

ABSOLUTELY CONTINUOUS S.R.B. MEASURES FOR RANDOM LASOTA-YORKE MAPS

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ABSTRACT. A. Lasota and J. A. Yorke proved that a piecewise expanding interval map admits finitely many ergodic absolutely continuous invariant probability measures. We generalize this to the random composition of such maps under conditions which are natural and less restrictive than those previously studied by Morita and Pelikan. For instance our conditions are satisfied in the case of arbitrary random β -transformations, i.e., $x \mapsto \beta x \bmod 1$ on $[0, 1]$ where β is chosen according to any stationary stochastic process (in particular, not necessarily i.i.d.) with values in $]1, \infty[$.

RÉSUMÉ. A. Lasota et J. A. Yorke ont montré qu'une application de l'intervalle dilatante par morceaux admet un nombre fini de mesures de probabilité invariantes et ergodiques absolument continues. Nous généralisons ce résultat à la composition aléatoire de telles applications sous des conditions naturelles, moins restrictives que celles précédemment envisagées par Morita et Pelikan. Par exemple, nos conditions sont satisfaites par toute β -transformation aléatoire, i.e., $x \mapsto \beta x \bmod 1$ sur $[0, 1]$ avec β choisi selon un processus stochastique stationnaire quelconque (en particulier, non-nécessairement i.i.d.) à valeurs dans $]1, \infty[$.

0. INTRODUCTION

0.1. Setting. We are interested in the “chaotic” dynamics defined by maps of the following, well-known type:

Definition 0.1. $f : [0, 1] \rightarrow [0, 1]$ is a **Lasota-Yorke** map if there exists a finite subdivision $a_0 = 0 < a_1 < \dots < a_N = 1$ of the unit interval such that, for each restriction f_i of f to any subinterval $]a_{i-1}, a_i[$, $i = 1, \dots, N$:

1. f_i is a homeomorphism on its image;
2. f_i is non-singular with respect to Lebesgue measure.

We shall assume that N is minimal. We call the intervals $]a_{i-1}, a_i[$ the **intervals of f** , and define $N(f) = N$, $\delta(f) = \inf\{|f'(x)| : x \in [0, 1] \setminus \{a_0, \dots, a_N\}\}$ and $\text{var}(f') = \sup_a \sum_{i=1}^m |f'(a_i) - f'(a_{i-1})|$, $a = (a_0 = 0 < a_1 < \dots < a_m = 1)$, ranging over the finite subdivisions of $[0, 1]$, which is the variation of $|f'|$, a fixed jacobian of f .

By a well-known result of A. Lasota and J. A. Yorke [19], the iteration of any map of this type is “chaotic” whenever the map is expanding (i.e., $\delta(f) > 1$). By

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“chaotic” one means that finitely many absolutely continuous invariant probability measures describe Lebesgue almost all orbits.

On the other hand, in higher dimensions, while everyone seems to think that the analogous theorem is true, the currently known methods need a sufficient amount of expansion to kill the possible accumulation of singularities [12], or need to be restricted to generic situations [6], [9] where this accumulation does not take place (see however [8], [23] and, perhaps more surprisingly, [24]).

In this paper we consider an “intermediate case” of dimension “ $1 + \epsilon$ ”: the random composition of Lasota-Yorke maps. One can define a random dynamical system f on a space M to be a stationary stochastic process $f_1, f_2, \dots : M \rightarrow M$ (i.e., with values in a space of maps) (see for instance [15]). The associated random orbits are the stochastic processes

$$x_0, x_1, \dots \text{ where } x_0 \in X \text{ and } x_{n+1} = f_{n+1}(x_n).$$

One can then ask about the asymptotic behavior of these random orbits. Observe that in general there is no measure simultaneously invariant under all the maps $f(\omega)$, $\omega \in \Omega$. One considers instead S.R.B. measures (Sinai-Ruelle-Bowen measures), defined below, and one asks whether Lebesgue almost all orbits are described by “chaotic”, i.e., absolutely continuous, S.R.B. measures.

We are going to prove that it is indeed the case in this “ $1 + \epsilon$ -dimensional” Lasota-Yorke problem, as in the classical case.

0.2. Results. The stationary process f_1, f_2, \dots can be seen as given by $f_{n+1} = f \circ T^n$, where f is a measurable map defined on some abstract probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and T an automorphism. Write δ_x for the Dirac measure at a point x , f_ω for $f(\omega)$, $\delta(\omega)$ for $\delta(f(\omega))$, etc. Then:

Definition 0.2. A probability measure μ on M is **S.R.B.** for the random dynamical system $(\Omega, \mathcal{A}, \mathbb{P}, T, f)$ (or simply f) if, for \mathbb{P} -almost all $\omega \in \Omega$, the set $B_\omega(\mu)$ of points $x \in M$ such that

$$\frac{1}{n} \sum_{k=0}^{n-1} \delta_{f_{T^k \omega} \circ \dots \circ f_\omega(x)} \xrightarrow{n \rightarrow \infty} \mu \quad \text{vaguely}$$

has positive Lebesgue measure. We call $B_\omega(\mu)$ the (random) **basin** of μ .

In the case where f_ω is independent of ω this reduces to one of the classical definitions of an S.R.B. measure for a deterministic map.

Assumptions. A random Lasota-Yorke map f is **admissible** if the following conditions are satisfied:

$$(A1) \quad \lim_{K \rightarrow \infty} \int_{\Omega} \log \delta(\omega) \wedge K d\mathbb{P} > 0 \quad (\text{possibly } +\infty),$$

$$(A2) \quad \log^+ \frac{N(\omega)}{\delta(\omega)} \text{ is } \mathbb{P}\text{-integrable,}$$

$$(A3) \quad \log^+ \text{var}(1/|f'_\omega|) \text{ is } \mathbb{P}\text{-integrable.}$$

$$(x \wedge y \stackrel{\text{def}}{=} \min(x, y)).$$

We also assume $(\Omega, \mathcal{A}, \mathbb{P}, T)$ to be ergodic (otherwise one has to require (A1) for almost all ergodic components).

Observe that we do not assume any kind of uniformity with respect to ω , except the one which results from the fact that we are looking through an invariant measure \mathbb{P} . It should be noted in this connection that our results do not follow from analogous statements about arbitrary sequences of maps. Indeed, there are sequences of maps f_1, f_2, \dots such that $\delta(f_n) \geq \delta_*$, $N(f_n)/\delta(f_n) \leq K$ and $\text{var}(1/|f'_n|) \leq K$ for some $\delta_* > 1$ and $K < \infty$, and nevertheless, $(f_n \circ \dots \circ f_1)_* m$ converges to a singular measure (e.g., a Dirac measure). This is in contrast to settings like [2].

Moreover, our conditions are natural in the sense that if we suppress any one of the three conditions (A1), (A2) or (A3) then we can find even deterministic counter-examples to the existence of absolutely continuous invariant measures (this is easy for (A1) or (A2); see [13] for (A3)).

As usual, one associates to the random map f the skew-product:

$$\begin{aligned} F : \Omega \times [0, 1] &\longrightarrow \Omega \times [0, 1] \\ (\omega, x) &\longmapsto (T\omega, f_\omega x). \end{aligned}$$

Theorem 0.3. *Let $(\Omega, \mathcal{A}, \mathbb{P}, T, f)$ be an admissible random Lasota-Yorke map.*

*Then the set of F -invariant probability measures absolutely continuous with respect to $\mathbb{P} \times m$ (m being Lebesgue measure), or, for short, **a.c.i.m.**'s, is a **non-empty finite-dimensional simplex**, generated by its ergodic elements.*

Moreover the density of each such measure μ satisfies (after modification on a $\mathbb{P} \times m$ -null set) the condition

$$x \mapsto \frac{d\mu}{d\mathbb{P} \times m}(\omega, x) \text{ has bounded variation for all } \omega \in \Omega,$$

and we have strong convergence to equilibrium in the sense that, for each probability measure ν absolutely continuous with respect to $\mathbb{P} \times m$,

$$\frac{1}{n} \sum_{k=0}^{n-1} \frac{dT^k(\nu)}{d\mathbb{P} \times m} \xrightarrow{n \rightarrow \infty} \frac{d\mu}{d\mathbb{P} \times m} \quad \text{in } L^1(\mathbb{P} \times m),$$

where μ is an a.c.i.m.

Finally, the number r of ergodic a.c.i.m.'s is explicitly bounded by the non-trivial condition

$$(0.4) \quad \int_{\Omega} \log \frac{1 + (N(\omega) - 1)/r}{\delta(\omega)} d\mathbb{P} \geq 0.$$

We formulate a probabilistic corollary of this.

Theorem 0.5. *Any admissible random Lasota-Yorke map admits a finite number of S.R.B. measures, bounded by (0.4).*

Moreover, the following hold:

1. *These finitely many measures together describe almost all orbits in the sense that the union of their random basins has \mathbb{P} -almost surely total Lebesgue measure in $[0, 1]$.*
2. *These S.R.B. measures are absolutely continuous.*

0.3. Discussion of the results. Absolutely continuous S.R.B. measures for random maps have been mostly considered with respect to the related, but different problem of small random perturbations of a given deterministic dynamical system (see [16], [3] and the references there) or in the case of expanding mappings without singularities (see [15], [17], [5], [14], [1] and the references there). Random Lasota-Yorke maps have been considered by S. Pelikan [21] and T. Morita [20].

However, our conditions on the maps are much less restrictive than the ones in these two articles: in both papers the maps f_1, f_2, \dots are given by an i.i.d. process and in [20] the lengths of the intervals of the map are not too small whereas in [21] the maps must even be chosen from a finite set (we note that this last paper includes some situations outside the random composition of maps, as the probability of picking a given map is allowed to depend on the current position in $[0, 1]$).

In particular, one of the motivations of this work was to be able to treat the following example, which could not be done by [21], [20].

0.4. Example. $f(x) = \beta x \pmod{1}$, where β_1, β_2, \dots is an arbitrary stationary and ergodic stochastic process with values in $]1, \infty[$.

This example turns out to be important for some applications (see [6]); for instance, it allows the reduction of the higher dimensional Lasota-Yorke problem to the one restricted to maps with constant Jacobian.

In this example, every critical point c satisfies $f(c+) = f(0) = 0$. Hence, it is easy to see (cf. section 3) that there is **at most one a.c.i.m.**

We also remark that the natural symbolic dynamics defined by our random maps are random subshifts which are *not of finite type*, contrary to the case of random smooth expanding maps investigated by T. Bogenschütz and V.M. Gundlach [4], [5].

0.5. Outline of proof. In the first section, we give **two Lasota-Yorke inequalities** estimating the variation of the density after the application of one given Lasota-Yorke map: one inequality, due to M. Rychlik, is well-known and gives a fine control over the variation provided that the intervals of the map are not too small; the other one is very crude but does not depend on the size of these intervals. Then, by applying these inequalities (the first one most of the time, the second one only when some interval of the map is very small), we prove that, for $h \in \mathcal{H}$, a dense subset of $L^1(\mathbb{P} \times m)$, the functions $\frac{1}{n} \sum_{k=0}^{n-1} \mathcal{L}^k h$ for $n = 1, 2, \dots$ have their variation on each fiber $\omega \times [0, 1]$ uniformly bounded with respect to n .

The second section builds an a.c.i.m. for the skew-product F as an integrable function invariant under the **transfer operator** \mathcal{L} defined by F and $\mathbb{P} \times m$:

$$(\mathcal{L}h)(\omega, x) = \sum_{(\alpha, y) \in F^{-1}(\omega, x)} \frac{1}{|f'_\alpha(y)|} h(\alpha, y)$$

\mathcal{L} satisfies the fundamental property $\int h \cdot \mathcal{L}g \, d\mathbb{P} \times m = \int h \circ F \cdot g \, d\mathbb{P} \times m$ for every $h \in L^\infty(\mathbb{P} \times m)$ and every $g \in L^1(\mathbb{P} \times m)$. Recall that \mathbb{P} is preserved by T .

From the results of the first section, we deduce that the sequence $\frac{1}{n} \sum_{k=0}^{n-1} \mathcal{L}^k h$, $n \geq 1$, is weakly compact in $L^1(\mathbb{P} \times m)$ for each h in a dense subset \mathcal{H} . We can therefore apply the **Mean Ergodic Theorem** and conclude the proof of existence.

Remark 0.6. The transfer operator \mathcal{L} is not even bounded in $L^\infty(\mathbb{P} \times m)$, as we do not assume uniformity of expansion in (A1). Thus we cannot apply the Ionescu-Tulcea-Marinescu Theorem as it is usually done.

In the third section, the **explicit bound on the multiplicity** of the a.c.i.m. will be deduced from the following geometric considerations. Because of the bounded variation property, the support of every a.c.i.m. must contain intervals in almost every fiber $\omega \times [0, 1]$. Now the image of an interval has its length increased unless it contains some discontinuity. Working from this remark, one can see that every

a.c.i.m. must “use” a certain number of the discontinuities. The bound then follows from assumption (A2). Finally we remark that finiteness can be proven much more abstractly.

We conclude by deducing from the theorem about the skew-product (Theorem 0.3) its probabilistic corollary (Theorem 0.5).

0.6. Perspectives. 1. Except for the explicit bound on the number of S.R.B. measures, the results and proofs presented in this paper only depend on having good Lasota-Yorke inequalities for a reasonable notion of variation. Hence they can be readily generalized, in particular, to some *multi-dimensional settings*. See [7].

2. One would like to know the *ergodic and statistical properties* of the S.R.B. measures constructed here. Under an additional and non-trivial assumption of strong topological mixing (“random covering”), we have proved [7] that there is indeed exponential decay of random correlations like in smooth settings [1], [2], [5], [14], [17]. The proof is by Birkhoff cones. But this more precise result does not cover all the maps studied here and leaves open the (exciting if unlikely) possibility that some of those random maps, despite their uniform expansion, would have weak ergodic properties.

3. Are the S.R.B. measures constructed here stable with respect to (suitably defined) small perturbations?

4. Under what hypothesis would one be able to make statements (e.g., quasi-compactness) about a global transfer operator and therefore hope for much more precise results like those of [1].

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1. BOUNDS ON VARIATION

This section establishes fiberwise bounds on the variation of $\mathcal{L}^n h$ for good functions h .

1.1. One-dimensional bounds. We can write $(\mathcal{L}h)(\omega, \cdot) = L_{f(T^{-1}\omega)}h(T^{-1}\omega, \cdot)$, where L_f is the transfer operator associated to the interval map f :

$$L_f h(x) = \sum_{y \in f^{-1}(x)} \frac{1}{|f'_\omega(y)|} h(y) \quad \text{for } h : [0, 1] \rightarrow \mathbb{R}$$

(recall that T is an automorphism).

The basic ingredient is, as in the case of iteration of a fixed map, the following well-known inequality, proved by Rychlik (extending the inequality proved by Lasota and Yorke [19] for maps piecewise C^2 to absolutely continuous maps with bounded variation of the derivative).

Lemma 1.1 (M. Rychlik [22]). *Let $f : [0, 1] \rightarrow [0, 1]$ be a Lasota-Yorke map with $\text{var}(1/|f'|) < \infty$. Let $h : [0, 1] \rightarrow \mathbb{R}$ be a function of bounded variation. Then*

$$\text{var}(L_f h) \leq \frac{3}{\delta(f)} \text{var}(h) + \beta(f) \|h\|_1,$$

where $\beta(f) < \infty$ depends measurably on f .

A key feature of this estimate is the factor of 3 in front of the first term. It could be replaced by any number greater than 2 but not by 1: this is related to the possibility, if $1 < \delta(f) < 2$, to have some interval mapped into itself. A recent paper of M. Blank and G. Keller [3] shows that this can produce unexpected behavior for random perturbations of some Lasota-Yorke maps in the limit when the size of the perturbation goes to zero.

In our setting, however, this factor of 3 turns out to be only a technical problem. We shall overcome it as in the case of the iteration of a fixed map by replacing our map with some high order iterate. However, in contrast to this classical case, the factor in front of the second term may then explode: $\beta(f \circ g)$ is not bounded by a function of $\beta(f)$ and $\beta(g)$. To control this we shall use the following more robust estimate:

Lemma 1.2. *Let $f : [0, 1] \rightarrow [0, 1]$ be a Lasota-Yorke map. Let $h : [0, 1] \rightarrow \mathbb{R}$ be a function of bounded variation. Then*

$$\text{var}(L_f h) \leq \left[\frac{2N(f)}{\delta(f)} + \text{var}(1/|f'|) + \frac{1}{\delta(f)} \right] \text{var}(h) + \left[\frac{2N(f)}{\delta(f)} + \text{var}(1/|f'|) \right] \|h\|_1.$$

Proof. We have

$$\begin{aligned} \text{var}(L_f h) &\leq \sum_{i=1}^N \text{var} \left(\frac{h}{f'} \circ f_i^{-1} \cdot 1_{f([a_{i-1}, a_i])} \right) \\ &\leq \sum_{i=1}^N \left| \frac{h}{f'}(a_{i-1}) \right| + \left| \frac{h}{f'}(a_i) \right| + \max \left(\frac{1}{|f'|} \right) \text{var}_{]a_{i-1}, a_i[}(h) \\ &\quad + \text{var}_{]a_{i-1}, a_i[} \left(\frac{1}{|f'|} \right) \max(|h|), \end{aligned}$$

using the bound $\text{var}(fg) \leq \text{var}(f) \max(|g|) + \max(|f|) \text{var}(g)$. Hence,

$$\begin{aligned} \text{var}(L_f h) &\leq \left[\frac{2N(f)}{\delta(f)} + \text{var} \left(\frac{1}{|f'|} \right) \right] \max(|h|) + \frac{1}{\delta(f)} \text{var}(h) \\ &\leq \left[\frac{2N(f)}{\delta(f)} + \text{var} \left(\frac{1}{|f'|} \right) \right] (\|h\|_1 + \text{var}(h)) + \frac{1}{\delta(f)} \text{var}(h) \\ &\leq \left[\frac{2N(f)}{\delta(f)} + \text{var} \left(\frac{1}{|f'|} \right) + \frac{1}{\delta(f)} \right] \text{var}(h) + \left[\frac{2N(f)}{\delta(f)} + \text{var} \left(\frac{1}{|f'|} \right) \right] \|h\|_1. \end{aligned}$$

□

The above inequality trivially implies

$$\begin{aligned} \text{var}(L_f h) &\leq 6 \left(\frac{N(f)}{\delta(f)} \vee 1 \right) (\text{var}(1/|f'|) \vee 1) \left(\frac{1}{\delta(f)} \vee 1 \right) \cdot \text{var}(h) \\ &\quad + 4 \left(\frac{N(f)}{\delta(f)} \vee 1 \right) (\text{var}(1/|f'|) \vee 1) \cdot \|h\|_1, \end{aligned}$$

where $x \vee y = \max(x, y)$.

An easy induction gives, after some simplifications,

Corollary 1.3. *Let $F = f_1 \circ \dots \circ f_n$ be a composition of Lasota-Yorke maps. Then*

$$\text{var}(L_F h) \leq n6^n \prod_{k=1}^n \left(\frac{N(f_k)}{\delta(f_k)} \vee 1 \right) (\text{var}(1/|f'_k|) \vee 1) \left(\frac{1}{\delta(f_k)} \vee 1 \right) \cdot (\text{var}(h) + \|h\|_1).$$

1.2. Random bound. We are going to deduce from the previous Lasota-Yorke type inequalities the following fact.

Proposition 1.4. *Let f be an admissible random Lasota-Yorke map. Then there exists a \mathbb{P} -almost-everywhere finite function M on Ω such that, and for all $h \in L^1(m)$ with bounded variation, for all $\omega \in \Omega$,*

$$(1.5) \quad \text{var}(L_{f(T^{-1}\omega)} \circ \dots \circ L_{f(T^{-n}\omega)} h) \leq M(\omega)(\text{var}(h) + \|h\|_1) \quad \forall n = 1, 2, \dots$$

and

$$(1.6) \quad \limsup_{n \rightarrow \infty} \text{var}(L_{f(T^{-1}\omega)} \circ \dots \circ L_{f(T^{-n}\omega)} h) \leq M(\omega)\|h\|_1.$$

Proof. Define σ , λ and ϵ according to

$$(1.7) \quad \begin{aligned} 0 \leq \sigma &= \int_{\Omega} \log^+ \frac{1}{\delta} + \log^+ \frac{N}{\delta} + \log^+ \text{var}(1/|f'_\omega|) d\mathbb{P} < \infty, \\ 0 < \lambda &< \lim_{K \rightarrow \infty} \int_{\Omega} \log \delta \wedge K d\mathbb{P}, \\ 0 < \epsilon &< \frac{\lambda}{2(\sigma + \lambda + 3 + \log 12)}. \end{aligned}$$

Write $K(\omega) = \left(\frac{N(\omega)}{\delta(\omega)} \vee 1 \right) (\text{var}(1/|f'_\omega|) \vee 1) (\delta(\omega)^{-1} \vee 1)$. Let R be so large that the following properties hold for all ω in a subset Ω_1 of \mathbb{P} -measure greater than $1 - \epsilon/2$:

$$(1.8) \quad \begin{aligned} \frac{1}{R} \log \delta(\omega) \cdots \delta(T^{R-1}\omega) &\geq \lambda - \epsilon/2, \\ \frac{1}{R} \log K(\omega) \cdots K(T^{R-1}\omega) &\in [\sigma - \epsilon, \sigma + \epsilon]. \end{aligned}$$

We also assume that $R \leq 2^R$, $e^{\epsilon R/2} \geq 3$.

Fix $\beta_* < \infty$ so large that we may assume that $\beta(f) \leq \beta_*$ for $\omega \in \Omega_1$ by reducing Ω_1 by a measure at most $\epsilon/2$. Hence $\mathbb{P}(\Omega_1) > 1 - \epsilon$.

Deleting a null set in Ω , we may assume that for all $\omega \in \Omega$ and for all large n

$$(1.9) \quad \begin{aligned} \frac{1}{n} \#\{k = 1, \dots, n : T^{-kR+j}\omega \notin \Omega_1\} &< \epsilon, \\ \frac{1}{nR} \sum_{k=1}^{nR} \log K(T^{-k}\omega) &\in [\sigma - \epsilon, \sigma + \epsilon], \end{aligned}$$

where $0 \leq j = j(\omega) < R$ (note that T but not T^R is assumed to be ergodic). We set $j = 0$, as this only introduces a finite factor (depending on ω) in the desired bounds (1.5) and (1.6) (this follows from Lemma 1.2).

Let $\omega \in \Omega$ and $n \geq 1$. According to Lemmas 1.1 and 1.2, we have

$$\text{var}(L_\omega h) \leq \alpha(\omega) \text{var}(h) + \beta(\omega)\|h\|_1$$

if we set $\alpha(\omega) = e^{-(\lambda-\epsilon)R} \geq 3e^{-(\lambda-\epsilon/2)R}$ and $\beta(\omega) = \beta_*$ for $\omega \in \Omega_1$, and $\alpha(\omega) = \beta(\omega) = 12^R \prod_{i=0}^{R-1} K(T^i\omega)$ otherwise.

Let $h : [0, 1] \rightarrow \mathbb{R}$ be an integrable function of bounded variation (which we think of as a density on $T^{-nR}\omega \times [0, 1]$). Set $v_0 = \text{var}(h)$ and

$$v_{k+1} = \alpha(T^{-(n-k)R}\omega)v_k + \beta(T^{-(n-k)R}\omega)\|h\|_1 \quad \text{for } k = 0, 1, \dots, n - 1.$$

We have $\text{var}(L_{f(T^{-(n-k)R-1}\omega) \circ \dots \circ f(T^{-nR}\omega)}h) \leq v_k$. By induction we get from the definition of the v_k 's that

$$(1.10) \quad v_{k+1} = \alpha(T^{-nR}\omega) \dots \alpha(T^{-(n-k)R}\omega) \text{var}(h) + \sum_{j=0}^k \beta(T^{-(n-j)R}\omega) \cdot \alpha(T^{-(n-j-1)R}\omega) \dots \alpha(T^{-(n-k)R}\omega) \cdot \|h\|_1.$$

We compute (using (1.8) and (1.9))

$$\begin{aligned} \prod_{\substack{1 \leq k \leq n \\ T^{-kR}\omega \notin \Omega_1}} \alpha(T^{-kR}\omega) &= \prod_{1 \leq k \leq n} 12^R K(T^{-kR}\omega) \dots K(T^{-kR+R-1}\omega) \\ &\quad / \prod_{\substack{1 \leq k \leq n \\ T^{-kR}\omega \in \Omega_1}} 12^R K(T^{-kR}\omega) \dots K(T^{-kR+R-1}\omega) \\ &\leq 12^{nR} e^{(\sigma+\epsilon)nR} / \left[12^R e^{(\sigma-\epsilon)R} \right]^{(1-\epsilon)n} \\ &\leq e^{\epsilon(\sigma+2+\log 12)nR}. \end{aligned}$$

Hence

$$\begin{aligned} \alpha(T^{-nR}\omega) \dots \alpha(T^{-R}\omega) &\leq e^{-(\lambda-\epsilon)(1-\epsilon)nR} e^{\epsilon(\sigma+2+\log 12)nR} \\ &\leq e^{-(\lambda-\epsilon(\sigma+\lambda+\log 12+3))nR}. \end{aligned}$$

This holds for n large enough. Equivalently, for all $n \geq 0$,

$$\alpha(T^{-nR}\omega) \dots \alpha(T^{-R}\omega) \leq C_1(\omega) e^{-(\lambda-\epsilon(\sigma+\lambda+\log 12+3))nR}$$

for some measurable and \mathbb{P} -almost-everywhere finite function C_1 . Remark that $\lambda - \epsilon(\sigma + \lambda + \log 12 + 3) > \lambda/2 > \epsilon$, because of (1.7).

On the other hand, the function $\log K$ is \mathbb{P} -integrable. Therefore, the convergence in the Birkhoff ergodic theorem implies that $\lim_{n \rightarrow \infty} \frac{1}{n} \log K(T^{-n}\omega) = 0$ a.e. Hence, there exists a measurable and \mathbb{P} -almost-everywhere finite function C_2 such that

$$\beta(T^{-kR}\omega) \leq C_2(\omega) e^{\epsilon k} \quad \forall \omega \in \Omega \text{ and } \forall k = 0, 1, 2, \dots$$

(1.10) gives

$$\begin{aligned} \text{var}(L_{f(T^{-1}\omega) \circ \dots \circ f(T^{-nR}\omega)}h) &\leq v_n \\ &\leq C_1(\omega) e^{-\frac{\lambda}{2}nR} \text{var}(h) + \sum_{j=0}^{n-1} C_2(\omega) e^{(n-j)\epsilon} \cdot C_1(\omega) e^{-\frac{\lambda}{2}(n-j-1)R} \cdot \|h\|_1. \end{aligned}$$

Proposition 1.4 is now obvious. □

2. EXISTENCE OF AN A.C.I.M. FOR THE SKEW-PRODUCT

This section is devoted to the construction of one F -invariant $\mathbb{P} \times m$ -density. Obviously, h is an invariant density iff it is a fixed point for the transfer operator \mathcal{L} . Therefore we are going to build an invariant density by trying to take some weak limit of $\frac{1}{n} \sum_{k=0}^{n-1} \mathcal{L}^k 1$.

We shall use the following “fibered” Mean Ergodic Theorem:

Proposition 2.1. *Let (Ω, \mathbb{P}) and (I, m) be two probability spaces and let T be an automorphism of (Ω, \mathbb{P}) . Let \mathcal{L} be a positive contraction on $L^1(\mathbb{P} \times m)$ (i.e., \mathcal{L} maps nonnegative functions to nonnegative functions and $\|\mathcal{L}\| \leq 1$) with the following properties:*

1. *For some measurable $M : \Omega \rightarrow [0, \infty[$, for all h in some subset $\mathcal{H} \subset L^\infty(\mathbb{P} \times m)$ dense in $L^1(\mathbb{P} \times m)$, there exists $C(h) < \infty$ such that for all $\omega \in \Omega$ and all $n = 1, 2, \dots$*

$$\|(\mathcal{L}^n h)(\omega, \cdot)\|_\infty \leq C(h)M(\omega),$$

2. *For all $\omega \in \Omega$ we have $m((\mathcal{L}h)(\omega, \cdot)) = m(h(T^{-1}\omega, \cdot))$ for all $h \in L^1(\mathbb{P} \times m)$.*

Then the averages $\frac{1}{n} \sum_{k=0}^{n-1} \mathcal{L}^k$ converge in $L^1(\mathbb{P} \times m)$ to Q , the projection on the fixed space of \mathcal{L} with kernel the closure of $(\text{Id} - \mathcal{L})L^1(\mathbb{P} \times m)$.

If $\|M\|_\infty < \infty$, condition (1) is simply that the $\mathcal{L}^n h$ are bounded uniformly in n , and this proposition reduces to the Mean Ergodic Theorem [10, VIII.5.5]. But, because of the lack of uniformity in our assumptions, even \mathcal{L} by itself is not a bounded operator on $L^\infty(\mathbb{P} \times m)$; this extension is therefore necessary.

Condition (2) means that m is preserved by \mathcal{L} fiberwise above T . Observe that this condition, together with positivity and the T -invariance of \mathbb{P} , implies that Q is an isometry on positive L^1 -functions. Hence

Corollary 2.2. *Let \mathcal{L} be as above. Then there exists $h \in L^1(\mathbb{P} \times m)$ with $h \geq 0$ and $\|h\|_1 = 1$ such that $\mathcal{L}h = h$.*

Before proving this, we apply it to our problem:

Construction of an a.c.i.m. Proposition 1.4 shows that condition (1) is satisfied by our transfer operator \mathcal{L} on $L^1(\mathbb{P} \times m)$ with

\mathcal{H} is the set of $h \in L^\infty(\mathbb{P} \times m)$ such that the variation on fibers is uniformly bounded (i.e., $\sup_\omega \text{var}(h(\omega, \cdot)) < \infty$).

We may set $C(h) = 2\|h\|_\infty + \sup_\omega \text{var}(h(\omega, \cdot))$ (assuming $M(\omega) \geq 1$ for all ω). To approximate a function in L^1 by an element of \mathcal{H} , we use conditional expectation with respect to a generating sequence of finer and finer finite partitions into subsets of the form $A \times I$, where $A \subset \Omega$ is Borel and I is an interval. Condition (2), the invariance of Lebesgue measure under each one-dimensional transfer operator L_ω , is clear. Thus Proposition 2.1 applies and its corollary gives the first point of our Main Theorem: the existence of an a.c.i.m. for F . \square

We turn to the proof of the proposition. To take care of the unboundedness of M , we renormalize and work with a new reference measure, $\lambda = M \cdot \mathbb{P} \times m$. λ is a σ -finite positive measure. It is not necessarily finite, forcing us to clear up some things with the next lemma.

Recall that a set of measures μ on $\Omega \times I$ is λ -**tight** if for every $\epsilon > 0$ there exists X_ϵ with finite λ -measure such that $|\mu|(\Omega \times I \setminus X_\epsilon) < \epsilon$ for every measure μ in the set.

Lemma 2.3. *Let K be a bounded subset of $L^\infty(\lambda)$ such that the collection of measures $h \cdot \lambda$, $h \in K$, is λ -tight.*

Then K is a conditionally weakly compact subset of $L^1(\lambda)$, i.e., it has compact closure in the topology defined by the functionals

$$h \mapsto \int hg \, d\lambda \quad (g \in L^\infty(\lambda))$$

(remember that $L^\infty(\lambda)$ is the dual of $L^1(\lambda)$).

Proof of the lemma. By the Banach-Alaoglu theorem, the unit ball of $L^\infty(\lambda)$ is compact in the weak star topology defined by the functionals: $h \mapsto \int hg \, d\lambda$, where g ranges over $L^1(\lambda)$. This implies the statement of the lemma, but restricted to $g \in L^\infty(\lambda)$ such that $\lambda(g \neq 0) < \infty$. We extend it to general g . It is enough to prove the following for any sequence h_1, h_2, \dots of positive or negative parts of elements of K : if this sequence converges in the weak star topology to some limit $h \in L^\infty(\lambda)$, then, for each non-negative $g \in L^\infty(\lambda)$, $\lim_{n \rightarrow \infty} \int h_n g \, d\lambda = \int hg \, d\lambda$.

We first consider the case where $g = 1_E$ for some measurable $E \subset \Omega \times I$. Write E as an increasing union of subsets E_1, E_2, \dots , each one with finite λ -measure. We have $\lambda(E) = \sup_m \lambda(E_m)$, and the weak star convergence gives, for each $m \geq 1$, $(h \cdot \lambda)(E_m) = \lim_{n \rightarrow \infty} (h_n \cdot \lambda)(E_m) \leq \liminf_{n \rightarrow \infty} (h_n \cdot \lambda)(E)$. Hence

$$(2.4) \quad (h \cdot \lambda)(E) \leq \liminf_{n \rightarrow \infty} (h_n \cdot \lambda)(E) \quad \text{for all measurable } E.$$

Assume that for some subset E_0 this is a strict inequality. Let ϵ be the “loss” in measure, i.e., $\epsilon = \liminf_{n \rightarrow \infty} (h_n \cdot \lambda)(E_0) - (h \cdot \lambda)(E_0) > 0$. Observe that tightness of the $h_n \cdot \lambda$ together with L^∞ -boundedness of h_1, h_2, \dots implies that $\sup_n (h_n \cdot \lambda)(\Omega \times I) < \infty$, so that $0 < \epsilon < \infty$. In particular, by writing $\Omega \times I = E_0 \cup (\Omega \times I \setminus E_0)$ and applying (2.4) to $\Omega \times I \setminus E_0$, we see that $\Omega \times I$ must exhibit at least the same loss ϵ ; hence we may take $E_0 = \Omega \times I$.

By the tightness hypothesis again, there exists a subset Z of finite λ -measure such that $(h_n \cdot \lambda)(\Omega \times I \setminus Z) \leq \epsilon/2$ for all $n = 1, 2, \dots$. Now

$$\liminf_{n \rightarrow \infty} (h_n \cdot \lambda)(\Omega \times I) \leq (h \cdot \lambda)(Z) + \epsilon/2 \leq (h \cdot \lambda)(\Omega \times I) + \epsilon/2.$$

This contradicts the definition of $0 < \epsilon < \infty$. Hence (2.4) is an equality.

Now consider an arbitrary non-negative $g \in L^\infty(\lambda)$. Let $\epsilon > 0$. By tightness, there exists Y with λ -finite measure such that $(h_n \cdot \lambda)(\Omega \times I \setminus Y) \leq \epsilon/3 \|g\|_\infty$ for all n (implying, by (2.4), that the same inequality holds for the limit, h). Now $g1_Y$ is in $L^1(\lambda)$; hence, for large n , $|\int h_n g 1_Y \, d\lambda - \int hg 1_Y \, d\lambda| \leq \epsilon/3$. Therefore

$$\begin{aligned} \left| \int h_n g \, d\lambda - \int hg \, d\lambda \right| &\leq \left| \int h_n g (1 - 1_Y) + (h_n - h)g 1_Y + hg(1_Y - 1) \, d\lambda \right| \\ &\leq \|g\|_\infty (h_n \cdot \lambda)(\Omega \times I \setminus Y) + \epsilon/3 \\ &\quad + \|g\|_\infty (h \cdot \lambda)(\Omega \times I \setminus Y) \\ &\leq \epsilon \end{aligned}$$

for all large n . As $\epsilon > 0$ was arbitrarily small, the weak convergence and hence the lemma are proved. \square

Proof of the proposition. Let $h \in \mathcal{H}$. We set

$$h_n = \frac{1}{M} \left(\frac{1}{n} \sum_{k=0}^{n-1} \mathcal{L}^k h \right), \quad n = 1, 2, \dots$$

The normalization by M ensures that $\sup_{n \geq 1} \|h_n\|_\infty < \infty$ by condition (1). Using condition (2), each $h_n \cdot \lambda$ projects onto Ω to a measure dominated by $\|h\|_\infty \cdot \mathbb{P}$. Indeed, for any measurable $E \subset \Omega$

$$\begin{aligned} |(h_n \cdot \lambda)(E \times I)| &\leq \frac{1}{n} \sum_{k=0}^{n-1} \int_{E \times I} \frac{1}{M} \|\mathcal{L}^k h(\omega, \cdot)\|_1 d\lambda = \frac{1}{n} \sum_{k=0}^{n-1} \int_E \|\mathcal{L}^k h(\omega, \cdot)\|_1 d\mathbb{P} \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \int_E \|h(T^{-k}\omega, \cdot)\|_1 d\mathbb{P} \leq \|h\|_\infty \mathbb{P}(E). \end{aligned}$$

Therefore the sequence $h_1 \cdot \lambda, h_2 \cdot \lambda, \dots$ is λ -tight and we may apply the previous lemma. Now the weak $L^1(\lambda)$ -convergence of h_n is the same as the weak $L^1(\mathbb{P} \times m)$ -convergence of $M \cdot h_n$. Therefore the lemma shows that

$$\left\{ \frac{1}{n} \left(\sum_{k=0}^{n-1} \mathcal{L}^k h \right) \right\}_{n=1,2,\dots} \text{ is } L^1(\mathbb{P} \times m)\text{-conditionally weakly compact.}$$

This holds for all $h \in \mathcal{H}$. \mathcal{H} is dense in $L^1(\mathbb{P} \times m)$. Therefore, by [10, VIII.5.3] we have that the operators $\frac{1}{n} \sum_{k=0}^{n-1} \mathcal{L}^k$ converge in L^1 to the projection Q with range the set of fixed points of \mathcal{L} and kernel the closure of $(\text{Id} - \mathcal{L})L^1(\mathbb{P} \times m)$, by [10, VIII.5.2].

This finishes the proof of the proposition. \square

3. BOUND ON THE MULTIPLICITY OF THE A.C.I.M.

To prove the explicit bound (0.4), we first remark that the support of any a.c.i.m. contains intervals by establishing the regularity of the invariant densities. From this we deduce that assumption (A2) implies that there can be only a finite number of mutually singular a.c.i.m.'s. This immediately implies that there exist finitely many ergodic a.c.i.m.'s such that any a.c.i.m. is a convex combination of those: the set of a.c.i.m.'s is, as claimed in our Main Theorem, a finite-dimensional simplex.

We also remark that simple finiteness can be proved more abstractly, by a compactness argument—we learnt this from G. Keller in the deterministic case.

3.1. Regularity of arbitrary a.c.i.m.'s. We first prove that the densities of the a.c.i.m.'s constructed as above have finite variation on fibers:

Lemma 3.1. *Let $h \in \mathcal{H}$. For $\mu = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathcal{L}^k h \cdot \mathbb{P} \times m$, we have*

$$\text{var} \left(\frac{d\mu}{d\mathbb{P} \times m}(\omega, \cdot) \right) \leq M(\omega) \sup_{\alpha} \|h_{\alpha}\|_1 < \infty \quad \text{for } \mathbb{P}\text{-a.e. } \omega$$

(after modification on a null set).

Proof. First note that the convergence in $L^1(\mathbb{P} \times m)$ of $\left(\frac{1}{n} \sum_{k=0}^{n-1} \mathcal{L}^k h \right)_{n \geq 1}$ implies that there exist $n_1 < n_2 < \dots$ such that $\left(\frac{1}{n_i} \sum_{k=0}^{n_i-1} \mathcal{L}^k h \right) (\omega, \cdot)$ converges in $L^1(m)$ to $\frac{d\mu}{d\mathbb{P} \times m}(\omega, \cdot)$ for \mathbb{P} -almost every ω .

Recall (1.6): $\limsup_{n \rightarrow \infty} \text{var}((\mathcal{L}^n h)(\omega, \cdot)) \leq M(\omega) \sup_{\alpha} \|h_{\alpha}\|_1 \leq M(\omega) \|h\|_{\infty} < \infty$, independently of the finite variation on fibers of h . But, $\{g \in L^1(m) : \|g\|_1 \leq \|h\|_{\infty} \text{ and } \text{var}(g) \leq M(\omega) \sup_{\alpha} \|h_{\alpha}\|_1 + \epsilon\}$ is a compact subset of $L^1(m)$, for each $\epsilon > 0$. Hence $\frac{d\mu}{d\mathbb{P} \times m}(\omega, \cdot)$ must belong to it. But this is the statement of the lemma. \square

We extend this bounded variation property to the general case:

Proposition 3.2. *Let ν be an invariant probability measure such that $\nu \ll \mathbb{P} \times m$. Let $M : \Omega \rightarrow [1, \infty[$ be as in (1.6). Then*

$$\text{var} \left(\frac{d\nu}{d\mathbb{P} \times m}(\omega, \cdot) \right) \leq M(\omega) < \infty \quad \text{for } \mathbb{P}\text{-a.e. } \omega$$

(after modification on a null set).

Proof. We write ν as $h \cdot \mathbb{P} \times m$ with $h \in L^1(\mathbb{P} \times m)$, $h \geq 0$, $\int_{[0,1]} h_{\omega} dm = 1$ for \mathbb{P} -a.a. ω and with $\mathcal{L}h = h$. Using the density of \mathcal{H} in $L^1(\mathbb{P} \times m)$, we first find $h^{\epsilon} \in \mathcal{H}$ such that $\|h - h^{\epsilon}\|_1 \leq \epsilon$. We apply the preceding section to it and see that $\frac{1}{n} \sum_{k=0}^{n-1} \mathcal{L}^k h^{\epsilon}$ converges in L^1 to an invariant density g^{ϵ} . As $\|\mathcal{L}\|_1 = 1$,

$$\left\| h - \frac{1}{n} \sum_{k=0}^{n-1} \mathcal{L}^k h^{\epsilon} \right\|_1 \leq \|h - h^{\epsilon}\|_1 \leq \epsilon.$$

The inequality goes to the limit $n \rightarrow \infty$: $\|h - g^{\epsilon}\|_1 \leq \epsilon$. That is, g^{ϵ} goes to h as ϵ goes to zero.

But $\text{var}(g^{\epsilon}(\omega, \cdot)) \leq M(\omega)$ for every $\omega \in \Omega$. From the L^1 -compactness of $\{g \in L^1(m) : \text{var}(g) \vee \|g\|_1 \leq K\}$ for finite K , we get that $\text{var}(h(\omega, \cdot))$ must be finite \mathbb{P} -a.e. \square

Bound on the multiplicity. Assume that there exist r mutually singular a.c.i.m.'s μ_1, \dots, μ_r . In particular there exist invariant and pairwise disjoint measurable subsets $E_1, \dots, E_r \subset \Omega \times [0, 1]$ such that $\mu_i(E_i) = 1$.

It is obvious that any non-negative function of bounded variation which has positive Lebesgue integral admits a non-trivial interval on which it is positive. Hence, for \mathbb{P} -a.a. ω , there exist non-trivial intervals I_1, \dots, I_r such that $F^n(\omega \times I_i) \subset E_i \cap (T^n \omega \times [0, 1])$ modulo m , for all $n = 0, 1, \dots$. We fix such an ω .

Set $I_i(0) = I_i$ and, for $n \geq 0$, $I_i(n+1) = f_{T^n \omega} I_i(n) \cap J$, where J is the open interval of $f_{T^n \omega}$ which maximizes the length of $I_i(n+1)$. Recall that the intervals of a Lasota-Yorke map are simply the largest intervals on which the map is continuous and monotonic.

Define $N_i(n)$ to be the number of intervals of $f_{T^n \omega}$ meeting $I_i(n)$. Obviously,

$$m(I_i(n+1)) \geq \frac{\delta(T^n \omega)}{N_i(n)} m(I_i(n)).$$

As $m(I_i(n)) \leq 1$ in all cases, we get

$$(3.3) \quad \sum_{k=0}^{n-1} \log \frac{N_i(k)}{\delta(T^k \omega)} \geq \log m(I_i(0)) \quad \forall n = 1, 2, \dots$$

$N_i(k) - 1$ is the number of singularities of $f_{T^k\omega}$ contained in the open interval $I_i(k)$. Hence, $I_1(k), \dots, I_r(k)$ being disjoint, $N_1(k) - 1 + \dots + N_r(k) - 1 \leq N(T^k\omega) - 1$. Thus

$$N_1(k) \cdots N_r(k) \leq \left(\frac{N(T^k\omega) + r - 1}{r} \right)^r.$$

Hence, summing (3.3) over $i = 1, \dots, r$ and dividing by r , we get

$$\sum_{k=0}^{n-1} \log \left(\frac{N(T^k\omega) + r - 1}{r\delta(T^k\omega)} \right) \geq \frac{1}{r} \sum_{i=1}^r \log m(I_i(0)).$$

Thus

$$\frac{1}{n} \sum_{k=0}^{n-1} \log \frac{1 + (N(T^k\omega) - 1)/r}{\delta(T^k\omega)} \geq \frac{1}{n} \text{const.}$$

Now $(1/r)N \leq 1 + (N - 1)/r \leq (1 + 1/r)N$, as $N \geq 1$ and $r \geq 1$. Hence, $\log \frac{1+(N-1)/r}{\delta} - \log N/\delta$ is a bounded function. But $\log N/\delta$ is integrable by assumption (A2) and the fact that $N/\delta \geq 1$ (otherwise the image of some interval of the map would not fit into $[0, 1]$). Thus we may let $n \rightarrow \infty$ and apply the ergodic theorem, to find that

$$\int_{\Omega} \log \frac{1 + (N(\omega) - 1)/r}{\delta(\omega)} d\mathbb{P} \geq 0.$$

The integrand above is a non-increasing function of r . It is integrable for $r = 1$. Hence, by the monotone convergence theorem, the integral decreases to $\int_{\Omega} \log \frac{1}{\delta} d\mathbb{P}$, which is strictly negative, when $r \rightarrow \infty$. Thus the above inequality gives a (non-trivial) bound on r , and in particular we obtain finiteness.

Abstract proof of finiteness. The above explicit bound uses the “geometry” of the interval. We now give an abstract and simple proof of the finiteness (without any bound), suitable for generalization to systems only assumed to satisfy “good Lasota-Yorke inequalities” (see [7]). We learnt this argument (applied to the deterministic case) from G. Keller.

Assume that there are infinitely many a.c.i.m.’s for the skew-product. Choose a countable subset h_1, h_2, \dots of invariant, normalized, true functions which are pairwise distinct in $L^1(\mathbb{P} \times m)$. Consider the vector space \mathcal{D} generated by these, i.e., the set of all finite linear combination of h_n ’s. For each $\omega \in \Omega$, the restriction to $\omega \times [0, 1]$ defines a linear map $h \mapsto h_\omega$. This map is injective for a.a. ω . Indeed, consider for instance the set of ω ’s which are typical with respect to the countably many subsets $\{\omega \in \Omega : \|\alpha_1 h_1 + \dots + \alpha_n h_n|_\omega\|_1 \leq r_n\}$ for all $n = 1, 2, \dots$ and all rationals $\alpha_1, \dots, \alpha_n, r_n$. For such an ω , for any $h \in \mathcal{D}$, for any $h \in \mathcal{D}$, $\frac{1}{n} \sum_{k=0}^{n-1} \|h_{T^k\omega}\|_1 \xrightarrow{n \rightarrow \infty} \|h\|_1$, which is not zero provided $h \neq 0$. But $\|h_{T^k\omega}\|_1 = \|L_\omega^k h_\omega\|_1 \leq \|h_\omega\|_1$. Hence $h_\omega \neq 0$ for $h \neq 0$, proving that $h \mapsto h_\omega$ is one-to-one.

Thus the image \mathcal{D}_ω of \mathcal{D} must also be infinite-dimensional for a.a. ω .

Now Proposition 1.4 implies that, for a.a. ω , the unit ball of $\mathcal{D}_\omega \subset L^1(m)$ is made of functions with uniformly bounded variation. Hence this unit ball is relatively compact as a subset of $L^1(m)$. But this means that the space \mathcal{D}_ω is finite-dimensional. The contradiction proves the finiteness.

4. S.R.B. MEASURES FOR THE RANDOM MAP

We prove Theorem 0.5, restated as:

Proposition 4.1. *Let μ_1, \dots, μ_r be the finite collection of all ergodic a.c.i.m.'s for the skew-product F on $\Omega \times [0, 1]$. Define ν_i as the projection on $[0, 1]$ of the a.c.i.m. μ_i . Then each ν_i is an absolutely continuous S.R.B. measure. Moreover, \mathbb{P} -almost surely, the union of their basins has total Lebesgue measure.*

Proof. The Birkhoff Ergodic Theorem says that the set $B_i \subset \Omega \times [0, 1]$ of points (ω, x) such that $\frac{1}{n} \sum_{k=0}^{n-1} \delta_{F^k(\omega, x)}$ converges vaguely to μ_i has μ_i -measure 1 (we endow the Lebesgue space Ω with the topology defined by some identification of it with $([0, 1], m)$). Projecting this to $[0, 1]$, we get the defining property of an S.R.B. measure provided that we check that $B_\omega(\nu_i) \stackrel{\text{def}}{=} B_i \cap (\omega \times [0, 1])$ has positive Lebesgue measure for \mathbb{P} -almost all ω . Now $\mathbb{P} \times m(B_i) > 0$, as μ_i is an a.c.i.m. It is now enough to recall that B_i is F -invariant and (T, \mathbb{P}) is ergodic, to see that $m(B_\omega(\nu_i)) > 0$, \mathbb{P} -almost surely in ω .

It remains to check that $\bigcup_{i=1}^r B_\omega(\nu_i) = [0, 1] \pmod{m}$, \mathbb{P} -almost surely. We have to prove that the following set E is a $\mathbb{P} \times m$ -null set:

$$E = \left\{ (\omega, x) \in \Omega \times [0, 1] : \forall i = 1, \dots, r \quad \frac{1}{n} \sum_{k=0}^{n-1} \delta_{F^k(\omega, x)} \not\rightarrow \mu_i \right\}.$$

Note that $F(E) \subset E$. Hence $\frac{1}{n} \sum_{k=0}^{n-1} \mathcal{L}^k 1_E$ is zero outside E for each $n = 1, 2, \dots$. Using L^1 -convergence, this implies the same for the limit $Q1_E$. Now, $\|Q1_E\|_1 = \|1_E\|_1 = \mathbb{P} \times m(E)$. Hence, if $\mathbb{P} \times m(E) > 0$, $\frac{1}{\mathbb{P} \times m(E)} Q1_E \cdot \mathbb{P} \times m$ should be an a.c.i.m., hence with ergodic components μ_1, \dots, μ_r . In particular, applying the ergodic theorem, for $\mathbb{P} \times m$ -almost all points of $\{Q1_E > 0\}$, the empirical measures should converge to one of the μ_i 's. This contradicts $\{Q1_E > 0\} \subset E$, unless E is a null set. \square

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