_m]$ denote the polynomial ring over $\mathcal{A}$ in the commuting variables $Y_1, \dots, Y_m$. For $p(y) = \sum_{i=0}^{N} a_i y^i \in \mathcal{A}[y]$, define the formal integral

$$\int_0^1 p(y) dy \equiv \sum_{i=0}^{N} \frac{a_i}{i+1} \in \mathcal{A}.$$

If $f(t, y) = \sum_{n=0}^{\infty} p_n(y) t^n$ is a formal power series in the commuting variable t with coefficients in $\mathcal{A}[y]$, define

$$\int_0^1 f(t, y) dy \equiv \sum_{n=0}^{\infty} \left(\int_0^1 p_n(y) dy \right) t^n.$$

Given a sequence $a_n \in \mathcal{A}$, $n = 1, 2, \ldots$, the associated *generating function* is the formal power series in the commuting variable s:

$$\rho(s) \equiv \sum_{n=1}^{\infty} a_n s^n.$$

For $k \geq 2$, define a formal power series over $\mathcal{A}[y_1, \ldots, y_{k-1}]$ in the commuting variables s, t by

$$\begin{aligned} V_k(s,t;y_1,\ldots,y_{k-1}) &\equiv (y_1 \cdots y_{k-1})^{-1} \rho(y_1 s) \left(\prod_{j=2}^{k-1} \rho(y_j y_{j-1} t) \right) \rho(y_{k-1} t) \\ &= \sum_{n=k}^{\infty} \sum_{n_1+\cdots+n_k=n} a_{n_1} \cdots a_{n_k} y_1^{n_1+n_2-1} \cdots y_{k-1}^{n_{k-1}+n_k-1} s^{n_1} t^{n-n_1}. \end{aligned}$$

The second sum is over all compositions of n into k parts.

Set $\phi_1(s,t) \equiv \rho(s)$ and, for $k \geq 2$, define

$$\begin{aligned} \phi_k(s,t) &\equiv \int_0^1 \cdots \int_0^1 V_k(s,t;y_1,\ldots,y_{k-1}) dy_1 \cdots dy_{k-1} \\ &= \sum_{n=k}^{\infty} \sum_{n_1+\cdots+n_k=n} \frac{a_{n_1} \cdots a_{n_k}}{\prod_{j=1}^{k-1}(n_j + n_{j+1})} s^{n_1} t^{n-n_1}. \end{aligned} \tag{2.1}$$

Note that $\phi_k(s,t) = \int_0^1 \frac{\rho(ys)}{y} \phi_{k-1}(ty,t) dy$ for $k > 1$, and consequently, if we define

$$\Phi(s,t) \equiv \sum_{k=1}^{\infty} (-1)^{k-1} \phi_k(s,t), \tag{2.2}$$

then $\Phi(s,t)$ satisfies the basic integral equation

$$\Phi(s,t) + \int_0^1 \frac{\rho(ys)}{y} \Phi(ty,t) dy = \rho(s). \tag{2.3}$$

From (2.1) and (2.2) we have that

$$\Phi(s,t) = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} (-1)^{k-1} \sum_{n_1+\cdots+n_k=n} \frac{a_{n_1} \cdots a_{n_k}}{\prod_{j=1}^{k-1}(n_j + n_{j+1})} s^{n_1} t^{n-n_1} \right), \tag{2.4}$$

and thus

$$\Phi(t,t) = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} (-1)^{k-1} \sum_{n_1+\cdots+n_k=n} \frac{a_{n_1} \cdots a_{n_k}}{\prod_{j=1}^{k-1}(n_j + n_{j+1})} \right) t^n.$$

If $\Phi(s,t) \equiv \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} A_{ij} s^i t^j$ satisfies the integral equation (2.3), then a comparison of the coefficients on both sides of (2.3) yields the following recursion formula for the coefficients A_{ij}:

$$A_{i0} = a_i, \qquad i = 1, 2, \ldots, \tag{2.5}$$

$$A_{ij} = -a_i \sum_{q=0}^{j-1} \frac{A_{j-q,q}}{j-q+i}, \qquad j > 0. \tag{2.6}$$

Conversely, if A_{ij} are elements in $\mathcal{A}$ defined by (2.5) and (2.6), then $\Phi(s,t) \equiv \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} A_{ij}s^i t^j$ is a solution to (2.3). Since (2.5) and (2.6) uniquely define the A_{ij}'s, this formal power series solution to (2.3) is unique.

2.2. **Contour integral equations.** Let $\mathbb{F}$ be a field of characteristic 0, let $\mathbb{F}((x))$ be the field of formal power series over $\mathbb{F}$ in the variable x and let $\mathcal{A}$ be a (not necessarily commutative) algebra over $\mathbb{F}((x))$. Define the *formal contour integral* of a polynomial $f(y) = \sum_{n=1}^{N} a_n y^n \in \mathcal{A}[y]$ by

$$\oint f(y)\eta(x,y)dy \equiv \sum_{n=1}^{N} \frac{a_n}{1-x^n}.$$

Remark 2.1. We justify the terminology "formal contour integral" as follows. Consider the series

$$\eta(u,v) = \frac{1}{2\pi i}\sum_{k=0}^{\infty} \frac{u^k}{v(v-u^k)}$$

in the complex variables u, v. This series converges uniformly on compacta in the domain

$$\Omega = \{(u,v) \in \mathbb{C}^2 \mid v \neq 0,\ |u| < 1,\ v \neq u^k, k = 0,1,\dots\}.$$

Calculating residues, we obtain

$$\operatorname*{Res}_{v=u^k} \frac{v^n u^k}{v(v-u^k)} = \operatorname*{Res}_{v=u^k} \frac{v^{n-1}u^k}{v-u^k} = u^{kn}, \qquad n = 1,2,\dots,$$

and hence, by the residue theorem,

$$\oint v^n \eta(u,v)dv = \sum_{k=0}^{\infty} u^{kn} = \frac{1}{1-u^n}, \qquad n = 1,2,\dots$$

(contour integral over the unit circle).

Recall that, given a sequence $Q_n \in \mathcal{A}$, $n = 1,2,\dots$, the associated generating function is the formal power series:

$$\rho(u) \equiv \sum_{n=1}^{\infty} Q_n u^n.$$

For $k \geq 2$, define a formal power series over $\mathcal{A}[y_1,\dots,y_{k-1}]$ in the commuting variables s, t by

$$\begin{aligned}\widetilde{V}_k(s,t;y_1,\dots,y_{k-1}) &\equiv \rho(y_1 s)\left(\prod_{j=2}^{k-1}\rho(y_j y_{j-1}t)\right)\rho(y_{k-1}t)\\ &= \sum_{n=k}^{\infty}\sum_{n_1+\cdots+n_k=n} Q_{n_1}\cdots Q_{n_k} y_1^{n_1+n_2-1}\cdots y_{k-1}^{n_{k-1}+n_k-1} s^{n_1} t^{n-n_1}.\end{aligned}$$

The second sum is over all compositions of n into k parts.

Set $\widetilde{\phi}_1(s,t) \equiv \rho(s)$ and, for $k \geq 2$, define

$$\begin{aligned} \widetilde{\phi}_k(s,t) &\equiv \oint \cdots \oint \widetilde{V}_k(s,t;y_1,\ldots,y_{k-1}) \prod_{j=1}^{k-1} \eta(x,y_j)dy_1 \cdots dy_{k-1} \\ &= \sum_{n=k}^{\infty} \sum_{n_1+\cdots+n_k=n} \frac{Q_{n_1}\cdots Q_{n_k}}{\prod_{j=1}^{k-1}(1-x^{n_j+n_{j+1}})}\, s^{n_1}t^{n-n_1}. \end{aligned} \tag{2.7}$$

Note that $\widetilde{\phi}_k(s,t) = \oint \rho(sy)\widetilde{\phi}_{k-1}(ty,y)\eta(x,y)dy$ for $k > 1$, and consequently, if we define

$$\widetilde{\Phi}(s,t) \equiv \sum_{k=1}^{\infty}(-1)^{k-1}\widetilde{\phi}_k(s,t), \tag{2.8}$$

then $\widetilde{\Phi}(s,t)$ satisfies the basic integral equation

$$\widetilde{\Phi}(s,t) + \oint \rho(sy)\widetilde{\Phi}(ty,t)\eta(x,y)dy \;=\; \rho(s). \tag{2.9}$$

From (2.7) and (2.8) we see that

$$\widetilde{\Phi}(s,t) \;=\; \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(-1)^{k-1}\sum_{n_1+\cdots+n_k=n}\frac{Q_{n_1}\cdots Q_{n_k}}{\prod_{j=1}^{k-1}(1-x^{n_j+n_{j+1}})}\, s^{n_1}t^{n-n_1}\right). \tag{2.10}$$

If $\widetilde{\Phi}(s,t) \equiv \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\widetilde{A}_{ij}s^it^j$ satisfies the integral equation (2.9), then a comparison of the coefficients on both sides of (2.9) yields the following recursion formula for the coefficients $\widetilde{A}_{ij}$:

$$\widetilde{A}_{i0} \;=\; Q_i, \qquad i = 1,2,\ldots, \tag{2.11}$$

$$\widetilde{A}_{ij} \;=\; -Q_i\sum_{q=0}^{j-1}\frac{\widetilde{A}_{j-q,q}}{1-x^{j-q+i}}, \qquad j > 0. \tag{2.12}$$

Conversely, if $\widetilde{A}_{ij}$ are the elements in $\mathcal{A}$ defined by (2.11) and (2.12), then $\Phi(s,t) \equiv \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\widetilde{A}_{ij}s^it^j$ is a solution to (2.9). Since (2.11) and (2.12) uniquely define the $\widetilde{A}_{ij}$'s, this formal power series solution to (2.9) is unique.

Let Q_n and $Q_{n,d}$ ($n > 0$, $d \in \mathbb{Z}/n\mathbb{Z}$) be elements of a not necessarily commutative algebra over $\mathbb{F}((x))$ which are related by

$$Q_{n,d} \;=\; \sum_{k=1}^{n}\sum_{n_1+\cdots+n_k=n}\;\sum_{\substack{d_1/n_1>\cdots>d_k/n_k\\ d_1+\cdots+d_k=d}} x^{N(n_1,\ldots,n_k;d_1,\ldots,d_k)}Q_{n_1}\cdots Q_{n_k}, \tag{2.13}$$

where

$$N(n_1,\ldots,n_k;d_1,\ldots,d_k) \;=\; \sum_{1\leq i<j\leq k}(d_in_j - d_jn_i).$$

A recursive formula of this type was given by Harder and Narasimhan [11] (implicitly, but made explicit in [6]) and by Atiyah and Bott [1] for computing the Poincaré polynomials of the moduli space of semistable rank n, degree d holomorphic vector bundles over a given Riemann surface of genus g in the case when n and d are relatively prime. Frohman showed [8] that the same recursion can be used

to compute the Lefschetz polynomials $L_{n,\omega^d}(t;K)$ (cf. Proposition 1.6 from §1.2) when n and d are relatively prime. Explicitly, if Q_n and $Q_{n,d}$ are defined by

$$Q_{n,d} = \frac{t^{n^2(1-g)}c(t)}{1-t^2}\, L_{n,\omega^d}(t), \tag{2.14}$$

$$Q_n = \frac{t^{n^2(1-g)}c(t)c(t^3)\cdots c(t^{2n-1})}{(1-t^2)^2\cdots(1-t^{2n-2})^2(1-t^{2n})}, \tag{2.15}$$

where $c(t) = t^g\Delta_K(t)$ is the (unbalanced) Alexander polynomial of K (this agrees with (1.1) of §1.2), then, for $x = t^2$, Q_n and $Q_{n,d}$ satisfy (2.13).

Zagier discovered the following summation formula (Theorem 2 of [15]) for solving the recursion given by (2.13).

Theorem 2.2. *For Q_n and $Q_{n,d}$ as in (2.13)*

$$Q_{n,d} = \sum_{k=1}^{n}(-1)^{k-1}\sum_{n_1+\cdots+n_k=n}\frac{x^{M(n_1,\dots,n_k;d/n)}Q_{n_1}\cdots Q_{n_k}}{\prod_{j=1}^{k-1}(1-x^{n_j+n_{j+1}})},$$

where $M(n_1,\dots,n_k;s)$ is defined as in equation (1.2) from §1.2.

Since $M(n_1,\dots,n_k;1/n) = n - n_1$, for Q_n and $Q_{n,d}$ as in (2.13), we have

$$Q_{n,1} = \sum_{k=1}^{n}(-1)^{k-1}\sum_{n_1+\cdots+n_k=n}\frac{x^{n-n_1}Q_{n_1}\cdots Q_{n_k}}{\prod_{j=1}^{k-1}(1-x^{n_j+n_{j+1}})},$$

and a comparison with (2.10) yields the following result.

Proposition 2.3. *For Q_n and $Q_{n,d}$ as in (2.13) and $\widetilde{\Phi}(s,t)$ as in (2.10), we have that*

$$\sum_{n=1}^{\infty} Q_{n,1}\, u^n = \widetilde{\Phi}(u,ux),$$

i.e., $\widetilde{\Phi}(u,ux)$ is the generating function for the sequence $\{Q_{n,1} \mid n = 1,2,\dots\}$.

In particular, this proposition can be applied to obtain an efficient new recursive algorithm for computing the Lefschetz polynomials $L_{n,\omega^d}(t;K)$:

(i) Let $Q_{n,1}$ and Q_n be given by (2.14) and (2.15) respectively.
(ii) Compute $\widetilde{\Phi}(s,t)$ via the recursive formula given by (2.11) and (2.12).
(iii) Obtain $Q_{n,1}$ as the coefficient of u^n in $\widetilde{\Phi}(u,ux)$.

2.3. **Evaluation of sums via integral equations: An example.** For $n = 1,2,\dots$ define the rational numbers β_n by

$$\beta_n \equiv \frac{n}{4^{2n}}\binom{2n}{n}^2.$$

Consider the formal power series

$$\rho(t) \equiv \sum_{n=1}^{\infty}\beta_n t^n.$$

Then $\rho(t)$ can be characterized in terms of hypergeometric functions; in fact $\rho(t) = \frac{t}{4}F(\frac{3}{2},\frac{3}{2},2;t)$, where $F(a,b,c;t)$ is a power series solution to Gauss's hypergeometric differential equation $t(1-t)y'' + [c-(a+b+1)t]y' - aby = 0$.

We are interested in the integral equation

$$\Phi(s,t) + \int_0^1 \frac{\rho(xs)}{x}\Phi(tx,t)dx \;=\; \rho(s). \tag{2.16}$$

Proposition 2.4. $\Phi(s,t) = \frac{s}{s-t}\rho\left(\frac{s-t}{1-t}\right)$ *is the unique power series solution to (2.16).*

Before giving its proof, we observe the following consequence of Proposition 2.4, which asserts that the sums denoted by B_n in (‡) of the introduction evaluate to 1.

Theorem 2.5. *Let* $b_k \equiv 4^k k!(k-1)!$. *For* $n \geq 1$,

$$4\sum_{k=1}^{n}(-1)^{k-1}\sum_{n_1+\cdots+n_k=n}\frac{\prod_{i=1}^{k}\prod_{j=1}^{n_i}(2j-1)^2}{\prod_{i=1}^{k}b_{n_i}\prod_{j=1}^{k-1}(n_j+n_{j+1})} = 1.$$

Proof. By Proposition 2.4 and (2.4),

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}(-1)^{k-1}\sum_{n_1+\cdots+n_k=n}\frac{\beta_{n_1}\cdots\beta_{n_k}}{\prod_{j=1}^{k-1}(n_j+n_{j+1})}t^n = \Phi(t,t) = \frac{t}{4(1-t)}.$$

The conclusion follows from the fact that

$$\beta_n = \frac{\prod_{i=1}^{n}(2i-1)^2}{4^n n!(n-1)!}.$$

□

Remark 2.6. We sketch a topological proof that B_n from (‡) of the introduction (i.e., the sum considered in Theorem 2.5) equals ± 1, assuming Conjecture 1.9. If K is a fibered knot of genus 1, then $\nabla_K(z) = C_0 + C_2 z^2$, where $C_0 = \pm 1$ and $C_2 = 1$. Moreover, for $(n,d) = 1$, it follows that R_{n,ω^d} is connected and 0-dimensional, hence just a point. Thus $L_{n,\omega^d}(t;K) = \pm 1$. Now Conjecture 1.9 implies that

$$\pm 1 = \lambda_{n,\omega}(K) = \nu_n(C_0, C_2, 0, \ldots, 0) = B_n C_2^{n-1} = B_n.$$

Proof of Proposition 2.4. Expanding $\Phi(s,t) = \frac{s}{s-t}\rho\left(\frac{s-t}{1-t}\right)$ as a power series and using the binomial theorem and the formula $\frac{1}{(1-t)^{n+1}} = \sum_{k=n}^{\infty}\binom{k}{n}t^{k-n}$, we observe that

$$\begin{aligned}
\Phi(s,t) &= \sum_{n=0}^{\infty} s\beta_{n+1}\frac{(s-t)^n}{(1-t)^{n+1}} \\
&= \sum_{n=0}^{\infty} s\beta_{n+1}\left(\sum_{\ell=0}^{n}(-1)^{\ell}\binom{n}{\ell}s^{n-\ell}t^{\ell}\right)\left(\sum_{k=n}^{\infty}\binom{k}{n}t^{k-n}\right) \\
&= \sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\sum_{\ell=0}^{n}(-1)^{\ell}\beta_{n+1}\binom{n}{\ell}\binom{k}{n}s^{n-\ell+1}t^{k+\ell-n} \\
&= \sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\sum_{m=0}^{n}(-1)^{n-m}\beta_{n+1}\binom{n}{n-m}\binom{k}{n}s^{m+1}t^{k-m} \\
&= \sum_{k=0}^{\infty}\sum_{m=0}^{k}\left[\sum_{n=m}^{k}(-1)^{n-m}\beta_{n+1}\binom{n}{m}\binom{k}{n}\right]s^{m+1}t^{k-m}, \qquad (2.17)
\end{aligned}$$

where the fourth line follows by making the substitution $m = n - \ell$.

Substituting $i = m+1$ and $j = k-m$ into (2.17) shows that

$$\Phi(s,t) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} A_{i,j} s^i t^j,$$

where $A_{i,j}$ is defined for $i \geq 1$ and $j \geq 0$ by the term in brackets in (2.17), i.e.,

$$\begin{aligned} A_{i,j} &= \sum_{n=i-1}^{i+j-1} (-1)^{n-i+1} \binom{n}{i-1}\binom{i+j-1}{n}\beta_{n+1} \\ &= \sum_{q=0}^{j} (-1)^q \binom{i+q-1}{i-1}\binom{i+j-1}{i+q-1}\beta_{i+q}. \end{aligned} \tag{2.18}$$

Now define $\widetilde{A}_{i,j}$ for $i \geq 1$ and $j \geq 0$ by setting

$$\begin{aligned} \widetilde{A}_{i,0} &= \beta_i \\ \widetilde{A}_{i,j} &= -\beta_i \sum_{q=0}^{j-1} (-1)^q \binom{j-1}{q} \frac{q!\,\beta_{q+1}}{(i+1)\cdots(i+q+1)}. \end{aligned} \tag{2.19}$$

□

Lemma 2.7. (i) *For $i \geq 1$ and $j \geq 0$, $A_{i,j} = \widetilde{A}_{i,j}$.*
(ii) *For $i, j \geq 1$, $\widetilde{A}_{i,j}$ satisfies the recursion*

$$\widetilde{A}_{i,j} = -\beta_i \sum_{q=0}^{j-1} \frac{\widetilde{A}_{j-q,q}}{(i+j-q)}.$$

First, observe that parts (i) and (ii) of the lemma establish the theorem, since (ii) is exactly the recursion formula that the solution to (2.16) must satisfy. Moreover, part (i) actually implies part (ii), as we now explain.

Part (ii) is equivalent to the statement that, for $i, j \geq 1$,

$$\begin{aligned} &\sum_{k=0}^{j-1} (-1)^k \binom{j-1}{k} \frac{k!\,\beta_{k+1}}{(i+1)\cdots(i+k+1)} = \frac{\beta_j}{i+j} - \\ &\sum_{q=1}^{j-1} \frac{\beta_{j-q}}{i+j-q} \left(\sum_{r=0}^{q-1} (-1)^r \binom{q-1}{r} \frac{r!\,\beta_{r+1}}{(j-q+1)\cdots(r+j-q+1)} \right). \end{aligned} \tag{2.20}$$

Applying the method of partial fractions to $1/(x+1)\cdots(x+k+1)$, we see that

$$\frac{k!\,\beta_{k+1}}{(i+1)\cdots(i+k+1)} = \sum_{s=0}^{k} (-1)^s \binom{k}{s} \frac{\beta_{k+1}}{i+s+1}.$$

Substituting this into the left hand side of (2.20), interchanging the order of summation, and equating coefficients of $1/(i+s+1)$ on the left with those of $1/(i+j-q)$ on the right (so $s = j-q-1$), we now see that (2.20) is equivalent to the statement

that if $j > q > 0$ then

$$\sum_{k=j-q-1}^{j-1} (-1)^{j+k-q-1} \binom{j-1}{k} \binom{k}{j-q-1} \beta_{k+1}$$

$$= \beta_{j-q} \sum_{r=0}^{q-1} (-1)^r \binom{q-1}{r} \frac{r!\,\beta_{r+1}}{(j-q+1)\cdots(r+j-q+1)}. \tag{2.21}$$

(The $1/(i+j)$ term can be handled separately.) Comparing with (2.18) and (2.19), we see that (2.21) is equivalent to the statement that if $j > q > 0$ then $A_{j-q,q} = \widetilde{A}_{j-q,q}$. Thus (i) implies (ii).

It remains to prove part (i) of the lemma. If $j = 0$, this is obvious, and for $j > 0$, define

$$B_{i,j} = \sum_{k=1}^{j} (-1)^k \binom{j-1}{k-1} A_{i,k} \quad \text{and} \quad \widetilde{B}_{i,j} = \sum_{k=1}^{j} (-1)^k \binom{j-1}{k-1} \widetilde{A}_{i,k}. \tag{2.22}$$

We prove (i) by showing that $B_{i,j} = \widetilde{B}_{i,j}$ for all $i, j > 0$. From (2.19), we see that

$$\begin{aligned}
\widetilde{B}_{i,j} &= -\beta_i \sum_{k=1}^{j} \sum_{q=0}^{k-1} (-1)^{k+q} \binom{j-1}{k-1} \binom{k-1}{q} \frac{q!\,\beta_{q+1}}{(i+1)\cdots(i+q+1)!} \\
&= -\beta_i \sum_{q=0}^{j-1} \left[\sum_{k=q+1}^{j} (-1)^{k+q} \binom{j-q-1}{j-k} \right] \frac{i!\,(j-1)!\,\beta_{q+1}}{(j-q-1)!\,(i+q+1)!} \\
&= \frac{i!\,(j-1)!}{(i+j)!} \beta_i \beta_j.
\end{aligned}$$

The last step uses $\sum_{k=q+1}^{j} (-1)^{k+q} \binom{j-q-1}{j-k} = -\delta_{j-1,q}$, where $\delta_{i,j}$ is the Kronecker symbol. This formula is an easy consequence of the binomial theorem.

We claim that the same is true of $B_{i,j}$, i.e., we claim that

$$B_{i,j} = \frac{i!\,(j-1)!}{(i+j)!} \beta_i \beta_j. \tag{2.23}$$

This requires induction and is more difficult to prove.

Using (2.18) and the identity $\binom{j}{q} = \binom{j-1}{q} + \binom{j-1}{q-1}$, it follows that

$$\begin{aligned}
A_{i,j} &= \sum_{q=0}^{j} (-1)^q \binom{i+q-1}{i-1} \binom{i+j-1}{i+q-1} \beta_{i+q} \\
&= \sum_{q=0}^{j} (-1)^q \binom{i+j-1}{i-1} \binom{j-1}{q} \beta_{i+q} \\
&= \sum_{q=0}^{j} (-1)^q \binom{i+j-1}{i-1} \left[\binom{j}{q} - \binom{j-1}{q-1} \right] \beta_{i+q} \\
&= \sum_{q=0}^{j-1} (-1)^q \binom{i+j-1}{i-1} \binom{j-1}{q} \left(\beta_{i+q} - \beta_{i+q+1} \right). \qquad (2.24)
\end{aligned}$$

Substituting (2.24) into (2.22) and using Identity A.1, we obtain that

$$\begin{aligned}
B_{i,j} &= \sum_{k=1}^{j}\sum_{q=0}^{k-1}(-1)^{k+q}\binom{j-1}{k-1}\binom{i+k-1}{i-1}\binom{k-1}{q}(\beta_{i+q}-\beta_{i+q+1}) \\
&= \sum_{q=0}^{j-1}\sum_{k=q+1}^{j}(-1)^{k+q}\binom{j-1}{q}\binom{i+k-1}{i-1}\binom{j-q-1}{k-q-1}(\beta_{i+q}-\beta_{i+q+1}) \\
&= \sum_{q=0}^{j-1}(-1)^{q}\binom{j-1}{q}\left[\sum_{k=q+1}^{j}(-1)^{k}\binom{i+k-1}{i-1}\binom{j-q-1}{k-q-1}\right](\beta_{i+q}-\beta_{i+q+1}) \\
&= \sum_{q=0}^{j-1}(-1)^{j+q}\binom{j-1}{q}\binom{i+q}{j}(\beta_{i+q}-\beta_{i+q+1}).
\end{aligned}$$

(Recall the convention that $\binom{n}{k}=0$ if $n<k$.) Claim (2.23) now follows directly from Identity A.6. □

2.4. **Methods for solving integral equations.** Let $b_n = 4^n n!(n-1)!$ and define the power series

$$\gamma(z) \equiv \sum_{n=1}^{\infty}\frac{z^n}{b_n}.$$

This series is given by the Bessel function $\gamma(u^2) = -\frac{iu}{2}J_1(iu)$. It will also be useful to define

$$\mu(z) \equiv \frac{\gamma(z)}{z} = \sum_{n=0}^{\infty}\frac{z^n}{b_{n+1}}.$$

We will be interested in solving integral equations of the form

$$\Phi(s,t) + \int_0^1 \frac{\gamma(xs)}{x}\Phi(tx,t)dx \;=\; f(s,t), \tag{2.25}$$

where $f(s,t)$ is a formal power series in the variables s,t. We first consider the case $f(s,t)=\gamma(s)$:

$$\Phi(s,t) + \int_0^1 \frac{\gamma(xs)}{x}\Phi(tx,t)dx \;=\; \gamma(s). \tag{2.26}$$

Proposition 2.8. *The unique formal power series solution to the equation (2.26) is given by $\Phi(s,t)=\mu(s-t)s$.*

Proof. For $\Phi(s,t)\equiv\mu(s-t)s$ the left side of (2.26) is

$$\mu(s-t)s + t\int_0^1 \gamma(xs)\mu(t(x-1))dx.$$

This expression is 0 when $s=0$, and for $k\geq 1$ the coefficient of s^k is

$$\frac{\mu^{(k-1)}(-t)}{(k-1)!} + \frac{t}{b_k}\int_0^1 x^k\mu(t(x-1))dx. \tag{2.27}$$

For $n \geq 1$ and $k \geq 1$,

$$\begin{aligned}\binom{k+n-1}{k}(k+n)b_k b_n &= \frac{(k+n-1)!}{k!\ (n-1)!}(k+n)4^k k!\ (k-1)!\ 4^n n!\ (n-1)! \\ &= 4^{k+n}(k+n)!\ n! = \frac{n!\ b_{n+k}}{(k+n-1)!},\end{aligned}$$

and thus

$$\frac{(k+n-1)!}{b_{n+k}n!}(-1)^n + \frac{(-1)^{n-1}}{b_k b_n}\binom{k+n-1}{k}^{-1}(k+n)^{-1} = 0. \tag{2.28}$$

The coefficient of t^n in (2.27), $n \geq 1$, is precisely the left side of the identity (2.28) and is consequently equal to 0. Hence (2.27) is equal to its value at $t = 0$, namely $1/b_k$, which coincides with the coefficient of s^k in $\gamma(s)$. □

The following result is another useful special case of (2.25).

Proposition 2.9. *The unique formal power series solution to the equation*

$$P_q(s,t) + \int_0^1 \frac{\gamma(xs)}{x} P_q(tx,t)dx \ = \ s^q$$

is given by

$$P_q(s,t) = s^q + (-1)^q 4^q q! \sum_{n=q+1}^{\infty} \sum_{k=0}^{n-q-1} \frac{(-1)^{n+k}(2n-k-q-2)!}{b_n\ (n-q)!(n-k-1-q)!}\binom{n-1}{k} s^{k+1}t^{n-k-1}.$$

Proof. A comparison of coefficients shows that the series

$$P_q(s,t) \equiv \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} A_{ij}s^i t^j$$

as defined in the statement of the proposition satisfies the integral equation $P_q(s,t) + \int_0^1 \frac{\gamma(xs)}{x}P_q(tx,t)dx \ = \ s^q$ if and only if the A_{ij}'s satisfy the recursion

$$A_{i0} = \delta_{iq}, \qquad i = 1, 2, \dots, \tag{2.29}$$

$$A_{ij} = -\frac{1}{b_i}\sum_{\ell=0}^{j-1}\frac{A_{j-\ell,\ell}}{j-\ell+i}, \qquad j > 0, \tag{2.30}$$

where $\delta_{iq} = 1$ if $i = q$ and $\delta_{iq} = 0$ otherwise. Observe that (2.29) is equivalent to the condition $P_q(s,0) = s^q$, which is clearly satisfied. In the case $0 < j \leq q$, it is also easily seen that (2.30) is satisfied. When $j > q$, after dividing by $(-1)^q 4^q q!$, (2.30) becomes the assertion that

$$\frac{(-1)^{j-1}(i+2j-q-1)!}{b_{i+j}\ (i+j-q)!(j-q)!}\binom{i+j-1}{i-1}$$

is equal to

$$\frac{-1}{b_i}\sum_{\ell=q}^{j-1}\frac{(-1)^{\ell-1}(j+\ell-q-1)!}{b_j\ (j-q)!(\ell-q)!(j-\ell+i)}\binom{j-1}{j-\ell-1}.$$

This is equivalent to the assertion that

$$(-1)^{j-1}\binom{i+2j-q-1}{j-1} = \sum_{\ell=q}^{j-1}(-1)^{\ell}\binom{j+\ell-q-1}{j-1}\binom{i+j}{\ell}\binom{i+j-\ell-1}{i},$$

which is Identity A.5. □

Given a formal power series $f(t) = \sum_{n=0}^{\infty} a_n t^n$ and a non-negative integer p, the *p–th derivative of* $f(t)$, denoted by $f^{(p)}(t)$, is the formal power series:

$$f^{(p)}(t) \equiv \sum_{\ell=0}^{\infty} \frac{(\ell+p)!}{\ell!} a_{\ell+p} t^{\ell}.$$

Proposition 2.10. $P_q(t,t) = 4^q q!\ t^q \mu^{(q-1)}(-t)$.

Proof. By Proposition 2.9, $P_q(t,t)$ equals

$$t^q + (-1)^q 4^q q! \sum_{n=q+1}^{\infty} \frac{(-1)^n (n-1)!}{b_n (n-q)!} \left(\sum_{k=0}^{n-q-1} \frac{(-1)^k (2n-k-q-2)!}{(n-q-1-k)!\ (n-1-k)!\ k!} \right) t^n.$$

By Identity A.3,

$$\begin{aligned} &\sum_{k=0}^{n-q-1} \frac{(-1)^k (2n-k-q-2)!}{(n-q-1-k)!\ (n-1-k)!\ k!} \\ &\quad = \sum_{k=0}^{n-q-1} (-1)^k \binom{n-1}{k}\binom{2n-k-q-2}{n-1} = 1, \end{aligned}$$

and thus

$$\begin{aligned} P_q(t,t) &= t^q + (-1)^q 4^q q! \sum_{n=q+1}^{\infty} \frac{(-1)^n (n-1)!}{b_n (n-q)!} t^n \\ &= t^q + 4^q q!\ t^q \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell}(\ell+q-1)!}{b_{\ell+q}\ell!} t^{\ell} \\ &= 4^q q!\ t^q \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}(\ell+q-1)!}{b_{\ell+q}\ell!} t^{\ell} \ = \ 4^q q!\ t^q \mu^{(q-1)}(-t). \end{aligned}$$

□

Consider a power series of the form $f(s,t) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} F_{ij} s^i t^j$, which we write as $\sum_{n=1}^{\infty} f_n(t) s^n$, where $f_n(t) = \sum_{j=0}^{\infty} F_{nj} t^j$, $n = 1, 2, \ldots$.

Proposition 2.11. *The solution to (2.25) is* $\Phi(s,t) = \sum_{n=1}^{\infty} f_n(t) P_n(s,t)$.

Proof. Let $\mathfrak{m}$ be the maximal ideal of formal power series in the variables s, t with vanishing constant term. Since $P_n(s,t) \in \mathfrak{m}^n$, $n \geq 1$, the expression $\Phi(s,t) \equiv$

$\sum_{n=1}^{\infty} f_n(t)P_n(s,t)$ is valid as a formal power series; furthermore,

$$\begin{aligned}\Phi(s,t) &+ \int_0^1 \frac{\gamma(xs)}{x}\Phi(tx,t)dx \\ &= \sum_{n=1}^{\infty} f_n(t)P_n(s,t) + \int_0^1 \frac{\gamma(xs)}{x}\left(\sum_{n=1}^{\infty} f_n(t)P_n(tx,t)\right)dx \\ &= \sum_{n=1}^{\infty} f_n(t)\left(P_n(s,t) + \int_0^1 \frac{\gamma(xs)}{x}P_n(tx,t)dx\right).\end{aligned}$$

The conclusion now follows from Proposition 2.9. □

2.5. **Some facts about power sums.** In this subsection we collect some facts for use in the sequel concerning sums of powers of odd integers.

The *Bernoulli numbers* are the rational numbers B_n, $n = 0, 1, \ldots$, defined by the exponential generating function:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n.$$

The *Euler-Maclaurin polynomials* are defined by

$$p_k(x) = \sum_{i=0}^{k} \frac{B_i}{k+1-i}\binom{k}{i} x^{k+1-i},$$

and their associated exponential generating function is given by

$$G(x,t) \equiv \sum_{k=0}^{\infty} p_k(x)\frac{t^k}{k!} \;=\; \frac{e^{xt} - e^t}{e^t - 1}.$$

Using this generating function, it is easy to deduce (see [12]) the well known *Euler-Maclaurin formula*:

$$p_k(n) = 1^k \;+\; 2^k \;+\; \cdots \;+\; (n-1)^k, \qquad n = 1, 2, \ldots .$$

Define polynomials

$$u_k(x) \equiv p_k(2x) - 2^k p_k(x), \qquad k = 0, 1, \ldots .$$

Then for each positive integer n, the Euler-Maclaurin formula yields the formula

$$u_k(n) = 1^k \;+\; 3^k \;+\; 5^k \;+\; \cdots \;+\; (2n-1)^k. \tag{2.31}$$

The exponential generating function $H(x,t) \equiv \sum_{k=0}^{\infty} u_k(x)t^k/k!$ is given by

$$H(x,t) \;=\; G(2x,t) - G(x,2t) = e^{xt}\frac{\sinh(xt)}{\sinh(t)}.$$

Note that

$$\sum_{k=0}^{\infty} u_{2k}(x)\frac{t^{2k}}{(2k)!} \;=\; \tfrac{1}{2}(H(x,t) + H(x,-t)) = \frac{\sinh(2xt)}{2\sinh(t)},$$

from which it follows that

$$u_{2k}(x) = \sum_{j=0}^{k} \frac{4^j D_{2k-2j}}{2j+1}\binom{2k}{2j} x^{2j+1}, \tag{2.32}$$

where the rational numbers D_n are given by the exponential generating function:

$$\frac{t}{\sinh(t)} \;=\; \sum_{n=0}^{\infty} \frac{D_n}{n!}\, t^n \;=\; 1 - \frac{1}{6}t^2 + \frac{7}{360}t^4 - \cdots . \tag{2.33}$$

2.6. **Proof of Conjecture 1.16 for $d = 1$.** Let δ_{ij} be the Kronecker symbol, i.e.,

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

For $m \geq 1$, define the power series $W_m(s,t)$ by

$$W_m(s,t) \;\equiv\; \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} (-1)^{k-1} \sum_{n_1+\cdots+n_k=n} \frac{\sum_{j=1}^{k} \delta_{mn_j}}{\prod_{j=1}^{k} b_{n_j} \prod_{j=1}^{k-1} (n_j + n_{j+1})}\, s^{n_1} t^{n-n_1} \right). \tag{2.34}$$

The power series $W_m(s,t)$ arises from an integral equation of the type (2.3) as follows. Define the power series $\rho(s;\lambda)$, where λ is a complex parameter, by

$$\rho(s;\lambda) \;\equiv\; \sum_{n=1}^{\infty} (1 + \lambda\delta_{mn}) \frac{s^n}{b_n} \;=\; \gamma(s) + \lambda \frac{s^m}{b_m},$$

and let $W_m(s,t;\lambda)$ be the solution to the integral equation

$$W_m(s,t;\lambda) + \int_0^1 \frac{\rho(xs;\lambda)}{x} W_m(tx,t;\lambda)dx \;=\; \rho(s;\lambda). \tag{2.35}$$

By (2.4), $W_m(s,t;\lambda)$ is given by the power series:

$$W_m(s,t;\lambda) = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} (-1)^{k-1} \sum_{n_1+\cdots+n_k=n} \frac{\prod_{j=1}^{k} (1 + \lambda\delta_{mn_j})}{\prod_{j=1}^{k} b_{n_j} \prod_{j=1}^{k-1} (n_j + n_{j+1})}\, s^{n_1} t^{n-n_1} \right). \tag{2.36}$$

Note that $\rho(s;0) = \gamma(s)$, so that the substitution of $\lambda = 0$ into (2.35) yields the equation (2.26), and thus by Proposition 2.8 we have that $W_m(s,t;0) = \mu(s-t)s$.

From the power series (2.36) we observe that

$$\frac{d}{d\lambda} W_m(s,t;\lambda)\Big|_{\lambda=0} = W_m(s,t).$$

Differentiating the equation (2.35) at $\lambda = 0$ yields

$$W_m(s,t) + \int_0^1 \frac{\gamma(xs)}{x} W_m(tx,t)dx + \int_0^1 \frac{s^m}{b_m} x^{m-1} \Phi_0(tx,t)dx \;=\; \frac{s^m}{b_m}, \tag{2.37}$$

where $\Phi_0(s,t) \equiv W_m(s,t;0) = \mu(s-t)s$. By (2.26) and Proposition 2.8, the expression

$$\int_0^1 \frac{-x^{m-1}}{b_m} \Phi_0(tx,t)dx + \frac{1}{b_m}$$

is the coefficient of s^m in $\mu(s-t)s$, which by Taylor's theorem is $\dfrac{\mu^{(m-1)}(-t)}{(m-1)!}$, and so (2.37) is equivalent to

$$W_m(s,t) + \int_0^1 \frac{\gamma(xs)}{x} W_m(tx,t)dx \;=\; \frac{\mu^{(m-1)}(-t)}{(m-1)!} s^m. \tag{2.38}$$

The solution to (2.38) is given by the next result, which follows from Proposition 2.11.

Proposition 2.12.

$$W_m(s,t) \;=\; \frac{\mu^{(m-1)}(-t)}{(m-1)!}P_m(s,t). \quad \square$$

Consequently, using Proposition 2.10, we draw the following conclusion.

Corollary 2.13.

$$W_m(t,t) \;=\; 4^m m\; t^m \left(\mu^{(m-1)}(-t)\right)^2. \quad \square$$

Define the power series $\Phi_1(s,t;z)$, where z is a complex variable, by

$$\Phi_1(s,t;z) \;\equiv\; \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(-1)^{k-1}\sum_{n_1+\cdots+n_k=n}\frac{\sum_{j=1}^{k} z^{n_j}}{\prod_{j=1}^{k} b_{n_j}\prod_{j=1}^{k-1}(n_j+n_{j+1})}\; s^{n_1}t^{n-n_1}\right).$$

Observe from (2.34) that, for $m \geq 1$,

$$\frac{1}{2\pi i}\oint \frac{\Phi_1(s,t;z)}{z^{m+1}}dz = W_m(s,t)$$

(contour integral over the unit circle), and thus

$$\Phi_1(s,t;z) = \sum_{m=1}^{\infty} W_m(s,t)z^m. \tag{2.39}$$

Define polynomials $\Psi_n(z)$ by

$$\Psi_n(z) = \sum_{k=1}^{n}(-1)^{k+n}\frac{2k\;(2n-1)!}{(n+k)!\;(n-k)!}z^k, \qquad n = 1,2,\ldots. \tag{2.40}$$

Proposition 2.14.

$$\Phi_1(t,t;z) = \sum_{n=1}^{\infty}\Psi_n(z)\frac{t^n}{b_n}.$$

Proof. By (2.39) and Corollary 2.13,

$$\Phi_1(t,t;z) = \sum_{m=1}^{\infty} W_m(t,t)z^m = \sum_{m=1}^{\infty} 4^m m t^m \left(\mu^{(m-1)}(-t)\right)^2 z^m.$$

Using Identity A.4,

$$\sum_{k=0}^{\ell}\frac{(k+m-1)!}{k!\;b_{k+m}}\frac{(\ell-k+m-1)!}{(\ell-k)!\;b_{\ell-k+m}} = \frac{1}{4^{2m+\ell}\ell!\;(2m+\ell)!}\binom{2m+2\ell}{m+\ell}, \tag{2.41}$$

we have that

$$\begin{aligned}\left(\mu^{(m-1)}(-t)\right)^2 &= \left(\sum_{k=0}^{\infty}\frac{(k+m-1)!}{k!\;b_{k+m}}(-1)^k t^k\right)^2\\ &= \sum_{\ell=0}^{\infty}\left(\sum_{k=0}^{\ell}\frac{(k+m-1)!}{k!\;b_{k+m}}\frac{(\ell-k+m-1)!}{(\ell-k)!\;b_{\ell-k+m}}\right)(-1)^\ell t^\ell\\ &= \sum_{\ell=0}^{\infty}\frac{1}{4^{2m+\ell}\ell!\;(2m+\ell)!}\binom{2m+2\ell}{m+\ell}(-1)^\ell t^\ell,\end{aligned}$$

and thus

$$\begin{aligned}\Phi_1(t,t;z) &= \sum_{m=1}^{\infty}\sum_{\ell=0}^{\infty} 4^m m \frac{1}{4^{2m+\ell}\ell!\ (2m+\ell)!}\binom{2m+2\ell}{m+\ell}(-1)^\ell t^{\ell+m} z^m \\ &= \sum_{n=1}^{\infty}\left(\sum_{m=1}^{n} m\frac{z^m}{4^n(n-m)!\ (n+m)!}\binom{2n}{n}(-1)^{n-m}\right)t^n \\ &= \sum_{n=1}^{\infty}\Psi_n(z)\frac{t^n}{b_n}.\end{aligned}$$

□

Corollary 2.15. *For $m \geq 0$ and $n \geq 1$,*

$$\sum_{k=1}^{n}(-1)^{k-1}\sum_{n_1+\cdots+n_k=n}\frac{\sum_{j=1}^{k} n_j^m}{\prod_{j=1}^{k} b_{n_j}\prod_{j=1}^{k-1}(n_j+n_{j+1})} = \frac{1}{b_n}\sum_{k=1}^{n}\frac{(-1)^{k+n}2k^{m+1}(2n-1)!}{(n+k)!(n-k)!}.$$

Proof. By Proposition 2.14,

$$\sum_{k=1}^{n}(-1)^{k-1}\sum_{n_1+\cdots+n_k=n}\frac{\sum_{j=1}^{k} z^{n_j}}{\prod_{j=1}^{k} b_{n_j}\prod_{j=1}^{k-1}(n_j+n_{j+1})} = \frac{\Psi_n(z)}{b_n}.$$

Making the substitution $z = e^y$ and then taking the m-th derivative at $y = 0$ of the resulting equation yields the conclusion of the corollary. □

Proposition 2.16. *For $n \geq 1$,*

$$2\sum_{k=1}^{n}(-1)^k k^{2p}\binom{2n}{n-k} = \begin{cases} 0 & \text{if } 1 \leq p < n, \\ (-1)^n(2n)! & \text{if } p = n. \end{cases}$$

Proof. By the binomial theorem,

$$(-1)^n x^{-n}(1-x)^{2n} = \binom{2n}{n} + \sum_{k=1}^{n}(-1)^k\binom{2n}{n-k}(x^k + x^{-k}).$$

Making the substitution $x = e^y$, we obtain that

$$(-1)^n e^{-ny}(1-e^y)^{2n} = \binom{2n}{n} + 2\sum_{k=1}^{n}(-1)^k\binom{2n}{n-k}\cosh(ky). \tag{2.42}$$

Since $e^{-ny}(1-e^y)^{2n} = y^{2n} + o(y^{2n+1})$, the $2p$-th derivative of the left side of (2.42) at $y = 0$ equals 0 for $0 \leq p < n$ and equals $(-1)^n(2n)!$ for $p = n$. For $p \geq 1$, the $2p$-th derivative of the right side of (2.42) at $y = 0$ equals $2\sum_{k=1}^{n}(-1)^k k^{2p}\binom{2n}{n-k}$. □

Combining Corollary 2.15 and Proposition 2.16 yields the next proposition.

Proposition 2.17. *For $n \geq 1$,*

$$\sum_{k=1}^{n}(-1)^{k-1}\sum_{n_1+\cdots+n_k=n}\frac{\sum_{j=1}^{k} n_j^{2p-1}}{\prod_{j=1}^{k} b_{n_j}\prod_{j=1}^{k-1}(n_j+n_{j+1})} = \begin{cases} 0 & \text{if } 1 \leq p < n, \\ 4^{-n}\binom{2n-1}{n} & \text{if } p = n. \end{cases}$$ □

The following theorem verifies Conjecture 1.16 in the case $d = 1$.

Theorem 2.18. *For $n \geq 1$,*

$$4\sum_{k=1}^{n}(-1)^{k-1}\sum_{n_1+\cdots+n_k=n}\frac{\sum_{j=1}^{k}\sum_{\ell=1}^{n_j}(2\ell-1)^{2p}}{\prod_{j=1}^{k}b_{n_j}\prod_{j=1}^{k-1}(n_j+n_{j+1})} = \begin{cases} 0 & \text{if } 1 \leq p < n-1, \\ \frac{1}{n}\binom{2n-2}{n-1} & \text{if } p = n-1. \end{cases}$$

Proof. For $n \geq 1$ and $m \geq 0$ define

$$g(n,m) \equiv \sum_{k=1}^{n}(-1)^{k-1}\sum_{n_1+\cdots+n_k=n}\frac{\sum_{j=1}^{k}n_j^m}{\prod_{j=1}^{k}b_{n_j}\prod_{j=1}^{k-1}(n_j+n_{j+1})}.$$

By (2.31) and (2.32), $\sum_{\ell=1}^{n_j}(2\ell-1)^{2p} = u_{2p}(n_j)$, where

$$u_{2p}(x) = \sum_{j=0}^{p}\frac{4^j D_{2p-2j}}{2j+1}\binom{2p}{2j}x^{2j+1}$$

and the rational numbers D_i are given by (2.33). Thus

$$\sum_{k=1}^{n}(-1)^{k-1}\sum_{n_1+\cdots+n_k=n}\frac{\sum_{j=1}^{k}\sum_{\ell=1}^{n_j}(2\ell-1)^{2p}}{\prod_{j=1}^{k}b_{n_j}\prod_{j=1}^{k-1}(n_j+n_{j+1})} = \sum_{j=0}^{p}\frac{4^j D_{2p-2j}}{2j+1}\binom{2p}{2j}g(n,2j+1).$$

By Proposition 2.17, $g(n,2j+1) = 0$ if $0 \leq j < n-1$ and $g(n,2n-1) = 4^{-n}\binom{2n-1}{n}$. Thus the right side of the above identity is equal to 0 if $1 \leq p < n-1$ and is equal to

$$\frac{4^{n-1}}{2n-1}g(n,2n-1) = \left(\frac{4^{n-1}}{2n-1}\right)4^{-n}\binom{2n-1}{n} = \frac{1}{4n}\binom{2n-2}{n-1}$$

if $p = n-1$. □

Proposition 2.17 verifies the following conjecture in the case $d = 1$.

Conjecture 2.19. *For $n \geq 1$ and $m_1, \ldots, m_d \geq 0$,*

$$\sum_{k=1}^{n}(-1)^{k-1}\sum_{n_1+\cdots+n_k=n}\frac{\prod_{\ell=1}^{d}\left(\sum_{j=1}^{k}n_j^{2m_\ell+1}\right)}{\prod_{j=1}^{k}b_{n_j}\prod_{j=1}^{k-1}(n_j+n_{j+1})} = \begin{cases} 0 & \text{if } \sum_{\ell=1}^{d}m_\ell < n-1, \\ n^{d-2}4^{-n}\prod_{\ell=1}^{d}\binom{2m_\ell}{m_\ell}(2m_\ell+1) & \text{if } \sum_{\ell=1}^{d}m_\ell = n-1. \end{cases}$$ □

Remark 2.20. It is not difficult to see that Conjecture 2.19 implies Conjecture 1.16. In fact, the two conjectures are equivalent.

3. Topological Consequences

In this section, we indicate two topological consequences of the previous results. These corollaries are independent of Conjectures 1.9 and 1.16. The first result is an immediate consequence of Theorem 2.18. The only additional fact one needs is that the Conway polynomial of a fibered knot of genus g is a polynomial in z^2 of degree g.

Corollary 3.1. *Let $K \subset N$ be a fibered knot with Conway polynomial*

$$\nabla_K(z) = C_0 + C_2 z^2 + \cdots C_{2g} z^{2g}$$

and let $G_K = \pi_1(N \setminus \tau(K))$ denote the knot group and $\ell \in G_K$ the element represented by the longitude of K. If C_{2n} is the first nonzero coefficient of $\nabla_K(z)$ with $n \geq 1$, then

$$\lambda_{n+1,\omega}(K) = \frac{1}{n+1}\binom{2n}{n} C_{2n} C_0^{n-1}.$$

If N is a rational homology 3-sphere, then $C_0 \neq 0$, and it follows that there exists an irreducible representation $\rho : G_K \to \mathrm{SU}(n+1)$ such that $\rho(\ell) = \omega$.

Remark 3.2. Our computation of the universal polynomials and the hypotheses above imply that $\lambda_{m,\omega}(K) = 0$ for $2 \leq m \leq n$. Theorem 1.7 of [8] states that, for N a rational homology sphere, $\lambda_{m,\omega}(K) \neq 0$ for some $2 \leq m \leq g+1$. Corollary 3.1 identifies the first nonvanishing invariant and shows that it can be easily computed from the Conway polynomial of K.

To put our results into perspective, it is helpful to further compare them to the results in [8]. Theorem 1.7 of [8] can also be deduced from Corollary 3.1, since $C_{2n} \neq 0$ for some n with $1 \leq n \leq g$. Moreover, using the notation of [8], Theorem 2.18 shows that the coefficient of $x_0^{n-2} x_{2n-2}$ in $p_{n,\omega}$ is $1/n!(n-1)!$. Proposition 13.3 (b) of [8] claims correctly that this coefficient is nonzero, but the proof of 13.3 (b) incorrectly asserts that $r = r(n) - r_1$, i.e. that $r_i = 0$, and direct computation shows that this is not true. Theorem 2.18 should be viewed as filling the gap in Proposition 13.3 (b), which is important because Theorems 1.6 and 1.7 of [8] depend on 13.3 (b) in an essential way.

Next, we consider the situation where N is not a rational homology sphere. In this case, we can determine the invariants $\lambda_{n,\omega}$ for all n from Theorem 2.5.

Corollary 3.3. *If $K \subset N$ is a fibered knot and $H_1(N;\mathbb{Q}) \neq 0$, then the Conway polynomial has the form $\nabla_K(z) = C_2 z^2 + \cdots + C_{2g} z^{2g}$. In particular, since $C_0 = 0$, it follows that $\lambda_{n,\omega}(K) = C_2^{n-1}$.*

Appendix A. Identities

In this appendix, we collect a number of combinatorial identities upon which some of our previous results depend. For the most part, the proofs are elementary applications of the residue theorem and are included for the reader's convenience.

Identity A.1. For $i, j, q > 0$,

$$\sum_{k=q+1}^{j} (-1)^k \binom{i+k-1}{i-1}\binom{j-q-1}{k-q-1} = (-1)^j \binom{i+q}{j}.$$

Remark A.2. This is true for $i + q < j$ provided $\binom{n}{\ell}$ is defined to be 0 when $n < \ell$.

Identity A.3. For $n > q \geq 0$,

$$\sum_{k=0}^{n-q-1} (-1)^k \binom{n-1}{k} \binom{2n-k-q-2}{n-1} = 1.$$

Identity A.4. For $k, m > 0$,

$$\sum_{k=0}^{\ell} \binom{2m+2\ell}{m+k} \binom{\ell}{k} = \binom{2m+2\ell}{m+\ell}.$$

Identity A.5. For $i > 0$ and $j > q \geq 0$,

$$\sum_{\ell=q}^{j-1} (-1)^\ell \binom{j+\ell-q-1}{j-1} \binom{i+j}{\ell} \binom{i+j-\ell-1}{i} = (-1)^{j-1} \binom{i+2j-q-1}{j-1}.$$

Identity A.6. Set $\beta_n \equiv \frac{n}{4^{2n}} \binom{2n}{n}^2$. Then

$$\sum_{q=0}^{j-1} (-1)^{j+q} \binom{j-1}{q} \binom{i+q}{j} (\beta_{i+q} - \beta_{i+q+1}) = \frac{i!\,(j-1)!}{(i+j)!} \beta_i \beta_j.$$

Proofs. All but the last identity are exhibited using the method of residues [7]. Recall that for a meromorphic function $f(z)$, its residue at infinity is defined as

$$\operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0} f(\tfrac{1}{z})/z^2.$$

In particular, if $f(z) = p(z)/q(z)$ is a rational function, where $p(z)$ and $q(z)$ are polynomials with $\deg q \geq \deg p + 2$, then one can easily verify that $\operatorname{Res}_{z=\infty} f(z) = 0$.

Proof of Identity A.1. Assume $i, j, q > 0$. Substituting $\operatorname{Res}_{x=0} (1+x)^n / x^{\ell+1}$ for $\binom{n}{\ell}$, summing the resulting geometric series, and applying the residue theorem to the incident rational function, we have

$$\begin{aligned}
&\sum_{k=q+1}^{j} (-1)^k \binom{i+k-1}{i-1} \binom{j-q-1}{k-q-1} \\
&\quad = \operatorname*{Res}_{x=0} \operatorname*{Res}_{y=0} \sum_{k=q+1}^{j} \frac{(1-x)^{i+k-1}}{x^{k+1}} \frac{(1+y)^{j-q-1}}{y^{k-q}} \\
&\quad = \operatorname*{Res}_{x=0} \operatorname*{Res}_{y=0} \sum_{k=-\infty}^{j} \frac{(1-x)^{i+k-1}}{x^{k+1}} \frac{(1+y)^{j-q-1}}{y^{k-q}} \\
&\quad = \operatorname*{Res}_{x=0} \operatorname*{Res}_{y=0} \frac{(1-x)^{i+j}(1+y)^{j-q-1}}{x^{j+1} y^{j-q} (1-x-xy)} \\
&\quad = \operatorname*{Res}_{x=0} \operatorname*{Res}_{y=\left[\frac{1-x}{x}\right]} \frac{-(1-x)^{i+j}(1+y)^{j-q-1}}{x^{j+1} y^{j-q} (1-x-xy)} \\
&\quad = \operatorname*{Res}_{x=0} \frac{(1-x)^{i+q}}{x^{j+1}} = (-1)^j \binom{i+q}{j}.
\end{aligned}$$

□

Proof of Identity A.3. Assume $n > q \geq 0$ and argue as before. We get

$$\begin{aligned}
&\sum_{k=0}^{n-q-1} (-1)^k \binom{2n-k-q-2}{n-1}\binom{n-1}{k} \\
&\quad = \operatorname*{Res}_{x=0}\operatorname*{Res}_{y=0} \sum_{k=-\infty}^{n-q-1} \frac{(1-x)^{n-1}}{x^{k+1}}\frac{(1+y)^{2n-q-k-2}}{y^n} \\
&\quad = \operatorname*{Res}_{x=0}\operatorname*{Res}_{y=0} \frac{(1-x)^{n-1}(1+y)^{n-1}}{x^{n-q}y^n(1-x-xy)} = 1.
\end{aligned}$$

□

Proof of Identity A.4. Assume $k, m > 0$. Then

$$\begin{aligned}
\sum_{k=0}^{\ell} \binom{2m+2\ell}{m+k}\binom{\ell}{k} &= \operatorname*{Res}_{x=0}\operatorname*{Res}_{y=0} \sum_{k=-\infty}^{\ell} \frac{(1+x)^{2m+\ell}}{x^{m+k+1}}\frac{(1+y)^{\ell}}{y^{k+1}} \\
&= \operatorname*{Res}_{x=0}\operatorname*{Res}_{y=0} \frac{(1+x)^{2m+\ell}(1+y)^{\ell}}{x^{\ell+m+1}y^{\ell+1}(1-xy)} = \binom{2m+2\ell}{m+\ell}.
\end{aligned}$$

□

Proof of Identity A.5. By reindexing, Identity A.5 is easily seen to be equivalent to

$$\binom{i+2j-q-1}{j-1} = \sum_{k=0}^{j-q-1} (-1)^k \binom{2j-k-q-2}{j-1}\binom{i+j}{j-k-1}\binom{i+k}{k}.$$

We prove the above formula by making use of Identity A.3. First, notice that

$$\begin{aligned}
&\sum_{k=0}^{j-q-1} (-1)^k \binom{2j-k-q-2}{j-1}\binom{i+j}{j-k-1}\binom{i+k}{k} \\
&\quad = \operatorname*{Res}_{x=0}\operatorname*{Res}_{y=0}\operatorname*{Res}_{z=0} \sum_{k=-\infty}^{j-q-1} \frac{(1+x)^{2j-k-q-2}}{x^j}\frac{(1+y)^{i+j}}{y^{i+k+2}}\frac{(1-z)^{i+k}}{z^{k+1}} \\
&\quad = \operatorname*{Res}_{x=0}\operatorname*{Res}_{y=0}\operatorname*{Res}_{z=0} \frac{(1+x)^{j-1}(1+y)^{i+j}(1-z)^{i+j-q}}{x^j y^{i+j-q+1} z^{j-q}(1-xyz-yz-z)} \qquad \text{(A.1)} \\
&\quad = \operatorname*{Res}_{y=0}\operatorname*{Res}_{z=0} \frac{(1+y)^{i+j}(1-z)^{i+j-q}}{y^{i+j-q+1}z^{j-q}} \left(\operatorname*{Res}_{x=\left[\frac{1-z}{yz}-1\right]} \frac{-(1+x)^{j-1}}{x^j(1-xyz-yz-z)} \right) \\
&\quad = \operatorname*{Res}_{y=0}\operatorname*{Res}_{z=0} \frac{(1+y)^{i+j}}{y^{i+j-q+1}}\frac{(1-z)^{i+2j-q-1}}{z^{j-q}(1-z-yz)^j}.
\end{aligned}$$

Using the binomial theorem and the equation: $\frac{1}{(1-w)^{n+1}} = \sum_{m=0}^{\infty}\binom{n+m}{n}w^m$, where $w = z(1+y)$, we find that

$$\begin{aligned}
&\frac{(1-z)^{i+2j-q-1}}{(1-z-yz)^j} \\
&\quad = \sum_{\ell=0}^{i+2j-q-1}\sum_{m=0}^{\infty}(-1)^{\ell}\binom{i+2j-q-1}{\ell}\binom{j+m-1}{j-1}(1+y)^m z^{\ell+m}.
\end{aligned}$$

Hence

$$\operatorname*{Res}_{z=0} \frac{(1-z)^{i+2j-q-1}}{z^{j-q}(1-z-yz)^j} = \sum_{\ell=0}^{j-q-1} (-1)^\ell \binom{i+2j-q-1}{\ell} \binom{2j-q-\ell-2}{j-1} (1+y)^{j-q-\ell-1}.$$

Inserting this into equation (A.2), we see that that expression equals

$$\begin{aligned}\sum_{\ell=0}^{j-q-1} (-1)^\ell \binom{i+2j-q-1}{\ell} \binom{2j-q-\ell-2}{j-1} \binom{i+2j-q-\ell-1}{i+j-q} \\ = \binom{i+2j-q-1}{j-1} \sum_{\ell=0}^{j-q-1} (-1)^\ell \binom{2j-q-\ell-2}{j-q-1} \binom{j-q-1}{\ell} \\ = \binom{i+2j-q-1}{j-1}.\end{aligned}$$

This used the equation

$$\sum_{\ell=0}^{j-q-1} (-1)^\ell \binom{j-q-1}{\ell} \binom{2j-q-\ell-2}{j-q-1} = 1,$$

which is a reformulation of Identity A.3. □

Proof of Identity A.6. For notational convenience, set

$$B_{i,j} = \sum_{q=0}^{j-1} (-1)^{j+q} \binom{j-1}{q} \binom{i+q}{j} (\beta_{i+q} - \beta_{i+q+1}).$$

The proof proceeds by induction on $i > 0$. Using the relation

$$n(n+1)(\beta_n - \beta_{n+1}) = \beta_1 \beta_n, \tag{A.2}$$

which is easily verified, it is not difficult to establish Identity A.6 for $i = 1$ and for all $j > 0$. This gets the induction started.

To prove the inductive step, we use the formula

$$(i-1)B_{i,j} = (i-j-1)B_{i-1,j} + (j+1)B_{i-1,j+1}, \tag{A.3}$$

which is valid for $i > 1$ and $j > 0$. We first show that equation (A.3) and the inductive hypothesis imply Identity A.6, and afterwards we will prove equation (A.3). Fix $i > 1$ and assume that $B_{i-1,k} = (i-1)!\,(k-1)!\beta_{i-1}\beta_k/(i+k-1)!$ for

all $k > 0$. Inserting this into the right hand side of equation (A.3), we find that

$$\begin{aligned}
(i-1)B_{i,j} &= (i-j-1)\left(\frac{(i-1)!\,(j-1)!}{(i+j-1)!}\right)\beta_{i-1}\beta_j \\
&\quad + (j+1)\left(\frac{(i-1)!\,j!}{(i+j)!}\right)\beta_{i-1}\beta_{j+1} \\
&= [(i+j)(i-j-1)\beta_j + j(j+1)\beta_{j+1}]\,\frac{(i-1)!\,(j-1)!}{(i+j)!}\beta_{i-1} \\
&= \left[(i+j)(i-j-1)\beta_j + (j^2+j+\tfrac{1}{4})\beta_j\right]\left(\frac{(i-1)!\,(j-1)!}{(i+j)!}\right)\beta_{i-1} \qquad \text{(A.4)} \\
&= \left(\frac{(i-1)!\,(j-1)!}{(i+j)!}\right)(i^2-i+\tfrac{1}{4})\beta_{i-1}\beta_j \\
&= \left(\frac{(i-1)!\,(j-1)!}{(i+j)!}\right)i(i-1)\beta_i\beta_j.
\end{aligned}$$

This uses the relation

$$(n^2+n+\tfrac{1}{4})\beta_n = n(n+1)\beta_{n+1},$$

which follows directly from equation (A.2). It is applied twice to the equation above; once to the term in the brackets in the second line with $n = j$, and again to the fourth line with $n = i-1$. Now Identity A.6 is a direct consequence of equation (A.4) and induction, and it only remains to prove equation (A.3). Since this fact makes no reference to the special properties of the β_i's, we will prove a slightly more general version of equation (A.3).

With this in mind, define polynomials (cf. the formula for $B_{i,j}$ given at the end of the proof of Lemma 2.7) by setting

$$B_{i,j}(x) = \sum_{q=0}^{j-1}(-1)^{j+q}\binom{j-1}{q}\binom{i+q}{j}x^{i+q},$$

and consider the analog of equation (A.3) given by

$$(i-1)B_{i,j}(x) = (i-j-1)B_{i-1,j}(x) + (j+1)B_{i-1,j+1}(x). \qquad \text{(A.5)}$$

It is clear that this assertion implies equation (A.3). Writing

$$\begin{aligned}
B_{i-1,j}(x) &= \sum_{q=0}^{j-1}(-1)^{j+q}\binom{j-1}{q}\binom{i+q-1}{j}x^{i+q-1}, \\
B_{i-1,j+1}(x) &= \sum_{q=0}^{j}(-1)^{j+q+1}\binom{j}{q}\binom{i+q-1}{j+1}x^{i+q-1},
\end{aligned}$$

and equating coefficients of x^{i+q-1}, we see that equation (A.5) follows from the collection of equations, valid for $q \geq 0$:

$$\begin{aligned}
&(j-i+1)\binom{j-1}{q}\binom{i+q-1}{j} + (j+1)\binom{j}{q}\binom{i+q-1}{j+1} \\
&\quad = \binom{i+q-1}{j}\left[(j-i+1)\binom{j-1}{q} + (i+q-j-1)\binom{j}{q}\right] \\
&\quad = \binom{i+q-1}{j}\left[(j-i+1)\left[\binom{j-1}{q} - \binom{j}{q}\right] + q\binom{j}{q}\right] \\
&\quad = \binom{i+q-1}{j}\left[(i-j-1)\binom{j-1}{q-1} + q\binom{j}{q}\right] \\
&\quad = \binom{i+q-1}{j}\left[(i-1)\binom{j-1}{q-1} - j\binom{j-1}{q-1} + q\binom{j}{q}\right] \\
&\quad = (i-1)\binom{i+q-1}{j}\binom{j-1}{q-1}.
\end{aligned}$$

The last step uses the relation $j\binom{j-1}{q-1} = q\binom{j}{q}$, and this completes the proof. □

Appendix B. Table of ν_n for $2 \leq n \leq 10$

$$\nu_2 = y_2$$

$$\nu_3 = 2y_0y_4 + y_2^2$$

$$\nu_4 = 5y_0^2y_6 + 7y_0y_2y_4 + y_2^3$$

$$\nu_5 = 14y_0^3y_8 + 26y_0^2y_2y_6 + 11y_0^2y_4^2 + 16y_0y_2^2y_4 + y_2^4$$

$$\nu_6 = 42y_0^4y_{10} + y_0^3(98y_2y_8 + 78y_4y_6) + y_0^2(82y_2^2y_6 + 68y_2y_4^2) + 30y_0y_2^3y_4 + y_2^5$$

$$\begin{aligned}
\nu_7 = {} & 132y_0^5y_{12} + y_0^4(372y_2y_{10} + 288y_4y_8 + 134y_6^2) + y_0^3(398y_2^2y_8 + 620y_2y_4y_6) \\
& + 86y_0^3y_4^3 + y_0^2(202y_2^3y_6 + 247y_2^2y_4^2) + 50y_0y_2^4y_4 + y_2^6
\end{aligned}$$

$$\begin{aligned}
\nu_8 = {} & 429y_0^6y_{14} + y_0^5(1419y_2y_{12} + 1083y_4y_{10} + 971y_6y_8) + 1857y_0^4y_2^2y_{10} \\
& + y_0^4(2818y_2y_4y_8 + 1305y_2y_6^2 + 1097y_4^2y_6) + y_0^3(1223y_2^3y_8 + 2805y_2^2y_4y_6) \\
& + 767y_0^3y_2y_4^3 + y_0^2(427y_2^4y_6 + 686y_2^3y_4^2) + 77y_0y_2^5y_4 + y_2^7
\end{aligned}$$

$$\begin{aligned}
\nu_9 = {} & 1430y_0^7y_{16} + y_0^6(5434y_2y_{14} + 4114y_4y_{12} + 3610y_6y_{10} + 1735y_8^2) \\
& + y_0^5(8426y_2^2y_{12}12628y_2y_4y_{10} + 11256y_2y_6y_8 + 4776y_4^2y_8) \\
& + y_0^4(6862y_2^3y_{10} + 15346y_2^2y_4y_8 + 7079y_2^2y_6^2 + 11756y_2y_4^2y_6 + 807y_4^4) \\
& + 4418y_0^4y_4y_6^2 + y_0^3(3148y_2^4y_8 + 9472y_2^3y_4y_6 + 3836y_2^2y_4^3) \\
& + y_0^2(812y_2^5y_6 + 1610y_2^4y_4^2) + 112y_0y_2^6y_4 + y_2^8
\end{aligned}$$

$$\begin{aligned}
\nu_{10} \quad = \quad & 4862y_0^8y_{18} + y_0^7(12778y_8y_{10} + 13618y_6y_{12} + 20878y_2y_{16} + 15730y_4y_{14}) \\
& + y_0^6(37466y_2^2y_{14} + 55792y_2y_4y_{12} + 48664y_2y_6y_{10} + 23352y_2y_8^2) \\
& + y_0^6(20810y_4^2y_{10} + 37012y_4y_6y_8 + 5714y_6^3) \\
& + y_0^5(36454y_2^3y_{12} + 80622y_2^2y_4y_{10}) \\
& + y_0^5(71510y_2^2y_6y_8 + 60038y_2y_4^2y_8 + 55346y_2y_4y_6^2 + 15350y_4^3y_6) \\
& + y_0^4(20876y_2^4y_{10} + 61284y_2^3y_4y_8 + 28178y_2^3y_6^2 + 69420y_2^2y_4^2y_6) \\
& + y_0^3(7152y_2^5y_8 + 26520y_2^4y_4y_6 + 14160y_2^3y_4^3 + 9425y_2y_4^4) \\
& + y_0^2(1428y_2^6y_6 + 3360y_2^5y_4^2) + 156y_0y_2^7y_4 + y_2^9
\end{aligned}$$

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