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p-CENTRAL GROUPS AND POINCARÉ DUALITY

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Abstract. In this note we investigate the mod p cohomology ring of finite p-central groups with a certain extension property. For p odd it turns out that the structure of the cohomology ring characterizes this class of groups up to extensions by p'-groups. For certain examples the cohomology ring can be calculated explicitly. As a by-product one gets an alternative proof of a theorem of M.Lazard which states that the Galois cohomology of a uniformly powerful pro-p-group of rank n is isomorphic to $\Lambda[x_1, .., x_n]$.

1. Introduction

One problem in group cohomology is to relate the structure of a finite group G with its mod p cohomology ring $H^\bullet(G) := H^\bullet(G, \mathbb{F}_p)$ and to determine the isomorphism type of $H^\bullet(G)$. In this note we characterize the finite groups G for which the *reduced mod p cohomology ring* $H^\bullet(G)_{red}$, p odd, is a polynomial algebra generated in degree 2, where $A^\bullet_{red} = A^\bullet/nil(A^\bullet)$ and $nil(\ )$ denotes the ideal consisting of all nilpotent elements in the commutative[1] $\mathbb{N}_0$-graded $\mathbb{F}_p$-algebra $A^\bullet$. We know that, apart from trivial examples, these are the easiest reduced cohomology rings which might occur (cf. [QU1]).

For a group G and a fixed prime number p one defines

$$\Omega_k(G) = \langle\, g \in G \mid g^{p^k} = 1 \,\rangle.$$

The group G is called *p^k-central* if $\Omega_k(G) \leq Z(G)$. We say that the p-central group has the *Ω-extension property* ($=$ ΩEP) if there exists a p-central group H such that $G = H/\Omega_1(H)$. Using Quillen's stratification theorem, one deduces that for p odd, $H^\bullet(G)_{red}$ is a polynomial algebra generated in degree 2 if and only if G is p-nilpotent and $P \in \mathrm{Syl}_p(G)$ is p-central and has the ΩEP (cf. Thm. 2.1).

Some effort is spent to determine the isomorphism type of the $\mathbb{F}_p$-cohomology ring $H^\bullet(G)$ for G a p-central finite group with ΩEP. A finite-dimensional connected commutative $\mathbb{N}_0$-graded $\mathbb{F}_p$-algebra $A^\bullet$ satisfies *Poincaré duality in dimension d* if $\dim_{\mathbb{F}_p}(A^d) = 1$, $\dim_{\mathbb{F}_p}(A^k) = 0$ for $k > d$, and the bilinear mapping $A^k \otimes A^{d-k} \to A^d$ induced by the product is non-degenerate for $k = 0, .., d$.

Using a splitting theorem for comodule algebras (cf. Thm.3.1) and the Lyndon-Hochschild-Serre spectral sequence for certain group extensions (cf. §4), we will prove the following.

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[1] This definition follows [MM]. Sometimes these algebras are also called *anti-commutative*.

Theorem A. *Let G be a finite group and p odd. Then the following are equivalent:*

1. $H^\bullet(G)_{red} \simeq \mathbb{F}_p[y_1,..,y_r]$, *where y_i are homogeneous generators of degree* 2,
2. $H^\bullet(G) \simeq \mathbb{F}_p[y_1,..,y_r] \otimes C^\bullet$, *where y_i are homogeneous generators of degree* 2 *and $C^\bullet$ is a connected commutative $\mathbb{F}_p$-algebra satisfying Poincaré duality in dimension r,*
3. *G is p-nilpotent, $P \in \mathrm{Syl}_p(G)$ is p-central with* ΩEP*, and* $r = \dim_{\mathbb{F}_p}(\Omega_1(P))$.

Note that the isomorphism in (1) and (2) is an isomorphism as $\mathbb{F}_p$-algebras. Recall that a finite p-group P is called *powerful*, if $[P,P] \leq P^p$ for p odd, respectively $[P,P] \leq P^4$ for $p = 2$ (cf. [DD]). Theorem A has also the following consequence, which can be seen as the classification of finite groups with mod p cohomology ring isomorphic to the mod p cohomology ring of an abelian group, p odd.

Theorem B. *Let G be a finite group and p odd. Then the following are equivalent:*

1. $H^\bullet(G) \simeq \Lambda[x_1,..,x_r] \otimes \mathbb{F}_p[y_1,..,y_r]$, *where x_i, y_j are homogeneous generators with* $|x_i| = 1$, $|y_j| = 2$.
2. *G is p-nilpotent, $P \in \mathrm{Syl}_p(G)$ is powerful and p-central with* ΩEP*, and* $r = \dim_{\mathbb{F}_p}(\Omega_1(P))$.

The mod p cohomology of p-central groups has been studied by many authors. It is known that for G p-central the mod p cohomology ring $H^\bullet(G)$ is a Cohen-Macaulay algebra (cf. [Du], [BrH]). Hence $H^\bullet(G)$ satisfies a weak form of Poincaré duality which was introduced by D.Benson and J.F.Carlson (cf. [BC]). In particular, the Poincaré series

$$p(t) := p_{H^\bullet(G)}(t) = \sum_{k\in\mathbb{N}_0} \dim_{\mathbb{F}_p}(H^k(G)).t^k$$

satisfies the functional equation

$$p(1/t) = (-t)^r.p(t),$$

where $r = \dim_{\mathbb{F}_p}(\Omega_1(G))$. If G is a finite group and $P \in \mathrm{Syl}_p(G)$ is p-central, p odd, then G is a *Swan group* (cf. [HP], [MP]), i.e., the canonical mapping

$$res\colon H^\bullet(G) \longrightarrow H^\bullet(N_G(P))$$

is an isomorphism.

What are the standard examples of p-central p-groups with ΩEP? For W a finite $\mathbb{Z}_p$-module let

$$PL(W) = \{\, \alpha \in \mathrm{End}_{\mathbb{Z}_p}(W) \mid \exists\beta \in \mathrm{End}_{\mathbb{Z}_p}(W),\ \alpha = id_W + \mathbf{p}.\beta \,\},$$

where $\mathbf{p} = \mathbf{p}(p) = p$ for p odd, and $\mathbf{p}(2) = 4$. Then $PL(W)$ is p-central for p odd, respectively 4-central for $p = 2$. Furthermore, $PL(W)$ is powerful and has the ΩEP (cf. [W]). Note that if W is a finitely generated free $\mathbb{Z}/p^k.\mathbb{Z}$-module, then $PL(W)$ coincides with $ker(GL(W) \to GL(W \otimes \mathbb{F}_p))$. A p-group P is called *p-linear* if it is a subgroup of $PL(W)$ for some finite $\mathbb{Z}_p$-module W. For p odd, p-linear groups are p-central with ΩEP (cf. [W]). Furthermore, if $p \geq 5$, every powerful p-central group with ΩEP is in fact p-linear (cf. [W]).

Although the class of groups we consider seems to be rather small, one can use these results to compute the mod p Galois cohomology of finitely generated uniformly powerful pro-p-groups $\hat{G}$, reproving a remarkable result of M.Lazard (cf. Remark 5.2).

2. Finite groups with polynomial reduced cohomology

For $p = 2$ we have to consider a condition which is stronger than 4-centrality. A finite 2-group G is called 4^**-central*, if $\Omega_2(G)$ is a homocyclic group of exponent 4, i.e., $\Omega_2(G) = (\mathbb{Z}/4.\mathbb{Z})^r$ for some $r \in \mathbb{N}_0$. For a p-central p-group (resp. a 4^*-central 2-group) G we put for short $l\text{-}\mathrm{rk}_p(G) := \dim_{\mathbb{F}_p}(\Omega_1(G))$ and $\Omega(G) := \Omega_1(G)$ for p odd, respectively $\Omega(G) = \Omega_2(G)$ for $p = 2$. The following theorem shows that the ΩEP is closely related to the structure of the $\mathbb{F}_p$-cohomology ring.

Theorem 2.1. *Let G be a finite p-central p-group, p odd, or a finite 4^*-central 2-group. Then the following are equivalent:*

1. *$H^\bullet(G)_{red}$ is a polynomial $\mathbb{F}_p$-algebra of Krull dimension $l\text{-}\mathrm{rk}_p(G)$ generated in degree 2.*
2. *The reduced restriction mapping $j_{red}\colon H^\bullet(G)_{red} \to H^\bullet(\Omega(G))_{red}$ is surjective.*
3. *The reduced restriction mapping j_{red} is surjective in degree 2.*
4. *G has the ΩEP.*

Proof. Since G is p-central, $j_{red}\colon H^\bullet(G)_{red} \to H^\bullet(\Omega(G))_{red}$ is an F-isomorphism (cf. [QU1]), in particular, j_{red} is injective. Thus (1), (2) and (3) are obviously equivalent. Hence it suffices to show that (3) and (4) are equivalent.

For any group K and any 2-cocycle $c \in C^2(K, M)$, where

$$C^2(K,M) = \{\, f \in \mathfrak{F}(K^2, M) \mid f(y,z) + f(x,yz) = f(xy,z) + f(x,y),\ x,y,z \in K \,\}$$

and M a trivial $\mathbb{F}_pK$-module, we denote by $K_c := K \times M$ the group with multiplication given by

$$(k_1, m_1).(k_2, m_2) := (k_1.k_2, m_1 + m_2 + c(k_1, k_2)). \tag{2.1}$$

Let S be an elementary abelian p-group, p odd, or a homocyclic 2-group of exponent 4, $l\text{-}\mathrm{rk}_p(S) = r$, and $M = \mathbb{F}_p^r$ a trivial $\mathbb{F}_pS$-module. For $c \in C^2(S, M)$ one checks easily that $\Omega_1(S_c) = M$ if and only if $\tau(c_1)$,..,$\tau(c_r)$ span $H^2(S)_{red}$, where $\tau\colon C^2(S, \mathbb{F}_p) \to H^2(S)_{red}$ denotes the canonical projection, and $c = (c_1, .., c_r)$.

Assume that G satisfies (3), and let $c_1, .., c_r \in C^2(G, \mathbb{F}_p)$ be such that $\tau(j_{red}(c_1))$, .., $\tau(j_{red}(c_r))$ span $H^2(\Omega(G))_{red}$. Then $H = G_c$, $c = (c_1, .., c_r)$, is p-central and $G = H/\Omega_1(H)$. Hence G has the ΩEP. Assume that G has the ΩEP and that H is p-central satisfying $H/\Omega_1(H) = G$. We may also assume that $l\text{-}\mathrm{rk}_p(G) = l\text{-}\mathrm{rk}_p(H) = r$ (cf. [W]). Then $H \simeq G_c$ for some $c \in C^2(G, \mathbb{F}_p^r)$. Let $\bar{c} = (\bar{c}_1, .., \bar{c}_r) \in H^2(G)_{red}$ denote the image of c under the projection to the reduced cohomology. Then the previous remark shows that $j_{red}(\bar{c}_1)$,.., $j_{red}(\bar{c}_r)$ span $H^2(\Omega(G))_{red}$. This completes the proof. ∎

Theorem 2.1 yields the following property of p-central groups with ΩEP.

Proposition 2.2. *Let p be odd, and let G be a finite p-central p-group with ΩEP. Let $A \le \Omega_1(G)$ be such that $A \cap \Omega_2(G)^p = 1$. Then $\overline{G} := G/A$ is a p-central p-group with ΩEP.*

Proof. By [W, Prop. 4.1.(d)], $\overline{G}$ is a p-central p-group and satisfies $l\text{-}\mathrm{rk}_p(G) = l\text{-}\mathrm{rk}_p(\overline{G}) + l\text{-}\mathrm{rk}_p(A)$. Consider the Lyndon-Hochschild-Serre spectral sequence associated to the extension

$$1 \longrightarrow A \longrightarrow G \longrightarrow \overline{G} \longrightarrow 1.$$

By Theorem 2.1, $\mathbf{Tot}(E_\infty)_{red}$ is a polynomial $\mathbb{F}_p$-algebra of Krull dimension $r = l\text{-}\mathrm{rk}_p(G)$ generated in degree 2. $H^\bullet(A)$ is a polynomial $\mathbb{F}_p$-algebra of Krull dimension $l\text{-}\mathrm{rk}_p(A)$ generated in degree 2. Since $\mathbf{Tot}(E_\infty)_{red}$ is a subquotient of $\mathbf{Tot}(E_2)_{red}$, it follows that $\mathbf{Tot}(E_2)_{red}$ has to contain a polynomial $\mathbb{F}_p$-subalgebra of Krull dimension r generated in degree 2. But $\mathbf{Tot}(E_2) \simeq H^\bullet(\overline{G}) \otimes H^\bullet(A)$, and therefore $\mathbf{Tot}(E_2)_{red} \simeq H^\bullet(\overline{G})_{red} \otimes H^\bullet(A)_{red}$. Since $\overline{G}$ is p-central, $H^\bullet(\overline{G})_{red}$ is a polynomial $\mathbb{F}_p$-algebra of Krull dimension $l\text{-}\mathrm{rk}_p(\overline{G})$ generated in degree $2.p^{\alpha_i}$, $\alpha_i \in \mathbb{N}_0$, $i = 1,..,l\text{-}\mathrm{rk}_p(\overline{G})$ (cf. [BrH]). Hence, $H^\bullet(\overline{G})_{red}$ must be generated in degree 2, and the claim follows from Theorem 2.1. □

For p-groups one has the following p-centrality criterium.

Lemma 2.3. *Let P be a finite p-group, and assume that $A \le P$ is a maximal elementary abelian p-subgroup of P such that $N_P(A) = C_P(A)$. Then $N_P(A) = C_P(A) = P$, i.e., P is p-central and $A = \Omega_1(P) = \Omega_1(Z(P))$.*

Proof. Assume the assertion is false and let P denote a minimal counterexample (with respect to $|P|$), i.e., $N_P(A) \neq P$. Let $M \lneqq P$ denote a maximal subgroup of P containing $N_P(A)$. Then M is normal in P, A is a maximal elementary abelian p-subgroup of M and

$$N_M(A) = M \cap N_P(A) = M \cap C_P(A) = C_M(A). \tag{2.2}$$

Hence the assertion holds for M, i.e., $A = \Omega_1(Z(M))$. In particular, A is characteristic in M and thus normal in P, a contradiction. □

The following theorem is a direct consequence of the stratification theorem [QU1] and the characterization of p-nilpotency given in [QU2].

Theorem 2.4. *Let G be a finite group and let $A \le G$ be an elementary abelian p-subgroup of G such that*

$$res_{G,A}\colon H^\bullet(G) \longrightarrow H^\bullet(A)$$

is an F-isomorphism. Then A is a maximal elementary abelian p-subgroup of G, $N_G(A) = C_G(A)$, and $\{\, A^g \mid g \in G \,\}$ is the unique G-conjugacy class of maximal elementary abelian p-subgroups of G. Furthermore, $P \in \mathrm{Syl}_p(G)$ is a p-central p-group. If $p \neq 2$, G is p-nilpotent.

Proof. Obviously, $H^\bullet(G)$ and $H^\bullet(A)$ must have the same Krull dimension. Hence, A has to be a maximal elementary abelian p-subgroup (cf. [QU2, Cor.3.3]), and since $H^\bullet(G)_{red}$ is a domain, there is a unique G-conjugacy class of maximal elementary abelian p-subgroups of G (cf. [QU2, Cor.3.2]). From [QU2, Th.3.7], it follows that $W(A) := N_G(A)/C_G(A) = 1$. Assume $P \in \mathrm{Syl}_p(G)$ contains A. As in (2.2.), one obtains $N_P(A) = C_P(A)$, and thus P must be p-central (cf. Lemma 2.3).

Let p be odd. As $res_{P,A}\colon H^\bullet(P) \to H^\bullet(A)$ is an F-isomorphism and since $res_{G,P}\colon H^\bullet(G) \to H^\bullet(P)$ is injective, $res_{G,P}$ is an F-isomorphism. Hence G is p-nilpotent (cf. [QU2]). □

Note that for $p = 2$ a finite group as claimed in Theorem 2.4 does not have to be 2-nilpotent (cf. [QU2], Ex. after Th. 1.4.).

Corollary 2.5. *Let G be a finite group and p be odd. Then*

$$H^\bullet(G)_{red} \simeq \mathbb{F}_p[y_1,..,y_n], \tag{2.3}$$

where all generators y_i have degree 2, if and only if G is p-nilpotent, $P \in Syl_p(G)$ is a p-central group with ΩEP, and $l\text{-}\mathrm{rk}_p(P) = n$.

Proof. If G is p-nilpotent, one has $H^\bullet(G) = H^\bullet(P)$. Hence (2.3) follows from Theorem 2.1.

Assume that G is a finite group such that (2.3) holds. Hence $H^\bullet(G)_{red}$ is a domain and G has a unique G-conjugacy class of maximal elementary abelian p-subgroups $\{ A^g \mid g \in G \}$ (cf. [QU2, Cor.3.2]). Thus $j_{red}\colon H^\bullet(G)_{red} \to H^\bullet(A)_{red}$ must be injective (cf. [QU2, Th.3.1.]). Since $H^\bullet(G)_{red}$ and $H^\bullet(A)_{red}$ are generated in degree 2 and have the same Krull dimension (cf. [QU2, Cor.3.3.]), j_{red} must be bijective. In particular, j_{red} is an F-isomorphism. Hence the claim follows from Theorems 2.1 and 2.4. □

3. A splitting theorem for comodule algebras

Throughout this section let K denote a field. Let $(A, \phi_A, \Delta_A, \eta_A, \varepsilon_A)$ be a K-Hopf algebra, $B = \coprod_{k\in\mathbb{N}_0} B_k$ a graded K-module and

$$\phi_B\colon B\otimes B \longrightarrow B, \qquad \eta_B\colon K \longrightarrow B,$$
$$\Delta_B\colon B \longrightarrow A\otimes B, \qquad \varepsilon_B\colon B \longrightarrow K,$$

be morphisms of graded K-modules such that

1. (B, Δ_B) is a left A-comodule,
2. $(B, \phi_A, \eta_B, \varepsilon_B)$ is an augmented K-algebra,
3. $\phi_B\colon B\otimes B \to B$ is a morphism of left A-comodules,
4. $\eta_B\colon K \to B$ is a morphism of left A-comodules,

Then $(B, \phi_B, \Delta_B, \eta_B, \varepsilon_B)$ is called an *augmented left A-comodule algebra*. For an augmented left A-comodule algebra B one has a canonical mapping $j\colon B \to A$ given by

$$j\colon B \xrightarrow{\Delta_B} A\otimes B \xrightarrow{id_A\otimes\varepsilon_B} A\otimes K \xrightarrow{\phi_A\circ(id_A\otimes\eta_A)} A \tag{3.1}$$

which is a morphism of graded K-modules. If one gives $A\otimes B$ the left A-comodule structure defined by $\Delta_{A\otimes B} := \Delta_A\otimes id_B$, one sees easily that all morphisms are morphisms of left A-comodules. Hence $j\colon B\to A$ is a morphism of left A-comodules.

Let $C := K\,\square_A B$ denote the cotensor product of the A-bicomodule K with the left A-comodule B. Note that C is a trivial left A-subcomodule of $K\otimes B$. Let $\iota_C\colon C \to K\otimes B$ denote the canonical embedding. Then one has a canonical mapping $i\colon C\to B$ given by

$$i\colon\ C \xrightarrow{\iota_C} K\otimes B \xrightarrow{\eta_B\otimes id_B} B\otimes B \xrightarrow{\phi_B} B \tag{3.2}$$

which is a morphism of left A-comodules. There are uniquely determined mappings of graded K-modules

$$\phi_C\colon C\otimes C \longrightarrow C, \qquad \eta_C\colon K \longrightarrow C$$

making $(C, \phi_C, \eta_C, \varepsilon_C)$ a K-algebra and $i\colon C\to B$ an injective morphism of augmented K-algebras. (cf. [MM, Prop.4.6]).

Let A be a K-Hopf algebra. We call A *of affine type*, if:

1. As a K-algebra A is isomorphic to a free commutative K-algebra,
$$A \simeq \Lambda[\, x_\mu \mid \mu \in S_1 \,] \otimes K[\, x_\lambda \mid \lambda \in S_2 \,]$$
for some sets S_1, S_2, where for $char(K) \neq 2$, x_μ, $\mu \in S_1$, are homogeneous elements of odd degree, x_λ, $\lambda \in S_2$, are homogeneous elements of even degree; for $char(K) = 2$, $S_1 = \emptyset$ and x_λ, $\lambda \in S_2$, are homogeneous elements of positive degree.
2. For all $\lambda \in S_1 \cup S_2$, x_λ are primitive elements, i.e.,
$$x_\lambda \in P(A) := \{\, a \in A \mid \Delta_A(a) = 1_A \otimes a + a \otimes 1_A \,\}.$$

For K-Hopf algebras of affine type Theorem 4.7 in [MM] specializes to the following:

Theorem 3.1. *Let A be a K-Hopf algebra of affine type and B a connected augmented left A-comodule algebra. Assume further that the canonical homomorphism of A-comodules $j\colon B \to A$ is surjective and additionally also a morphism of K-algebras. Then:*

(a) *There exists a section $\sigma\colon A \to B$ which is an injective homomorphism of K-algebras and A-comodules satisfying $j \circ \sigma = id_A$.*

(b) *Let $C := K \,\square_A B$. Then there exists an isomorphism*
$$h^* = \phi_B \circ (\sigma \otimes i)\colon A \otimes C \longrightarrow B$$
which is simultaneously an isomorphism of left A-comodules, right C-modules and left A-modules, where the left A-module structure on B and $A \otimes C$ is given by
$$\begin{aligned} \psi_B = \phi_B \circ (\sigma \otimes id_B)&\colon A \otimes B \longrightarrow B, \\ \psi_{A\otimes C} = \phi_A \otimes id_C&\colon \quad A \otimes A \otimes C \longrightarrow A \otimes C. \end{aligned}$$

(c) *If B is a commutative K-algebra, then h^* as given in* (b) *is an isomorphism of K-algebras.*

Proof. (a) Let
$$P_j(B) := \{\, b \in B \mid \Delta_B(b) = 1_A \otimes b + j(b) \otimes 1_B \,\}$$
denote the j-primitive elements in the left A-comodule B. As j is a morphism of K-algebras, one has $j(1_B) = 1_A$. Since j is a morphism of left A-comodules, it follows that
$$j_* := j|_{P_j(B)}\colon P_j(B) \longrightarrow P(A).$$
We claim that j_* is surjective. By Theorem 4.7 in [MM] (cf. proof of Thm.4.4. in [MM]), there exists a morphism of A-comodules $g\colon A \to B$ satisfying $j \circ g = id_A$. As A and B are connected, one obtains $g(1_A) = 1_B$. Furthermore, the composition
$$h\colon\ A \otimes C \xrightarrow{g\otimes i} B \otimes B \xrightarrow{\phi_B} B$$
is an isomorphism of left A-comodules. Hence $h(A \otimes 1_C)$ is a left A-subcomodule of B with
$$j \circ h(a \otimes 1_C) = a, \ \text{for all } a \in A. \tag{3.3}$$

Let $a \in P(A)$ be a primitive element in A. Hence it follows that

$$\begin{aligned}\Delta_B(h(a \otimes 1_C)) &= (id_A \otimes h) \circ \Delta_{A\otimes C}(a \otimes 1_C)\\ &= (id_A \otimes h)(a \otimes 1_A \otimes 1_C + 1_A \otimes a \otimes 1_C)\\ &= a \otimes h(1_A \otimes 1_C) + 1 \otimes h(a \otimes 1_C)\\ &= a \otimes 1_B + 1_A \otimes h(a \otimes 1_C)\end{aligned}$$

By (3.3), one concludes that $h(a \otimes 1) \in P_j(B)$. Hence, j_* is surjective.

By hypothesis, $A = \Lambda[x_\mu | \mu \in S_1] \otimes K[x_\lambda | \lambda \in S_2]$, where all x_λ, $\lambda \in S_1 \cup S_2$, are primitive homogeneous elements of positive degree. Let $t_\lambda \in P_j(B)$ be a homogeneous preimage of x_λ under j, i.e., $j(t_\lambda) = x_\lambda$ for all $\lambda \in S_1 \cup S_2$. Let $D := \langle t_\lambda | \lambda \in S_1 \cup S_2\rangle_{K\text{-}alg} \leq B$ denote the K-subalgebra of B generated by the elements t_λ and $\sigma\colon A \to D$ denote the induced homomorphism of graded K-algebras, which exists, since A is a free commutative K-algebra. Then by construction $\sigma \circ j|_D = id_D$ and $j|_D \circ \sigma = id_A$. Hence σ is an isomorphism of graded K-algebras. Since D is generated by j-primitive elements t_λ, one concludes easily by induction that D is a left A-subcomodule of B.

(b) By the proof of Theorem 4.7 in [MM], one knows that for any morphism $g\colon A \to B$ of left A-comodules with $j \circ g = id_A$, the composition

$$h\colon\ A \otimes C \xrightarrow{g\otimes i} B \otimes B \xrightarrow{\phi_B} B$$

is an isomorphism of left A-comodules and right C-modules. Let

$$h^*\colon\ A \otimes C \xrightarrow{\sigma\otimes i} B \otimes B \xrightarrow{\phi_B} B$$

with $\sigma\colon A \to B$ as in part (a). Hence h^* is an isomorphism of left A-comodules and right C-modules. Furthermore, for $a', a \in A$ and $c \in C$ one obtains

$$h^* \circ \psi_{A\otimes C}(a' \otimes a \otimes c) = h^*(a'.a \otimes c) = \sigma(a').\sigma(a).i(c) = \psi_B(a' \otimes h^*(a \otimes c)).$$

Hence h^* is also an isomorphism of left A-modules.

(c) Since $h^*(1_A \otimes 1_C) = 1_B$, it suffices to show that the diagram

$$\begin{array}{ccc} A\otimes C\otimes A\otimes C & \xrightarrow{\phi_{A\otimes C}} & A\otimes C \\ {\scriptstyle h^*\otimes h^*}\big\downarrow & & \big\downarrow{\scriptstyle h^*} \\ B\otimes B & \xrightarrow{\phi_B} & B \end{array}$$

commutes. Let $a_1, a_2 \in A$, $c_1, c_2 \in C$ be homogeneous elements with a_2 of degree p and c_1 of degree q. Hence one obtains

$$\phi_B \circ (h^* \otimes h^*)(a_1 \otimes c_1 \otimes a_2 \otimes c_2) = \sigma(a_1).i(c_1).\sigma(a_2).i(c_2)$$

and

$$h^* \circ \phi_{A\otimes C}(a_1 \otimes c_1 \otimes a_2 \otimes c_2) = (-1)^{pq}.\sigma(a_1).\sigma(a_2).i(c_1).i(c_2).$$

Hence the commutativity of the considered diagram follows from the commutativity of the K-algebra B, and this completes the proof of the theorem. □

If G is a group and $A \leq Z(G)$ is a central subgroup of G, the product mapping $A \times G \to G$ is a homomorphism of groups. This implies that $H^\bullet(G)$ becomes an augmented left $H^\bullet(A)_{red}$-comodule algebra. Furthermore, the mapping $j\colon H^\bullet(G) \to H^\bullet(A)_{red}$ coincides with the "reduced restriction mapping" and hence is a morphism of algebras. Hence one obtains

Corollary 3.2. *Let p be odd and P be a finite p-central p-group, respectively $p = 2$ and P be a 4^*-central 2-group. Assume further that P has the* ΩEP. *Then $H^\bullet(P) \simeq C \otimes H^\bullet(\Omega(P))_{red}$ for some finite dimensional $\mathbb{F}_p$-algebra C, and the isomorphism is an isomorphism as $\mathbb{F}_p$-algebras.*

Proof. From Theorem 2.1 and Theorem 3.1 one obtains a splitting $H^\bullet(P) \simeq C \otimes H^\bullet(\Omega(P))_{red}$ for some $\mathbb{F}_p$-algebra C consisting of nilpotent elements. Since $H^\bullet(P)$ is finitely generated as an algebra (cf. [B, Cor.4.2.2]), so is C. Hence C is finite dimensional. $\square$

4. Spectral sequences of p-central extensions

Let G be a finite p-central p-group, p odd. In this section we investigate the Lyndon-Hochschild-Serre spectral sequence (LHS spectral sequence) associated to the extension

$$(\dagger) \qquad 1 \longrightarrow A \longrightarrow G \longrightarrow \overline{G} \longrightarrow 1$$

where $A := \Omega_1(G)$ and $\overline{G} := G/A$. One knows by [W, Prop.5.4.(a)] that $\overline{G}$ is again a p-central group and $\overline{G}$ has the ΩEP. Thus

$$H^\bullet(\overline{G}) \simeq C \otimes \mathbb{F}_p[y_1, .., y_n], \tag{4.1}$$

$n = l\text{-}\mathrm{rk}(\overline{G})$, $|y_i| = 2$, and C is a finite dimensional commutative $\mathbb{F}_p$-algebra (cf. Corollary 3.2). Since A is central in G, the E_2-term of the LHS spectral sequence is given by $E_2^{r,s} = H^r(\overline{G}) \otimes H^s(A)$. The higher terms are described in the following theorem.

Theorem 4.1. *Let G be a finite p-central p-group, p odd. Assume further that $l\text{-}\mathrm{rk}(\overline{G}) = l\text{-}\mathrm{rk}(G) = n$. Then*

$$E_3^{r,s} = C_r \otimes S_s,$$

where S_s denotes the s^{th} homogeneous component of the $\mathbb{F}_p$-algebra $\mathbb{F}_p[z_1, .., z_n]$, $|z_i| = 2$, and $C = \coprod_{r \in \mathbb{N}_0} C_r$ is given as in (4.1). The product on E_3 is the naturally given one. Furthermore, $E_3 = E_\infty$ if and only if G has the ΩEP. *In this case one has*

$$H^\bullet(G) \simeq im(inf_{\overline{G},G}) \otimes im(res_{G,A}), \tag{4.2}$$

and this isomorphism is an isomorphism as $\mathbb{F}_p$-algebras.

Proof. As ($\dagger$) is a central extension, all the characteristc classes v^N vanish (cf. [CV, §7]). Hence, the differential $d_2 \colon E_2 \to E_2$ depends only on the element $\alpha \in H^2(\overline{G}, A)$ representing the extension ($\dagger$) (cf. [CV, Main Theorem]). Let $H^\bullet(A) = \Lambda[a_1, .., a_n] \otimes \mathbb{F}_p[z_1, .., z_n]$. Since $A = \Omega_1(G)$, we may assume that $\alpha \in H^2(\overline{G}, A) = H^2(\overline{G}, \mathbb{F}_p) \otimes H_1(A, \mathbb{F}_p)$ is given by

$$\alpha = \sum_{i=1}^{n} y_i \otimes a_i^*, \tag{4.3}$$

where $\{ a_i^* \mid i = 1, .., n \} \subset H_1(A, \mathbb{F}_p)$ is the basis dual to $\{ a_i \mid i = 1, .., n \}$. From [CV, §5] one deduces that $P_*(\alpha) \in H^2(\overline{G}, \mathbb{F}_p) \otimes \mathrm{Hom}_{\mathbb{F}_p}(H^s(A), H^{s-1}(A))$ is given by

$$P_*(\alpha) = \sum_{i=1}^{n} y_i \otimes \partial_i, \tag{4.4}$$

where ∂_i are the derivations satisfying

$$\partial_i(a_j) = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j, \end{cases} \quad \text{and} \quad \partial_i(z_j) = 0 \text{ for all } i, j.$$

Note that $\partial_i \circ \partial_j = -\partial_j \circ \partial_i$ for all i, j. For $u \otimes v \in E_2^{r,s}$ the image under d_2 is given by

$$d_2(u \otimes v) = -\sum_{i=1}^{n} (-1)^r . u . y_i \otimes \partial_i(v). \tag{4.5}$$

In particular, $(\mathbb{F}_p[y_1, .., y_n] \otimes \Lambda[a_1, .., a_n], d_2)$ coincides with the Koszul complex for the polynomial algebra $\mathbb{F}_p[y_1, .., y_n]$ and thus is a acyclic complex (cf. [HS]). Hence, $E_3 = H_\bullet(E_2, d_2) \simeq C \otimes \mathbb{F}_p[z_1, .., z_n]$, and the algebra structure on E_3 is the natural given one.

The group G has the ΩEP if and only if $E_k^{0,s} \leq ker(d_k)$ for all $k \geq 3$. As $d_k \colon E_k \to E_k$ are derivations for $k \geq 2$ (cf. [Be]), this is equivalent to $E_3 = E_\infty$. Statement (4.2) follows from Corollary 3.2. □

Corollary 4.2. *Let G be a powerful p-central p-group with ΩEP, p odd, and put $n := l\text{-}\mathrm{rk}(G)$. Then*

$$H^\bullet(G) \simeq \Lambda[x_1, .., x_n] \otimes \mathbb{F}_p[y_1, .., y_n], \tag{4.6}$$

where $|x_i| = 1$, $|y_j| = 2$.

Proof. We procceed by induction on $|G|$. For G abelian the assertion is obvious. Hence assume that the assertion holds for all powerful p-central p-groups with ΩEP which have order smaller than G.

If $\Omega_1(G) \not\leq G^p$, we choose a complement B for $\Omega_1(G) \cap G^p$ in $\Omega_1(G)$. Hence

$$B \simeq B^* = \Omega_1(G).G^p/G^p.$$

We choose a complement H^* to B^* in G/G^p, and let $H \leq G$ denote its preimage under the canonical projection $\pi \colon G \to G/G^p$. By construction $H^p = G^p$, and hence H is powerful p-central with ΩEP (cf. [W, Prop.5.4.(d)]). Furthermore, $G = H \times B$. Thus the assertion follows in this case.

Assume that $\Omega_1(G) \leq G^p$. Hence, $l\text{-}\mathrm{rk}(G) = l\text{-}\mathrm{rk}(\overline{G})$, where $\overline{G} := G/\Omega_1(G)$. By [W, Prop.5.4.(a)], $\overline{G}$ is powerful p-central with ΩEP. Hence the desired result follows by induction and Theorem 4.1. □

Next we consider another type of extension and the associated LHS spectral sequence. Let G be p-central with ΩEP and $A \leq \Omega_1(G)$, $|A| = p$ such that $A \cap \Omega_2(G)^p = 1$. Hence by Proposition 2.2 one knows that $\overline{G} := G/A$ is also p-central with ΩEP. Let $n = l\text{-}\mathrm{rk}(G) = l\text{-}\mathrm{rk}(\overline{G}) + 1$. Hence,

$$H^\bullet(\overline{G}) = D \otimes \mathbb{F}_p[y_1, .., y_{n-1}], \quad H^\bullet(A) = \Lambda[x] \otimes \mathbb{F}_p[y]. \tag{4.7}$$

We consider the extension

$$1 \longrightarrow A \longrightarrow G \longrightarrow \overline{G} \longrightarrow 1 \tag{‡}$$

and the associated LHS spectral sequence. As in the previous case we may assume that the cocycle $\gamma \in H^2(\overline{G}, A) = H^2(\overline{G}) \otimes H_1(A)$ representing the extension (‡) is given by $\gamma = c \otimes x^*$ for some element $c \in D_2$. This implies

$$P_*(\gamma) = c \otimes \partial, \tag{4.8}$$

where $\partial\colon H^\bullet(A) \to H^\bullet(A)$ denotes the derivation given by $\partial(x) = 1$, $\partial(y) = 0$. Hence, for $u \otimes v \in E_2^{r,s}$ the image under d_2 is given by

$$d_2(u \otimes v) = -(-1)^r.u.c \otimes \partial(v). \tag{4.9}$$

Since $\mathbb{F}_p \otimes \mathbb{F}_p[y_1, .., y_{n-1}] \otimes \mathbb{F}_p \otimes \mathbb{F}_p[y]$ is contained in the kernel of d_2, it follows that

$$E_3 = \mathbb{F}_p[y_1, .., y_{n-1}] \otimes H_\bullet(D \otimes \Lambda[x], \partial) \otimes \mathbb{F}_p[y],$$

where ∂ denotes the induced differential on the algebra $D \otimes \Lambda[x]$. More precisely we obtain

Theorem 4.3. *Let G be a p-central p-group, p odd, with* ΩEP *and $A \leq \Omega_1(G)$, $|A| = p$, such that $A \cap \Omega_2(G)^p = 1$. Let $n := l\text{-}\mathrm{rk}(G)$ and $\overline{G} := G/A$. Let $H^\bullet(\overline{G}) = D \otimes T$, where T is a polynomial $\mathbb{F}_p$-algebra of Krull dimension $n-1$ generated in degree 2, and D is a finite-dimensional commutative $\mathbb{F}_p$-algebra (cf. Cor. 3.2). Let $c \in D_2$ be given as above, and let $C^{\bullet,\bullet}$ denote the bigraded $\mathbb{F}_p$-algebra*

$$C^{\bullet,0} = D \otimes 1/D.c \otimes 1, \quad C^{\bullet,1} = Ann_D(c) \otimes x, \quad C^{\bullet,k} = 0 \text{ for } k \geq 2, \tag{4.10}$$

with the natural $\mathbb{F}_p$-algebra structure. Then

$$E_3^{r,s} = \sum_{\substack{i+j=r\\ l+m=s}} T_i \otimes C^{j,l} \otimes S_m,$$

where $S = \coprod_{m \in \mathbb{N}_0} S_m = \mathbb{F}_p[y]$. Furthermore, $E_3^{\bullet,\bullet} = E_\infty^{\bullet,\bullet}$.

If D satisfies Poincaré duality in dimension $n-1$, the algebra $\mathbf{Tot}(C^{\bullet,\bullet})$ satisfies Poincaré duality in dimension n.

Proof. From (4.9.) one concludes that

$$ker(\partial) = D \otimes 1 + Ann_D(c) \otimes x$$

and

$$im(\partial) = D.c \otimes 1.$$

Hence $C^{\bullet,\bullet} = H_\bullet(D \otimes \Lambda[x], \partial)$. If D satisfies Poincaré duality in dimension $n-1$, it is easily seen that the product on $\mathbf{Tot}(C^{\bullet,\bullet})$ induces a non-degenerate pairing between $D \otimes 1/D.c \otimes 1$ and $Ann_D(c) \otimes x$. Hence, $\mathbf{Tot}(C^{\bullet,\bullet})$ satisfies Poincaré duality in dimension n.

Let $d_k\colon E_k^{r,s} \to E_k^{r+k,s-k+1}$, $k \geq 3$, denote the higher differentials. Then one concludes as in the proof of Proposition 2.2 that $T \otimes 1 \otimes 1$, $1 \otimes 1 \otimes S \leq ker(d_k)$. Since $C^{\bullet,\bullet}$ is a bigraded algebra concentrated on the 0^{th} and 1^{st} bottom rows, one has also $C^{\bullet,\bullet} \leq ker(d_k)$, $k \geq 3$. This shows that $E_3^{\bullet,\bullet} = E_\infty^{\bullet,\bullet}$. □

Note that the product on $H^\bullet(G)$ does not necessarily coincide with the product on $\mathbf{Tot}(E_3^\infty) = \mathbf{Tot}(E_\infty^{\bullet,\bullet})$. However, if $H^\bullet(G) = R \otimes \mathbb{F}_p[y_1, .., y_n]$ one has $R \simeq \mathbf{Tot}(C^{\bullet,\bullet})$ as $\mathbb{F}_p$-modules. With the natural identification one gets for $a \in C^{r,s}$, $b \in C^{u,v}$

$$a \circ b \equiv a.b + \bigoplus_{\substack{i+j=r+s+u+v\\ j \leq s+v-1}} C^{i,j}, \tag{4.11}$$

where "." denotes the product in $\mathbf{Tot}(C^{\bullet,\bullet})$ and "$\circ$" denotes the product in $R \leq H^\bullet(G)$ (cf. [B, Th. 3.9.3]). This yields:

Corollary 4.4. *Let G and $\overline{G}$ be as in Theorem 4.3, $H^\bullet(G) = C \otimes \mathbb{F}_p[y_1, .., y_n]$, $H^\bullet(\overline{G}) = D \otimes \mathbb{F}_p[y_1, .., y_{n-1}]$. If D satisfies Poincaré duality in dimension $n-1$, then C satisfies Poincaré duality in dimension n.*

Proof. This follows from Theorem 4.3 and (4.11). □

5. The Proofs of Theorems A and B

Proof of Theorem A. In Theorem 2.2 it was shown that (1) and (3) are equivalent. (2) implies (3). Hence, it remains to show that (3) implies (2). Without loss of generality we may assume that G is a p-central p-group with ΩEP. We proceed by induction on $|G|$. If G is elementary abelian there is nothing to prove. Thus we may assume that $\Omega_1(G) \lneq G$. The group $G/\Omega_1(G)$ is again p-central with ΩEP (cf. [W, Prop.5.4.(a)]). If $l\text{-}\mathrm{rk}(G) = l\text{-}\mathrm{rk}(G/\Omega_1(G))$, the claim follows by induction and Theorem 4.1. Hence we may assume that

$$n := l\text{-}\mathrm{rk}(G) \gneq l\text{-}\mathrm{rk}(G/\Omega_1(G)).$$

This implies that $\Omega_2(G)^p \lneq \Omega_1(G)$ (cf. [W, Prop.4.1.(d)]). Let $A \leq \Omega_1(G)$, $|A| = p$, be such that $A \cap \Omega_2(G)^p = 1$. Then $\overline{G} := G/A$ is again a p-central group with ΩEP (cf. Prop.2.2). By induction, $H^\bullet(\overline{G}) = D \otimes \mathbb{F}_p[y_1, .., y_{n-1}]$, where D is a finite dimensional $\mathbb{F}_p$-algebra satisfying Poincaré duality in dimension $n-1$. Then Theorem 4.3 and Corollary 4.4 imply that $H^\bullet(G) = C \otimes \mathbb{F}_p[y_1, .., y_n]$, $|y_i| = 2$, and C is a finite-dimensional $\mathbb{F}_p$-algebra satisfying Poincaré duality in dimension n. □

Remark 5.1. (a) The Poincaré duality of C can be shown also by the spectral sequence introduced by D.Benson and J.Carlson (cf. [BC, Th.5.5.]).

(b) The proof of Theorem A shows also that $\chi_C = 0$, where $\chi_C = p_C(-1)$ denotes the Euler characteristic.

Proof of Theorem B. Corollary 4.2 yields that (2) ⇒ (1). On the other hand if G is a finite group such that $H^\bullet(G) = \Lambda[x_1, ..x_n] \otimes \mathbb{F}_p[y_1, .., y_n]$, $|x_i| = 1$, $|y_j| = 2$, p odd, then G is p-nilpotent, $P \in \mathrm{Syl}_p(G)$ is p-central with ΩEP, and $n = l\text{-}\mathrm{rk}(P)$ (cf. Cor. 2.5). Furthermore, one has also

$$P/[P,P].P^p = H_1(P, \mathbb{F}_p) = \mathbb{F}_p^n.$$

But this implies that P is in addition powerful (cf. [W, Prop.4.3.]), and this completes the proof of the theorem. □

Remark 5.2. Theorem B can also be used to compute the Galois cohomology of a uniformly powerful pro-p-group $\hat{P}$, p odd, of rank n. Then $\hat{P}/\hat{P}^n$ is a powerful p-central p-group with ΩEP. Then it follows from Theorem 4.1 that

$$H^\bullet_{gal}(\hat{P}, \mathbb{F}_p) = \varinjlim H^\bullet(\hat{P}/\hat{P}^n, \mathbb{F}_p) \simeq \Lambda[x_1.., x_n]$$

(cf. [La, Chap.V, §2.2.]).

Remark 5.3. Let $\mathcal{A}_p$ denote the Steenrod algebra for the prime p and $\mathcal{A}_p^* \leq \mathcal{A}_p$ the subalgebra generated by the reduced power actions. It can be shown easily that for every powerful p-central group G with ΩEP, $l\text{-}\mathrm{rk}(G) = n$, p odd, one has

$$H^\bullet(G) \simeq H^\bullet(\mathbb{F}_p^n)$$

as $\mathcal{A}_p^*$-module. However, this isomorphism is not an isomorphism as $\mathcal{A}_p$-module (cf. [BP]). One can speculate whether the isomorphism type of G is determined by $H^\bullet(G)$ as $\mathcal{A}_p$-module with the Bockstein spectral sequence. Some groups are known which have this property (cf. [BrL]). W.Browder and J.Pakianathan have shown that the isomorphism type of a uniformly powerful p-group of depth 2 with ΩEP is uniquely determined by the isomorphism type of $H^{\bullet}(G)$ as $\mathcal{A}_p$-module (cf. [BP]).

References

[B] D.Benson, *Representations and Cohomology II*, Cohomology of groups and modules, Cambridge University Press, 1991. MR **93g:**20099

[BC] D.Benson, J.F.Carlson, *Projective resolutions and Poincaré duality complexes*, Trans. Amer. Math. Soc. **342** (1994), 447–488. MR **94f:**20100

[Be] F.R.Beyl, *The spectral sequence of a group extension*, Bull. Sci. Math. (2) **105** (1981), 407–434. MR **83g:**20052

[BrH] C.Broto, H.-W.Henn, *Some remarks on central elementary abelian p-subgroups and cohomology of classifying spaces*, Quart. J. Math Oxford (2) **44** (1993), 155–163. MR **94c:**57060

[BrL] C.Broto, R.Levi, *On the homotopy type of BG for certain finite* 2*-groups*, Trans. Amer. Math. Soc. **349** (1997), 1487–1502. MR **97h:**55016

[BP] W.Browder, J.Pakianathan, *Cohomology of p-power exact groups*, preprint.

[CV] L.Charlap, A.Vasquez, *The cohomology of group extensions*, Trans. Amer. Math. Soc. **124** (1966), 24–40. MR **35:**5514

[DD] J.D.Dixon, M.P.F.du Sautoy, A.Mann, D.Segal, *Analytic pro-p-groups*, LMS Lecture Note Series **157**, Cambridge Univ. Press, 1991. MR **94e:**20037

[Du] J.Duflot, *Depth and equivariant cohomology*, Comm. Math. Helvetici **57** (1981), 137–141. MR **83g:**57029

[HP] H.-W.Henn, S.Priddy, *p-nilpotence, classifying space indecomposability, and other properties of almost all finite groups*, Comm. Math. Helv. **69** (1994), 335–350. MR **95e:**20033

[HS] P.J.Hilton, U.Stammbach, *A course in homological algebra*, Springer-Verlag, New York, 1971. MR **49:**10751

[La] M.Lazard, *Groupes analytiques p-adiques*, IHES Publ. Math. **26** (1965). MR **35:**188

[MP] J.Martino, S.Priddy, *On the cohomology and homotopy of Swan groups*, Math. Z. **225** (1997), 277–288. CMP 97:16

[MM] J.W.Milnor, J.C.Moore, *On the structure of Hopf algebras*, Ann. of Math. (2) **81** (1965), 211–264. MR **30:**4259

[QU1] D.Quillen, *The spectrum of an equivariant cohomology ring I*, Ann. of Math. **94** (1971), 549–572; II, Ann. of Math. **94** (1971), 573–602. MR **45:**7743

[QU2] ———, *A cohomological criterion for p-nilpotency*, J. Pure Appl. Algebra **1** (1971), 361–372. MR **47:**6886

[W] T.Weigel, *Combinatorial properties of p-central groups*, preprint.

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