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A REDUCED TITS QUADRATIC FORM AND TAMENESS OF THREE-PARTITE SUBAMALGAMS OF TILED ORDERS

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Dedicated to Klaus Roggenkamp on the occasion of his 60th birthday

Abstract. Let D be a complete discrete valuation domain with the unique maximal ideal $\mathfrak{p}$. We suppose that D is an algebra over an algebraically closed field K and $D/\mathfrak{p} \cong K$. Subamalgam D-suborders $\Lambda^\bullet$ of a tiled D-order Λ are studied in the paper by means of the integral Tits quadratic form $q_{\Lambda^\bullet} : \mathbb{Z}^{n_1+2n_3+2} \longrightarrow \mathbb{Z}$. A criterion for a subamalgam D-order $\Lambda^\bullet$ to be of tame lattice type is given in terms of the Tits quadratic form $q_{\Lambda^\bullet}$ and a forbidden list $\Omega_1, \dots, \Omega_{17}$ of minor D-suborders of $\Lambda^\bullet$ presented in the tables.

1. Introduction

Throughout this paper K is an algebraically closed field and D is a complete discrete valuation domain which is a K-algebra such that $D/\mathfrak{p} \cong K$, where $\mathfrak{p}$ is the unique maximal ideal of D. We denote by $F = D_0$ the field of fractions of D.

We recall that a **D-order** Λ in a finite dimensional semisimple F-algebra C is a subring Λ of C which is a finitely generated free D-submodule of C and Λ contains an F-basis of C [5]. We denote by latt(Λ) the category of right Λ-lattices, that is, finitely generated right Λ-modules which are free as D-modules. It is well-known that any D-order is a semiperfect ring and the category latt(Λ) has the finite unique decomposition property [32, Section 1.1].

A D-order Λ is said to be of **finite lattice type** if the category latt(Λ) has finitely many isomorphism classes of indecomposable modules. A D-order Λ is said to be of **tame lattice type** if the indecomposable Λ-lattices of any fixed D-rank form a finite set of at most one-parameter families (see [9], [34, Section 3], [39, Section 7]). The definitions are presented at the end of this section.

It was shown by the author in [40] that the weak positivity of the reduced Tits quadratic form (1.4) associated with the subamalgam D-order $\Lambda^\bullet$ (1.3) of tiled D-order Λ (1.1) is a necessary and sufficient condition for finite lattice type.

Our main result of this paper is the characterization given in Theorem 1.5 below of D-orders $\Lambda^\bullet$ (1.3) of tame lattice type in terms of the associated Tits quadratic form (1.4) defined below, and by presenting in Section 1A a list of minimal forbidden minor D-suborders of $\Lambda^\bullet$.

We shall use here the terminology and notation introduced in [40]. We denote by $\mathbb{M}_m(D)$ the full $m \times m$ matrix ring with coefficients in D. We suppose that

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$n, n_1, n_2 > 0$ and $n_3 \geq 0$ are natural numbers and Λ is a tiled D-suborder of $\mathbb{M}_n(D)$ of the form

$$\Lambda = \left.\begin{pmatrix} D & {}_1D_2 & \dots & {}_1D_n \\ \mathfrak{p} & D & \dots & {}_2D_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{p} & \mathfrak{p} & \dots & {}_{n-1}D_n \\ \mathfrak{p} & \mathfrak{p} & \dots & D \end{pmatrix}\right\} n \tag{1.1}$$

such that

(a) ${}_iD_j$ is either D or $\mathfrak{p}$, and

(b) Λ admits a three-partition

$$\Lambda = \left(\begin{array}{c|c|c} \Lambda_1 & \mathcal{X} & \mathbb{M}_{n_1}(D) \\ \hline \mathbb{M}_{n_3\times n_1}(\mathfrak{p}) & \Lambda_3 & \mathcal{Y} \\ \hline \mathbb{M}_{n_1}(\mathfrak{p}) & \mathbb{M}_{n_1\times n_3}(\mathfrak{p}) & \Lambda_2 \end{array}\right) \begin{array}{l} \}n_1 \\ \}n_3 \\ \}n_2 \end{array} \tag{1.2}$$

where $\Lambda_2 = \Lambda_1$, $n_1 = n_2$, $n_1 + n_2 + n_3 = n$ and Λ_3 is a hereditary $n_3 \times n_3$ matrix D-order

$$\Lambda_3 = \left.\begin{pmatrix} D & D & \dots & D & D \\ \mathfrak{p} & D & \dots & D & D \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathfrak{p} & \mathfrak{p} & \dots & D & D \\ \mathfrak{p} & \mathfrak{p} & \dots & \mathfrak{p} & D \end{pmatrix}\right\} n_3$$

In particular, ${}_iD_j = D$ holds in Λ for $1 \leq i \leq n_1$ and $n_1 + n_3 + 1 \leq j \leq n$.

Note that $1 = \varepsilon_1 + \varepsilon_3 + \varepsilon_2$, where ε_1, ε_3 and ε_2 are the matrix idempotents of Λ corresponding to the identity elements of Λ_1, Λ_3 and Λ_2, respectively. By a **three-partite subamalgam** of Λ we shall mean the D-suborder

$$\Lambda^\bullet = \big\{\lambda = [\lambda_{ij}]; \quad \varepsilon_1\lambda\varepsilon_1 - \varepsilon_2\lambda\varepsilon_2 \in \mathbb{M}_{n_1}(\mathfrak{p})\big\} \tag{1.3}$$

of Λ consisting of all matrices $\lambda = [\lambda_{ij}]$ of Λ such that the left upper corner $n_1 \times n_1$ submatrix $\varepsilon_1\lambda\varepsilon_1$ of λ is congruent modulo $\mathbb{M}_{n_1}(\mathfrak{p})$ to the right lower corner $n_1 \times n_1$ submatrix $\varepsilon_2\lambda\varepsilon_2$ of λ.

To any such D-order $\Lambda^\bullet$ we have associated in [40] the reduced Tits quadratic form

$$q_{\Lambda^\bullet} : \mathbb{Z}^{n_1+2n_3+2} \longrightarrow \mathbb{Z} \tag{1.4}$$

in the indeterminates $x_*, x_+, x_1, \dots, x_{n_1+n_3}, \overline{x}_{n_1+1}, \dots, \overline{x}_{n_1+n_3}$ defined by the formula

$$\begin{aligned} q_{\Lambda^\bullet}&(x_1, \dots, x_{n_1+n_3}, \overline{x}_{n_1+1}, \dots, \overline{x}_{n_1+n_3}, x_*, x_+) \\ &= x_*^2 + x_+^2 + \sum_{j=1}^{n_1+n_3} x_j^2 + \sum_{j=n_1+1}^{n_1+n_3} \overline{x}_j^2 \\ &\quad + \sum_{\substack{{}_iD_j = D \\ 1\leq i<j\leq n_1+n_3}} x_i x_j + \sum_{s<t} \overline{x}_s\overline{x}_t + \sum_{\substack{{}_tD_s = D \\ n_1<t\leq n_1+n_3<s}} x_{s-n_1-n_3}\overline{x}_t \\ &\quad - x_+\Big(\sum_{j=1}^{n_1+n_3} x_j\Big) - x_*\Big(\sum_{j=1}^{n_1} x_j + \sum_{j=n_1+1}^{n_1+n_3} \overline{x}_j\Big). \end{aligned}$$

Our main result of this paper is the following theorem.

Theorem 1.5. *Let K be an algebraically closed field and D a complete discrete valuation domain which is a K-algebra such that $D/\mathfrak{p} \cong K$, where $\mathfrak{p}$ is the unique maximal ideal of D.*

Let Λ be a three-partite D-order of the form (1.2) *and let $\Lambda^\bullet$ be the subamalgam* (1.3) *of $\Lambda \subseteq \mathbb{M}_n(D)$, where $\Lambda_1 = \Lambda_2 \subseteq \mathbb{M}_{n_1}(D)$, $\Lambda_3 \subseteq \mathbb{M}_{n_3}(D)$ and n_1, n_3 are as above. If the part $\mathcal{X}$ or the part $\mathcal{Y}$ of the D-order Λ in* (1.2) *consists of matrices with coefficients in $\mathfrak{p}$, then the following conditions are equivalent.*

(a) *The D-order $\Lambda^\bullet$ is of tame lattice type.*

(b) *The integral reduced Tits quadratic form $q_{\Lambda^\bullet} : \mathbb{Z}^{n_1+2n_3+2} \longrightarrow \mathbb{Z}$* (1.4) *is weakly non-negative, that is, $q_{\Lambda^\bullet}(z) \geq 0$ for any vector $z \in \mathbb{N}^{n_1+2n_3+2}$.*

(c) *Either $n_3 = 0$ and the D-order Λ_1 in* (1.2) *does not contain minor D-suborders of one of the forms*

$$\Delta_0 = \begin{pmatrix} D & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & D & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & D \end{pmatrix}, \qquad \Delta_1 = \begin{pmatrix} D & \mathfrak{p} & D \\ \mathfrak{p} & D & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & D \end{pmatrix},$$

$$\Delta_2 = \begin{pmatrix} D & D & \mathfrak{p} \\ \mathfrak{p} & D & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & D \end{pmatrix}, \qquad \Delta_3 = \begin{pmatrix} D & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & D & D \\ \mathfrak{p} & \mathfrak{p} & D \end{pmatrix},$$

or else $n_3 \geq 1$, Λ_1 is hereditary of the form

$$\begin{pmatrix} D & D & \dots & D & D \\ \mathfrak{p} & D & \dots & D & D \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathfrak{p} & \mathfrak{p} & \dots & D & D \\ \mathfrak{p} & \mathfrak{p} & \dots & \mathfrak{p} & D \end{pmatrix} \tag{1.6}$$

and the three-partite subamalgam D-orders $\Lambda^\bullet$ and $\mathrm{rt}(\Lambda)^\bullet$ (1.7) *do not contain three-partite minor D-suborders dominated by any of the* 17 *three-partite subamalgam D-orders listed in the tables of Section* 1A.

(d) *The two-peak poset $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ with zero-relations associated with $\Lambda^\bullet$ in* (3.3) *does not contain as a two-peak subposet with zero-relations any of the* 13 *forms shown in Figure* 1 (*the dotted line in $\widehat{\mathcal{F}}_4$ means a zero-relation*).

We recall from [40] that, given a matrix $\lambda \in \mathbb{M}_n(D)$, we define the **reflection transpose** of λ to be the transpose matrix $\mathrm{rt}(\lambda) \in \mathbb{M}_n(D)$ of λ with respect to the non-main diagonal. Given any D-order Λ, we define the **reflection transpose** of Λ (resp. of $\Lambda^\bullet$) to be the D-orders

$$\mathrm{rt}(\Lambda) = \{\mathrm{rt}(\lambda);\ \lambda \in \Lambda\} \quad (\text{resp. } \mathrm{rt}(\Lambda^\bullet) = \{\mathrm{rt}(\lambda);\ \lambda \in \Lambda^\bullet\}). \tag{1.7}$$

It is easy to see that $\mathrm{rt}(\Lambda^\bullet) = \mathrm{rt}(\Lambda)^\bullet$ and the map $\lambda \mapsto \mathrm{rt}(\lambda)$ defines the ring anti-isomorphisms $\Lambda \xrightarrow{\simeq} \mathrm{rt}(\Lambda)$ and $\Lambda^\bullet \xrightarrow{\simeq} \mathrm{rt}(\Lambda^\bullet)$.

If $1 \leq i_1 < \dots < i_s \leq n_1$, we say that the order Δ is an $(i_1, \dots, i_s)$-**minor** D-**suborder** of Λ_1 in (1.2) if Δ is obtained from Λ_1 by omitting the i_jth row and the i_jth column for $j = 1, \dots, s$.

A three-partite order Ω is said to be a **three-partite minor** D-**suborder** of $\Lambda^\bullet$ if Ω is a minor D-suborder of $\Lambda^\bullet$ obtained by omitting rows and columns simultaneously in parts Λ_1 and Λ_2; that is, we omit any i-th row and any i-th column

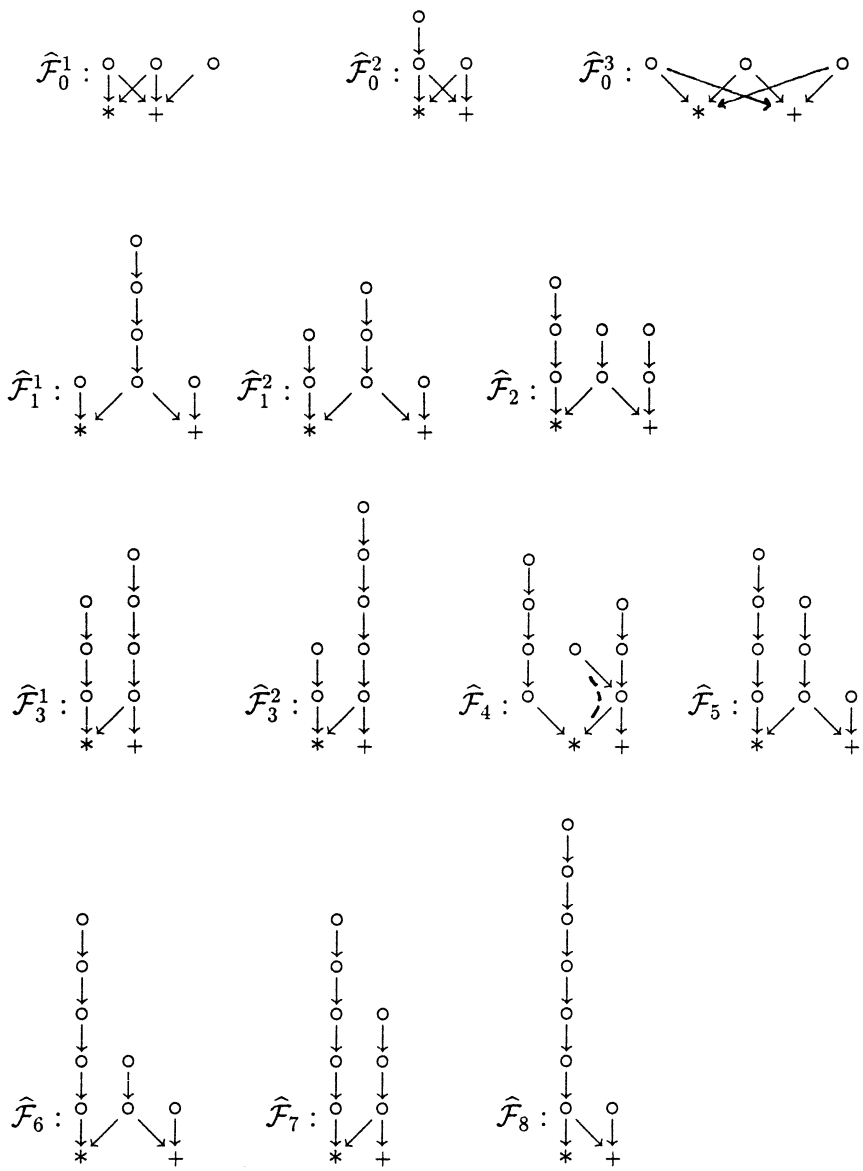


FIGURE 1.

of $\Lambda^\bullet$, where $1 \leq i \leq n_1$, and simultaneously we omit the $(n_1 + n_3 + i)$-th row and the $(n_1 + n_3 + i)$-th column of $\Lambda^\bullet$.

A three-partite subamalgam D-order $\Lambda^\bullet$ (1.3) is said to be **dominated** by a three-partite subamalgam D-order $\overline{\Lambda}^\bullet$ if $\Lambda^\bullet$ is a three-partite D-suborder of $\overline{\Lambda}^\bullet$ of the same size (1.2) and $\Lambda_1 = \overline{\Lambda}_1$, $\Lambda_2 = \overline{\Lambda}_2$, $\Lambda_3 = \overline{\Lambda}_3$, $\mathcal{X} \subseteq \overline{\mathcal{X}}$, $\mathcal{Y} \subseteq \overline{\mathcal{Y}}$ (see [40], [44, p. 69]).

Let us recall from [9], [32, Section 15.12] and [34, Section 3] the definition of an order of tame lattice type. Let Ω be an arbitrary D-order in a semisimple D_0-algebra C, where D is a complete discrete valuation domain which is an algebra over an algebraically closed field K and $D/\mathfrak{p} \cong K$. Then Ω is said to be of **tame lattice type** (or the category $\mathrm{latt}(\Omega)$ is said to be of tame representation type) if for any number $r \in \mathbb{N}$ there exist a non-zero polynomial $h \in K[y]$ and a family of additive functors

$$(1.8) \qquad (-) \otimes_A M^{(1)}, \ldots, (-) \otimes_A M^{(s)} : \mathrm{ind}_1(A) \longrightarrow \mathrm{latt}(\Omega),$$

where $A = K[y, h^{-1}]$, $\mathrm{ind}_1(A)$ is the full subcategory of $\mathrm{mod}\,(A)$ consisting of one dimensional A-modules, and $M^{(1)}, \ldots M^{(s)}$ are A-Ω-bimodules satisfying the following conditions:

(P0) The left A-modules ${}_AM^{(1)}, \ldots, {}_AM^{(s)}$ are flat.
(P1) All but finitely many indecomposable Ω-lattices of D-rank r are isomorphic to lattices in $\mathrm{Im}\,(-) \otimes_A M^{(1)} \cup \ldots \cup \mathrm{Im}\,(-) \otimes_A M^{(s)}$.
(P2) $M^{(1)}_{\Omega}, \ldots, M^{(s)}_{\Omega}$ viewed as D-modules are torsion-free.
(P3) ${}_AM^{(1)}_{\Omega}, \ldots, {}_AM^{(s)}_{\Omega}$ are finitely generated as A-Ω-bimodules.

This means that the functors (1.8) form an almost parameterizing family (see [32, Definition 14.13]) for the category $\mathrm{ind}_r(\mathrm{latt}(\Omega))$ of indecomposable Ω-lattices of D-rank r.

Given an integer $r \geq 1$, we define $\boldsymbol{\mu}^1_{\mathrm{latt}(\Omega)}(r)$ to be the minimal number s of functors (1.8) satisfying the above conditions. The D-order Ω of tame lattice type is defined to be of **polynomial growth** [34, Section 3] if there exists an integer $g \geq 1$ such that $\boldsymbol{\mu}^1_{\mathrm{latt}(\Omega)}(r) \leq r^g$ for all integers $r \geq 2$ (compare with [32, p. 291]).

It was proved in [9] that the tame-wild dichotomy holds for D-orders Ω under the assumption on D made above. The reader is referred to [9], [34, Section 3], [39, Section 7] for various definitions and discussion of orders of tame lattice type and of wild lattice type.

Our main result, Theorem 1.5, is proved in Section 4 by applying a technique developed in [35] and [40]. In particular, we apply the covering technique for bipartite stratified posets developed by the author in [31], and a reduction functor $\mathbb{H}$ (3.5) from $\mathrm{latt}(\Lambda^\bullet)$ to K-linear socle projective representations of a two-peak poset $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ (3.3) with zero-relations associated with $\Lambda^\bullet$ in [40]. Then we apply a criterion for tame prinjective type of two-peak posets given in [17] and [18].

In Section 2 we collect basic facts on K-linear socle projective representations of a multi-peak posets with zero-relations we need in this paper.

In Section 3 we associate with $\Lambda^\bullet$ a two-peak poset $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ with zero-relations (see (3.3)), and we prove in Theorem 3.4 main properties of our reduction functor $\mathbb{H} : \mathrm{latt}(\Lambda^\bullet) \to (I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})\text{-spr}$.

By applying [40, Theorem 6.1] we get a structure of the Auslander-Reiten quiver $\Gamma(\mathrm{latt}(\Lambda^\bullet))$ of $\mathrm{latt}(\Lambda^\bullet)$ (see Remark 3.12).

A simple criterion for a tame lattice type D-order $\Lambda^\bullet$ (1.3) to be of polynomial growth is given by the author in [42, Theorem 1.5]. Tame lattice type subamalgam D-orders $\Lambda^\bullet$ (1.3) of non-polynomial growth are completely described in [42, Theorem 6.2 and Corollary 6.3].

The main results of this paper were presented at the AMS-IMS-SIAM Joint Summer Research Conference "Trends in the Representation Theory of Finite Dimensional Algebras" at the University of Washington, Seattle, in July 1997 (see [41, Theorem 4.2]). They were also presented at the Euroconference "Interactions between Ring Theory and Representations of Algebras", Murcia, 12-17 January 1998 (see [10] and [43, Section 8]).

1A. Tables of Minimal Three-Partite Subamalgams of Tiled D-orders of Wild Lattice Type

We use here the notation introduced in Section 1; that is, we present three-partite tiled D-orders Ω_j, and the three-partition is indicated by vertical and horizontal lines. The subamalgam $\Omega_j^\bullet$ is obtained from Ω_j by the identification modulo $\mathfrak{p} = \mathrm{rad}(D)$ the left upper corner block of Ω_j with the right lower corner block of Ω_j.

Type $\widehat{\mathcal{F}}_1^1$:

$$\Omega_1 = \left(\begin{array}{cccc|c|cccc}
D & D & D & D & \mathfrak{p} & D & D & D & D \\
\mathfrak{p} & D & D & D & \mathfrak{p} & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & D & D & D & D \\ \hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) \quad n_3 = 1,$$

$$\Omega_2 = \left(\begin{array}{cccc|cc|cccc}
D & D & D & D & \mathfrak{p} & D & D & D & D & D \\
\mathfrak{p} & D & D & D & \mathfrak{p} & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & D & D & D & D & D \\ \hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) \quad n_3 = 2,$$

Type $\widehat{\mathcal{F}}_1^2$:

$$\Omega_3 = \left(\begin{array}{ccc|cc|ccc}
D & D & D & \mathfrak{p} & D & D & D & D \\
\mathfrak{p} & D & D & \mathfrak{p} & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & D & D & D & D \\ \hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) \quad n_3 = 2,$$

$$\Omega_4 = \left(\begin{array}{ccc|ccc|ccc}
D & D & D & \mathfrak{p} & D & D & D & D & D \\
\mathfrak{p} & D & D & \mathfrak{p} & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & D & D & D & D & D \\ \hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) \quad n_3 = 3,$$

Type $\widehat{\mathcal{F}}_2$:

$$\Omega_5 = \left(\begin{array}{cc|ccc|cc} D & D & \mathfrak{p} & \mathfrak{p} & D & D & D \\ \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & D & D & D \\ \hline \mathfrak{p} & \mathfrak{p} & D & D & D & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} \\ \hline \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D \end{array}\right) n_3 = 3,$$

$$\Omega_6 = \left(\begin{array}{cc|cccc|cc} D & D & \mathfrak{p} & \mathfrak{p} & D & D & D & D \\ \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & D & D & D & D \\ \hline \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} \\ \hline \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D \end{array}\right) n_3 = 4,$$

$$\Omega_7 = \left(\begin{array}{cc|ccccc|cc} D & D & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D \\ \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D \\ \hline \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} \\ \hline \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D \end{array}\right) n_3 = 5,$$

Type $\widehat{\mathcal{F}}_3^1$:

$$\Omega_8 = \left(\begin{array}{cccc|ccc|cccc} D & D & D & D & D & D & D & D & D & D & D \\ \mathfrak{p} & D & D & D & D & D & D & D & D & D & D \\ \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D & D \\ \hline \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \hline \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D \end{array}\right) n_3 = 3,$$

Type $\widehat{\mathcal{F}}_3^2$:

$$\Omega_9 = \left(\begin{array}{ccccc|cc|ccccc} D & D & D & D & D & D & D & D & D & D & D & D \\ \mathfrak{p} & D & D & D & D & D & D & D & D & D & D & D \\ \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D & D & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D & D \\ \hline \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \hline \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D \end{array}\right) n_3 = 2,$$

Type $\widehat{\mathcal{F}}_4$:

$$\Omega_{10} = \left(\begin{array}{ccc|cccc|ccc}
D & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & D \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) n_3 = 4,$$

$$\Omega_{11} = \left(\begin{array}{ccc|ccccc|ccc}
D & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D \\
\mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right);$$

Type $\widehat{\mathcal{F}}_5$:

$$\Omega_{12} = \left(\begin{array}{ccc|cccc|ccc}
D & D & D & D & D & D & D & D & D & D \\
\mathfrak{p} & D & D & \mathfrak{p} & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & D & D & D & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & D & D & D & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) n_3 = 4,$$

$$\Omega_{13} = \left(\begin{array}{ccc|ccccc|ccc}
D & D & D & D & D & D & D & D & D & D & D \\
\mathfrak{p} & D & D & \mathfrak{p} & D & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & D & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right);$$

Type $\widehat{\mathcal{F}}_6$:

$$\Omega_{14}=\left(\begin{array}{cc|ccccc|cc}
D & D & \mathfrak{p} & D & D & D & D & D & D\\
\mathfrak{p} & D & \mathfrak{p} & D & D & D & D & D & D\\
\hline
\mathfrak{p} & \mathfrak{p} & D & D & D & D & D & \mathfrak{p} & \mathfrak{p}\\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & \mathfrak{p} & \mathfrak{p}\\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & \mathfrak{p} & \mathfrak{p}\\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p}\\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p}\\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D\\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) n_3=5,$$

$$\Omega_{15}=\left(\begin{array}{cc|cccccc|cc}
D & D & \mathfrak{p} & D & D & D & D & D & D & D\\
\mathfrak{p} & D & \mathfrak{p} & D & D & D & D & D & D & D\\
\hline
\mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D & D\\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & \mathfrak{p} & \mathfrak{p}\\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & \mathfrak{p} & \mathfrak{p}\\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & \mathfrak{p} & \mathfrak{p}\\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p}\\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p}\\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D\\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) n_3=6$$

Type $\widehat{\mathcal{F}}_7$:

$$\Omega_{16}=\left(\begin{array}{ccc|ccccc|ccc}
D & D & D & D & D & D & D & D & D & D & D\\
\mathfrak{p} & D & D & D & D & D & D & D & D & D & D\\
\mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D & D & D\\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p}\\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p}\\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p}\\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p}\\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p}\\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D\\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D\\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) n_3=5,$$

Type $\widehat{\mathcal{F}}_8$:

$$\Omega_{17}=\left(\begin{array}{ccccccc|c|ccccccc}
D & D & D & D & D & D & D & \mathfrak{p} & D & D & D & D & D & D & D\\
\mathfrak{p} & D & D & D & D & D & D & \mathfrak{p} & D & D & D & D & D & D & D\\
\mathfrak{p} & \mathfrak{p} & D & D & D & D & D & \mathfrak{p} & D & D & D & D & D & D & D\\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & \mathfrak{p} & D & D & D & D & D & D & D\\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & \mathfrak{p} & D & D & D & D & D & D & D\\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & D & D & D & D & D & D & D\\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & D & D & D & D & D & D & D\\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D & D\\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D\\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D\\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D\\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D\\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D\\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D\\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) n_3=1$$

2. Filtered Socle Projective Representations of Posets with Zero-Relations

We recall from [46] and [32, Chapter 13] that the study of tiled orders is reduced to the study of representations of infinite posets having a unique maximal element. A similar idea applies in the study of some categories of Cohen-Macaulay modules and of abelian groups (see [1], [2], [3], [38], [43]).

We shall prove the main theorems of the paper by reducing the problem for lattices over three-partite subamalgams of tiled D-orders to a corresponding problem for K-linear socle projective representations of two-peak posets (that is, exactly two maximal elements) with zero-relations that was studied in [31] and [40], where $K = D/\mathfrak{p}$. Our reduction extends the reduction given in [35, Section 2] and involves the reduction functors defined in [11] and [27], and the covering technique for bipartite stratified posets developed by the author in [31].

Throughout we shall denote by $(I;\preceq)$ a finite **poset**, that is, a finite partially ordered set $(I;\preceq)$ with the partial order $\preceq$. We shall write $i \prec j$ if $i \preceq j$ and $i \neq j$. For the sake of simplicity we write I instead of $(I,\preceq)$. We denote by $\max I$ the set of all maximal elements of I and I will be called an r-**peak poset** if $|\max I| = r$.

Given a poset I, we denote by KI the incidence algebra of I [32], that is, the subalgebra of the full matrix algebra $\mathbb{M}_I(K)$ consisting of all $I \times I$ square matrices $\lambda = [\lambda_{pq}]_{p,q\in I}$ such that $\lambda_{pq} = 0$ if $p \not\preceq q$ in $(I;\preceq)$.

For $i \preceq j$ we denote by $e_{ij} \in KI$ the matrix having 1 at the i-j-th position and zeros elsewhere. Given j in I, we denote by $e_j = e_{jj}$ the standard primitive idempotent of KI corresponding to j.

The algebra KI is basic, and the standard matrix idempotents e_i, $i \in I$, form a complete set of primitive orthogonal idempotents of KI. Moreover, KI is of finite global dimension and the right socle of KI is isomorphic to a direct sum of copies of the right ideals e_pKI, $p \in \max I$, called the **right peaks** of KI [33].

We shall denote by $\mathrm{mod}_{\mathrm{sp}}(KI)$ the category of **socle projective right KI-modules** [28], that is, the full subcategory of $\mathrm{mod}(KI)$ consisting of modules X such that the socle $\mathrm{soc}(X)$ of X is projective and isomorphic to a direct sum of copies of the right ideals e_pKI, $p \in \max I$.n

In our definition of a main reduction functor we shall also need a notion of a poset with zero-relations (see [40]), as follows.

Definition 2.1. A **zero-relation** in a poset I is a pair (i_0, j_0) of elements of I such that $i_0 \prec j_0$.

A **set of zero-relations** in I is a set $\mathfrak{Z}$ satisfying the following two conditions:

(Z1) $\mathfrak{Z}$ consists of zero-relations (i_0, j_0) of I.

(Z2) If $(i_0, j_0) \in \mathfrak{Z}$ and $i_1 \preceq i_0 \preceq j_0 \preceq j_1$, then $(i_1, j_1) \in \mathfrak{Z}$.

A **right multipeak (or precisely r-peak) poset with zero-relations** is a pair $(I, \mathfrak{Z})$, where I is a poset, $r = |\max I|$, and $\mathfrak{Z}$ is a set of zero-relations satisfying the following condition (see [30, p. 118]):

(Z3) For every $i \in I \setminus \max I$ there exists $p \in \max I$ such that $(i, p) \notin \mathfrak{Z}$.

In case the set $\mathfrak{Z}$ is empty we shall write I instead of $(I, \mathfrak{Z})$.

A right multipeak poset $(I', \mathfrak{Z}')$ with zero-relations is said to be a **peak subposet** of $(I, \mathfrak{Z})$ if I' is a subposet of I, $\mathfrak{Z}'$ is the restriction of $\mathfrak{Z}$ to I' and $\max I' = I' \cap (\max I)$.

Given a right r-peak poset $(I, \mathfrak{Z})$ with zero-relations, we define the **incidence K-algebra** of $(I, \mathfrak{Z})$ to be the K-algebra

$$K(I, \mathfrak{Z}) = \{\lambda = [\lambda_{ij}]_{i,j \in I} \in KI;\ \lambda_{ij} = 0, \text{ for } (i,j) \in \mathfrak{Z}\} \subseteq KI \tag{2.2}$$

consisting of all $I \times I$ square matrices $\lambda = [\lambda_{ij}]_{i,j\in I} \in \mathbb{M}_I(K)$ such that $\lambda_{ij} = 0$ if $i \not\preceq j$ in $(I; \preceq)$, or if $(i,j) \in \mathfrak{Z}$. The addition in $K(I, \mathfrak{Z})$ is the usual matrix addition, whereas the multiplication of two matrices $\lambda = [\lambda_{ij}]_{i,j\in I}$ and $\lambda' = [\lambda'_{ij}]_{i,j\in I}$ in $K(I, \mathfrak{Z})$ is the matrix $\lambda'' = [\lambda''_{ij}]_{i,j\in I}$, where

$$\lambda''_{ij} = \begin{cases} \sum\limits_{i \preceq s \preceq j} \lambda_{is}\lambda'_{sj} & \text{if } i \preceq j \text{ and } (i,j) \notin \mathfrak{Z}, \\ 0 & \text{if } i \not\preceq j \text{ or } (i,j) \in \mathfrak{Z}. \end{cases}$$

In case the set $\mathfrak{Z}$ is empty we get $KI = K(I, \mathfrak{Z})$.

Note that in case the set $\mathfrak{Z}$ is not empty the algebra $K(I, \mathfrak{Z})$ is not a subalgebra of the matrix algebra $KI \subseteq \mathbb{M}_I(K)$.

The incidence algebra $K(I, \mathfrak{Z})$ is basic, and the standard matrix idempotents e_i, $i \in I$, form a complete set of primitive orthogonal idempotents of $K(I, \mathfrak{Z})$. It is easy to see that $K(I, \mathfrak{Z})$ is a factor K-algebra of KI modulo the ideal generated by all matrices $e_{ij} \in KI$ such that $(i,j) \in \mathfrak{Z}$. It follows that the global dimension of $K(I, \mathfrak{Z})$ is finite (see [33, Lemma 2.1]) and, in view of **(Z3)**, the right socle of $K(I, \mathfrak{Z})$ is isomorphic to a direct sum of copies of the right ideals $e_pK(I, \mathfrak{Z})$, $p \in \max I$, called the **right peaks** of $K(I, \mathfrak{Z})$ (see [28]).

We shall denote by $\operatorname{mod}_{\text{sp}} K(I, \mathfrak{Z})$ the category of **socle projective right $K(I, \mathfrak{Z})$-modules**, that is, the full subcategory of $\operatorname{mod} K(I, \mathfrak{Z})$ consisting of modules X such that the socle $\operatorname{soc}(X)$ of X is projective and isomorphic to a direct sum of copies of the right ideals $e_pK(I, \mathfrak{Z})$, $p \in \max I$ (see [28]).

The category $\operatorname{mod}_{\text{sp}} K(I, \mathfrak{Z})$ is closed under extensions, direct sums and summands in $\operatorname{mod} K(I, \mathfrak{Z})$, and has Auslander-Reiten sequences, source maps and sink maps, enough relative projective and enough relative injective objects (see [23]).

Throughout we shall denote by $\operatorname{rep}_K(I, \mathfrak{Z})$ the category of K-linear representation of $(I, \mathfrak{Z})$, that is, the systems

$$(X_i, {}_jh_i)_{i,j \in I,\ i \prec j}$$

of finite dimensional K-vector spaces X_j connected by K-linear maps ${}_jh_i : X_i \to X_j$ satisfying the following conditions:

- ${}_ih_i$ is the identity map on X_i for any $i \in I$,
- ${}_jh_i = 0$ if $(i,j) \in \mathfrak{Z}$,
- ${}_th_j \cdot {}_jh_i = {}_th_i$ if $i \preceq j \preceq t$.

It is well known that there exists a K-linear equivalence of categories

$$\operatorname{mod} K(I, \mathfrak{Z}) \xrightarrow{\ \simeq\ } \operatorname{rep}_K(I, \mathfrak{Z}) \tag{2.3}$$

defined as follows. If X is a module in $\operatorname{mod} K(I, \mathfrak{Z})$, we define the representation $(X_i, {}_jh_i)_{i,j\in I, i\prec j}$ in $\operatorname{rep}_K(I, \mathfrak{Z})$ by setting $X_i = Xe_i$, and we take for ${}_jh_i : X_i \to X_j$ the K-linear map defined by the multiplication by $e_{ij} \in K(I, \mathfrak{Z})$. Conversely, if the system $(X_i, {}_jh_i)_{i,j\in I, i\prec j}$ in $\operatorname{rep}_K(I, \mathfrak{Z})$ is given, we set $X = \bigoplus_{i\in I} X_i$ and we define the multiplication $\cdot : X \times K(I, \mathfrak{Z}) \to X$ by $x_i \cdot e_{ij} = {}_jh_i(x_i)$ for $x_i \in X_i$ and $i \preceq j$, $(i,j) \notin \mathfrak{Z}$. Throughout we shall identify the categories $\operatorname{mod} K(I, \mathfrak{Z})$ and $\operatorname{rep}_K(I, \mathfrak{Z})$ along the functor $X \mapsto (X_i, {}_jh_i)_{i,j\in I, i\prec j}$ (2.3).

The module X is socle projective if and only if X viewed as a K-linear representation $X = (X_i, {}_jh_i)_{i,j\in I, i\prec j}$ of $(I, \mathfrak{Z})$ is socle projective, that is, if $\bigcap_{p\in\max I} \operatorname{Ker} {}_ph_i = 0$ for any $i \in I \setminus \max I$ (see [28]). It is often useful to deal with filtered forms of socle projective K-linear representations of $(I, \mathfrak{Z})$. For this purpose we introduced in [40] the following definition (see also [47], [48]) .

Definition 2.4. Let K be a field and let $(I, \mathfrak{Z})$ be a right multipeak poset with zero-relations. A **peak $(I, \mathfrak{Z})$-space** (or **a filtered socle projective representation** of $(I, \mathfrak{Z})$) over the field K is the system $\mathbf{M} = (M_j)_{j\in I}$ of finite dimensional K-vector spaces M_j satisfying the following four conditions.

(a) For any $j \in I$ the K-space M_j is a K-subspace of $M^\bullet = \bigoplus_{p\in\max I} M_p$.

(b) The inclusion $M_p \subseteq M^\bullet$ is the standard p-coordinate embedding for any $p \in \max I$.

(c) $\pi_j(M_i) \subseteq M_j$ for all $i \prec j$ in I, where $\pi_j : M^\bullet \to M^\bullet$ is the composed K-linear endomorphism

$$M^\bullet \xrightarrow{\pi'_j} \bigoplus_{j\preceq p\in\max I} M_p \hookrightarrow M^\bullet$$

of $M^\bullet$ and π'_j is the direct summand projection.

(d) If $p \in \max I$ and either $i \not\preceq p$ or $i \prec p$ and $(i, p) \in \mathfrak{Z}$, then $\pi_p(M_i) = 0$.

A morphism $f : \mathbf{M} \to \mathbf{M}'$ from $\mathbf{M}$ to $\mathbf{M}'$ is a system $f = (f_p)_{p\in\max I}$ of K-linear maps $f_p : M_p \to M'_p$, $p \in \max I$, such that $\big(\bigoplus_{p\in\max I} f_p\big)(M_j) \subseteq M'_j$ for all $j \in I$.

We denote by $(I, \mathfrak{Z})$-spr the **category of peak I-spaces** (or filtered socle projective representations of $(I, \mathfrak{Z})$) over the field K. The direct sum and the indecomposablity in the category $(I, \mathfrak{Z})$-spr are defined in an obvious way.

A sequence $0 \to \mathbf{M}' \to \mathbf{M} \to \mathbf{M}'' \to 0$ in the category $(I, \mathfrak{Z})$-spr is said to be **exact** if the sequence $0 \to M'_j \to M_j \to M''_j \to 0$ of vector spaces is exact for every $j \in I$.

In case the set $\mathfrak{Z}$ is empty the category $(I, \mathfrak{Z})$-spr is the category I-spr of peak I-spaces (or socle projective representations of I) introduced in [33].

Let us present an alternative definition of peak $(I, \mathfrak{Z})$-spaces. For this purpose we assume that $\mathbf{M} = (M_j)_{j\in I}$ is system of finite dimensional K-vector spaces M_j. We associate with $\mathbf{M} = (M_j)_{j\in I}$ the K-linear representation

$$\mathbf{M}^\bullet = (M^\bullet_j, {}_i\pi^\bullet_j)_{j\in I, i\preceq j} \tag{2.5}$$

of the poset I, where

$$M^\bullet_j = \bigoplus_{\substack{j\prec p\in\max I\\ (j,p)\notin\mathfrak{Z}}} M_p \subseteq M^\bullet = \bigoplus_{j\prec p\in\max I} M_p \tag{2.6}$$

and if the relation $i \preceq j$ holds in I we define ${}_i\pi^\bullet_j : M^\bullet_i \longrightarrow M^\bullet_j$ to be the composed K-linear map

$$M^\bullet_i \subseteq M^\bullet \xrightarrow{\pi'_j} M^\bullet_j,$$

where π'_j is the direct summand projection.

The following useful fact is easily verified.

Lemma 2.7. *Let K be a field and let $(I, \mathfrak{Z})$ be a right multipeak poset with zero-relations. Assume that $\mathbf{M} = (M_j)_{j\in I}$ is a system of finite dimensional K-vector spaces M_j and $\mathbf{M}^\bullet = (M^\bullet_j, {}_i\pi^\bullet_j)_{j\in I, i\preceq}$ is the K-linear representation associated with $\mathbf{M}$ above.*

(a) *If* $i \in I$ *and* $j \preceq t \preceq s$, *then* ${}_i\pi_i^\bullet = \mathrm{id}$ *and* ${}_s\pi_t^\bullet \cdot {}_t\pi_j^\bullet = {}_s\pi_j^\bullet$. *If* $(i,j) \in \mathfrak{Z}$ *then* ${}_i\pi_j^\bullet = 0$.

(b) *The system* $\mathbf{M} = (M_j)_{j\in I}$ *is a peak* $(I,\mathfrak{Z})$*-space if and only if the following two conditions are satisfied:*

(i) $M_j \subseteq M_j^\bullet \subseteq M^\bullet$ *for all* $j \in I$, *and*

(ii) ${}_j\pi_i^\bullet(M_i) \subseteq M_j$ *if* $i \preceq j$*—that is, there is a unique factorisation* ${}_j\pi_i : M_i \to M_j$ *making the diagram*

$$\begin{array}{ccc} M_i & \subseteq & M_i^\bullet \\ \big\downarrow {\scriptstyle {}_j\pi_i} & & \big\downarrow {\scriptstyle {}_j\pi_i^\bullet} \\ M_j & \subseteq & M_j^\bullet \end{array} \tag{2.8}$$

commutative.

It is easy to see that $(I,\mathfrak{Z})$-spr is an additive category with the finite unique decomposition property [32, Section 1.1], and the K-linear functor

$$\boldsymbol{\rho} : (I,\mathfrak{Z})\text{-spr} \xrightarrow{\ \simeq\ } \mathrm{mod_{sp}}\ K(I,\mathfrak{Z}), \tag{2.9}$$

$\mathbf{M} \mapsto \widehat{\mathbf{M}} = (M_j;\ {}_j\pi_i)_{i\prec j}$, is an equivalence of categories, where ${}_j\pi_i : M_i \to M_j$ is the unique K-linear map making the diagram (2.8) commutative. The quasi-inverse of $\boldsymbol{\rho}$ is the restriction to the category $\mathrm{mod_{sp}}\ K(I,\mathfrak{Z})$ of the **adjustment functor** (see [28], [23], [32, (11.32)], [33])

$$\boldsymbol{\theta} : \mathrm{mod}\ K(I,\mathfrak{Z}) \longrightarrow K(I,\mathfrak{Z})\text{-spr} \tag{2.10}$$

associating to $X = (X_i, {}_jh_i)_{i,j\in I, i\prec j}$ the peak $(I,\mathfrak{Z})$-space $\mathbf{M}(X) = (M(X)_j)_{j\in I}$, where

$$M(X)_j = \begin{cases} X_j & \text{for } j \in \max I, \\ \mathrm{Im}\,[({}_ph_j)_{p\in\max I} : X_j \to \bigoplus_{p\in\max\ I} X_p] & \text{for } j \in I \setminus \max I. \end{cases}$$

Corollary 2.11. (a) *The category* $(I,\mathfrak{Z})$-spr *is an additive* K*-category with the finite unique decomposition property* [32, Section 1.1].

(b) *Every object in* $(I,\mathfrak{Z})$-spr *has a projective cover.*

(c) *The category* $(I,\mathfrak{Z})$-spr *has Auslander-Reiten sequences, source maps and sink maps, enough projective objects and enough relative injective objects* [23].

Proof. In view of the equivalence (2.9) the corollary follows by applying the results in [23] to the bipartite algebra $R = K(I,\mathfrak{Z})$ equipped with the bipartition

$$R = K(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet}) = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix},$$

where

$$A = K(I^{*+}_{\Lambda^\bullet} \setminus \max I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet}) \cong e_-Re_- \cong R/\mathrm{soc}(R_R),$$
$$B = K(\max I^{*+}_{\Lambda^\bullet}) \cong e_+Re_+ \cong K \times K \times \cdots \times K \quad (|\max I^{*+}_{\Lambda^\bullet}|\text{-times}),$$
$$e_- = \sum_{j\in I^{*+}_{\Lambda^\bullet}\setminus\max\ I^{*+}_{\Lambda^\bullet}} e_j, \qquad e_+ = \sum_{p\in\max\ I^{*+}_{\Lambda^\bullet}} e_p,$$

and the vector space

$$M = \bigoplus_{p\in\max I^{*+}_{\Lambda^\bullet}} \ \bigoplus_{\substack{j\prec p \\ (j,p)\notin\mathfrak{Z}_{\Lambda^\bullet}}} e_{ip}K \ \cong\ e_-Re_+$$

is viewed as an A-B-bimodule in an obvious way and multiplication is given by the usual matrix multiplication formula

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \begin{pmatrix} a' & m' \\ 0 & b' \end{pmatrix} = \begin{pmatrix} aa' & am' + mb' \\ 0 & bb' \end{pmatrix},$$

for $a, a' \in A$, $b, b' \in B$ and $m, m' \in M$.

Note that $(I, \mathfrak{Z})$-spr $\cong \operatorname{mod}_{\rm sp} K(I, \mathfrak{Z})$ is the category $\operatorname{mod}_{ic}(R)$ of injectively cogenerated modules in the notation of [23]. □

Following [32, Section 14.4], we say that the categories $\operatorname{mod}_{\rm sp} K(I, \mathfrak{Z}) \cong (I, \mathfrak{Z})$-spr are of **tame representation type** if for any number $r \in \mathbb{N}$ there exist a non-zero polynomial $h \in K[y]$ and a family of additive functors

$$(2.12) \qquad (-) \otimes_S N^{(1)}, \dots, (-) \otimes_S N^{(s)} : \operatorname{ind}_1(S) \longrightarrow \operatorname{mod}_{\rm sp} K(I, \mathfrak{Z}),$$

where $S = K[y, h^{-1}]$, $N^{(1)}, \dots, N^{(s)}$ are A-$K(I, \mathfrak{Z})$-bimodules satisfying the following conditions:

(T0) The left A-modules ${}_S N^{(1)}, \dots, {}_S N^{(s)}$ are finitely generated.

(T1) All but finitely many indecomposable modules in $\operatorname{mod}_{\rm sp} K(I, \mathfrak{Z})$ of K-dimension r are isomorphic to modules in $\operatorname{Im}(-) \otimes_S N^{(1)} \cup \dots \cup \operatorname{Im}(-) \otimes_S N^{(s)}$.

This means that the functors (2.12) form an almost parameterizing family (see [32, Definition 14.13]) for the category $\operatorname{ind}_r(\operatorname{mod}_{\rm sp} K(I, \mathfrak{Z}))$ of indecomposable modules X in $\operatorname{mod}_{\rm sp} K(I, \mathfrak{Z})$ such that $\dim_K X = r$.

Given an integer $r \geq 1$, we define $\boldsymbol{\mu}^1_{\operatorname{mod}_{\rm sp} K(I,\mathfrak{Z})}(r)$ to be the minimal number s of functors (2.12) satisfying the above conditions. The categories $\operatorname{mod}_{\rm sp} K(I, \mathfrak{Z}) \cong (I, \mathfrak{Z})$-spr of tame representation type are defined to be of **polynomial growth** [34] if there exists an integer $g \geq 1$ such that $\boldsymbol{\mu}^1_{\operatorname{mod}_{\rm sp} K(I,\mathfrak{Z})}(r) \leq r^g$ for all integers $r \geq 2$ (compare with [32, p. 291]).

Following [23] and [33, (3.1)] we associate to any r-peak poset $(I, \mathfrak{Z})$ with zero-relations the integral bilinear form $b_{(I,\mathfrak{Z})} : \mathbb{Z}^I \times \mathbb{Z}^I \longrightarrow \mathbb{Z}$,

$$(2.13) \qquad b_{(I,\mathfrak{Z})}(x, y) = \sum_{j \in I} x_j y_j + \sum_{\substack{i \prec j \notin \max I \\ (i,j) \notin \mathfrak{Z}}} y_i x_j - \sum_{p \in \max I} \sum_{\substack{i \prec p \\ (i,p) \notin \mathfrak{Z}}} x_i y_p.$$

and the integral Tits quadratic form $q_{(I,\mathfrak{Z})} : \mathbb{Z}^I \longrightarrow \mathbb{Z}$, $q_{(I,\mathfrak{Z})}(z) = b_{(I,\mathfrak{Z})}(z, z)$.

The following result is useful in applications.

Theorem 2.14. *Let $(I, \mathfrak{Z})$ be a multi-peak poset with zero-relations and let $b_{(I,\mathfrak{Z})} : \mathbb{Z}^I \times \mathbb{Z}^I \to \mathbb{Z}$ be the bilinear form* (2.13).

(a) *For any pair X and Y of modules in* prin $K(I, \mathfrak{Z})$ (*see* [40, Section 3]) *the following equality holds:*

$$(2.15) \qquad b_{(I,\mathfrak{Z})}(\mathbf{cdn}\, X, \mathbf{cdn}\, Y) = \dim_K \operatorname{Hom}_{K(I,\mathfrak{Z})}(X, Y) - \dim_K \operatorname{Ext}^1_{K(I,\mathfrak{Z})}(X, Y)$$

(b) *If the category $(I, \mathfrak{Z})$*-spr *is of tame representation type then the Tits quadratic form $q_{(I,\mathfrak{Z})} : \mathbb{Z}^I \to \mathbb{Z}$ (see* (2.13)) *is weakly non-negative.*

Proof. The statement (a) follows from [23, Proposition 4.4].

(b) We recall from [41, Proposition 2.7] that there exists an adjustment functor

$$\boldsymbol{\theta}_I : \operatorname{prin} K(I, \mathfrak{Z}) \longrightarrow \operatorname{mod}_{\rm sp} K(I, \mathfrak{Z}) \cong (I, \mathfrak{Z})\text{-spr}$$

which preserves and reflects tame representation type. Hence the tameness of the category $(I,\mathfrak{Z})$-spr yields the tameness of $\mathrm{mod}_{\mathrm{sp}}\, K(I,\mathfrak{Z})$, and consequently the tameness of the category $\mathrm{prin}\, K(I,\mathfrak{Z})$. Then (b) is a consequence of [23, Proposition 4.2] (see also [15, Theorem 3.18]). $\square$

We shall often use a reflection duality for the category $(I,\mathfrak{Z})$-spr introduced for socle projective modules in [30]. We shall present it here in a more convenient form.

For this purpose we associate with any s-peak poset $(I,\mathfrak{Z})$ with zero-relations, $s \geq 1$, the reflection-dual s-peak poset $(I^{\bullet},\mathfrak{Z}^{\bullet})$ with zero-relations defined as follows.

Definition 2.16. Assume that $(I,\mathfrak{Z})$ is an s-peak poset with zero-relations, and let $\max I = \{p_1,\dots,p_s\}$.

(a) We define a left-right s-peak poset with zero-relations $(\widehat{I},\widehat{\mathfrak{Z}})$, where

$$\widehat{I} = \{p_1^-,\dots,p_s^-\} \cup I$$

is a poset enlargement of I by a set $\{p_1^-,\dots,p_s^-\}$ of minimal elements. The partial order $\preceq$ in $\widehat{I}$ is an extension of the partial order $\preceq$ in I by the relations

$$p_h^- \prec j \quad \Leftrightarrow \quad \text{there exists } i \preceq j \text{ in } I \text{ such that } i \prec p_h \text{ in } I \text{ and } (i,p_h) \notin \mathfrak{Z}$$

for any $p_h \in \max I$. We define the set $\widehat{\mathfrak{Z}}$ of zero-relations in $\widehat{I}$ to be the set generated by the union of $\mathfrak{Z}$ and the set consisting of the following relations:

- (p_h^-, p_t) for all $h \neq t$, and
- (p_h^-, j), where $p_h^- \prec j$ holds in $(\widehat{I},\preceq)$, whereas $j \prec p_h$ does not hold in $(I,\preceq)$.

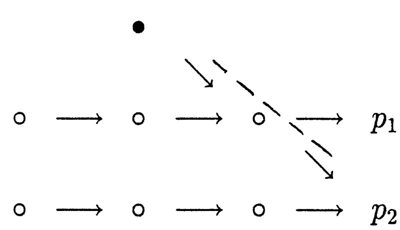
(b) We define the **reflection-dual s-peak poset** with zero-relations

$$(I,\mathfrak{Z})^{\bullet} = (I^{\bullet},\mathfrak{Z}^{\bullet}) \tag{2.17}$$

to the poset $(I,\mathfrak{Z})$ to be the poset $I^{\bullet} = (\widehat{I} \setminus \max I)^{\mathrm{op}}$ dual to $(\widehat{I} \setminus \max I, \preceq)$. We take for $\mathfrak{Z}^{\bullet}$ the dual of the restriction of $\widehat{\mathfrak{Z}}$ to $\widehat{I} \setminus \max I$.

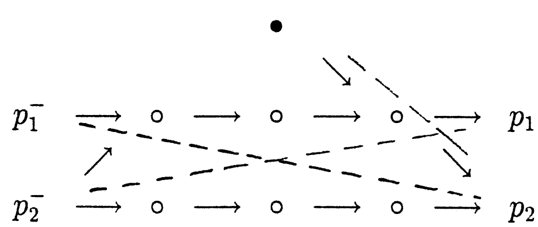
Example 2.18. Let

$(I,\mathfrak{Z})$:



where $\mathfrak{Z} = \{(\bullet, p_2)\}$ consists of one zero-relation $(\bullet, p_2)$. Then

$(\widehat{I},\widehat{\mathfrak{Z}})$:



where $\widehat{\mathfrak{Z}} = \{(\bullet, p_2), (p_1^-, p_2), (p_2^-, p_1)\}$ and the dotted lines mean zero-relations. The reflection-dual $(I^{\bullet},\mathfrak{Z}^{\bullet})$ of $(I,\mathfrak{Z})$ is the two-peak poset without zero-relations

$$
(I^\bullet, \emptyset): \qquad
\begin{array}{ccccccc}
 & & & & \bullet & & \\
 & & & & & \nwarrow & \\
p_1^- & \longleftarrow & \circ & \longleftarrow & \circ & \longleftarrow & \circ \\
 & \swarrow & & & & & \\
p_2^- & \longleftarrow & \circ & \longleftarrow & \circ & \longleftarrow & \circ
\end{array}
$$

that is, the set $\mathfrak{Z}^\bullet$ is empty. Note also that the two-peak poset $\widehat{\mathcal{F}}_5$ in Theorem 1.5 (d) is a reflection-dual to $\widehat{\mathcal{F}}_4$. $\square$

Following [30, 2.6] and [32, Chapter 5], we define a pair of **reflection duality functors**

$$
(I, \mathfrak{Z})\text{-spr} \underset{D^\bullet}{\overset{D^\bullet}{\rightleftarrows}} (I, \mathfrak{Z})^\bullet\text{-spr} \tag{2.19}
$$

as follows. Given $\mathbf{M} = (M_i)_{i\in I}$ in $(I, \mathfrak{Z})$-spr, we define $D^\bullet(\mathbf{M}) = (\widetilde{M}_i)_{i\in I^\bullet}$, where $\widetilde{M}_{p^-} = M_p^* = \mathrm{Hom}_K(M_p, K)$ for $p \in \max I$, and $\widetilde{M}_j$ is the image of the K-dual vector space to the cokernel $\overline{M}_j$ of the embedding

$$
u_j : M_j \hookrightarrow M_j^\bullet = \bigoplus_{\substack{j\preceq p\in\max I \\ (j,p)\notin\mathfrak{Z}}} M_p
$$

under the composed map

$$
(\mathrm{Coker}\ u_j)^* \xrightarrow{v_j^*} (M_j^\bullet)^* \hookrightarrow \widetilde{M}^\bullet = \bigoplus_{p^-\in\max I^\bullet} \widetilde{M}_{p^-}
$$

and v_j^* is the K-dual map to the cokernel epimorphism $v_j : M_j^\bullet \to \mathrm{Coker}\ u_j$ for $j \in I \setminus \max I$. The functor $D^\bullet$ is defined on morphisms in a natural way.

One has to note that $D^\bullet(\mathbf{M})$ is an object of $(I, \mathfrak{Z})^\bullet$-spr. This easily follows by applying Lemma 2.7 and the following equalities:

$$
M_j^\bullet = \bigoplus_{\substack{j\preceq p\in\max I \\ (j,p)\notin\mathfrak{Z}}} M_p = \bigoplus_{\substack{j\succeq p^-\in\min\widehat{I} \\ (p^-,j)\notin\widehat{\mathfrak{Z}}}} M_p = (\widetilde{M}_j^\bullet)^* \tag{*}
$$

It follows that the K-dual space to $M_j^\bullet$ is just the space $\widetilde{M}_j^\bullet$, and the exact sequence

$$
0 \longrightarrow M_j \xrightarrow{u_j} M_j^\bullet \xrightarrow{v_j} \overline{M}_j \longrightarrow 0 \tag{**}
$$

with $\overline{M}_j = \mathrm{Coker}\ u_j$ induces the embedding $\widetilde{M}_j \cong \overline{M}_j^* \xrightarrow{v_j^*} (M_j^\bullet)^* = \widetilde{M}_j^\bullet$ required in Lemma 2.7 (i). Moreover, if $i \preceq j$ in I, then according to Lemma 2.8 (ii) there is a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M_i & \xrightarrow{u_i} & M_i^\bullet & \xrightarrow{v_i} & \overline{M}_i & \longrightarrow & 0 \\
 & & \big\downarrow {\scriptstyle {}_j\pi_i} & & \big\downarrow {\scriptstyle {}_j\pi_i^\bullet} & & \big\downarrow {\scriptstyle {}_j\overline{\pi}_i} & & \\
0 & \longrightarrow & M_j & \xrightarrow{u_j} & M_j^\bullet & \xrightarrow{v_j} & \overline{M}_j & \longrightarrow & 0
\end{array}
$$

By the above equalities the dual diagram defines just the commutative diagram (2.8) for the system $D^\bullet(\mathbf{M}) = (\widetilde{M}_i)_{i\in I^\bullet}$. This proves that $D^\bullet(\mathbf{M}) = (\widetilde{M}_i)_{i\in I^\bullet}$ is an object of $(I, \mathfrak{Z})^\bullet$-spr, as we required.

Since there is a natural isomorphism $((I,\mathfrak{Z})^\bullet)^\bullet \cong (I,\mathfrak{Z})$ of posets with zero-relations, then the above construction applied to $(I,\mathfrak{Z})^\bullet$ defines the inverse reflection duality functor $D^\bullet : (I,\mathfrak{Z})^\bullet\text{-spr} \longrightarrow (I,\mathfrak{Z})\text{-spr}$.

Let us summarize the main facts about the reflection dualities in the following proposition.

Proposition 2.20. *Let $(I,\mathfrak{Z})$ be a poset with zero-relations and let $(I,\mathfrak{Z})^\bullet$ be its reflection-dual poset with zero-relations* (2.17). *Then the following statements hold.*

(a) *There is an isomorphism $((I,\mathfrak{Z})^\bullet)^\bullet \cong (I,\mathfrak{Z})$ of posets with zero-relations.*

(b) *If $v \in \mathbb{N}^I$ is given and $v^\bullet \in \mathbb{N}^{I^\bullet}$ is such that $v^\bullet_j = v_j$ for $j \in I \setminus \max I$, and $v^\bullet_j = v_p$ for $j = p^-$, $p \in \max I$, then $q_{(I,\mathfrak{Z})^\bullet}(v^\bullet) = q_{(I,\mathfrak{Z})}(v)$.*

(c) *The reflection duality functors* (2.19) *are dualities of categories inverse to each other. Moreover, they have the following properties:*

(i) *A sequence $0 \to \mathbf{M} \to \mathbf{N} \to \mathbf{L} \to$ is exact in the category $(I,\mathfrak{Z})$-spr if and only if $0 \to D^\bullet(\mathbf{M}) \to D^\bullet(\mathbf{N}) \to D^\bullet(\mathbf{L}) \to$ is an exact sequence in the category $(I,\mathfrak{Z})^\bullet$-spr.*

(ii) *The functor $D^\bullet$ carries relative injective objects to projective objects.*

(iii) *If $\mathbf{M} = (M_j)_{j\in I}$ is an object of $(I,\mathfrak{Z})$-spr and $\mathbf{dim}\,\mathbf{M} = (\dim_K M_j)_{j\in I}$, then*

$$\mathbf{dim}\, D^\bullet(\mathbf{M}) = \mathbf{s}^\bullet(\mathbf{dim}\,\mathbf{M})$$

where $\mathbf{s}^\bullet : \mathbb{Z}^I \longrightarrow \mathbb{Z}^{I^\bullet} \cong \mathbb{Z}^I$ is the group isomorphism defined by the formula

$$\mathbf{s}^\bullet(w)_j = \begin{cases} -w_j + \sum\limits_{j \prec p \in \max I} w_p & \text{if } j \in I \setminus \max I, \\ w_p & \text{if } j = p^-,\ p \in \max I. \end{cases}$$

(d) *The category $(I,\mathfrak{Z})$-spr is of tame (resp. wild) representation type if and only if the category $(I,\mathfrak{Z})^\bullet$-spr is of tame representation type.*

Proof. Statements (a) and (b) follow from the definitions. For (b) we note that the relation $j \prec p \in \max I$ holds in I and $(j,p) \notin \mathfrak{Z}$ if and only if the relation $j \prec p^- \in \max I^\bullet$ holds in $I^\bullet$ and $(j,p^-) \notin \mathfrak{Z}^\bullet$.

(c) For any $j \in I$ look at the exact sequence $0 \to M_j \xrightarrow{u_j} M^\bullet_j \xrightarrow{v_j} \overline{M}_j \to 0$ with $\overline{M}_j = \operatorname{Coker} u_j$. Since the K-dual to the map u_j induces the embedding

$$(***) \qquad \widetilde{M}_j \cong \overline{M}^*_j \xrightarrow{v^*_j} (M^\bullet_j)^* = \widetilde{M}^\bullet_j$$

required in Lemma 2.8 (i), then by applying the definition of $D^\bullet$ to $D^\bullet(\mathbf{M}) = (\widetilde{M}_j)_{j\in I}$ we easily conclude that $D^\bullet(D^\bullet(\mathbf{M})) \cong \mathbf{M}$, and the isomorphism is functorial with respect to morphisms $\mathbf{M} \to \mathbf{N}$.

The proof of (i) and (ii) is routine, we leave it to the reader. For the proof of (iii) we recall from $(*)$ and $(***)$ above that $\widetilde{M}_{p^-} = M^*_p$ for $p \in \max I$ and $\widetilde{M}_j \cong \overline{M}^*_j$, $\widetilde{M}^\bullet_j = (M^\bullet_j)^* = \bigoplus_{\substack{j \preceq p \in \max I \\ (j,p)\notin\mathfrak{Z}}} M^*_p$ for $j \in I \setminus \max I$, where $D^\bullet(\mathbf{M}) = (\widetilde{M}_j)_{j\in I}$. Let $w = \mathbf{dim}\,\mathbf{M} = (\dim_K M_j)_{j\in I}$. It follows that $\dim_K \widetilde{M}_{p^-} = \dim_K M_p = \mathbf{s}^\bullet(w)_{p^-}$ for $p \in \max I$. In view of the exact sequence $(*)$ above, given $j \in I \setminus \max I$ we get

$$\begin{aligned} \dim_K \widetilde{M}_j &= \dim_K \overline{M}_j = -\dim_K M_j + \dim_K M^\bullet_j \\ &= -\dim_K M_j + \sum_{\substack{j \preceq p \in \max I \\ (j,p)\notin\mathfrak{Z}}} \dim_K M_p = -w_j + \sum_{\substack{j \preceq p \in \max I \\ (j,p)\notin\mathfrak{Z}}} w_p = \mathbf{s}^\bullet(w)_j. \end{aligned}$$

This proves (iii) and finishes the proof of (c).

(d) Assume to the contrary that the category $(I,\mathfrak{z})$-spr is of tame representation type, whereas the category $(I,\mathfrak{z})^\bullet$-spr is not of tame representation type. Since the tame-wild dichotomy holds, then $(I,\mathfrak{z})^\bullet\text{-spr} \cong \operatorname{mod}_{\rm sp} K(I,\mathfrak{z})$ is of wild representation type and there exists a representation embedding functor $F: \operatorname{mod}\ \Gamma_3(K) \to (I,\mathfrak{z})^\bullet$-spr in the sense of [34], where

$$\Gamma_3(K) = \begin{pmatrix} K & K^3 \\ 0 & K \end{pmatrix}$$

is a generalized Kronecker K-algebra. Since $\Gamma_3(K)$ is self-dual and according to (c) the functor $D^\bullet$ is an exact equivalence of categories, then the composed functor

$$\begin{array}{ccccc} \operatorname{mod}\Gamma_3(K) & \xrightarrow{D} & (\operatorname{mod}\Gamma_3(K)^{\rm op})^{\rm op} & \cong & (\operatorname{mod}\Gamma_3(K))^{\rm op} \\ & & & & \Big\downarrow {\scriptstyle F^{\rm op}} \\ & & & & ((I,\mathfrak{z})^\bullet\text{-spr})^{\rm op} \xrightarrow{D^\bullet} (I,\mathfrak{z})\text{-spr} \end{array}$$

is a representation embedding functor, where $D = \operatorname{Hom}_K(-,K)$ is the standard duality functor. It follows from [34, Theorem 2.7] that the category $(I,\mathfrak{z})$-spr is of wild representation type, and according to the tame-wild dichotomy the category $(I,\mathfrak{z})$-spr is not of tame representation type, contrary to our assumption. The remaining part of (d) follows from the tame-wild dichotomy (see [9], [39]). This finishes the proof. □

Remark 2.21. It follows from [30, Proposition 2.5(c) and (2.6)] or from a straightforward analysis that the reflection duality functor $D^\bullet$ (2.19) can be alternatively described as follows.

Any object $\mathbf{M}$ of $(I,\mathfrak{z})$-spr can be viewed as an object of the category $(\widehat{I},\widehat{\mathfrak{z}})$-spr via the obvious embedding functor $(I,\mathfrak{z})\text{-spr} \subseteq (\widehat{I},\widehat{\mathfrak{z}})\text{-spr} \cong \operatorname{mod}_{\rm sp} K(\widehat{I},\widehat{\mathfrak{z}})$. It is easy to see that the injective envelope $\widehat{E}(\mathbf{M})$ of $\mathbf{M}$ in $\operatorname{mod} K(\widehat{I},\widehat{\mathfrak{z}})$ is a socle projective module and is isomorphic to an object of $(\widehat{I},\widehat{\mathfrak{z}})$-spr. Consider the short exact sequence $0 \to \mathbf{M} \to \widehat{E}(\mathbf{M}) \to \overline{\mathbf{M}} \to 0$ in $\operatorname{mod}\ K(I,\mathfrak{z}) \xrightarrow{\simeq} \operatorname{rep}_K(I,\mathfrak{z})$ (see (2.3)). It is clear that $\overline{M}_p = 0$ for all $p \in \max \widehat{I}$, and therefore the system $\overline{\mathbf{M}}^* = (\overline{M}_j^*)$ of K-dual vector spaces $\overline{M}_j^*$ is a peak $(I,\mathfrak{z})^\bullet$-space isomorphic with $D^\bullet(\mathbf{M})$.

Remark 2.22. The class of multi-peak posets with zero-relations defined above is the smallest subclass in the class of all multi-peak bound quivers [30] containing multi-peak posets without zero-relations and closed under the reflection duality operation (2.17).

3. A Reduction to Two-Peak Poset Representations

With any D-order $\Lambda^\bullet$ (1.3) we associate in (3.3) below (see [40, Section 4]) a two-peak poset $(I^{*+}_{\Lambda^\bullet},\mathfrak{z})$ with zero-relations, and we shall reduce the study of the category $\operatorname{latt}(\Lambda^\bullet)$ to the study of the category $(I^{*+}_{\Lambda^\bullet},\mathfrak{z})$-spr.

Suppose that Λ, Λ_1, Λ_2 and Λ_3 are tiled D-orders in (1.2). In order to define $(I^{*+}_{\Lambda^\bullet},\mathfrak{z}_{\Lambda^\bullet})$ we consider the poset $(I_\Lambda;\preceq)$ (see [45]), where

$$I_\Lambda = \{1,\dots,n\} \quad \text{and} \quad i \prec j \Leftrightarrow {}_iD_j = D. \tag{3.1}$$

First we associate with $\Lambda^\bullet$ the combinatorial object

$$I_{\Lambda^\bullet,\sigma} = (I_\Lambda, \preceq, I', C, I'', \sigma : I' \to I'') \tag{3.2}$$

where $(I_\Lambda, \preceq)$ is the poset (3.1), $C = I_{\Lambda_3} = \{n_1+1 \prec \ldots \prec n_1+n_3-1 \prec n_1+n_3\}$, $I' = I_{\Lambda_1} = \{1, 2, \ldots, n_1\}$ and $I'' = I_{\Lambda_2} = \{n_1+n_3+1, \ldots, n-1, n\}$ are viewed as subposets of I_Λ such that $I_\Lambda = I' \cup C \cup I''$ is a splitting decomposition of I_Λ in the sense of [32, Section 8.1], and $\sigma : I' \to I''$ is the poset isomorphism defined by the formula $\sigma(j) = n_1 + n_3 + j$. It is clear that $I_{\Lambda^\bullet,\sigma}$ is a bipartite stratified poset in the sense of [29], [31] and [32, Section 17.8], or a completed poset in the sense of [22].

Let $C' = \{c'; c \in C\}$ be a chain isomorphic with C. We construct two one-peak enlargements

$$(C \cup I'')^* = C \cup I'' \cup \{*\} \qquad \text{and} \qquad (I' \cup C)^+ = I' \cup C' \cup \{+\}$$

of the posets $C \cup I''$ and $I' \cup C \equiv I' \cup C'$ by the unique maximal points $*$ and $+$, and by the new relations $i \prec *$ and $s \prec +$ for all $i \in C \cup I''$ and all $s \in I' \cup C'$.

We associate with the D-order $\Lambda^\bullet$ (1.3) the two-peak poset with zero-relations

$$(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet}) = \Big((C \cup I'')^* \cup_{I'' \equiv I'} (I' \cup C)^+, \mathfrak{Z}_{\Lambda^\bullet}\Big) \tag{3.3}$$

where $I^{*+}_{\Lambda^\bullet}$ is obtained from the disjoint union $(C \cup I'')^* \cup (I' \cup C)^+$ of $(C \cup I'')^*$ and $(I' \cup C)^+$ by making the identification $j \equiv \sigma(j)$ for any element $j \in I' \subseteq (I' \cup C)^+$. The set $\mathfrak{Z}_{\Lambda^\bullet}$ consists of all a pairs (c, c_1') such that $c \in C \subseteq (C \cup I'')^*$, $c_1' \in C' \subseteq (I' \cup C)^+$ and the relations $c \prec s$, $\sigma(s) \prec c_1$ hold in I_Λ for some $s \in I'$. Here we use the convention $+' = +$.

It is easy to see that $I^{*+}_{\Lambda^\bullet}$ is a poset and $\max I^{*+}_{\Lambda^\bullet} = \{*, +\}$. We call $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ a **poset with zero-relations associated with the D-order $\Lambda^\bullet$**.

Now we are able to prove our main reduction theorem.

Theorem 3.4. *Let K be an algebraically closed field, D a complete discrete valuation domain which is a K-algebra, and $\mathfrak{p}$ is the unique maximal ideal of D. We assume that $D/\mathfrak{p} \cong K$. Let Λ be the D-order* (1.1) *with the three-partition* (1.2) *and $\Lambda_1 = \Lambda_2 \subseteq \mathbb{M}_{n_1}(D)$, $\Lambda_3 \subseteq \mathbb{M}_{n_3}(D)$ and n_1, n_3 as in Section 1. Let $\Lambda^\bullet$ be the subamalgam D-order* (1.3) *and let $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ be the two-peak poset with zero-relations* (3.3) *associated with $\Lambda^\bullet$.*

(a) *The Tits quadratic forms $q_{\Lambda^\bullet}$* (1.4) *and $q_{(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})}$ in* (2.13) *coincide.*

(b) *There exists an additive reduction functor*

$$\mathbb{H} : \mathrm{latt}(\Lambda^\bullet) \longrightarrow (I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})\text{-spr} \cong \mathrm{mod}_{\mathrm{sp}}\, K(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet}) \tag{3.5}$$

with the following properties:

(i) $\mathbb{H}$ *is full, reflects isomorphisms and preserves the indecomposability.*

(ii) *The image* $\mathrm{Im}\,\mathbb{H}$ *of* $\mathbb{H}$ *consists up to isomorphism of all objects of the category $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z})$-spr having no direct summand of one of the following two types:*

• *the simple projective representation $P_* = e_* K(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z})$ corresponding to the peak idempotent e_*, and*

• *any of the hereditary sp-injective representations $\mathbf{H}^-_{n_3} \hookrightarrow \mathbf{H}^-_{n_3-1} \hookrightarrow \cdots \hookrightarrow \mathbf{H}^-_0$ defined in* [40, (4.12)].

(iii) $\mathbb{H}$ *preserves and reflects tame representation type, wild representation type, and the polynomial growth property; that is,* $\mathrm{latt}(\Lambda^\bullet)$ *is of tame representation* *type (resp. wild, or of polynomial growth) if and only if* $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$-spr *is of tame representation type (resp. wild, or of polynomial growth).*

Proof. Statement (a) follows by a straightforward analysis.

(b) We take for the functor $\mathbb{H}$ the reduction functor constructed in [40, Definition 4.11] and defined to be the composed functor

$$\begin{array}{ccccc} \mathrm{latt}(\Lambda^\bullet) & \xrightarrow{\mathbb{F}} & \mathrm{mod}_{\mathrm{sp}}\ R & \xrightarrow{G} & \mathrm{mod}_{\mathrm{sp}}\ KJ_{\boldsymbol{\rho}} \\ & & & & \downarrow{\scriptstyle f^-} \\ & & & & \mathrm{mod}_{\mathrm{sp}}\ K(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet}) \xrightarrow{\boldsymbol{\rho}^{-1}} (I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})\text{-spr} \end{array} \tag{3.6}$$

where $\boldsymbol{\rho}$, f^-, G and $\mathbb{F}$ are the functors shown in (2.9), (3.7), (3.9) and in the diagram (3.11) below, and are defined as follows.

1° <u>The functor $\mathbb{F}$</u>. Consider the finite dimensional K-algebra

$$R = \begin{pmatrix} \Lambda^\bullet/\boldsymbol{\pi} & \Gamma/\boldsymbol{\pi} \\ 0 & \Gamma/\boldsymbol{\pi} \end{pmatrix},$$

where $\Gamma = \mathbb{M}_n(D)$ and $\boldsymbol{\pi} = \mathbb{M}_n(\mathfrak{p})$. Note that Γ is a hereditary D-order containing $\Lambda^\bullet$ and $\boldsymbol{\pi}$ is a two-sided ideal contained in the Jacobson radical $\mathrm{rad}(\Gamma)$ of Γ. It is also an ideal of $\Lambda^\bullet$ contained in $\mathrm{rad}(\Lambda^\bullet)$. It is easy to see that R is a right peak K-algebra, that is, R has a unique simple right ideal P_* up to isomorphism (see [32]). We take for $\mathbb{F}$ the reduction functor

$$\mathbb{F} : \mathrm{latt}(\Lambda^\bullet) \longrightarrow \mathrm{mod}_{\mathrm{sp}}\ R \tag{3.7}$$

defined [11] and [27] by the formula $\mathbb{F}(X) = (X/X\boldsymbol{\pi}, X\Gamma/X\boldsymbol{\pi}, u)$, where $X\Gamma$ is the Γ-submodule of $X \otimes_D F$ generated by X (see [24]), $F = D_0$ is the field of fractions of D and $u : X/X\boldsymbol{\pi} \to X\Gamma/X\boldsymbol{\pi}$ is the $\Lambda/\boldsymbol{\pi}$-monomorphism induced by the natural monomorphism $X \hookrightarrow X\Gamma$. We view $\mathbb{F}(X)$ as a right R-module in a natural way (see [11] and [27]).

By [11] and [27], the reduction functor $\mathbb{F}$ is full, reflects isomorphisms, preserves indecomposability, and Im $\mathbb{F}$ contains up to isomorphism all indecomposable objects of $\mathrm{mod}_{\mathrm{sp}}\ R$ except from the unique simple right ideal P_* of R.

It follows from [39, Theorem 7.19] that $\mathbb{F}$ preserves and reflects tame representation type, wild representation type, and the polynomial growth property. For note that [39, Theorem 7.19] applies, because in the case we consider here $\Gamma/\boldsymbol{\pi}$ is a simple K-algebra and according to [39, Proposition 4.5] the category $\mathrm{mod}_{\mathrm{sp}}\ R$ is equivalent with the category $\mathrm{mod}_{\ pr}\ R$ of projectively adjusted R-modules.

2° <u>The functor G</u>. Let $J = I^*_\Lambda = I_\Lambda \cup \{*\}$ be the poset obtained from I_Λ by adding the unique maximal element $*$ with new relations $i \prec *$ for all $i \in I_\Lambda$. Consider the set

$$\blacktriangle J := \{(i,j);\ i \preceq j \text{ in } J\} \subseteq J \times J$$

and define a binary equivalence relation $\boldsymbol{\rho}$ on $\blacktriangle J$ by setting

$$(i,j)\boldsymbol{\rho}(s,t) \Leftrightarrow (i,j) = (s,t) \text{ or } i,s \in I' = I_{\Lambda_1},\ j,t \in I'' = I_{\Lambda_2},\ j = \sigma(i),\ t = \sigma(s),$$

where $\sigma : I' \to I''$ given by $\sigma(i) = i + n_1 + n_3$ is a poset isomorphism. Then we have defined a bipartite stratified poset

$$J_{\boldsymbol{\rho}} = (J, \boldsymbol{\rho}) \tag{3.8}$$

in the sense of [29] and [31, Definition 4.1]. The bipartition $J = J' + C + J'''$ is given by taking $J' = I'$, $C = I_{\Lambda_3}$ and $J''' = (I'')^*$. We recall from [29] and [31] that the incidence K-algebra of $J_{\boldsymbol{\rho}}$ is the subalgebra $KJ_{\boldsymbol{\rho}}$ of KJ consisting of all matrices $\lambda = [\lambda_{pq}]_{p,q\in J}$ such that $\lambda_{ij} = \lambda_{st}$ if $(i,j),(s,t) \in \blacktriangle J$ and $(i,j)\boldsymbol{\rho}(s,t)$. It was shown in [31] that $KJ_{\boldsymbol{\rho}}$ is a basic right peak K-algebra and the right socle of $KJ_{\boldsymbol{\rho}}$ is isomorphic to a direct sum of the simple projective right ideal $P'_* = e_*KJ_{\boldsymbol{\rho}}$, called a right peak of $KJ_{\boldsymbol{\rho}}$. A simple analysis shows that the algebra R defined above is Morita equivalent with the incidence algebra $KJ_{\boldsymbol{\rho}}$. The idea of the proof of this fact is explained by Example 3.9 in [38, p. 95]. We define a K-linear functor

$$G : \mathrm{mod_{sp}}\ R \xrightarrow{\ \simeq\ } \mathrm{mod_{sp}}\ KJ_{\boldsymbol{\rho}} \tag{3.9}$$

to be the Morita equivalence restricted to socle projective modules. It is clear that G preserves and reflects finite representation type, tameness, wildness, and the polynomial growth property.

3° <u>The functor f^-</u>. Let $(Q,\Omega) = (Q(J_{\boldsymbol{\rho}}),\Omega(J_{\boldsymbol{\rho}}))$ be the bound quiver associated with $J_{\boldsymbol{\rho}}$ in [31, Definition 2.5]. It follows from [31, Proposition 2.8] that there exists a K-algebra isomorphism $K(Q,\Omega) \cong KJ_{\boldsymbol{\rho}}$. Let

$$f : (\widetilde{Q},\widetilde{\Omega}) \longrightarrow (Q,\Omega)$$

be the bound quiver Galois covering [31, (3.1)] of (Q,Ω). It follows from [31, Proposition 3.8] that f is a universal covering with the covering group $\mathbb{Z}$. Moreover, it follows from the construction that

$$(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet}) \cong J^{*+}_{\boldsymbol{\rho}}, \tag{3.10}$$

where $J^{*+}_{\boldsymbol{\rho}}$ is the two-peak bound subquiver of the quiver $(\widetilde{Q},\widetilde{\Omega})$ associated with $J_{\boldsymbol{\rho}}$ in [31, (4.3)] and $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ is the poset with zero-relations associated with $\Lambda^\bullet$ by the formula (3.3). By [31, Theorem 4.19] the push-down functor f_λ : $\mathrm{mod}\ K(\widetilde{Q},\widetilde{\Omega}) \to \mathrm{mod}\ K(Q,\Omega)$ induces the push-down functor

$$\mathrm{mod_{sp}}\ K(\widetilde{Q},\widetilde{\Omega}) \xrightarrow{\ f_{sp}\ } \mathrm{mod_{sp}}\ K(Q,\Omega) \cong \mathrm{mod_{sp}}\ KJ_{\boldsymbol{\rho}},$$

and we get the following diagram:

$$\begin{array}{ccc}
\mathrm{mod_{sp}}\,K(Q,\Omega) & \xleftarrow{\ f_{sp}\ } & \mathrm{mod_{sp}}\,K(\widetilde{Q},\widetilde{\Omega}) \\
\cong\uparrow & & \uparrow\Phi \\
\mathrm{mod_{sp}}\ KJ_{\boldsymbol{\rho}} & \underset{f^-}{\overset{f^+}{\leftrightarrows}} & \mathrm{mod_{sp}}\ K(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet}) \\
\uparrow G\circ\mathbb{F} & & \cong\uparrow\boldsymbol{\rho} \\
\mathrm{latt}(\Lambda^\bullet) & \xrightarrow{\ \mathbb{H}\ } & (I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})\text{-spr}
\end{array} \tag{3.11}$$

where (under the identification $J^{*+}_{\boldsymbol{\rho}} \equiv (I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$) f^+ is the glueing functor [31, (4.14)], $\Phi = T_v \circ L_\xi$ is the embedding defined in [31, Proposition 4.23], f^- is the section functor [31, (5.1)], and $\boldsymbol{\rho}$ is the equivalence of categories defined in (2.9). The idea of this construction is explained by Example 3.9 in [38, p. 95].

According to [31, Proposition 4.23], the category $\mathrm{mod_{sp}}\,K(\widetilde{Q},\widetilde{\Omega})$ is locally coordinate support finite and every indecomposable module of $\mathrm{mod_{sp}}\,K(\widetilde{Q},\widetilde{\Omega})$ is contained in the image of the functor Φ, up to a $\mathbb{Z}$-shift. It then follows from [31, Theorem 4.27] and the main results of Dowbor and Skowroński in [7] and [8] (see also [6]) that the push-down functor f_{sp}, and hence the functors f^+ and f^- as well, preserve and reflect tameness, wildness and the polynomial growth property (apply [31, Proposition 5.4 and Theorem 5.8]).

Hence we easily conclude that the composed functor $\mathbb{H}$ (3.5) has the properties stated in (i) and (iii) of the theorem. Since the statement (ii) was proved in [40, Theorem 4.14], the proof of the theorem is complete. $\square$

Remark 3.12. (a) It follows from [40, Theorem 6.1] that the Auslander-Reiten quiver of the category $(I^{*+}_{\Lambda^\bullet}, \mathfrak{z}_{\Lambda^\bullet})$-spr has the form presented in Figure 3.13. If $(I^{*+}_{\Lambda^\bullet}, \mathfrak{z}_{\Lambda^\bullet})$-spr is of finite lattice type then the part $\mathcal{R}$ in Figure 3.13 is empty, $\mathcal{P}(\Lambda^\bullet) = \mathcal{I}(\Lambda^\bullet)$, and $\mathcal{P}(\Lambda^\bullet)$ is finite.

(b) In view of [40, Theorem 6.1] we have a description of the Auslander-Reiten quiver $\Gamma(\mathrm{latt}(\Lambda^\bullet))$ of $\mathrm{latt}(\Lambda^\bullet)$. By applying the reduction functor

$$\mathbb{H} : \mathrm{latt}(\Lambda^\bullet) \to (I^{*+}_{\Lambda^\bullet}, \mathfrak{z}_{\Lambda^\bullet})\text{-spr}$$

(3.6) the Auslander-Reiten quiver $\Gamma(\mathrm{latt}(\Lambda^\bullet))$ is obtained from the Auslander-Reiten quiver $\Gamma((I^{*+}_{\Lambda^\bullet}, \mathfrak{z}_{\Lambda^\bullet})\text{-spr})$ of $(I^{*+}_{\Lambda^\bullet}, \mathfrak{z}_{\Lambda^\bullet})$-spr by the following two simple glueings:

1° The identification of a hereditary projective section

$$\mathbf{P}^+_{n_3} \hookrightarrow \mathbf{P}^+_{n_3-1} \hookrightarrow \cdots \hookrightarrow \mathbf{P}^+_0$$

of irreducible monomorphisms from the beginning of the unique preprojective component $\mathcal{P}(\Lambda^\bullet)$ in $(I^{*+}_{\Lambda^\bullet}, \mathfrak{z}_{\Lambda^\bullet})$-spr containing the simple projective module $\mathbf{P}^+_{n_3} \cong e_+K(I^{*+}_{\Lambda^\bullet}, \mathfrak{z}_{\Lambda^\bullet})$ with a hereditary sp-injective section

$$\mathbf{H}^-_{n_3} \hookrightarrow \mathbf{H}^-_{n_3-1} \hookrightarrow \cdots \hookrightarrow \mathbf{H}^-_0$$

of irreducible monomorphisms from the end the unique preinjective component $\mathcal{I}(\Lambda^\bullet)$ in $(I^{*+}_{\Lambda^\bullet}, \mathfrak{z}_{\Lambda^\bullet})$-spr containing the injective envelope $\mathbf{H}^-_0$ of the simple projective module $e_*K(I^{*+}_{\Lambda^\bullet}, \mathfrak{z}_{\Lambda^\bullet})$.

2° The identification of the simple projective module $P_* = e_*K(I^{*+}_{\Lambda^\bullet}, \mathfrak{z}_{\Lambda^\bullet})$ with the injective envelope $E(\mathbf{P}^+_{n_3})$ of the simple projective module

$$\mathbf{P}^+_{n_3} \cong e_+K(I^{*+}_{\Lambda^\bullet}, \mathfrak{z}_{\Lambda^\bullet})$$

in the category $(I^{*+}_{\Lambda^\bullet}, \mathfrak{z}_{\Lambda^\bullet})\text{-spr} \cong \mathrm{mod}_{\mathrm{sp}}\, K(I^{*+}_{\Lambda^\bullet}, \mathfrak{z}_{\Lambda^\bullet})$.

The reader is referred to [40, Section 6] for details. The glueing procedure of quiver 3.13 is explained in Example 6.6 of [40] (see also [25, pp. 451–455] and [26]).

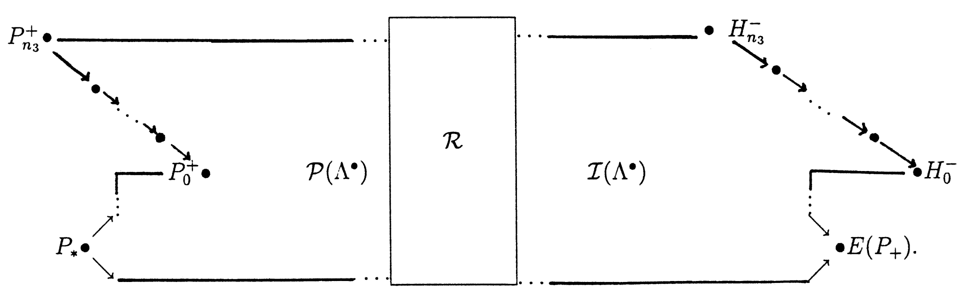


FIGURE 3.13. The shape of Auslander-Reiten quiver of the category $(I^{*+}_{\Lambda^\bullet}, \mathfrak{z}_{\Lambda^\bullet})$-spr

We finish this section by the following useful result concerning the existence of preprojective components.

Proposition 3.14. *Let* $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ *be the poset with zero-relations* (3.3) *associated with the three-partite subamalgam D-order* $\Lambda^\bullet$ (1.3).

(a) *Every point of the poset with zero-relations* $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ *is separating in the sense of Bongartz* [4] (*see also* [13], [31, Section 4]).

(b) *There exist a preprojective component* $\widetilde{\mathcal{P}}_{(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})}$ *in* prin $K(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ *and a preprojective component* $\mathcal{P}_{(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})}$ *in the category* $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$-spr *such that the adjustment functor* (2.10)

$$\boldsymbol{\theta} : \text{prin } K(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet}) \longrightarrow (I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})\text{-spr}$$

carries $\widetilde{\mathcal{P}}_{(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})}$ *to* $\mathcal{P}_{(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})}$.

(c) *The preprojective components* $\widetilde{\mathcal{P}}_{(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})}$ *and* $\mathcal{P}_{(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})}$ *can be constructed by Algorithm* 4.4 *in* [19].

Proof. The existence of a preprojective component $\widetilde{\mathcal{P}}_{(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})}$ in the category prin $K(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ and statement (a) follow from [31, Proposition 4.9] applied to the bipartite stratified poset $I_{\Lambda^\bullet,\sigma}$ (3.2), because the algebra $K(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ is obtained from $I_{\Lambda^\bullet,\sigma}$ by a construction required in [31, Proposition 4.9] and the arguments of Bongartz [4] apply (see also [13] and [19, Algorithm 4.4]). By [23, Lemma 3.12, Theorem 3.13] and properties of the adjustment functor $\boldsymbol{\theta}$ proved there, the image $\mathcal{P}_{(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})}$ of $\widetilde{\mathcal{P}}_{(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})}$ under $\boldsymbol{\theta}$ is a preprojective component in the category $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$-spr. Note that the arguments given for [19, Algorithm 4.4] and in [32, Theorem 11.68 and Corollary 11.76] for the case of one-peak posets extend to our situation. □

4. Proof of Main Results

Throughout this section K is an algebraically closed field and D is a complete discrete valuation domain which is a K-algebra such that $D/\mathfrak{p} \cong K$, where $\mathfrak{p}$ is the unique maximal ideal of D.

We start with the following useful reflection duality result.

Proposition 4.1. *Let* $\Lambda^\bullet$ *be a subamalgam D-suborder* (1.3) *of the tiled order* Λ (1.2)*, let* $\Gamma^\bullet = \text{rt}(\Lambda^\bullet)$ *be the reflection transpose order* (1.7) *of* $\Lambda^\bullet$*, let* $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ *be the two-peak poset with zero-relations* (3.3)*, and let* $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})^\bullet$ *be its reflection-dual* (2.17)*. Then the following statements hold.*

(a) *There is a D-algebra isomorphism* $\Gamma^\bullet \cong (\Lambda^\bullet)^{\text{op}}$ *and an isomorphism*

$$(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})^\bullet \cong (I^{*+}_{\Gamma^\bullet}, \mathfrak{Z}_{\Gamma^\bullet})$$

of two-peak posets with zero-relations.

(b) *There exists a commutative diagram*

$$\begin{array}{ccc} \text{latt}(\Lambda^\bullet) & \xrightarrow{\mathbb{H}} & (I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})\text{-spr} \\ \cong\big\downarrow D_\Lambda & & \cong\big\downarrow \widetilde{D}^\bullet \\ \text{latt}(\Gamma^\bullet) & \xrightarrow{\mathbb{H}} & (I^{*+}_{\Gamma^\bullet}, \mathfrak{Z}_{\Gamma^\bullet})\text{-spr} \end{array} \tag{4.2}$$

where $\mathbb{H}$ *is the composed reduction functor* (3.6), $D_\Lambda = \mathrm{Hom}_D(-, D)$ *is the standard* D*-duality, and* $\widetilde{D}^\bullet$ *is the composed duality functor*

$$(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})\text{-spr} \xrightarrow{\ D^\bullet\ } (I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})^\bullet\text{-spr} \cong (I^{*+}_{\Gamma^\bullet}, \mathfrak{Z}_{\Gamma^\bullet})\text{-spr}$$

induced by the reflection duality (2.19).

(c) *The* D*-order* $\Lambda^\bullet$ *is of tame lattice type if and only if the* D*-order* $\Lambda^\bullet$ *is of tame lattice type.*

Proof. Statements (a) and (b) follow directly from the definitions. The details are left to the reader. Statement (c) follows by applying the tame-wild dichotomy for D-orders proved in [9], because the arguments used in the proof of Proposition 2.20 (d) easily extend to our case. □

We shall need the following two simple lemmas.

Lemma 4.3. *Let* Ω *be a* D*-order in a semisimple* K*-algebra* C *and let* $e \in \Omega$ *be an idempotent. Then* $e\Omega e$ *is a* D*-order in the semisimple* K*-algebra* eCe*, and the following statements hold.*

(a) *The functors*

$$\mathrm{latt}(e\Omega e) \underset{\mathrm{res}_e}{\overset{L_e}{\rightleftarrows}} \mathrm{latt}(\Omega)$$

defined by the formulas $\mathrm{res}_e(X) = Xe$, $L_e(Y) = \mathrm{Hom}_{e\Omega e}(\Omega e, Y)$ *have the following properties:*

(i) *The functor* L_e *is a fully faithful embedding,* $\mathrm{res}_e L_e \cong \mathrm{id}$*, and* L_e *is right adjoint to* res_e*, that is, there is a natural isomorphism*

$$\mathrm{Hom}_\Omega(X, L_e(Y)) \cong \mathrm{Hom}_{e\Omega e}(\mathrm{res}_e(X), Y)$$

for every Ω*-lattice* X *and every* $e\Omega e$*-lattice* Y*.*

(ii) *The restriction functor* res_e *is exact, and* L_e *is left exact and preserves the indecomposability.*

(b) *If the* D*-order* Ω *is of finite lattice type, then the* D*-order* $e\Omega e$ *is of finite lattice type.*

(c) *If the* D*-order* Ω *is of tame lattice type* (*resp. tame of polynomial growth*)*, then the* D*-order* $e\Omega e$ *is of tame lattice type* (*resp. tame of polynomial growth*)*.*

(d) *If the* D*-order* $e\Omega e$ *is of wild lattice type, then the* D*-order* Ω *is of wild lattice type.*

Proof. Statement (a) is well-known and follows by the arguments applied in the proof of [32, Theorem 17.46]. The details are left to the reader. We only remark that the module $L_e(X)$ is finitely generated and D-torsionfree, if X is finitely generated and D-torsionfree.

It follows from (a) that the functor L_e carries indecomposable modules to indecomposable modules and carries nonisomomorphic modules to nonisomomorphic ones. Hence (b) easily follows.

(c) Assume that Ω is of tame lattice type and the functors

$$(-) \otimes_A M^{(1)}, \dots, (-) \otimes_A M^{(s)} : \mathrm{ind}_1(A) \longrightarrow \mathrm{latt}(\Omega)$$

(1.8) form an almost parameterizing family for the category $\mathrm{ind}_r(\mathrm{latt}(\Omega))$ of indecomposable Ω-lattices of D-rank r. Since the restriction functor $\mathrm{res}_e(-) = (-)e$ is exact, a simple analysis shows that the functors

$$(-)\otimes_A M^{(1)}e, \dots, (-)\otimes_A M^{(s)}e : \mathrm{ind}_1(A) \longrightarrow \mathrm{latt}(e\Omega e)$$

form an almost parameterizing family for the category $\mathrm{ind}_r(\mathrm{latt}(e\Omega e))$ of indecomposable $e\Omega e$-lattices of D-rank r. This proves that $e\Omega e$ is of tame lattice type. The polynomial growth version follows in a similar way.

The statement (d) follows immediately from (c) by applying the tame-wild dichotomy for D-orders proved in [9]. □

Lemma 4.4. *Assume that $\Lambda \subseteq \Omega$ are D-orders in a semisimple K-algebra C.*
(a) *If Λ is of finite lattice type, then Ω is of finite lattice type.*
(b) *If Λ is of tame lattice type, then Ω is of tame lattice type.*
(d) *If Ω is of wild lattice type, then Λ is of wild lattice type.*

Proof. It is easy to check that the forgetful functor $\mathrm{res}_\Lambda : \mathrm{latt}(\Omega) \longrightarrow \mathrm{latt}(\Lambda)$ (associating with any Γ-module X the vector space X viewed as Λ-module) is full, faithful and exact (see [5, p. 532, Ex. 2]). Hence (a) and (c) easily follow. The statement (b) follows immediately from (c), because of the tame-wild dichotomy for D-orders proved in [9]. □

Proof of Theorem 1.5. (a)⇒(b). It follows from Theorem 3.4 (a) that the Tits quadratic forms $q_{\Lambda^\bullet}$ (1.4) and $q_{(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})}(z) = b_{(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})}(z,z)$ in (2.13) coincide. Then the implication (a)⇒(b) follows from Theorem 3.4 (iii) and Theorem 2.14.

(b)⇒(d). Let $(L, \mathfrak{Z})$ be any of the two-peak posets with zero-relations listed in Theorem 1.5 (d). We claim that there exists a vector $v_{(L,\mathfrak{Z})} \in \mathbb{N}^L$ such that $q_{(L,\mathfrak{Z})}(v_{(L,\mathfrak{Z})}) < 0$. In case $\mathfrak{Z}$ is empty the claim follows from [16, Theorem 1.3], because the two-peak posets without zero-relations listed in Theorem 1.5 (d) are the hypercritical ones presented in Table 1 of [16, pp. 509–511]. It remains to prove the claim if $(L, \mathfrak{Z})$ is the poset $\widehat{\mathcal{F}}_4$ with one zero-relation. Since obviously $\widehat{\mathcal{F}}_4 = \widehat{\mathcal{F}}_5^\bullet$ is reflection-dual to the poset $\widehat{\mathcal{F}}_5$, then we can take for $v_{\widehat{\mathcal{F}}_4}$ the vector $v^\bullet_{\widehat{\mathcal{F}}_5}$ defined in Proposition 2.20 (b), because it is shown there that $q_{\widehat{\mathcal{F}}_4}(v^\bullet_{\widehat{\mathcal{F}}_5}) = q_{\widehat{\mathcal{F}}_5}(v_{\widehat{\mathcal{F}}_5}) < 0$. Since according to Theorem 3.4 (a) the quadratic forms $q_{\Lambda^\bullet}$ and $q_{(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})}$ coincide, the implication (b)⇒(d) follows.

(d)⇒(a). We consider three cases.

Case 1°. $n_3 = 0$. It follows that the sets C, C' and $\mathfrak{Z}_{\Lambda^\bullet}$ in the definition of $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ (3.3) are empty. By condition (d) of the theorem the two-peak poset $I^{*+}_{\Lambda^\bullet}$ does not contain as a two-peak subposet the posets $\widehat{\mathcal{F}}_0^2$ and $\widehat{\mathcal{F}}_0^3$. Thus $I^{*+}_{\Lambda^\bullet}$ is a peak subposet of a two-peak garland

$$(4.5) \qquad \mathcal{G}_m^{*+} : \quad \begin{array}{ccccccccccc} \circ & \longrightarrow & \circ \rightarrow & \cdots & \rightarrow \circ & \longrightarrow & \circ & \longrightarrow & * \\ & \times & & \cdots & & \times & & \times & \\ \circ & \longrightarrow & \circ \rightarrow & \cdots & \rightarrow \circ & \longrightarrow & \circ & \longrightarrow & + \end{array} \qquad \begin{array}{l} (2m\text{-points}), \\ m \geq 1. \end{array}$$

It follows from the proof of the implication (c)⇒(a) in [29, Proposition 4.13] or from [37, Theorem 5.2] (see also [36]) that the category $\mathcal{G}_m^{*+}$-spr is of tame representation type. Further, if $m \geq 3$, $\mathcal{G}_m^{*+}$-spr is of non-polynomial growth (see also [18, Lemma 3.1]). It follows from [40, Proposition 2.9] that the category $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$-spr is of tame representation type. Hence, in view of Theorem 3.4(iii), the category $\mathrm{latt}(\Lambda^\bullet)$ is of tame representation type, and (a) follows.

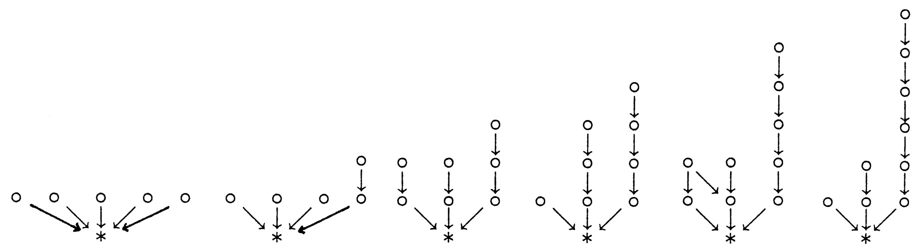


Table 4.6. One-peak enlargements of hypercritical posets of Nazarova

Case 2°. $n_3 \geq 1$ and the part $\mathcal{Y}$ of Λ in (1.2) consists of matrices with coefficients in $\mathfrak{p}$. It follows from the definition of $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ (3.3) that the chains C and C' are not empty, C is incomparable with all elements of $I' \equiv I''$, and the set $\mathfrak{Z}_{\Lambda^\bullet}$ of zero-relations is empty. Hence we conclude that the poset $I' \cong I''$ is linearly ordered, because otherwise the poset $C \cup I''$ contains a triple of incomparable points and therefore the poset $I^{*+}_{\Lambda^\bullet}$ contains a two-peak subposet isomorphic with $\widehat{\mathcal{F}}^1_0$, contrary to the assumption (d).

This shows that in this case the two-peak poset $I^{*+}_{\Lambda^\bullet}$ is thin in the sense of [18, Definition 3.1], and according to [18, Theorem 1.3] the following three statements are equivalent:

(a$'$) The category I^{*+}_{Λ}-spr is of tame representation type.

(b$'$) The Tits quadratic form $q_{I^{*+}_{\Lambda^\bullet}}$ is weakly non-negative.

(c$'$) The two-peak poset $I^{*+}_{\Lambda^\bullet}$ associated with $\Lambda^\bullet$ in (3.3) does not contain as a two-peak subposet any of the hypercritical two-peak posets presented in [18, Table 1], and does not contain as a peak subposet any of the one-peak enlargements $\mathcal{N}^*_1$, $\mathcal{N}^*_2$, $\mathcal{N}^*_3$, $\mathcal{N}^*_4$, $\mathcal{N}^*_5$, $\mathcal{N}^*_6$ of hypercritical Nazarova posets [21] shown in Table 4.6 (see also [32, Theorem 15.3]).

Note that the poset $I^{*+}_{\Lambda^\bullet} \setminus (I' \cup \{*, +\})$ is a disjoint union of two chains C and C'. Then a case by case inspection of the two peak posets in [17, Table 1] and [18, Table 1] shows that, for any three-partite subamalgam D-order $\Lambda^\bullet$ (1.3) such that the poset $I' = I_{\Lambda_1}$ is linearly ordered, the two-peak poset $I^{*+}_{\Lambda^\bullet}$ does not contain as a peak subposet any of the one-peak enlargements $\mathcal{N}^*_1$, $\mathcal{N}^*_2$, $\mathcal{N}^*_3$, $\mathcal{N}^*_4$, $\mathcal{N}^*_5$, $\mathcal{N}^*_6$ of hypercritical Nazarova posets, and $I^{*+}_{\Lambda^\bullet}$ could contain at most the nine hypercritical posets $\widehat{\mathcal{F}}^1_1$, $\widehat{\mathcal{F}}^2_1$, $\widehat{\mathcal{F}}_2$, $\widehat{\mathcal{F}}^1_3$, $\widehat{\mathcal{F}}^2_3$, $\widehat{\mathcal{F}}_5$, $\widehat{\mathcal{F}}_6$, $\widehat{\mathcal{F}}_7$ and $\widehat{\mathcal{F}}_8$ listed in Theorem 1.5 from the 41 posets presented in [18, Table 1]. It then follows that under the assumption we make in Case 2°, the condition (d) of Theorem 1.5 is equivalent with the condition (c$'$) above and therefore (d) implies the tameness of $I^{*+}_{\Lambda^\bullet}$-spr. Since $\mathfrak{Z}_{\Lambda^\bullet}$ is empty, then in view of Theorem 3.4, this implies that the order $\Lambda^\bullet$ is of tame lattice type, and (a) follows.

Case 3°. $n_3 \geq 1$ and the part $\mathcal{X}$ of Λ in (1.2) consists of matrices with coefficients in $\mathfrak{p}$. Let $\Gamma^\bullet = \mathrm{rt}(\Lambda^\bullet)$ be the reflection transpose of $\Lambda^\bullet$ (see (1.7)). Since the part $\mathcal{X}$ of Λ in (1.2) consists of matrices with coefficients in $\mathfrak{p}$, then the corresponding part $\mathcal{Y}$ of Γ in its three-partition (1.2) consists of matrices with coefficients in $\mathfrak{p}$ and by the arguments in Case 2° applied to $\Gamma^\bullet$ the set $\mathfrak{Z}_{\Gamma^\bullet}$ is empty. It follows from Proposition 4.1 that $I^{*+}_{\Gamma^\bullet} = (I^{*+}_{\Gamma^\bullet}, \mathfrak{Z}_{\Gamma^\bullet}) \cong (I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})^\bullet$, and according to (2.19) there exists a reflection duality functor

$$D^\bullet : I^{*+}_{\Gamma^\bullet}\text{-spr} \longrightarrow (I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})^\bullet\text{-spr}\,.$$

Since the two-peak poset $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ with zero-relations does not contains any of the following thirteen hypercritical posets with zero-relations $\widehat{\mathcal{F}}^1_0$, $\widehat{\mathcal{F}}^2_0$, $\widehat{\mathcal{F}}^3_0$, $\widehat{\mathcal{F}}^1_1$, $\widehat{\mathcal{F}}^2_1$, $\widehat{\mathcal{F}}_2$, $\widehat{\mathcal{F}}^1_3$, $\widehat{\mathcal{F}}^2_3$, $\widehat{\mathcal{F}}_4$, $\widehat{\mathcal{F}}_5$, $\widehat{\mathcal{F}}_6$, $\widehat{\mathcal{F}}_7$ and $\widehat{\mathcal{F}}_8$ presented in Theorem 1.5, and since it is easy to see that this list is closed under the reflection duality operation $(I, \mathfrak{Z}) \mapsto (I, \mathfrak{Z})^\bullet$ (2.17), then the Case 2° applies to I^{*+}_Γ and therefore the category $I^{*+}_{\Gamma^\bullet}$-spr is of tame representation type. It follows from Proposition 2.20 (d) that the category $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$-spr is of tame representation type and according to Theorem 3.4 the D-order $\Lambda^\bullet$ is of tame lattice type, and (a) follows.

Consequently we have proved that the statements (a), (b) and (d) of Theorem 1.5 are equivalent.

The proof of the equivalence (c)⇔(d) is divided into two parts.

<u>Case 1°</u>. $n_3 = 0$. From the construction $\Lambda^\bullet \mapsto (I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ in (3.3) the following three statements are easily derived:

- The sets C and C' in the definition of $I^{*+}_{\Lambda^\bullet}$ are empty, the set $\mathfrak{Z}_{\Lambda^\bullet}$ of zero-relations is empty, and $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ does not contain the following two-peak posets with zero-relations: $\widehat{\mathcal{F}}^1_0$, $\widehat{\mathcal{F}}^1_1$, $\widehat{\mathcal{F}}^2_1$, $\widehat{\mathcal{F}}_2$, $\widehat{\mathcal{F}}^1_3$, $\widehat{\mathcal{F}}^2_3$, $\widehat{\mathcal{F}}_4$, $\widehat{\mathcal{F}}_5$, $\widehat{\mathcal{F}}_6$, $\widehat{\mathcal{F}}_7$ and $\widehat{\mathcal{F}}_8$ presented in Theorem 1.5.
- If $\Lambda_1 = \Delta_0$, then $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet}) = \widehat{\mathcal{F}}^3_0$. If Λ_1 is one of the D-orders $\Delta_1, \Delta_2, \Delta_3$, then $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet}) \cong \widehat{\mathcal{F}}^2_0$.
- The D-order Λ_1 in (1.2) does not contain minor D-suborders of the form Δ_0 if and only if the two-peak poset $I^{*+}_{\Lambda^\bullet}$ does not contain as a two-peak subposet the two-peak poset $\widehat{\mathcal{F}}^3_0$ presented in Theorem 1.5.
- The D-order Λ_1 contains a minor D-suborder of one of the forms Δ_1, Δ_2, Δ_3 if and only if the two-peak poset $I^{*+}_{\Lambda^\bullet}$ contains as a two-peak subposet the two-peak poset $\widehat{\mathcal{F}}^2_0$ presented in Theorem 1.5.

Hence the equivalence (c)⇔(d) easily follows in case $n_3 = 0$.

<u>Case 2°</u>. $n_3 \geq 1$. First we note that the following four statements are equivalent:

(i) The D-order Λ_1 in (1.2) is hereditary of the form (1.6).

(ii) The poset $I' = I_{\Lambda_1}$ is linearly ordered.

(iii) The poset $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet}) \cong J^{*+}_{\boldsymbol{\rho}}$ (see 3.10) with zero-relations does not contain the poset

$$\mathcal{F}_0 : \begin{array}{ccc} \circ & & \circ \\ \downarrow & \times & \downarrow \\ * & & + \end{array}$$

as a two-peak subposet with zero-relations.

(iv) The poset $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ with zero-relations does not contain any of the posets $\widehat{\mathcal{F}}^1_0$, $\widehat{\mathcal{F}}^2_0$, $\widehat{\mathcal{F}}^3_0$ presented in Theorem 1.5 as a two-peak subposet with zero-relations.

The implications (i)⇔(ii)⇔(iii)⇒(iv) are immediate consequence of the construction $\Lambda^\bullet \mapsto (I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ in (3.3).

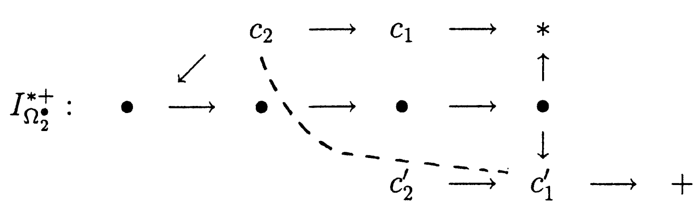
In order to prove (iv)⇒(iii), assume to the contrary that $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ contains the two-peak poset $\mathcal{F}_0$. Since $n_3 \geq 1$, each of the chains C and C' in the definition of $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ (3.3) is not empty. Further, since according to our assumption in the theorem the part $\mathcal{X}$ or the part $\mathcal{Y}$ of Λ in (1.2) consists of matrices with coefficients in $\mathfrak{p}$, then C or C' is incomparable with all elements of the subposet $I' \equiv I''$ of $(I^{*+}_{\Lambda^\bullet}, \mathfrak{z}_{\Lambda^\bullet})$. Since $\mathcal{F}_0$ is a two-peak subposet of $(I^{*+}_{\Lambda^\bullet}, \mathfrak{z}_{\Lambda^\bullet})$, then its extension by a point of C or a point of C' is a two-peak subposet of $(I^{*+}_{\Lambda^\bullet}, \mathfrak{z}_{\Lambda^\bullet})$ isomorphic with the poset $\mathcal{F}_0^1$, contrary to our assumption in (iv).

Consequently the conditions (i)–(iv) are equivalent, and therefore in order to finish the proof of (c)⇔(d) in the case $n_3 \geq 1$ it remains to show that, in case the D-order Λ_1 in (1.2) is hereditary of the form (1.6), the following two conditions are equivalent:

(c′) The three-partite subamalgam D-orders $\Lambda^\bullet$ and $\mathrm{rt}(\Lambda)^\bullet$ (1.7) do not contain three-partite minor D-suborders dominated by any of the 17 three-partite subamalgam D-orders listed in the tables of Section 1A.

(d′) The two-peak poset $(I^{*+}_{\Lambda^\bullet}, \mathfrak{z}_{\Lambda^\bullet})$ with zero-relations associated with $\Lambda^\bullet$ in (3.3) does not contain as a two-peak subposet with zero-relations any of the following ten hypercritical posets with zero-relations: $\widehat{\mathcal{F}}_1^1$, $\widehat{\mathcal{F}}_1^2$, $\widehat{\mathcal{F}}_2$, $\widehat{\mathcal{F}}_3^1$, $\widehat{\mathcal{F}}_3^2$, $\widehat{\mathcal{F}}_4$, $\widehat{\mathcal{F}}_5$, $\widehat{\mathcal{F}}_6$, $\widehat{\mathcal{F}}_7$ and $\widehat{\mathcal{F}}_8$ presented in Theorem 1.5.

Assume that the D-order Λ_1 in (1.2) is hereditary of the form (1.6). For the proof of (d′)⇒(c′) we note first that the two-peak poset with zero-relations $(I^{*+}_{\Lambda^\bullet}, \mathfrak{z}_{\Lambda^\bullet})$ associated with $\Lambda^\bullet$ in (3.3) contains as a two-peak subposet with zero-relations any of the hypercritical posets with zero-relations $\widehat{\mathcal{F}}_1^1$, $\widehat{\mathcal{F}}_1^2$, $\widehat{\mathcal{F}}_2$, $\widehat{\mathcal{F}}_3^1$, $\widehat{\mathcal{F}}_3^2$, $\widehat{\mathcal{F}}_4$, $\widehat{\mathcal{F}}_5$, $\widehat{\mathcal{F}}_6$, $\widehat{\mathcal{F}}_7$, $\widehat{\mathcal{F}}_8$ presented in Theorem 1.5 if $\Lambda^\bullet$ is one of the D-orders $\Omega_1^\bullet, \ldots, \Omega_{17}^\bullet$ presented in the tables of Section 1A. More precisely, if $\Omega_j^\bullet$ is of type $\widehat{\mathcal{F}}_j$ in the notation of Section 1, then $(I^{*+}_{\Omega_j^\bullet}, \mathfrak{z}_{\Omega_j^\bullet})$ contains $\widehat{\mathcal{F}}_j$. For example, $(I^{*+}_{\Omega_1^\bullet}, \mathfrak{z}_{\Omega_1^\bullet}) = \widehat{\mathcal{F}}_1^1$. The poset with zero-relations $(I^{*+}_{\Omega_2^\bullet}, \mathfrak{z}_{\Omega_2^\bullet})$ has the form



and $\mathfrak{z}_{\Omega_2^\bullet} = \{(c_2, c_1'), (c_2, +)\}$. It follows that the poset with zero-relations $(I^{*+}_{\Omega_2^\bullet}, \mathfrak{z}_{\Omega_2^\bullet})$ contains the poset $\widehat{\mathcal{F}}_1^1$ as the subposet with zero-relations obtained by omitting the points c_2 and c_1'. The proof in the remaining cases is left to the reader.

It follows from Theorem 3.4 (iii) that the D-orders $\Omega_1^\bullet, \ldots, \Omega_{17}^\bullet$ are of wild lattice type, because in view of the reflection duality (2.19), Proposition 2.20 (d) and [16, Theorem 1.3] the category $(I^{*+}_{\Omega_j^\bullet}, \mathfrak{z}_{\Omega_j^\bullet})$-spr is of wild representation type for $j = 1, \ldots, 17$.

In order to prove (d′)⇒(c′), assume to the contrary that the three-partite D-order $\Lambda^\bullet$ contains a three-partite minor D-suborder $\Gamma^\bullet = e\Lambda^\bullet e$, where $e \in \Lambda^\bullet$ is an idempotent, and $\Gamma^\bullet$ is dominated by $\Omega^\bullet \cong \Omega_j^\bullet$ for some j. Then $\Omega^\bullet$ is of wild lattice type, and according to Lemmas 4.3 and 4.4 the order $\Lambda^\bullet$ is also of wild lattice type. By the tame-wild dichotomy and the equivalences (a)⇔(d)⇔(d′) proved above, $(I^{*+}_{\Lambda^\bullet}, \mathfrak{z}_{\Lambda^\bullet})$ contains any of the hypercritical posets with zero-relations $\widehat{\mathcal{F}}_1^1$, $\widehat{\mathcal{F}}_1^2$, $\widehat{\mathcal{F}}_2$, $\widehat{\mathcal{F}}_3^1$, $\widehat{\mathcal{F}}_3^2$, $\widehat{\mathcal{F}}_4$, $\widehat{\mathcal{F}}_5$, $\widehat{\mathcal{F}}_6$, $\widehat{\mathcal{F}}_7$, $\widehat{\mathcal{F}}_8$, contrary to our assumption (d′).

Let us give an alternative and direct proof of the above fact. Since $\Gamma^\bullet$ is a three-partite minor suborder of $\Lambda^\bullet$, then $(I^{*+}_{\Lambda^\bullet}, \mathfrak{z}_{\Lambda^\bullet})$ contains $(I^{*+}_{\Gamma^\bullet}, \mathfrak{z}_{\Gamma^\bullet})$. Since $\Omega^\bullet = \Omega_j^\bullet$

dominates $\Gamma^\bullet$, then $(I^{*+}_{\Omega^\bullet}, \mathfrak{Z}_{\Omega^\bullet})$ is obtained from $(I^{*+}_{\Gamma^\bullet}, \mathfrak{Z}_{\Gamma^\bullet})$ by adding new relations of the forms $c \prec i$ and $j \prec c'$, where $i, j \in I' \equiv I''$, $c \in C$ and $c' \in C'$. Note that $(I^{*+}_{\Omega^\bullet_j}, \mathfrak{Z}_{\Omega^\bullet_j})$ has no relation of the above form for $j \notin \{4, 5\}$. It follows that in this case $(I^{*+}_{\Omega^\bullet_j}, \mathfrak{Z}_{\Omega^\bullet_j})$ is contained in $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$, contrary to our assumption. If $j = 4$ or $j = 5$, a simple analysis shows that either $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ contains $(I^{*+}_{\Omega^\bullet_j}, \mathfrak{Z}_{\Omega^\bullet_j})$, or else $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ contains the poset $\widehat{\mathcal{F}}^1_1$, contrary to our assumption. This finishes the proof of the implication (d′)⇒(c′).

The proof of the implication (c′)⇒(d′) reduces to pure combinatorial poset properties by applying the constructions

$$\Lambda^\bullet \mapsto I_{\Lambda^\bullet,\sigma} \mapsto (I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet}),$$

where $I_{\Lambda^\bullet,\sigma} = (I_\Lambda, \preceq, I', C, I'', \sigma : I' \to I'')$ is the bipartite stratified poset (3.2) and $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ is the two-peak poset with zero-relations (3.3).

The following properties of the constructions follow directly from the definitions.

(A) The D-order Λ together with its three-partition shown in (1.2) is uniquely determined by the bipartite stratified poset $I_{\Lambda^\bullet,\sigma}$. Hence the three-partite subamalgam D-order $\Lambda^\bullet$ (1.3) is uniquely determined by $I_{\Lambda^\bullet,\sigma}$.

(B) A three-partite subamalgam D-order $\Gamma^\bullet$ is a three-partite minor D-suborder of $\Lambda^\bullet$ if and only if $I_{\Gamma^\bullet,\tau}$ is a bipartite stratified subposet of $I_{\Lambda^\bullet,\sigma}$.

(C) For any bipartite stratified subposet $J_\tau = (J, \preceq, J', C, J'', \tau : J' \to J'')$ of $I_{\Lambda^\bullet,\sigma}$ there exists a unique three-partite minor D-suborder Γ of Λ such that $I_{\Gamma^\bullet,\tau} = J_\tau$.

(D) A three-partite D-order Λ' of the form (1.2) dominates a three-partite D-order Λ if and only if $I' = I_{\Lambda_1} = I_{\Lambda'_1}$, $I'' = I_{\Lambda_2} = I_{\Lambda'_2}$, $C = I_{\Lambda_3} = I_{\Lambda'_3}$ (a poset equality) and the partial order relation of $I_{\Lambda'}$ is obtained from the partial order relation of I_Λ by adding finitely many new relations $i' \preceq c_1$, $c_2 \preceq i''$, where $i' \in I'$, $i'' \in I''$ and $c_1, c_2 \in C$.

(E) If the two-peak poset with zero-relations $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ is given, then the poset $I' \equiv I''$ can be reconstructed as the subposet of $I^{*+}_{\Lambda^\bullet}$ consisting of all points s such that $s \preceq *$, $s \preceq +$ and each of the pairs $(s, *)$ and $(s, +)$ does not belong to the set $\mathfrak{Z}_{\Lambda^\bullet}$ of zero-relations. Moreover, $C \cup C' = I^{*+}_{\Lambda^\bullet} \setminus (I' \equiv I'')$ in the notation of (3.3).

It follows that the classification of minimal three-partite subamalgam D-orders $\Lambda^\bullet$ of wild lattice type can be given by means of bipartite stratified subposets of $I_{\Lambda^\bullet,\sigma}$.

In this way we shall show that if Λ is a three-partite D-order (1.2) and the associated two-peak poset with zero-relations $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ contains one of the hypercritical posets with zero-relations $\widehat{\mathcal{F}}^1_1$, $\widehat{\mathcal{F}}^2_1$, $\widehat{\mathcal{F}}_2$, $\widehat{\mathcal{F}}^1_3$, $\widehat{\mathcal{F}}^2_3$, $\widehat{\mathcal{F}}_4$, $\widehat{\mathcal{F}}_5$, $\widehat{\mathcal{F}}_6$, $\widehat{\mathcal{F}}_7$, $\widehat{\mathcal{F}}_8$ as a two-peak subposet with zero-relations, then the subamalgam D-order $\Lambda^\bullet$ (1.3) contains a three-partite minor D-suborder $\Gamma^\bullet$ which is dominated by any of the D-orders $\Omega^\bullet_1, \ldots, \Omega^\bullet_{17}$ shown in the tables of Section 1A.

For example we assume that Λ is a three-partite D-order of the form (1.2) such that $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ contains the poset

$$\begin{array}{cccccccccc} & & & & & & & c_1 & \longrightarrow & * \\ & & & & & & & & \nearrow & \\ \widehat{\mathcal{F}}^1_1: & a_4 & \longrightarrow & a_3 & \longrightarrow & a_2 & \longrightarrow & a_1 & & \\ & & & & & & & & \searrow & \\ & & & & & & & c'_2 & \longrightarrow & + \end{array}$$

and $(I^{*+}_{\Lambda^\bullet}, \mathfrak{z}_{\Lambda^\bullet})$ does not contain the poset $\widehat{\mathcal{F}}_2$. We shall show that the subamalgam D-order $\Lambda^\bullet$ contains a three-partite minor D-suborder $\Omega^\bullet$ which is dominated by the D-order $\Omega_1^\bullet$ or by $\Omega_2^\bullet$ shown in Section 1A.

Look at the bipartite stratified poset $I_{\Lambda^\bullet,\sigma} = (I_\Lambda, \preceq, I', C, I'', \sigma : I' \to I'')$ (3.2). Recall that C is a chain, the elements c_1, c_2 belong to C, and c_2' denotes a copy of c_2 in $C' \subseteq I^{*+}_{\Lambda^\bullet}$ (see (3.3)). Without loss of generality we may suppose that $a_4 \preceq a_3 \preceq a_2 \preceq a_1$ is a chain in I' and $a_4' \preceq a_3' \preceq a_2' \preceq a_1'$ is the image of $a_4 \preceq a_3 \preceq a_2 \preceq a_1$ under the poset isomorphism $\sigma : I' \to I''$. It follows from our assumption on the bipartition (1.2) that $a_1 \preceq a_4'$.

Let Γ be a three-partite minor of Λ (1.2) defined by the rows and columns numbered by the elements $a_4, a_3, a_2, a_1, a_4', a_3', a_2', a_1', c_1, c_2$. By our assumption

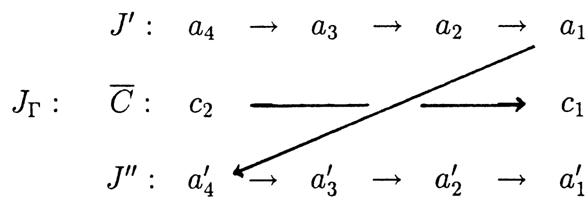
$$I_{\Gamma^\bullet,\sigma} = (J_\Gamma, \preceq, J', \overline{C}, J'', \sigma : J' \to J''),$$

where $J' = \{a_4 \preceq a_3 \preceq a_2 \preceq a_1\} \subset I'$, $J'' = \{a_4' \preceq a_3' \preceq a_2' \preceq a_1'\} \subset I''$, $\overline{C} = \{c_1, c_2\} \subseteq C$, and $\sigma : J' \to J''$ is given by $\sigma(a_1) = a_1'$, $\sigma(a_2) = a_2'$, $\sigma(a_3) = a_3'$ and $\sigma(a_4) = a_4'$.

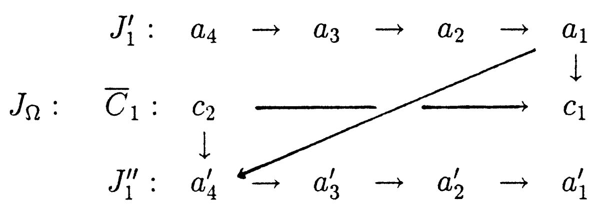
It follows from the shape of $\widehat{\mathcal{F}}_1^1$ that c_1 is not comparable with the chain $a_4' \to a_3' \to a_2' \to a_1'$ in the poset I_Λ and c_2 is not comparable with the chain $a_4 \to a_3 \to a_2 \to a_1$, and either $c_1 = c_2$ or else $c_2 \prec c_1$.

In case $c_1 = c_2$ we conclude from **(A)**–**(C)** and from the shape of the bipartite stratified poset $I_{\Gamma^\bullet,\sigma}$ that $\Gamma = \Omega_1$.

Now consider the case $c_2 \prec c_1$. Since $(I^{*+}_{\Gamma^\bullet}, \mathfrak{z}_{\Gamma^\bullet})$ does not contains the poset $\widehat{\mathcal{F}}_2$, it follows from the above observations and **(A)**–**(E)** that the poset $J_\Gamma = J' \cup \overline{C} \cup J''$ has the following structure:



with some relations from J' to $\overline{C}$ and from $\overline{C}$ to J''. It follows from **(D)** that in his class any D-order Γ is dominated by a unique three-partite D-order Ω corresponding to the bipartite stratified poset



(see the proof of the implication (d)⇒(c) in [40, Section 5]). It is clear that Ω is just the D-order Ω_2 in the tables of Section 1A.

It follows from the above analysis that, up to domination and minors, the minimal three-partite D-orders (1.2) such that $(I^{*+}_{\Lambda^\bullet}, \mathfrak{z}_{\Lambda^\bullet})$ contains the poset $\widehat{\mathcal{F}}_1^1$ and $(I^{*+}_{\Lambda^\bullet}, \mathfrak{z}_{\Lambda^\bullet})$ does not contain the poset $\widehat{\mathcal{F}}_2$ are just the D-orders Ω_1 and Ω_2 shown in Section 1A.

By the technique applied above we also prove that if $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ contains any of the hypercritical posets with zero-relations $\widehat{\mathcal{F}}^1_1$, $\widehat{\mathcal{F}}^2_1$, $\widehat{\mathcal{F}}_2$, $\widehat{\mathcal{F}}^1_3$, $\widehat{\mathcal{F}}^2_3$, $\widehat{\mathcal{F}}_4$, $\widehat{\mathcal{F}}_5$, $\widehat{\mathcal{F}}_6$, $\widehat{\mathcal{F}}_7$, $\widehat{\mathcal{F}}_8$ (see Theorem 1.5) as a two-peak subposet with zero-relations, then the three-partite order $\Lambda^\bullet$ contains a three-partite minor D-suborder $\Gamma^\bullet$ dominated by a D-order $\Omega^\bullet$ of one of the 17 forms shown in the tables of Section 1A. The details are left to the reader. This completes the proof of Theorem 1.5. □

Question 4.7. Does Theorem 1.5 remain valid if we remove the assumption that the part $\mathcal{X}$ or the part $\mathcal{Y}$ of the D-order Λ in (1.2) consists of matrices with coefficients in $\mathfrak{p}$?

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