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A COMPACTIFICATION OF A FAMILY OF DETERMINANTAL GODEAUX SURFACES

YONGNAM LEE

Abstract. In this paper, we present a geometric description of the compactification of the family of determinantal Godeaux surfaces, via the study of the bicanonical pencil and using classical Prym theory. In particular, we reduce the problem of compactifying the space of bicanonical pencils of determinantal Godeaux surfaces to the compactification of the family of twisted cubic curves in $\mathbb{P}^3$ with certain given tangent conditions.

0. Introduction

Let $\mathcal{X} \to \Delta$ be a flat family of projective surfaces over a small disk in $\mathbb{C}$. Assume that the general fiber X_t is a minimal projective surface of general type for $t \in \Delta^*$. Then an interesting and difficult question is how to find a distinguished *canonical model* for the central fiber after a possible base change, and to classify such central fibers. There are two main difficulties in studying this problem compared to a case of curves. One difficulty is that, in general, we cannot obtain a semistable reduction while preserving relative ampleness. The other difficulty is that the moduli space, $\mathcal{M}_{K^2,\chi}$, is unknown and the number of components is also unknown.

We study the special case in which the general fiber X_t is a *numerical Godeaux surface*, namely, those with

$$\chi(\mathcal{O}_X) = 1 \ \textit{and} \ K^2 = 1, \tag{0.1}$$

for $t \in \Delta^*$. $\chi(\mathcal{O}_X) = 1$ and $K^2 = 1$ imply that $p_g = q = 0$ via the construction of the cyclic unramified covering and some inequalities. The reason for choosing this kind of surface for our study is that, if $\chi(\mathcal{O}_{X_t})$ and $K^2_{X_t}$ are as small as possible, then the quadratic polynomial $h^0(X_t, mK_{X_t}) = P_{X_t}(m)$ increases slower than other cases. Also, if we consider the cohomology of the holomorphic tangent bundle, which is related with the first order deformation space of X_t, we have

$$-\chi(T_{X_t}) = 10\chi(\mathcal{O}_{X_t}) - 2K^2_{X_t}.$$

Furthermore by using Bogomolov's lemma ($h^0(\Omega_{X_t}(-mK_{X_t})) = 0$ if $m \geq 1$) and $4K_{X_t}$'s base point freeness, then $h^2(T_{X_t}) \leq 14K^2_{X_t}$ (cf. [C2, §5]). So

$$h^1(T_{X_t}) \leq 10\chi(\mathcal{O}_{X_t}) + 12K^2_{X_t}.$$

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In this paper, we study such degenerations for a special subfamily of Godeaux surfaces (i.e. quotients of a quintic in $\mathbb{P}^3$ under a free $\mathbb{Z}_5$-action). In the 8-dimensional family of Godeaux surfaces, there is a 4-dimensional subfamily for which the quintic is symmetric determinantal. We call these surfaces determinantal Godeaux surfaces. They were first studied systematically by Catanese in [C1]. This subfamily and its degenerations are the main object of our study. The purpose of this paper is to present a geometric description of a compactification of the family of determinantal Godeaux surfaces by studying its bicanonical pencil using classical Prym theory. The outline of our approach is as follows.

A Godeaux surface X has a bicanonical pencil, this determines a rational curve $\mathbb{P}^1_X$ in $\overline{\mathcal{M}_4}$, the Deligne-Mumford compactification of the moduli space of genus four curves. Let τ be the generators of the fundamental group of a Godeaux surface X. Then $H^0(K_X + i\tau)$ for $i = 1, \ldots, 4$ contains a unique curve C_i of genus two and C_i, C_j intersect transversally at one point $P(C_i, C_j)$. Denote

$$\begin{aligned} P_1 &= P(C_1, C_2), \quad P_2 = P(C_1, C_3),\\ P_3 &= P(C_4, C_2), \quad P_4 = P(C_4, C_3),\\ P &= P(C_1, C_4), \quad Q = P(C_2, C_3). \end{aligned}$$

Then $P_1 \ldots P_4$ are base points of $|2K_X|$ and P, Q are base points of $|3K_X|$. The canonical sheaf K_C of each member C in $|2K_X|$ can be given as the combination of $P_1 \ldots P_4$, and the two g^r_d's on C, $r = 1$, $d = 3$ (which we write g^1_3, h^1_3 with $g^1_3 + h^1_3 = K_C$) are also given in terms of these four points. Because these points are base points of the bicanonical pencil, g^1_3 and h^1_3 are both monodromy invariant. The general member C in the bicanonical pencil of a determinantal Godeaux surface X has an invariant unbranched double cover $\tilde{C}$ induced from a distinguished double cover of X. So determinantal Godeaux surface X determines two rational curves $\mathbb{P}^1_{X,g^1_3}$, $\mathbb{P}^1_{X,h^1_3}$ in $\overline{\mathcal{R}_7} \times_{\overline{\mathcal{M}_4}} \mathcal{W}^1_3$ where $\overline{\mathcal{R}_7}$ is the compactification of the space of unbranched double covers of curves of genus four, and $\mathcal{W}^1_3$ represents the family of the linear system of degree 3 with projective dimension one on curves of genus four. The Prym construction of Donagi gives a birational morphism Φ,

$$\begin{array}{c} \mathbb{P}^1_{X,g^1_3} \subseteq \overline{\mathcal{R}_7} \times_{\overline{\mathcal{M}_4}} \mathcal{W}^1_3 \supseteq \mathbb{P}^1_{X,h^1_3} \\ \Phi \big\downarrow \\ \mathbb{P}^1_{X,g^1_4} \subseteq Pic^4_3 \supseteq \mathbb{P}^1_{X,h^1_4} \end{array}$$

where Pic^4_3 is the family of degree 4 line bundles on curves of genus 3. Let $\tilde{X}$ be the blow up of X at the base points of $|2K_X|$ and $|3K_X|$. Then the geometry of Φ^{-1} allows us to describe the image of mapping

$$\tilde{X} \xrightarrow{|2K_X| \times |3K_X|} \mathbb{P}^1 \times \mathbb{P}^3$$

as the pull back surface induced by a degree 3 map

$$\mathbb{P}^1 \to \mathbb{P}(H^0(\textit{Segre cubic}, \mathcal{O}(2)))$$

passing through two fixed points with fixed skew tangent direction. Thus the degenerations of the surfaces are given by two possible degenerations of the corresponding twisted cubics.

It is hoped that further study will lead to similar geometric conclusions for other families of numerical Godeaux surfaces.

1. The family of Godeaux curves

Let (x, y, z, w) be the coordinates of $\mathbb{P}^3$, and consider a natural $\mathbb{Z}_5$-action on $\mathbb{P}^3$ via

$$(x, y, z, w) \to (\epsilon x, \epsilon^2 y, \epsilon^3 z, \epsilon^4 w)$$

where ϵ is the primitive 5-th root of unity. The $\mathbb{Z}_5$-invariant quintics form an irreducible 8-dimensional family $\mathcal{M}$ [Mi]:

$$x^5 + y^5 + z^5 + w^5 + t_1 x^3 zw + t_2 xyw^3 + t_3 xy^3 z + t_4 yz^3 w \\ + t_5 x^2 yz^2 + t_6 y^2 zw^2 + t_7 x^2 y^2 w + t_8 xz^2 w^2.$$

Since the four fixed points of $\mathbb{Z}_5$-action are not in Σ, the quotient X is a smooth projective surface of general type, with $p_g = q = 0$, $K^2 = 1$, and $\pi_1(X) = \mathbb{Z}_5$. We will denote these surfaces as *Godeaux surfaces*. For a Godeaux surface, the bicanonical pencil (resp. the tricanonical linear system) is given by $H^0(2K_\Sigma)^{\mathbb{Z}_5}$ (resp. $H^0(3K_\Sigma)^{\mathbb{Z}_5}$). $H^0(2K_\Sigma)^{\mathbb{Z}_5}$ is generated by xw, yz, and $H^0(3K_\Sigma)^{\mathbb{Z}_5}$ is generated by xy^2, x^2z, yw^2, z^2w. $|2K_\Sigma|^{\mathbb{Z}_5}$ has 20 base points on Σ:

$$(1, 0, -\epsilon^i, 0), (1, -\epsilon^i, 0, 0), (0, -\epsilon^i, 0, 1), (0, 0, -\epsilon^i, 1).$$

Also, $|3K_\Sigma|^{\mathbb{Z}_5}$ has 10 base points on Σ: $(1, 0, 0, -\epsilon^i), (0, 1, -\epsilon^i, 0)$.

The author proves the following three lemmas in [L2].

Lemma 1. *Choose a generic element X in the moduli space of Godeaux surfaces. Then $|2K_X|$ has no base components, and has four simple base points. Let $p : S \to X$ be the blow-up of four base points of $|2K_X|$. Then each fiber of $f : S \to \mathbb{P}^1$ is a stable curve of genus four, and has at most one node.*

Consider Deligne-Mumford compactification of the moduli space of curves of genus four, $\overline{\mathcal{M}_4}$. In this paper, $\overline{\mathcal{M}_4}$ is mostly a moduli functor instead of a moduli space, because we are usually interested in numerical data associated with the pull back to the pencil. The Hodge class λ and the boundary classes δ_0,δ_1,δ_2 are the standard basis of $Pic\,\overline{\mathcal{M}_4} \otimes \mathbb{Q}$ [AC]. $f : S \to \mathbb{P}^1$ induces a morphism $\tilde{q} : \mathbb{P}^1 \to \overline{\mathcal{M}_4}$. Then we obtain

$$f_*\omega_{S/\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(1)^4.$$

Lemma 2. *Let $f : S \to \mathbb{P}^1$ be the bicanonical pencil of a general Godeaux surface. Then $\lambda.\mathbb{P}^1 = 4$, $\delta_0.\mathbb{P}^1 = 25$, $\delta_1.\mathbb{P}^1 = 0$ and $\delta_2.\mathbb{P}^1 = 2$.*

Lemma 3. *Let us call a stable curve C of genus four a "Godeaux curve" if it occurs as a fiber of the bicanonical pencil for a smooth Godeaux surface. Then the family of Godeaux curves has five dimensions.*

A general curve C in the bicanonical pencil of a Godeaux surface is not a hyperelliptic curve [L1, §3.1]. In fact, there is no hyperelliptic curve in the bicanonical pencil of a Godeaux surface [L1, §4.3]. Let X be a numerical Godeaux surface (surface of general type, with $p_g = q = 0$, and $K^2 = 1$). Let τ_i be a nonzero torsion element of $H_2(X, \mathbb{Z})$. Then there is a unique effective divisor C_i corresponding to $K_X + \tau_i$. C_i, C_j intersect transversally at one point $P(C_i, C_j)$, by the following lemma of Reid in [Re].

Lemma 4. *Let X be a numerical Godeaux surface and $C_i = K_X + \tau_i$. Then*

1. $h^0(K_X + \tau_i) = 1$,

2. *C_i, C_j intersect transversally at one point $P(C_i, C_j)$.*
3. *If C_i, C_j, C_k are distinct, then $P(C_i, C_j)$ and $P(C_i, C_k)$ are distinct.*

For a Godeaux surface X, $H_2(X, \mathbb{Z})_{tor} = \mathbb{Z}_5$. So we have two special members of $|2K_X|$, related to δ_2. Write $D_1 = C_1 + C_4$ and $D_2 = C_2 + C_3$. Also denote

$$P_1 = P(C_1, C_2), \quad P_2 = P(C_1, C_3),$$
$$P_3 = P(C_4, C_2), \quad P_4 = P(C_4, C_3),$$
$$P = P(C_1, C_4), \quad Q = P(C_2, C_3).$$

Then $P_1 \dots P_4$ are base points of $|2K_X|$ and P, Q are base points of $|3K_X|$.

Lemma 5. *Let C be a general member of the bicanonical pencil of a Godeaux surface. Then K_C can be written as the combination of base points $P_1 \dots P_4$ in the following way:*

$$\begin{aligned} K_C &= 2P_1 + 3P_2 + P_4 \\ &= 3P_1 + P_2 + 2P_3 \\ &= 2P_2 + P_3 + 3P_4 \\ &= P_1 + 3P_3 + 2P_4. \end{aligned}$$

Also, we have g_3^1 and h_3^1 as the combination of base points, where $g_3^1 + h_3^1 = K_C$.

Proof. Since $K_X = 2(K_X + \tau_i) - (K_X + \tau_j)$ where $2\tau_i = \tau_j$, then we can write $K_X|_C$ as the combination of base points.

$$\begin{aligned} K_X|_C &= P_1 + 2P_2 - P_3 \\ &= 2P_1 + P_3 - P_4 \\ &= P_2 + 2P_4 - P_1 \\ &= 2P_3 + P_4 - P_2. \end{aligned}$$

So $K_C = 3K_X|_C = 2K_X|_C + K_X|C = P_1 + \cdots + P_4 + K_X|_C$.

Since $2P_1 + P_2 \sim (2P_3 + P_4 - P_2) + (P_4 - P_3) + P_2 = P_3 + 2P_4$, we have $2P_2 + P_4 \sim P_1 + 2P_3$, so

$$g_3^1 : 2P_1 + P_2 \sim P_3 + 2P_4,$$

$$h_3^1 : 2P_2 + P_4 \sim P_1 + 2P_3$$

and their sum is K_C. □

Theorem 6. *The generic curve that is in the bicanonical pencil of a Godeaux surface can be constructed in the following way (see Figure 1.1).*

Fix general four points $P_1 \dots P_4$ in $\mathbb{P}^3$. Draw four lines $L_{12} = \overline{P_1 P_2}$, $L_{13} = \overline{P_1 P_3}$, $L_{24} = \overline{P_4 P_2}$, $L_{34} = \overline{P_4 P_3}$, and consider the pencil of quadrics $\{Q\}$ containing these lines. Then consider the space $\{W\}$ of cubic surfaces in $\mathbb{P}^3$ which meet L_{ij} doubly at P_i. The intersection curves $Q \cap W$ are shown in Figure 1.1. So $2P_1 + P_2 \sim P_3 + 2P_4$ gives g_3^1 and $P_1 + 2P_3 \sim 2P_2 + P_4$ gives h_3^1, and their sum is K_C.

Proof. According to Lemma 5, the curves related with Godeaux surfaces satisfy the condition of Theorem 6. So we need to prove that the dimension of the family of curves satisfying the above condition is five.

Let C be a canonical curve with genus four in $\mathbb{P}^3$. Let $(s_1, \dots, s_4)$ be coordinates of $\mathbb{P}^3$ and $P_1 = (1, 0, 0, 0), P_2 = (0, 1, 0, 0), P_3 = (0, 0, 1, 0), P_4 = (0, 0, 0, 1)$. Since C

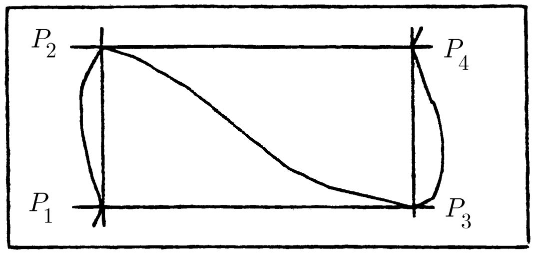

FIGURE 1.1. Generic curve in the bicanonical pencil

is a complete intersection of quadric Q and cubic F, we have the following exact sequence and commutative diagrams, between the cohomology of tangent sheaf and the normal sheaf (see (1.2)).

$$0 \to T_C \to T_{\mathbb{P}^3|C} \to \mathcal{O}_C(2) + \mathcal{O}_C(3) \to 0$$

$$\begin{array}{ccccccc} H^0(\mathcal{O}_F(1))^4 & & & & H^0(\mathcal{O}_Q(1))^4 & & \\ \downarrow & & & & \downarrow & & \\ H^0(T_{\mathbb{P}^3|F}) & \longrightarrow & H^0(\mathcal{O}_F(3)) & & H^0(T_{\mathbb{P}^3|Q}) & \longrightarrow & H^0(\mathcal{O}_Q(2)) \\ \| & & \downarrow & & \| & & \| \\ H^0(T_{\mathbb{P}^3|C}) & \longrightarrow & H^0(\mathcal{O}_C(3)) & & H^0(T_{\mathbb{P}^3|C}) & \longrightarrow & H^0(\mathcal{O}_C(2)) \end{array} \tag{1.2}$$

In (1.2), the composition of two maps from $H^0(\mathcal{O}_F(1))^4$ to $H^0(\mathcal{O}_F(3))$ and from $H^0(\mathcal{O}_Q(1))^4$ to $H^0(\mathcal{O}_Q(2))$ are the simple sum of s_i times partial derivatives of F and Q.

It is clear that the pencil of quadrics containing $\bigcup L_{ij}$ is generated by s_1s_4, s_2s_3. Fix one smooth quadric Q in the above pencil, and consider the cubics containing $\bigcup L_{ij}$. These spaces are parameterized by the $\mathbb{P}^7$, generated by $s_is_1s_4$, $s_is_2s_3$ for $i = 1 \dots 4$. Then for given Q and W, changing the cubic W by an element of this $\mathbb{P}^7$ provides a trivial deformation of C for the following reason.

Let F be a cubic containing four lines,

$$Q \cap F = (3,3) = (2,2) + (1,1)$$

as a divisor in Q. Consider the following long exact sequence:

$$0 \to H^0(T_{\mathbb{P}^3|C}) \to H^0(\mathcal{O}_C(2)) + H^0(\mathcal{O}_C(3)) \to H^1(T_C) \to .$$

We may assume that $Q = s_1s_4 + s_2s_3$. Then the partial derivative

$$(\frac{\partial Q}{\partial s_i}) = (Q_i) = (s_4, s_3, s_2, s_1).$$

So the image of the map from $H^0(T_{\mathbb{P}^3|C})$ to $H^0(\mathcal{O}_C(2)) + H^0(\mathcal{O}_C(3))$ is generated by

$$(s_is_4|_C, s_iF_1|_C),\ (s_is_3|_C, s_iF_2|_C),\ (s_is_2|_C, s_iF_3|_C),\ (s_is_1|_C, s_iF_4|_C)$$

where $F_i = \frac{\partial F}{\partial s_i} \cdot s_1s_4|_C = s_2s_3|_C = 3(P_1 + \cdots + P_4)$. Therefore there is a four-dimensional subspace of $H^0(T_{\mathbb{P}^3|C})$ which fixes $H^0(\mathcal{O}_C(3P_1 + \cdots + 3P_4))$. When we fix Q, and deform $Q \cap W$ by changing W in the direction of cubics containing four lines, we obtain the subspace $H^0(\mathcal{O}_C(3P_1+\cdots+3P_4+q_1+\cdots+q_6)$ of $H^0(\mathcal{O}_C(3))$, varied only by $q_1\dots q_6$. That is the same as $H^0(\mathcal{O}_C(K_C))$. So the image of the four-dimensional subspace of $H^0(T_{\mathbb{P}^3|C})$ goes surjectively to $H^0(\mathcal{O}_C(K_C))$. Therefore there is no deformation of C by varying W by an element of the space of cubics containing four lines.

Then g^1_3, h^1_3, the tangent conditions at P_i give a 13-dimensional projective space of cubics. In this case four-tangent conditions do not give eight conditions, they give seven conditions by g^1_3, h^1_3 (four points and three tangent directions determine the fourth tangent direction). So for fixing Q, we have a four-dimensional deformation of $H^1(T_C)$ by changing of cubics. By changing the quadrics in the pencil, we have a five-dimensional family of curves of genus four. □

2. The associated Prym curves

By Lemma 3, each curve in this five-dimensional family of Godeaux curves occurs in a four-dimensional family of Godeaux surfaces (because Godeaux surfaces have eight moduli). The author finds a four-dimensional subfamily of Godeaux surfaces on which each of these curves occurs with finite frequency in [L1, §3.2], namely Godeaux surfaces coming from $\mathbb{Z}_5$-invariant symmetric determinantal quintics in $\mathbb{P}^3$. A general symmetric determinantal quintic has an even set of twenty nodes [C1] (an even set of nodes means that there is a double cover branched of only those nodal points).

Each determinantal Godeaux surface X has a double covering. The four fixed points of a $\mathbb{Z}_2$-action are exactly the four nodes of X. Since these four nodes are not four base points of $|2K_X|$, the general member C of $|2K_X|$ has a double covering $\tilde{C}$ without fixed points. So then it is possible to construct a "Prym curve" D for $(\tilde{C}, C)$ [Mu]. The Prym curve D associated to $(\tilde{C}, C)$ is a curve of genus three with two distinguished linear series g^1_4, h^1_4 (associated to the two g^1_3's on C) for which $g^1_4 + h^1_4 = 2K_D$. We characterized this subfamily of curves in $\overline{\mathcal{M}_4}$ in Theorem 6 by using the base points of $|2K_X|$. In this section, we will characterize the subfamily of Prym curves in $\overline{\mathcal{M}_3}$ generated by this construction. Also we can construct reversely. These constructions are given in [Rc].

Let C be a curve of genus four with a g^1_3 and $\sigma : \tilde{C} \to C$ an unbranched double cover, then the 2^3 liftings of $P+Q+R \in g^1_3$ to a divisor of degree three on $\tilde{C}$ break up into two components according to the parity of how many come from the "top" sheet. This induces a curve $\tilde{D}$ in $\tilde{C}^{(3)}$. The involution on $\tilde{D}$ coming from parity gives a curve D of genus three and $\tilde{D} \to D$ is the reducible unbranched double covering. Also D has its natural g^1_4 coming from the four liftings. Conversely, let D be a curve of genus three with a g^1_4. Then let $\tilde{C}$ be the set of pairs $\{p, q\}$ on D such that there exist r, s on D with $p+q+r+s \in g^1_4$. Then $\tilde{C}$ has a natural involution $\{p,q\} \Leftrightarrow \{r,s\}$, and quotient C has genus four with a g^1_3, we have

$$Jac(D) = Prym(\tilde{C}, C).$$

The following theorem is due to Donagi in [Do].

Theorem 7. *The above construction gives a birational morphism between trigonal curves C of genus g with a double cover $\tilde{C}$ and tetragonal curves D of genus $g-1$.*

Consider C and g^1_3, h^1_3 in Lemma 5.

$$g^1_3 : 2P_1 + P_2 \sim P_3 + 2P_4,$$

$$h_3^1 : 2P_2 + P_4 \sim P_1 + 2P_3$$

and their sum is K_C. Let $P_1^1, P_1^2, \dots P_4^1, P_4^2$ denote the liftings of P_1, P_2, P_3, P_4 in $\tilde{C}$.

For convenience let us assume that the odd parity induces the curve D. We denote the points of D built from these lifting as follows:

$$\begin{aligned}
&(P_1^1, P_1^1, P_2^1) \to y_1, \quad (P_2^1, P_2^1, P_4^1) \to y_2,\\
&(P_1^1, P_1^2, P_2^2) \to x_1, \quad (P_2^1, P_2^2, P_4^2) \to x_2,\\
&(P_1^2, P_1^2, P_2^1) \to z_1, \quad (P_2^2, P_2^2, P_4^1) \to z_2,\\
&(P_4^1, P_4^1, P_3^1) \to y_4, \quad (P_3^1, P_3^1, P_1^1) \to y_3,\\
&(P_4^1, P_4^2, P_3^2) \to x_4, \quad (P_3^1, P_3^2, P_1^2) \to x_3,\\
&(P_4^2, P_4^2, P_3^1) \to z_4, \quad (P_3^2, P_3^2, P_1^1) \to z_3.
\end{aligned}$$

Denote D to be a canonical curve of genus three, i.e., smooth quartic plane curve. Then all g_4^1's can be understood in the following way. Consider four fixed points in D and a pencil of conics through these four fixed points. Then the other four intersection points give a g_4^1. By the bijectiveness of Theorem 7 and duality, we have g_4^1, $h_4^1 = 2K_D - g_4^1$ on D such that

$$g_4^1 : 2x_1 + y_1 + z_1 \sim 2x_4 + y_4 + z_4,$$

$$h_4^1 : 2x_2 + y_2 + z_2 \sim 2x_3 + y_3 + z_3.$$

And we have four conics $C_{12}, C_{13}, C_{24}, C_{34}$:

$$\begin{aligned}
C_{12}|_D &: 2x_1 + y_1 + z_1 + 2x_2 + y_2 + z_2,\\
C_{13}|_D &: 2x_1 + y_1 + z_1 + 2x_3 + y_3 + z_3,\\
C_{24}|_D &: 2x_2 + y_2 + z_2 + 2x_4 + y_4 + z_4,\\
C_{34}|_D &: 2x_3 + y_3 + z_3 + 2x_4 + y_4 + z_4.
\end{aligned}$$

There is a natural map from $D^{(2)}$ to $D^{(2)}$ by $K_D - x - y$ for $\{x, y\} \in D^{(2)}$. Consider two constructions $\tilde{C}_g$ in $D^{(2)}$ using g_4^1 and $\tilde{C}_h$ using h_4^1. Taking the quotients by the respective involutions we have curves C_g and C_h. The natural map $x + y \to K_D - x - y$ gives an isomorphism between C_g and C_h. Working out this correspondence explicitly with respect to the points P_i^j above, we conclude that $\overline{x_3 z_3}$ passes through x_1, $\overline{x_1 z_1}$ through x_2, $\overline{x_4 z_4}$ through x_3, $\overline{x_2 z_2}$ through x_4. Then this property and $2K_D = g_4^1 + h_4^1$ gives that each conic C_{ij} is the union of two lines,

$$C_{12} = \overline{x_1 x_2} + \overline{x_1 z_2},\ C_{34} = \overline{x_3 x_4} + \overline{x_4 z_3},$$

$$C_{24} = \overline{x_2 x_4} + \overline{x_2 z_4},\ C_{13} = \overline{x_1 x_3} + \overline{x_3 z_1}.$$

This induces the following theorem.

Theorem 8. *The family of Prym curves of determinantal Godeaux surfaces can be constructed in the following way. Fix four points $\{x_i\}$ and four lines in $\mathbb{P}^2$ as shown in Figure 2.1.*

On each line pick a point $\{z_i\}$. Consider four conics $C_{12}, C_{34}, C_{24}, C_{13}$ as above. Then consider the pencil of quartics,

$$\lambda(C_{12}C_{34}) + \mu(C_{13}C_{24}).$$

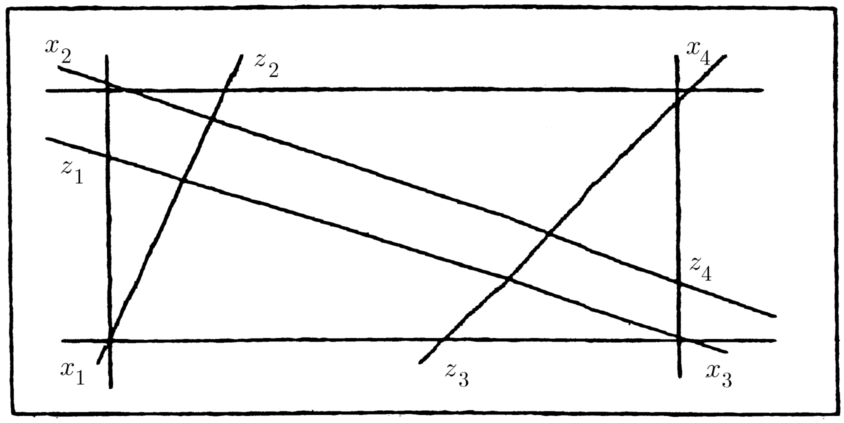

FIGURE 2.1. Associated Prym curves

The set of all such quartic plane curves, obtained by moving z_i on the line, forms a five-dimensional family.

Proof. Figure 2.1 determines four points $y_1, \dots y_4$ such that $y_1 \in \overline{x_1z_2} \cap \overline{x_3z_1}$, $y_2 \in \overline{x_1z_2} \cap \overline{x_2z_4}$, $y_3 \in \overline{x_4z_3} \cap \overline{x_3z_1}$, $y_4 \in \overline{x_4z_3} \cap \overline{x_2z_4}$.

Then $2x_1+y_1+z_1 \sim 2x_4+y_4+z_4$, $2x_2+y_2+z_2 \sim 2x_3+y_3+z_3$ are the elements of g_4^1, h_4^1 in Prym curve D. We show that these g_4^1, h_4^1 correspond to the elements of g_3^1 and h_3^1 in C (Theorem 6) for the determinantal Godeaux surfaces. Since the family of determinantal Godeaux curves has five dimensions and the family of curves constructed above has no more than five dimensions, we obtain Theorem 8. □

3. The reverse construction

Consider the linear system of conics in $\mathbb{P}^2$. Let Π be a general three-dimensional linear subspace of $\mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))) \cong \mathbb{P}^5$. Let V_4 denote the set of conics in $\mathbb{P}^2$ of rank≤ 2 (two lines). V_4 is an irreducible variety of dimension 4 and degree 3. Let V_2 denote the set of conics in $\mathbb{P}^2$ of rank ≤ 1 (double line). V_2 is an irreducible variety of dimension 2 and degree 4. Then $\Pi \cap V_4$ is a cubic surface with four nodes (Segre cubic surface). This cubic surface has a natural double covering that is branched over the four nodal points.

Consider four double lines in $\mathbb{P}^2$ (Figure 2.1): $2\overline{x_1x_2}$, $2\overline{x_1x_3}$, $2\overline{x_2x_4}$, $2\overline{x_3x_4}$. These four double lines give four points $P_{12} = P_{21}, P_{13} = P_{31}, P_{24} = P_{42}, P_{34} = P_{43}$ in V_2 and determine $\Pi = \mathbb{P}^3$ in $\mathbb{P}^5$. Also these four points are exactly the node points of Segre cubic $S_0 = \Pi \cap V_4$.

Consider a pencil of conics (union of two lines) in Figure 2.1, $\overline{x_1x_2} \cup \overline{x_1z_2}$ given by moving of z_2 and fixing of x_1, x_2. This gives a line that connects P_{12}, P_{13}. By the same construction of the other three lines, we have a configuration of four lines and four points. The quartic curve in the associated pencil of Prym curves determined by z_1, z_2, z_3, z_4 is given by $\lambda(C_{12}C_{34}) + \mu(C_{13}C_{24})$ for some $(\lambda, \mu) \in \mathbb{P}^1$(see Theorem 8). These four points z_i determine four points P_1, P_2, P_3, P_4 where P_i is on the line $\overline{P_{ij}P_{ik}}$. Then draw four lines,

$$\overline{P_1P_2},\ \overline{P_1P_3},\ \overline{P_2P_4},\ \overline{P_3P_4}$$

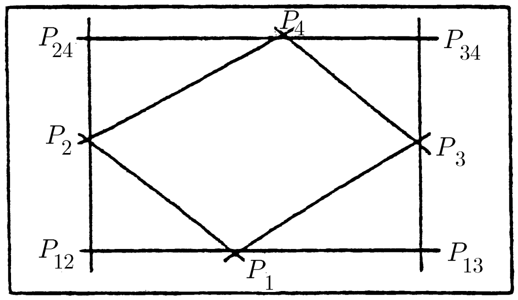

FIGURE 3.1. reverse construction in Segre cubic surface

(see Figure 3.1).

Choose the quadric of the pencil containing the four lines $\overline{P_iP_j}$ which is given by (λ, μ). Then $Q \cap S_0$ gives the reverse construction of the genus four curve. Consider now the quadric linear system in $\mathbb{P}^3$:

$$\mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}(2))) = \mathbb{P}^9.$$

Inside of the bicanonical pencil of a Godeaux surface or a determinantal Godeaux surface there are two special quadrics xw, yz. The general quadric associated to a bicanonical pencil lies in

$$s(\lambda_1 x + \lambda_2 y)(\lambda_3 z + \lambda_4 w) + t(\mu_1 x + \mu_2 z)(\mu_3 y + \mu_4 w).$$

Therefore we have a morphism

$$\varphi : \mathbb{P}^1 \to \mathbb{P}^5 \subset \mathbb{P}^9,$$

where $\mathbb{P}^5$ is given by xz, xw, yz, yw, xy, zw. Also we have a genus four fibration $\mathcal{V}$ over $\mathbb{P}^5$, whose fiber at $x \in \mathbb{P}^5$ is $S_0 \cap Q_x$. So we have the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{V}_{\mathbb{P}^1} & \longrightarrow & \mathcal{V} \subset \mathbb{P}^5 \times S_0 & \xrightarrow{p_2} & \mathbb{P}^3 \\ {\scriptstyle g}\downarrow & & {\scriptstyle p_1}\downarrow & & \\ \mathbb{P}^1 & \xrightarrow{\varphi} & \mathbb{P}^5 & & \end{array} \tag{3.2}$$

Lemma 9. *Let S_0 be a Segre cubic surface and ℓ a line through two nodes. Then the rank of the Hessian matrix at a generic point in the line is two.*

Proof. Let (x, y, z, w) be the coordinates of $\mathbb{P}^3$. Then the equation of Segre cubic is the determinant of

$$\begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix} + w \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Set $w = 1$ and choose the affine coordinates (x, y, z). The four nodal points are

$$(0,0,0),\ (1,0,0),\ (0,1,0),\ (0,0,1)$$

and the intersection at $x = 0$ is $yz = 0$. In the affine coordinate $(w = 1)$ quadric: $x + \lambda yz$ for degree of $\lambda yz = 3$, cubic$|_{x=0} = yz$.

So the Hessian of the intersection has rank two. □

Theorem 10. *Degree of $\varphi = 3$.*

Proof. Theorem 10 can be proved by an adjunction formula. Since $K_{\mathbb{P}^5\times\mathbb{P}^3} = p_1^*\mathcal{O}_{\mathbb{P}^5}(-6)\otimes p_2^*\mathcal{O}_{\mathbb{P}^3}(-4)$, so we have

$$K_{\mathbb{P}^5\times S_0} = p_1^*\mathcal{O}_{\mathbb{P}^5}(-6)\otimes p_2^*\mathcal{O}_{S_0}(-1).$$

Let N be the normal bundle of $\mathbb{P}^1$ in $\mathbb{P}^5$, and d the degree of φ. Then

$$\mathcal{O}_{\mathbb{P}^5}(-6)\otimes det\, N|_{\mathbb{P}^1} = -2,$$

we have $det\, N = 6d-2$ and

$$K_{\mathcal{V}_{\mathbb{P}^1}} = p_1^*\mathcal{O}_{\mathbb{P}^1}(d-2)\otimes p_2^*\mathcal{O}_{S_0}(1).$$

So $g_*K_{\mathcal{V}_{\mathbb{P}^1}} = \mathcal{O}_{\mathbb{P}^1}(d-2)^4$, and $g_*K_{\mathcal{V}_{\mathbb{P}^1}/\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(d)^4$. Let $p : S\to X$ be a blow-up of four base points of the bicanonical pencil and $p' : S'\to S$ denote a blow-up of the two base points of $p^*(3K_X)$. Consider the following diagram.

$$\begin{array}{ccc} S' & \xrightarrow{\omega_{S'/\mathbb{P}^1}\otimes\mathcal{O}(-1)} & \mathcal{V}_{\mathbb{P}^1} \\ \downarrow & & \\ \mathbb{P}^1 & & \end{array} \tag{3.3}$$

Because we have two multiple fibers with multiplicity two in $\mathcal{V}_{\mathbb{P}^1}\to\mathbb{P}^1$ and by Lemma 9, the tangent cone through the line is a generically normal crossing, so the relative dualizing sheaf of the normalization is the pull back of the relative dualizing sheaf tensored with the ideal sheaf of the two special fibers of the Godeaux surface. Therefore $g_*K_{\mathcal{V}_{\mathbb{P}^1}/\mathbb{P}^1}\otimes\mathcal{O}_{\mathbb{P}^1}(-2) = \mathcal{O}_{\mathbb{P}^1}(1)^4$, so $d=3$. □

According to Theorem 10, the degree of the map from the bicanonical pencil to each line $\overline{P_{ij}P_{ik}}$ is one, i.e.

$$K_{\mathcal{V}} = p_1^*\mathcal{O}_{\mathbb{P}^5}(-5)\otimes p_2^*\mathcal{O}_{\mathbb{P}^3}(1)|_{\mathcal{V}}.$$

This implies that the degree of the map from the pencil of associated Prym curves, related with a determinantal Godeaux surface, to each of the four lines

$$(\overline{x_1x_2}, \overline{x_1x_3}, \overline{x_4x_2}, \overline{x_4x_3})$$

is one (see Figure 2.1).

Remark. The family of twisted cubics in $\mathbb{P}^5$ through the two marked points, xw, yz, is via projection from these points, a bundle over the Grassmannian variety $\mathcal{G}$ of lines in $\mathbb{P}^3$. In affine coordinates the general twisted cubic coming from a determinantal Godeaux surface is of the form

$$(x+c_1ty)(c_2tz+w)+t(x+c_3tz)(c_4ty+w)$$

From this it is easy to check that these twisted cubics fiber over

$$H = \overline{\{xz,yw\}}\times\overline{\{xy,zw\}} = \mathbb{P}^1\times\mathbb{P}^1\subseteq\mathcal{G}\subseteq\mathbb{P}^5$$

where H is just the intersection of two tangent hyperplane sections of $\mathcal{G}$. Thus the problem of compactifying the space of “Godeaux twisted cubics” is reduced to studying the compactification of the following space:

Let $p=\{xw\}, q=\{yz\}, p'=\{c_2xz+c_1yw\}, = q'=\{c_4xy+c_3zw\}$. Let $L=\overline{pp'}$, $M=\overline{qq'}$. We compactify the surface of twisted cubics in $\mathbb{P}^3$ tangent to L at p and tangent to M at q with osculating plane at p given by $\langle p,p',q'\rangle$ and osculating plane at q given by $\langle q,q',p'\rangle$.

This compactification is worked out in [PS]. The associated relative dualizing sheaf can be computed easily. So if we let $\mathcal{S}^* \to \Delta^* \times \mathbb{P}^1$ be a family of four point blow-ups of determinantal Godeaux surfaces over the punctured disk, this family determines an associated family $\mathcal{T}^*$ of canonical curves

$$\mathcal{T}^* \subseteq \Delta^* \times \mathbb{P}^1 \times S_0 \subseteq \Delta^* \times \mathbb{P}^1 \times \mathbb{P}^3.$$

The relative dualizing sheaf of $\mathcal{T}^*$ over Δ^* is

$$\mathcal{O}_{\mathbb{P}^1}(3) \otimes \mathcal{O}_{\mathbb{P}^3}(1)$$

by Theorem 10, and by Lemma 9, the relative dualizing sheaf is

$$\mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{O}_{\mathbb{P}^3}(1).$$

Then we compactify $\Delta^* \times \mathbb{P}^1$ in the above space of twisted cubics and use the relative dualizing sheaf to compute the dualizing sheaf of the compactified families

$$\mathcal{S} \to \mathcal{T} \to \Delta.$$

We intend to study this compactification in detail in a future work.

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Korea Institute for Advanced Study, 207-43 Cheongryangri-dong, Dongdaemun-gu, Seoul 130-012, Korea

Current address: Department of Mathematics, Sogang University, Sinsu-dong, Mapo-gu, Seoul 121-742, Korea

E-mail address: `ynlee@ccs.sogang.ac.kr`