TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 352, Number 11, Pages 4911–4936
S 0002-9947(00)02480-6
Article electronically published on June 28, 2000

TRACES ON ALGEBRAS OF PARAMETER DEPENDENT PSEUDODIFFERENTIAL OPERATORS AND THE ETA–INVARIANT

MATTHIAS LESCH AND MARKUS J. PFLAUM

ABSTRACT. We identify Melrose's suspended algebra of pseudodifferential operators with a subalgebra of the algebra of parametric pseudodifferential operators with parameter space $\mathbb{R}$. For a general algebra of parametric pseudodifferential operators, where the parameter space may now be a cone $\Gamma \subset \mathbb{R}^p$, we construct a unique "symbol valued trace", which extends the L^2–trace on operators of small order. This construction is in the spirit of a trace due to Kontsevich and Vishik in the nonparametric case. Our trace allows us to construct various trace functionals in a systematic way. Furthermore, we study the higher–dimensional eta–invariants on algebras with parameter space $\mathbb{R}^{2k-1}$. Using Clifford representations we construct for each first order elliptic differential operator a natural family of parametric pseudodifferential operators over $\mathbb{R}^{2k-1}$. The eta–invariant of this family coincides with the spectral eta–invariant of the operator.

1. INTRODUCTION

Let M be a smooth compact Riemannian manifold without boundary. Furthermore, let E be a hermitian vector bundle over M. We denote by $\mathrm{CL}^*(M,E)$ the algebra of classical pseudodifferential operators acting on $L^2(M,E)$. It is well–known that up to a scalar factor $\mathrm{CL}^*(M,E)$ has a unique trace, the residue trace of Guillemin [8] and Wodzicki [20]. The residue trace vanishes on trace class operators and hence there is no trace on $\mathrm{CL}^*(M,E)$ which extends the L^2-trace. However, Kontsevich and Vishik [9, 10] constructed a functional

$$\mathrm{TR} : \bigcup_{\alpha \in \mathbb{R}\setminus\mathbb{Z}} \mathrm{CL}^\alpha(M,E) \to \mathbb{C},$$

which extends the L^2-trace and satisfies $\mathrm{TR}(AB) = \mathrm{TR}(BA)$ for $A, B \in \mathrm{CL}^*(M,E)$ with $\mathrm{ord}(A) + \mathrm{ord}(B) \notin \mathbb{Z}$. TR has further interesting properties (cf. [9, 10], [12, Sec. 5]).

In this paper we study traces on the algebra of parameter dependent pseudodifferential operators $\mathrm{CL}^*(M,E,\Gamma)$, where $\Gamma \subset \mathbb{R}^p$ is a conic set. These algebras play an important role in the study of the resolvent of an elliptic differential operator in which case Γ is a sector in $\mathbb{C}$ (cf. [19]).

Received by the editors September 15, 1998 and, in revised form, November 1, 1998.
2000 *Mathematics Subject Classification.* Primary 58G15.

Our first result shows that $\mathrm{CL}^*(M,E,\mathbb{R})$ contains a canonical isomorphic image of the algebra $\mathrm{CL}^*_{\mathrm{sus}}(M,E)$ introduced by Melrose [15]. This algebra appears naturally in an index theorem for manifolds with boundary [17, Sec. 12]. It should be thought of as pseudodifferential suspension of the algebra $\mathrm{CL}^*(M,E)$.

More generally, we then study the algebra $\mathrm{CL}^*(M,E,\Gamma)$ for a connected cone $\Gamma\subset\mathbb{R}^p$ with nonempty interior. Our main result is that for $H^1_{\mathrm{dR}}(\Gamma)\neq 0$ the algebra $\mathrm{CL}^*(M,E,\Gamma)$ has a unique "symbol valued trace". More precisely, there exists a unique linear map

$$\mathrm{TR}:\mathrm{CL}^*(M,E,\Gamma)\to \mathrm{PS}^*(\Gamma)/\mathbb{C}[\mu_1,...,\mu_p]$$

with the following properties:

(i) $\mathrm{TR}(AB)=\mathrm{TR}(BA)$, i.e. TR is a "trace",

(ii) $\mathrm{TR}(\partial_j A)=\partial_j\,\mathrm{TR}(A),\quad j=1,...,p,$

(iii) If $A\in\mathrm{CL}^m(M,E,\Gamma)$ and $m+\dim M<0$, then

$$\mathrm{TR}(A)(\mu)=\mathrm{tr}_{L^2}(A(\mu)).$$

Here $\mathrm{PS}^*(\Gamma)$ is the class of symbols having a complete asymptotic expansion in terms of homogeneous functions and log–powers.

So the parametric situation is different from the nonparametric one: the symbol valued L^2–trace can be extended, however, only modulo polynomials. TR should be viewed as the analogue of the Kontsevich-Vishik trace, since it can be constructed quite similarly (see Remark 5.4 below). Our main result allows us to construct various traces on the algebra $\mathrm{CL}^*(M,E,\Gamma)$ just by composing TR with linear functionals on $\mathrm{PS}^*(\Gamma)/\mathbb{C}[\mu_1,...,\mu_p]$.

The most important examples are the extended and the formal trace $\overline{\mathrm{Tr}}$ resp. $\widetilde{\mathrm{Tr}}$. For $A\in\mathrm{CL}^m(M,E,\mathbb{R}^p)$ the extended trace is given by

$$\overline{\mathrm{Tr}}(A)=\fint_{\mathbb{R}^p}\mathrm{TR}(A)(\mu)d\mu, \tag{1.1}$$

where $\fint_{\mathbb{R}^p}$ is a certain regularization of the integral (cf. 5.3). If $m+\dim M+p<0$, the function $\mathrm{TR}(A)$ is integrable, and

$$\overline{\mathrm{Tr}}(A)=\int_{\mathbb{R}^p}\mathrm{tr}_{L^2}(A(\mu))d\mu. \tag{1.2}$$

holds indeed.

From $\mathrm{CL}^*(M,E,\mathbb{R}^p)$ we can construct a de Rham complex $(\Omega^*\mathrm{CL}^*(M,E,\mathbb{R}^p),d)$ in a canonical way. Then $\overline{\mathrm{Tr}}$ extends to a graded trace on $\Omega^*\mathrm{CL}^*(M,E,\mathbb{R}^p)$

$$\overline{\mathrm{Tr}}(\omega):=\begin{cases}0, & \text{if } \deg\ \omega<p,\\ \overline{\mathrm{Tr}}(f), & \text{if } \omega=f dx_1\wedge\ldots\wedge dx_p.\end{cases} \tag{1.3}$$

However, the graded trace is not closed, but its derivative $\widetilde{\mathrm{Tr}}:=d\overline{\mathrm{Tr}}$ is a closed graded trace on $\Omega^*\mathrm{CL}^*(M,E,\mathbb{R}^p)$. If $p=1$, then $\overline{\mathrm{Tr}}$ and $\frac{1}{2\pi}\widetilde{\mathrm{Tr}}$ coincide with the corresponding traces introduced in [15].

Like in [15] $\widetilde{\mathrm{Tr}}$ is an analogue of the residue trace. It only depends on finitely many terms of the symbol expansion of the operator. One of the results of Melrose [15] was the construction of the eta–homomorphism. In our notation

$$\eta:\mathrm{CL}^*(M,E,\mathbb{R})^{-1}\longrightarrow\mathbb{C} \tag{1.4}$$

is a homomorphism from the group of invertible elements of $\mathrm{CL}^*(M,E,\mathbb{R})$ into the additive group $\mathbb{C}$. In some sense η generalizes the winding number. Namely, for $A \in \mathrm{CL}^*(M,E,\mathbb{R})^{-1}$ one has

$$\eta(A) = \frac{1}{\pi i}\overline{\mathrm{Tr}}(A^{-1}dA) = \frac{1}{\pi i}\int\!\!\!\!\!-_{\mathbb{R}} \mathrm{TR}(A^{-1}A')(x)dx. \tag{1.5}$$

In case A is a function on $\mathbb{R}$ taking values in the space of invertible matrices and which is constant outside a compact set, then $\frac{1}{2}\eta(A)$ is an integer equal to the winding number of A. Thus it is natural to expect a similar invariant for odd–dimensional parameter spaces. Indeed for $A \in \mathrm{CL}^*(M,E,\mathbb{R}^{2k-1})^{-1}$ we put

$$\eta_k(A) := 2c_k\overline{\mathrm{Tr}}((A^{-1}dA)^{2k-1}), \tag{1.6}$$

where c_k is a normalization constant. Again, if A is just a matrix valued function and constant outside a compact set, (1.6) is an even integer which actually classifies the $(2k-1)$th homotopy group of $\mathrm{GL}(\infty,\mathbb{C})$.

In contrast to its finite–dimensional analogue η_k is not a homotopy invariant. However, its variation is local which means for a smooth family A_s of invertible elements the equality

$$\frac{d}{ds}\eta_k(A_s) = 2(2k-1)c_k\widetilde{\mathrm{Tr}}\left((A_s^{-1}\partial_s A_s)(A_s^{-1}dA_s)^{2k-2}\right) \tag{1.7}$$

holds true. Unfortunately η_k is not a homomorphism for $k \geq 2$, instead we have

$$\eta_k(AB) - \eta_k(A) - \eta_k(B) = \widetilde{\mathrm{Tr}}(\omega(A,B)), \tag{1.8}$$

where $\omega(A,B)$ denotes a universal polynomial in the 1–forms $B^{-1}(A^{-1}dA)B$ and $B^{-1}dB$. So the defect of the additivity is a symbolic term.

Finally, we compare η_k with the spectral eta–invariant. For any first order invertible self–adjoint elliptic differential operator D we construct a natural family $\mathcal{D}(\mu) := D + c(\mu)$ in $\mathrm{CL}^1(M,E,\mathbb{R}^{2k-1})$ such that

$$\eta_k(\mathcal{D}) = -\eta(D), \tag{1.9}$$

where c is the standard Clifford representation and $\eta(D)$ the spectral eta–invariant of D.

We understand that some of our results also have been obtained by R. B. Melrose and V. Nistor [18].

Acknowledgement. The first-named author gratefully acknowledges the hospitality and financial support of the Erwin–Schrödinger Institute, Vienna, where part of this work was completed. Furthermore, the first-named author was supported by Deutsche Forschungsgemeinschaft.

2. Review of parametric pseudodifferential operators

The concept of parameter dependent symbols and pseudodifferential operators used in this article involves several different classes of symbol spaces. For the convenience of the reader and to fix the notation we briefly recall some basic facts about symbols and the corresponding operator calculus. As general references we mention the books Shubin [19] and Grigis-Sjøstrand [6].

A conic manifold is a smooth principal fiber bundle $\Gamma \to B$ with structure group $\mathbb{R}_+ := (0,\infty)$. It is always trivializable (cf. Duistermaat [4], §2.1). A subset $\Gamma \subset \dot{\mathbb{R}}^\nu := \mathbb{R}^\nu \setminus \{0\}$ which is a conic manifold by the natural $\mathbb{R}_+$-action on $\dot{\mathbb{R}}^\nu$ is called a conic set. The base manifold of a conic set $\Gamma \subset \dot{\mathbb{R}}^\nu$ is isomorphic to

$S\Gamma := \Gamma \cap S^{\nu-1}$. By a cone $\Gamma \subset \mathbb{R}^\nu$ we will always mean a conic set or the closure of a conic set in $\mathbb{R}^\nu$ such that Γ has nonempty interior. Thus $\mathbb{R}^n$ and $\dot{\mathbb{R}}^n$ are cones, but only the latter is a conic set.

Now let M be a smooth manifold, $m \in \mathbb{R}$, $\Gamma \subset \mathbb{R}^\nu$ a cone, and $0 < \rho \leq 1$. Then by $\mathrm{S}^m_\rho(X, \Gamma)$ we denote the space of all functions $a(x, \xi) \in C^\infty(M \times \Gamma)$ such that for every differential operator D on M, all compact $L \subset \Gamma$ and $K \subset M$, we have the uniform estimate

$$\left|D\, \partial_\xi^\alpha a(x,\xi)\right| \leq C_{K,L,D,\alpha}\, \langle \xi \rangle^{m-\rho|\alpha|}, \quad x \in K,\ \xi \in L^{\mathrm{c}},\ \alpha \in \mathbb{Z}_+^\nu. \tag{2.1}$$

Here, $L^{\mathrm{c}} := \{t\xi \,|\, \xi \in L, t \geq 1\}$, $C_{K,L,D,\alpha} > 0$ and $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$ for all $\xi \in \mathbb{R}^\nu$. In case $\rho = 1$ we write $\mathrm{S}^m(M, \Gamma)$ for $\mathrm{S}^m_\rho(M, \Gamma)$.

A symbol $a \in \mathrm{S}^m(M, \Gamma)$ is called classical polyhomogeneous of degree m or just classical, if it has an asymptotic expansion of the form $a \sim \sum_{j \geq 0} a_{m-j}$, where the $a_k(x, \xi) \in \mathrm{S}^k(M, \Gamma)$ are k-homogeneous in ξ of degree k. The space of classical polyhomogeneous symbols of order m is denoted by $\mathrm{CS}^m(M, \Gamma)$.

Now let $U \subset \mathbb{R}^n$ be an open set, and $a \in \mathrm{S}^m_\rho(U, \mathbb{R}^n \times \Gamma)$. For each fixed μ_0 we have $a(\cdot, \cdot, \mu_0) \in \mathrm{S}^m(U, \mathbb{R}^n)$, hence we obtain a family of pseudodifferential operators parametrized over Γ by putting

$$\begin{aligned} &\left[\mathrm{Op}(a(\mu))\, u\right](x) \\ &\quad := \left[A(\mu)\, u\right](x) := \int_{\mathbb{R}^n} \mathrm{e}^{i\langle x, \xi\rangle}\, a(x, \xi, \mu)\, \hat{u}(\xi)\, đ\xi, \quad đ\xi := (2\pi)^{-n} d\xi. \end{aligned} \tag{2.2}$$

Note that for $a \in \mathrm{S}^{-\infty}(U, \mathbb{R}^n \times \Gamma)$ the operator $A(\mu)$ has a kernel which lies in $\mathcal{S}(\Gamma, C^\infty(U \times U))$, the Schwartz space of $C^\infty(U \times U)$-valued functions. We denote by $\mathrm{L}^m_\rho(U, \Gamma)$ the set of all $A(\mu)$, where $a \in \mathrm{S}^m_\rho(U, \mathbb{R}^n \times \Gamma)$. In case $\Gamma = \{0\}$ we obtain the well-known space $\mathrm{L}^m_\rho(U)$ of pseudodifferential operators of order m and type ρ on $U \subset \mathbb{R}^n$.

For a smooth manifold M and vector bundles E, F over M we define the spaces $\mathrm{L}^m(M, E, F; \Gamma)$ of parameter dependent pseudodifferential operators between sections of E, F in the usual way by patching together local data.

The space of parameter dependent pseudodifferential operators with symbol lying in the space $\mathrm{CS}^m(U, \mathbb{R}^n \times \Gamma)$ will be denoted by $\mathrm{CL}^m(U, \Gamma)$. Its elements are the classical parameter dependent pseudodifferential operators over U. Following Grubb and Seeley [7] we also call these operators strongly polyhomogeneous.

Example 2.1. Let M be a compact manifold, $A \in \mathrm{CL}^m(M)$, and assume that $\Gamma \subset \mathbb{C} \setminus \{0\}$ is a cone such that $\sigma^m_A(x, \xi) - z$ is invertible for $z \in \Gamma$. If A is a differential operator and $\operatorname{spec} A \cap \Gamma = \emptyset$, then $(A - z)^{-1} \in \mathrm{L}^{-m}(M, \Gamma)$. However, in general this need not be true for pseudodifferential operators.

The following result is just a mild generalization of the classical resolvent expansion of a differential operator (see e.g. [5, Sec. 1.7]).

Theorem 2.2. *Let M be a compact manifold,* $\dim M =: n$*, and* $A \in \mathrm{L}^m(M, E, \Gamma)$ *(resp.* $\mathrm{CL}^m(M, E, \Gamma)$*). If* $m + \dim M < 0$*, then* $A(\mu)$ *is trace class for all* $\mu \in \Gamma$ *and*

$$\mathrm{tr}\ A(\cdot) \in \mathrm{S}^{m+\dim M}(\Gamma) \quad (\textit{resp.}\ \mathrm{CS}^{m+\dim M}(\Gamma)).$$

Proof. We present the proof for $A \in \mathrm{CL}$. For $A \in \mathrm{L}$ it is even a bit simpler. Choosing a suitable partition of unity it suffices to prove the claim for $E = \mathbb{C}$,

$M = U$ a coordinate patch, and A compactly supported, i.e.

$$(Au)(x) = \int_{\mathbb{R}^n} \sigma_A(x,\xi,\mu)\,\hat{u}(\xi)\,đ\xi = \int_{\mathbb{R}^n}\int_U \mathrm{e}^{i\langle x-y,\xi\rangle}\,\sigma_A(x,\xi,\mu)\,u(y)\,dy\,đ\xi, \tag{2.3}$$

where $\sigma_A \in \mathrm{CS}^m(U, \mathbb{R}^n \times \Gamma)$ and $\pi_1(\operatorname{supp}\sigma_A(\cdot,-,\mu)) \subset K \subset U$ is compact for every $\mu \in \Gamma$. For fixed μ we have $A(\mu) \in \mathrm{CL}^m(U)$, hence $A(\mu)$ is trace class since $m < -\dim M = -n$. Since $\sigma_A \in \mathrm{CS}^m(U, \mathbb{R}^n \times \Gamma)$ we have

$$\sigma_A \sim \sum_{j=0}^{\infty} a_{m-j} \tag{2.4}$$

with $a_{m-j}(x, \lambda\xi, \lambda\mu) = \lambda^{m-j}\, a_{m-j}(x,\xi,\mu)$ for $\lambda \geq 1$, $|(\xi,\mu)| \geq 1$. Thus we write

$$\sigma_A = \sum_{j=0}^{N-1} a_{m-j} + R_N \tag{2.5}$$

with $R_N \in \mathrm{CS}^{m-N}(U, \mathbb{R}^n \times \Gamma)$. Now pick $L \subset \Gamma$ compact and a multi-index α. Then

$$\begin{aligned} \left|\partial_\mu^\alpha \operatorname{tr}\, \mathrm{Op}(R_N(\mu))\right| &= \left|\int_K\int_{\mathbb{R}^n} \partial_\mu^\alpha R_N(x,\xi,\mu) đ\xi\, dx\right| \\ &\leq C_{\alpha,K,L}\int_K\int_{\mathbb{R}^n} (1 + (|\xi|^2 + |\mu|^2)^{1/2})^{m-|\alpha|-N}\, đ\xi\, dx \\ &\leq C_{\alpha,K,L}(1+|\mu|)^{m+n-|\alpha|-N}. \end{aligned} \tag{2.6}$$

Furthermore, let $\lambda \geq 1$, $|\mu| \geq 1$. Then

$$\begin{aligned} \operatorname{tr}\mathrm{Op}(a_{m-j}(\lambda\mu)) &= \int_U\int_{\mathbb{R}^n} a_{m-j}(x,\xi,\lambda\mu)\, đ\xi\, dx \\ &= \lambda^{m-j}\int_U\int_{\mathbb{R}^n} a_{m-j}(x,\lambda^{-1}\xi,\mu)\, đ\xi\, dx \\ &= \lambda^{m+n-j}\operatorname{tr}\mathrm{Op}(a_{m-j}(\mu)), \end{aligned} \tag{2.7}$$

and similar to (2.6) one shows that $\operatorname{tr}(\mathrm{Op}(a_{m-j})) \in \mathrm{S}^{m-j+n}(\Gamma)$, thus

$$\operatorname{tr} A(\mu) \sim \sum_{j=0}^{\infty} \operatorname{tr}\mathrm{Op}(a_{m-j}(\mu)) \tag{2.8}$$

in $\mathrm{CS}^{m+n}(\Gamma)$, where $\operatorname{tr}\mathrm{Op}(a_{m-j}(\mu))$ is homogeneous of degree $m+n-j$ for $\mu \geq 1$. □

The previous proof provides even more, namely

Theorem 2.3. *Let M be a smooth manifold. If $m + \dim M < 0$, then for any properly supported $A \in \mathrm{L}^m(M,E,F;\Gamma)$ (resp. $\mathrm{CL}^m(M,E,F;\Gamma)$) there is a density*

$$\omega_A \in \mathrm{S}^m(M, \mathrm{Hom}(E,F)\otimes|\Omega M|;\Gamma) \quad (\textit{resp.}\ \mathrm{CS}^m(M,\mathrm{Hom}(E,F)\otimes|\Omega M|;\Gamma))$$

with the following properties:

1. *For any local chart $\phi : M \supset U \to \phi(U) \subset \mathbb{R}^n$ we have*

$$\phi^*\omega_{\phi_* A} = \omega_A.$$

2. *If $E = F$, then*

$$\operatorname{tr} A(\mu) = \int_M \operatorname{tr}_{E_x} \omega_A(x, \mu).$$

Proof. The proof of Theorem 2.2 shows that we can put

$$\omega_A(x, \mu) = \int_{\mathbb{R}^n} \sigma_A(x, \xi, \mu)\, đ\xi\, |dx|.$$

Then 1 and 2 follow easily. □

Remark 2.4. For the preceding two theorems the assumption $m + \dim M < 0$ was essential. However, in this paper we will show that these theorems can be extended to arbitrary parametric operators.

3. Melrose's suspended algebra of pseudodifferential operators

In the paper [15] R. B. Melrose invented a "suspended" algebra of pseudodifferential operators on a compact manifold. He introduced trace functionals on this algebra and constructed the "η–homomorphism". In this section we will briefly recall the definition of the suspended algebra and we will show that it is isomorphic to a subalgebra of $\mathrm{CL}^*(M, \mathbb{R})$.

In the subsequent sections we will construct trace functionals on $\mathrm{CL}^*(M, \Gamma)$ which generalize the Melrose traces.

Let M be a compact manifold. Following [15] $\mathrm{CL}^m_{\mathrm{sus}}(M)$ consists of those operators $A \in \mathrm{CL}^m(M \times \mathbb{R})$ such that

(i) A acts as convolution in the second variable, i.e. by slight abuse of notation

$$(Au)(x,t) = \int_M \int_{\mathbb{R}} K_A(x, y, t-s) u(y, s)\, ds dy, \tag{3.1}$$

where K_A denotes the convolution kernel of A.

(ii) The kernel satisfies

$$K_A \in C_0^\infty(M \times M \times \mathbb{R}; |\Omega M| \boxtimes 1)' + \mathcal{S}(M^2 \times \mathbb{R}; 1 \boxtimes |\Omega M|), \tag{3.2}$$

where $C_0^\infty(M \times M \times \mathbb{R}; |\Omega M| \boxtimes 1)'$ denotes the space of distributions which act on the smooth compactly supported sections of the exterior tensor product of the density bundle $|\Omega M|$ with the trivial line bundle over $M \times \mathbb{R}$.

For vector bundles E and F, $\mathrm{CL}^m_{\mathrm{sus}}(M, E, F)$ is defined accordingly. $\mathrm{CL}^*_{\mathrm{sus}}(M, E)$ is an order filtered algebra. For $A \in \mathrm{CL}^*_{\mathrm{sus}}(M, E)$ Melrose introduced what he calls the indicial family $\widehat{A}$. This is the partial Fourier transform in the t–variable. Namely, for $\mu \in \mathbb{R}$ we obtain a pseudodifferential operator $\widehat{A}(\mu) \in \mathrm{CL}^m(M, E)$ by putting

$$\widehat{A}(\mu) g = e^{-it\mu} A(e^{i\mu(\cdot)} g)(\cdot, t), \quad g \in C^\infty(M, E). \tag{3.3}$$

Proposition 3.1. *The map*

$$\mathrm{CL}^*_{\mathrm{sus}}(M, E) \longrightarrow \mathrm{CL}^*(M, E, \mathbb{R}), \quad A \mapsto \widehat{A}(\cdot)$$

is an order preserving injective homomorphism of $*$-algebras.

Proof. We only have to check that for $A \in \mathrm{CL}^m_{\mathrm{sus}}(M,E)$ we have

$$\widehat{A}(\cdot) \in \mathrm{CL}^m(M,E,\mathbb{R}).$$

It suffices to check this locally for $E = \mathbb{C}$. Let U be a coordinate patch in M. Then for $u \in C_0^\infty(U \times \mathbb{R})$ we have

$$(Au)(x,t) = \int_{U\times\mathbb{R}} \sigma_A((x,t),(\xi,\mu))\hat{u}(\xi,\mu)e^{i(\langle x,\xi\rangle + t\mu)}đ\mu đ\xi, \tag{3.4}$$

with $\sigma_A \in \mathrm{CS}^m(U\times\mathbb{R},\mathbb{R}^n\times\mathbb{R})$. In view of (3.1), σ_A is independent of t, hence $\sigma_A(x,\xi,\mu) \in \mathrm{CS}^m(U,\mathbb{R}^n\times\mathbb{R})$ and this is the complete symbol of $\widehat{A}(\mu)$. $\square$

Summing up the suspended algebra can be viewed as an algebra of strongly polyhomogeneous parameter dependent pseudodifferential operators. We did not try to express (ii) in terms of the indicial family $\widehat{A}$. However, it turns out that the extended trace and the η–homomorphism can be constructed without using (ii). So one could equally well consider the algebra $\mathrm{CL}^*(M,E,\mathbb{R})$ as $\mathrm{CL}^*_{\mathrm{sus}}(M,E)$.

There is no reason to restrict the consideration to $\mathbb{R}$ as a parameter space. Therefore, we will deal with $\mathrm{CL}^*(M,\Gamma)$ in the sequel.

4. Tracial maps on $\mathrm{L}^*(M,E,\Gamma)$

During the whole section let $\Gamma \subset \mathbb{R}^p$ be a *connected* cone with nonempty interior.

Definition 4.1. We define the following spaces of functions on Γ:

$$\mathcal{P}^m = \mathcal{P}^m(\Gamma) := \{P{\restriction}\Gamma \mid P \in \mathbb{C}[x_1,\dots,x_p], \deg P \le m\}, \quad m \in \mathbb{R}_+,$$

$$\tilde{\mathrm{S}}^m(\Gamma) := \begin{cases} \mathrm{S}^m(\Gamma), \quad m \notin \mathbb{Z}_+, \\ \{f \in \bigcap_{\varepsilon>0} \mathrm{S}^{m+\varepsilon}(\Gamma) \mid \partial^\alpha f \in \mathrm{S}^{m-|\alpha|}(\Gamma), |\alpha| \ge m+1\}, \quad m \in \mathbb{Z}_+. \end{cases}$$

Note that $\tilde{\mathrm{S}}^*(\Gamma) = \bigcup_{m\in\mathbb{R}} \tilde{\mathrm{S}}^m(\Gamma) = \mathrm{S}^*(\Gamma)$.

Since Γ is assumed to be connected with nonempty interior, the restriction map $C^\infty(\mathbb{R}^p) \to C^\infty(\Gamma), f \mapsto f{\restriction}\Gamma$ induces an isomorphism $\mathcal{P}^m(\mathbb{R}^p) \to \mathcal{P}^m(\Gamma)$. This justifies the notation $\mathcal{P}^m$ for $\mathcal{P}^m(\Gamma)$.

We have the obvious inclusion

$$\mathcal{P}^m \subset \tilde{\mathrm{S}}^m(\Gamma). \tag{4.1}$$

Moreover, if $m < 0$, then

$$\mathrm{S}^m(\Gamma) \cap \mathcal{P} = \{0\}. \tag{4.2}$$

If $\mathcal{A} \subset C^\infty(\Gamma)$ is a vector space which is closed under ∂_j, $j = 1,...,p$, we put

$$\Omega^l\mathcal{A} := \Big\{ \sum f_I dx_I \in \Omega^l(\Gamma) \,\Big|\, f_I \in \mathcal{A}\Big\}. \tag{4.3}$$

If $\mathcal{A}$ is graded, then $\Omega^l\mathcal{A}$ is bigraded, namely

$$\Omega^l\mathcal{A}^m := \Big\{ \sum f_I dx_I \in \Omega^l(\Gamma) \,\Big|\, f_I \in \mathcal{A}^m\Big\}. \tag{4.4}$$

Since $\mathcal{A}$ is closed under ∂_j, $j = 1,...,p$, the exterior derivative maps $\Omega^l\mathcal{A}$ into $\Omega^{l+1}\mathcal{A}$, hence we obtain a complex $(\Omega^*\mathcal{A}, d)$. Obvious examples are $\mathrm{S}^*(\Gamma), \tilde{\mathrm{S}}^*(\Gamma), \mathcal{P}^*(\Gamma)$. Note that $d(\Omega^l\mathcal{P}^m) \subset \Omega^{l+1}\mathcal{P}^{m-1}$.

Lemma 4.2. *The homology of the complex* $(\Omega^*\mathcal{P}, d)$ *is given by*

$$H^l(\Omega^*\mathcal{P}, d) = \begin{cases} \mathbb{C}, & l = 0, \\ 0, & l \geq 1. \end{cases}$$

More precisely, if $\omega \in \Omega^l\mathcal{P}^m$ *with* $l \geq 1$ *is closed, then there exists* $\eta \in \Omega^{l-1}\mathcal{P}^{m+1}$ *with* $d\eta = \omega$.

Proof. $H^0(\Omega^*\mathcal{P}, d) = \mathbb{C}$ is obvious. We mimick the usual proof of the Poincaré Lemma (cf. [2, Sec. I.4]) and proceed by induction on $p = \dim \Gamma$. If $p = 1$ and $\omega = f(x)dx, f \in \mathcal{P}^m$, then we put $\eta(x) := \int_0^x f(t)dt \in \mathcal{P}^{m+1}$. Hence the assertion is true for $p = 1$.

Next we consider

$$\mathbb{R}^p \times \mathbb{R} \underset{s}{\overset{\pi}{\rightleftarrows}} \mathbb{R}^p, \quad \pi(x,t) = x, \ s(x) = (x, 0). \tag{4.5}$$

Obviously,

$$\begin{aligned} &\pi^*(\Omega^l\mathcal{P}^m(\mathbb{R}^p)) \subset \Omega^l\mathcal{P}^m(\mathbb{R}^p \times \mathbb{R}), \\ &s^*(\Omega^l\mathcal{P}^m(\mathbb{R}^p \times \mathbb{R})) \subset \Omega^l\mathcal{P}^m(\mathbb{R}^p) \end{aligned} \tag{4.6}$$

holds. Since $\pi \circ s = \mathrm{id}_{\mathbb{R}^p}$ we have $s^* \circ \pi^* = \mathrm{id}_{\Omega^l\mathcal{P}^m(\mathbb{R}^p)}$. An inspection of the construction of the usual homotopy operator K (cf. [2, Sec. I.4]) shows that it induces a map

$$K : \Omega^l\mathcal{P}^m(\mathbb{R}^p \times \mathbb{R}) \longrightarrow \Omega^{l-1}\mathcal{P}^{m+1}(\mathbb{R}^p \times \mathbb{R}). \tag{4.7}$$

Furthermore, K satisfies the identity

$$\mathrm{id} - \pi^* \circ s^* = dK + Kd \tag{4.8}$$

from which the assertion follows immediately. □

Proposition 4.3. *Let* $\omega \in \Omega^1\tilde{\mathrm{S}}^m(\Gamma), d\omega = 0$. *If the cohomology class of* ω *in* $H^1_{\mathrm{dR}}(\Gamma)$ *vanishes, then there exists* $F \in \tilde{\mathrm{S}}^{m+1}(\Gamma)$ *with* $dF = \omega$. *If* $m + 1 < 0$, *then* F *is uniquely determined.*

Proof. It is clear by assumption that ω has a primitive in $C^\infty(\Gamma)$. The point is to prove that one of its primitives already lies in $\tilde{\mathrm{S}}^{m+1}(\Gamma)$. We write

$$\omega =: \sum_{j=1}^p h_j dx_j.$$

If $m+1 < 0$, then the constant functions do not belong to $\tilde{\mathrm{S}}^{m+1}(\Gamma)$ which proves the uniqueness statement.

If $m + 1 > 0$, we fix $x_0 \in \Gamma$ and put

$$F(x) := \int_{x_0}^x \omega, \tag{4.9}$$

where $\int_{x_0}^x$ denotes integration along any path from x_0 to x. This makes sense since $[\omega]_{H^1_{\mathrm{dR}}(\Gamma)} = 0$. Since $\partial_j F = h_j \in \tilde{\mathrm{S}}^m(\Gamma)$, it suffices to prove the estimate

$$|F(x)| \leq C_{\varepsilon,L}(1 + |x|)^{1+\varepsilon+m}, \quad x \in L^c, \tag{4.10}$$

for compact $L \subset \Gamma$ and $\varepsilon > 0$ (resp. $\varepsilon = 0$ if $m + 1 \notin \mathbb{Z}_+$).

Since $h_j \in \tilde{\mathrm{S}}^m(\Gamma)$, we have

$$|h_j(x)| \le C_{\varepsilon,L}(1+|x|)^{m+\varepsilon}$$

and thus

$$|F(x)| = \big|F(\frac{x}{|x|}) + \int_{\frac{x}{|x|}}^{x} \omega\big| \le C + \big|\int_{\frac{x}{|x|}}^{x} \omega\big|.$$

Now

$$\int_{\frac{x}{|x|}}^{x} \omega = \sum_{j=1}^{p} \int_{1/|x|}^{1} h_j(tx)x_j dt$$

and consequently

$$\begin{aligned}\big|\int_{\frac{x}{|x|}}^{x} \omega\big| &\le C_{\varepsilon,L} \int_{1/|x|}^{1} (1+t|x|)^{m+\varepsilon}|x|dt \\ &\le C_{\varepsilon,L}(1+|x|)^{m+\varepsilon+1}.\end{aligned}$$

If $m=-1$, then $h_j(x) \le C(1+|x|)^{-1}$. Similarly,

$$\begin{aligned}|F(x)| &\le C(1+\int_{1/|x|}^{1}(1+t|x|)^{-1}|x|dt) \le C_1 + C_2\big|\log|x|\big| \\ &\le C_3(1+|x|)^{\varepsilon}\end{aligned}$$

holds for any $\varepsilon > 0$, hence we reach the conclusion in this case.

It remains to consider the case $m+1<0$. Then we put

$$F(x) := -\sum_{j=1}^{p} \int_{1}^{\infty} h_j(tx)x_j dt. \tag{4.11}$$

Since $m+1<0$, we may differentiate under the integral. Taking $\partial_j h_l = \partial_l h_j$ into account we find $\partial_j F = h_j$. Moreover, one easily checks

$$|F(x)| \le C_L(1+|x|)^{m+1},$$

which implies $F \in \tilde{\mathrm{S}}^{m+1}(\Gamma)$. □

The partial derivatives ∂_j, $j=1,...,p$, are well–defined on the quotient space $\tilde{\mathrm{S}}^*(\Gamma)/\mathcal{P}$, hence we can form the complex $(\Omega^*(\tilde{\mathrm{S}}^*(\Gamma)/\mathcal{P}), d)$, which is obviously isomorphic to the quotient complex $(\Omega^*\tilde{\mathrm{S}}^*(\Gamma), d)/(\Omega^*\mathcal{P}, d)$.

Proposition 4.4. *Let $\Gamma \subset \mathbb{R}^p$ be a connected cone. For $\omega \in \Omega^1(\tilde{\mathrm{S}}^m(\Gamma)/\mathcal{P})$, $d\omega = 0$, $[\omega]_{H^1_{\mathrm{dR}}(\Gamma)} = 0$ there exists $F \in \tilde{\mathrm{S}}^{m+1}(\Gamma)/\mathcal{P}$ with $dF = \omega$. F is the unique element in $\tilde{\mathrm{S}}^*(\Gamma)/\mathcal{P}$ with $dF = \omega$.*

Remark 4.5. Here $[\omega]_{H^1_{\mathrm{dR}}(\Gamma)} = 0$ means that $[\omega_1]_{H^1_{\mathrm{dR}}(\Gamma)} = 0$ for a closed representative $\omega_1 \in \Omega^1\tilde{S}^m(\Gamma)$ of ω. In view of Lemma 4.2 we then have $[\omega_2]_{H^1_{\mathrm{dR}}(\Gamma)} = 0$ for any closed representative ω_2 of ω.

Proof. We first prove that $\omega = \sum_{j=1}^{p} f_j dx_j$ has a closed representative in $\Omega^1\tilde{\mathrm{S}}^m(\Gamma)$. Namely, pick representatives $g_j \in \tilde{\mathrm{S}}^m(\Gamma)$ of f_j and put

$$\tilde{\omega} = \sum_{j=1}^{p} g_j dx_j.$$

Since $d\omega = 0$ we have $d\tilde{\omega} \in \Omega^2\mathcal{P}^{m-1}$. Since $d\tilde{\omega}$ is closed, in view of Lemma 4.2 there exists $\eta \in \Omega^1\mathcal{P}^m$ with $d\eta = d\tilde{\omega}$. Thus $\omega_1 := \tilde{\omega} - \eta$ is a closed representative of ω.

By Proposition 4.3 there exists $F_1 \in \tilde{\mathrm{S}}^{m+1}(\Gamma)$ with $dF_1 = \omega_1$. Then $F := F_1 \mod \mathcal{P}$ satisfies $dF = \omega_1 \mod \mathcal{P} = \omega$, which proves the existence of F.

If $\omega_2 \in \Omega^1\tilde{\mathrm{S}}^m(\Gamma)$ is another closed representative of ω and $F_2 \in \tilde{\mathrm{S}}^{m+1}(\Gamma), dF_2 = \omega_2$, then

$$d(F_1 - F_2) = \omega_1 - \omega_2 \in \Omega^1\mathcal{P}^m.$$

Since $d(\omega_1 - \omega_2) = d^2(F_1 - F_2) = 0$ we again invoke Lemma 4.2 and find a polynomial $P \in \mathcal{P}^{m+1}$ with $dP = \omega_1 - \omega_2$. Hence $d(F_1 - F_2 - P) = 0$ and thus $F_1 - F_2 = P + c \in \mathcal{P}^{m+1}$. This proves the uniqueness statement. ☐

Theorem 4.6. *Let M be a compact manifold, E a smooth vector bundle over M, and $\Gamma \subset \mathbb{R}^p$ a connected cone with nonempty interior and $H^1_{\mathrm{dR}}(\Gamma) = 0$. Then there exists a unique linear map*

$$\mathrm{TR} : \mathrm{L}^*(M, E, \Gamma) \to \mathrm{S}^*(\Gamma)/\mathcal{P}$$

with the following properties:

(i) $\mathrm{TR}(AB) = \mathrm{TR}(BA)$, *i.e.* TR *is a "trace".*
(ii) $\mathrm{TR}(\partial_j A) = \partial_j \mathrm{TR}(A), \quad j = 1, ..., p.$
(iii) *If $A \in \mathrm{L}^m(M, E, \Gamma)$ and $m + \dim M < 0$, then*

$$\mathrm{TR}(A)(\mu) = \mathrm{tr}_{L^2}(A(\mu)).$$

This unique TR *further satisfies:*

(iv) $\mathrm{TR}(\mu_j A) = \mu_j \mathrm{TR}(A), \quad j = 1, ..., p.$
(v) $\mathrm{TR}(\mathrm{L}^m(M, E, \Gamma)) \subset \tilde{\mathrm{S}}^{m+\dim M}(\Gamma)$.

Remark 4.7. By slight abuse of notation μ_j denotes the operator of multiplication by the j–th coordinate function. Note that ∂_j and μ_j is well–defined on the quotient $\tilde{\mathrm{S}}^*(\Gamma)/\mathcal{P}$ since both operators map $\mathcal{P}$ into itself. Furthermore,

$$\begin{aligned} \partial_j &: \mathrm{L}^m(M, E, \Gamma) \to \mathrm{L}^{m-1}(M, E, \Gamma), \\ \mu_j &: \mathrm{L}(M, E, \Gamma) \to \mathrm{L}^{m+1}(M, E, \Gamma). \end{aligned} \tag{4.12}$$

Proof. Uniqueness: Assume there are T_1, T_2 satisfying (i)–(iii). By (iii) T_1 and T_2 coincide on $\mathrm{L}^m(M, E, \Gamma)$ for $m < -\dim M$. By induction assume that T_1, T_2 coincide on $\mathrm{L}^{m_0}(M, E, \Gamma)$ and let $A \in \mathrm{L}^m(M, E, \Gamma), m \le m_0 + 1$, be given.

Consider the 1–form

$$\omega := \sum_{j=1}^p T_1(\partial_j A)(\mu) d\mu_j = \sum_{j=1}^p T_2(\partial_j A)(\mu) d\mu_j \ \in \Omega^1(\tilde{\mathrm{S}}^{m-1}(\Gamma)/\mathcal{P}). \tag{4.13}$$

In view of (ii) ω is closed and we have

$$dT_1(A) = \omega = dT_2(A). \tag{4.14}$$

Hence Proposition 4.4 implies $T_1(A) = T_2(A)$.

Next we assume that we have the unique TR with (i)–(iii) and prove that it also satisfies (iv), (v).

If $m + 1 + \dim M < 0$ and $A \in \mathrm{L}^m(M, E, \Gamma)$, then by (iii)

$$\mathrm{TR}(\mu_j A)(\mu) = \mathrm{tr}_{L^2}(\mu_j A(\mu)) = \mu_j \,\mathrm{tr}_{L^2}(A(\mu)) = \mu_j \,\mathrm{TR}(A)(\mu) \tag{4.15}$$

and

$$\mathrm{TR}(A) = \mathrm{tr}_{L^2}(A(\cdot)) \in \tilde{\mathrm{S}}^{m+\dim M}(\Gamma) \tag{4.16}$$

in view of Theorem 2.2.

By induction we assume that (4.15) and (4.16) are true for $m \leq m_0$. Now pick $A \in \mathrm{L}^m(M, E, \Gamma), m \leq m_0 + 1$. Then

$$\begin{aligned} d\,\mathrm{TR}(\mu_j A) &= \sum_{l=1}^{p} \mathrm{TR}(\mu_j \partial_l A) d\mu_l + \mathrm{TR}(A) d\mu_j \\ &= \sum_{l=1}^{p} \mu_j \partial_l \,\mathrm{TR}(A) d\mu_l + \mathrm{TR}(A) d\mu_j \\ &= d(\mu_j \,\mathrm{TR}(A)) \end{aligned} \tag{4.17}$$

and again by Proposition 4.4 we find $\mathrm{TR}(\mu_j A) = \mu_j \,\mathrm{TR}(A)$.

Similarly, if $\mathrm{TR}(\partial_j A) \in \tilde{\mathrm{S}}^{m-1+\dim M}(\Gamma)/\mathcal{P}$, then

$$d\,\mathrm{TR}(A) = \sum_{j=1}^{p} \mathrm{TR}(\partial_j A) d\mu_j \tag{4.18}$$

implies $\mathrm{TR}(A) \in \tilde{\mathrm{S}}^{m+\dim M}(\Gamma)/\mathcal{P}$. This proves (v) again by induction.

Existence: Existence is also proved by induction using Proposition 4.4. Assume we have constructed TR on $\mathrm{L}^{m_0}(M, E, \Gamma)$. For $A \in \mathrm{L}^m(M, E, \Gamma), m \leq m_0 + 1$, we let $\mathrm{TR}(A) \in \tilde{\mathrm{S}}^{m+\dim M}(\Gamma)/\mathcal{P}$ be the unique primitive of the closed 1–form

$$\omega := \sum_{j=1}^{p} \mathrm{TR}(\partial_j A) d\mu_j \in \Omega^1(\tilde{\mathrm{S}}^{m-1+\dim M}(\Gamma)/\mathcal{P}). \tag{4.19}$$

Obviously, in this way we obtain a linear map $\mathrm{TR} : \mathrm{L}^{m_0+1}(M, E, \Gamma) \to \tilde{\mathrm{S}}^*(\Gamma)/\mathcal{P}$.

Now let $A \in \mathrm{L}^m(M, E, \Gamma), B \in \mathrm{L}^{m'}(M, E, \Gamma), m + m' \leq m_0 + 1$. Then

$$\begin{aligned} &d\big[\,\mathrm{TR}(AB) - \mathrm{TR}(BA)\big] \\ &= \sum_{j=1}^{p} \big[\,\mathrm{TR}((\partial_j A)B) + \mathrm{TR}(A\partial_j B) - \mathrm{TR}((\partial_j B)A) - \mathrm{TR}(B\partial_j A)\big] d\mu_j = 0, \end{aligned} \tag{4.20}$$

hence $\mathrm{TR}(AB) = \mathrm{TR}(BA)$ again by Proposition 4.4.

Since (ii) is obvious by construction, we reach the conclusion. □

Remark 4.8. Similarly to Theorem 2.3 TR is given by integration of a canonically defined "density". See Remark 5.4 below.

The construction of TR is much simpler, and more concrete, if Γ is star-shaped:

Proposition 4.9. *Let $\Gamma \subset \mathbb{R}^p$ be a star-shaped cone with star-point μ_0. Given $A \in \mathrm{L}^m(M, E, \Gamma)$ we have for fixed $\mu \in \Gamma$*

$$A(\mu) - \sum_{|\alpha| \leq N-1} \frac{(\partial_\mu^\alpha A)(\mu_0)}{\alpha!} (\mu - \mu_0)^\alpha \in \mathrm{L}^{m-N}(M, E), \tag{4.21}$$

and

$$\mathrm{TR}(A)\,(\mu) = \mathrm{tr}_{L^2} \Big(A(\mu) - \sum_{|\alpha| \leq N-1} \frac{(\partial_\mu^\alpha A)(\mu_0)}{\alpha!} (\mu - \mu_0)^\alpha \Big) \bmod \mathcal{P}, \tag{4.22}$$

where N is large enough.

Proof. Taylor's formula implies

$$\begin{aligned} &A(\mu) - \sum_{|\alpha|\le N-1} \frac{(\partial_\mu^\alpha A)(\mu_0)}{\alpha!}\,(\mu-\mu_0)^\alpha \\ &= \frac{1}{(N-1)!} \sum_{|\alpha|=N} \int_0^1 (1-t)^{N-1}\,(\partial_\mu^\alpha A)(\mu_0+t(\mu-\mu_0))\,dt\,(\mu-\mu_0)^\alpha \\ &\in \mathrm{L}^{m-N}(M,E), \end{aligned}$$

hence the right hand side of (4.22) is well-defined. Now it is easy to check that the right hand side of (4.22) defines a (well–defined, i.e. independent of N) map with the properties (i), (ii) and (iii) of Theorem 4.6. By uniqueness we reach the conclusion. $\square$

We tried hard to prove that for the algebra $\mathrm{CL}(M,\Gamma)$ the properties (i), (ii), and (iv) of Theorem 4.6 already determine TR up to a scalar factor. This would be nicer than assuming (iii), which prescribes TR on a large class of operators.

We state this as a conjecture:

Conjecture 4.10. *Let M be a compact manifold and $\Gamma \subset \mathbb{R}^p$ be a connected cone with nonempty interior and $H^1_{\mathrm{dR}}(\Gamma) = 0$. Let*

$$\tau : \mathrm{CL}^*(M,\Gamma) \to \tilde{\mathrm{S}}^*(\Gamma)/\mathcal{P}$$

be a linear map satisfying (i), (ii), *and* (iv) *of Theorem* 4.6*. Then there is a constant c such that $\tau = c\ \mathrm{TR}$.*

But we have a partial result:

Proposition 4.11. *Let M and Γ be as in the preceding conjecture. Let*

$$\tau : \mathrm{L}^{-\infty}(M,\Gamma) \to \mathrm{S}^{-\infty}(\Gamma)$$

be a linear map satisfying (i), (ii), *and* (iv) *of Theorem* 4.6*. Then there is a constant c such that $\tau = c\ \mathrm{TR}\restriction \mathrm{L}^{-\infty}(M,\Gamma)$, i.e. for $A \in \mathrm{L}^{-\infty}(M,\Gamma)$*

$$\tau(A)(\mu) = c\ \mathrm{tr}_{L^2}(A(\mu)).$$

Proof. τ is local, e.g. $\tau(A(\cdot))(\mu) = 0$ if $A(\mu) = 0$. To see this let $A(\mu_0) = 0$. We write

$$A(\mu) = \sum_{j=1}^p A_j(\mu)(\mu_j - \mu_{0,j})$$

with $A_j \in \mathrm{L}^{-\infty}(M,\Gamma)$. Then in view of (iv) we have $\tau(A)(\mu_0) = 0$.

Next we pick $\mu_0 \in \Gamma$ and choose a function $f \in C_0^\infty(\Gamma)$ with $f(\mu_0) = 1$. For $K \in \mathrm{L}^{-\infty}(M)$ we have $f(\cdot)K \in \mathrm{L}^{-\infty}(M,\Gamma)$. Consequently,

$$\tau_{\mu_0} : \mathrm{L}^{-\infty}(M) \to \mathbb{C}, \quad K \mapsto \tau(f(\cdot)K)(\mu_0) \tag{4.23}$$

is a trace on $\mathrm{L}^{-\infty}(M)$.

Indeed, from the locality of τ we conclude that τ_{μ_0} is independent of the f chosen and in view of (i) τ_{μ_0} is a trace.

Since each trace on $\mathrm{L}^{-\infty}(M)$ is a scalar multiple of the L^2–trace (see e.g. [8, Appendix]), there is a constant $c(\mu_0)$ such that

$$\tau_{\mu_0} = c(\mu_0)\,\mathrm{tr}_{L^2}\,. \tag{4.24}$$

For $f \in C_0^\infty(\Gamma)$ with $f(\mu_0) \neq 0$ we find in view of (4.24)

$$\tau(f(\cdot)K)(\mu_0) = f(\mu_0)c(\mu_0)\,\mathrm{tr}_{L^2}(K). \tag{4.25}$$

By the locality of τ (4.25) also holds if $f(\mu_0) = 0$. Since $\tau(f(\cdot)K) \in \mathrm{S}^{-\infty}(\Gamma)$, we infer from (4.25) that the function $\mu \mapsto c(\mu)$ is smooth.

Now loc. cit. gives for $f \in C_0^\infty(\Gamma)$

$$\partial_j \tau(f(\cdot)K) = ((\partial_j f)c + f\partial_j c)\,\mathrm{tr}_{L^2}(K). \tag{4.26}$$

On the other hand, using (ii) and (4.25)

$$\partial_j \tau(f(\cdot)K) = \tau((\partial_j f)(\cdot)K) = c(\partial_j f)\,\mathrm{tr}_{L^2}(K). \tag{4.27}$$

Hence $\partial_j c = 0$, $j = 1, ..., p$, and thus c is a constant.

Finally, we invoke again the locality of τ. Let $A \in \mathrm{L}^{-\infty}(M, \Gamma)$ and $f \in C_0^\infty(\Gamma)$ with $f(\mu_0) = 1$. Then

$$\begin{aligned}\tau(A)(\mu_0) =& \tau(A - f(\cdot)A(\mu_0))(\mu_0) + \tau(f(\cdot)A(\mu_0))(\mu_0) \\ =& \tau_{\mu_0}(A(\mu_0)) = c\ \mathrm{tr}_{L^2}(A(\mu_0)),\end{aligned} \tag{4.28}$$

since $A - f(\cdot)A(\mu_0)$ vanishes at μ_0. □

This proof cannot be extended to prove the Conjecture 4.10, at least not in an obvious way. The main reason is that if $K \in \mathrm{L}^m(M, \Gamma)$ (resp. $\mathrm{CL}^m(M, \Gamma)$), $m > -\infty$, and $f \in C_0^\infty(\Gamma) \setminus \{0\}$, then $f(\cdot)K \notin \mathrm{L}^*(M, \Gamma)$ (resp. $\mathrm{CL}^m(M, \Gamma)$).

Another idea of proving Conjecture 4.10 is to mimick the uniqueness proof for the noncommutative residue. If every $A \in \mathrm{CL}^m(M, \Gamma)$ could be written

$$A = \sum_{j=1}^{N} [P_j, Q_j] + R, \tag{4.29}$$

with $R \in \mathrm{L}^{-\infty}(M, \Gamma)$ and $P_j, Q_j \in \mathrm{CL}(M, \Gamma)$ (as it is the case for $\Gamma = \{0\}$), then the Conjecture 4.10 would immediately follow from the previous Proposition 4.11.

However, there is some evidence that (4.29) actually is wrong in general.

5. Exotic traces on $\mathrm{CL}^*(M, E, \mathbb{R}^p)$

Let TR be the map defined in Theorem 4.6. First we want to characterize the space $\mathrm{TR}(\mathrm{CL}^*(M, E, \Gamma))$. We know from Theorem 2.2 that $\mathrm{TR}(\mathrm{CL}^m(M, E, \Gamma)) \subset \mathrm{CS}^{m+\dim M}(\Gamma)$ if $m + \dim M < 0$.

Lemma 5.1. *Let $\Gamma \subset \mathbb{R}^p$ be a connected cone with nonempty interior and $H^1_{\mathrm{dR}}(\Gamma) = 0$. If $A \in \mathrm{CL}^m(M, E, \Gamma)$, then $\mathrm{TR}(A)$ has a representative $f \in \mathrm{S}^*(\Gamma)$ which has an asymptotic expansion in $\mathrm{S}^*(\Gamma)$*

$$f \sim \sum_{j=0}^{\infty} f_j + \sum_{k=0}^{[m+\dim M]} g_k + \sum_{k=0}^{[m+\dim M]} h_k, \tag{5.1}$$

where $f_j(\mu)$ is homogeneous of degree $m - j$ for $|\mu| \geq 1$, $g_k(\mu)$ is homogeneous of degree k for $|\mu| \geq 1$, and $h_k(\mu) = \tilde{h}_k(\mu/|\mu|)\,|\mu|^k \log|\mu|$ for $|\mu| \geq 1$ and a smooth function $\tilde{h}_k \in C^\infty(\Gamma \cap S^{p-1})$.

Proof. If $m + \dim M < 0$, then the assertion follows from Theorem 2.2. Assume by induction that the assertion is true for $m \leq m_0$ and consider $A \in \mathrm{CL}^m(M, E, \Gamma)$ with $m \leq m_0 + 1$. Then $\mathrm{TR}(\partial_j A)$, $j = 1, ..., p$, has a representative of the form (5.1). Integrating the representative of $d\,\mathrm{TR}(A)$ we reach the conclusion. □

Definition 5.2. Let $\Gamma \subset \mathbb{R}^p$ be a cone. We denote by $\mathrm{PS}^m(\Gamma)$ the set of all functions $a \in \mathrm{S}^*(\Gamma)$ which admit an asymptotic expansion $a \sim \sum_{j\geq 0} a_{m_j}$ in $\mathrm{S}^*(\Gamma)$, where $a_{m_j} \in C^\infty(\Gamma)$, $m \geq m_j \searrow -\infty$ and

$$a_{m_j} = \sum_{l=0}^{k_j} g_{lj},$$

with $g_{lj}(\xi) = \tilde{g}_{lj}(\xi/|\xi|)|\xi|^{m_j} \log^l |\xi|$ for $|\xi| \geq 1$. We call these functions log–*polyhomogeneous* (cf. [12]).

By Lemma 5.1 $\mathrm{TR}(A)$ is log–polyhomogeneous for $A \in \mathrm{CL}^*(M, E, \Gamma)$.

From now on we are content with $\Gamma = \mathbb{R}^p$. First we are going to introduce a regularized integral for log–polyhomogeneous functions. Let us consider a log–polyhomogeneous function f and write

$$f = \sum_{m_j \geq -N} f_{m_j} + g,$$

where

$$g(\xi) = O(|\xi|^{-N}), \quad |\xi| \to \infty.$$

Thus we have an asymptotic expansion

$$\int_{|\xi|\leq R} f(\xi) d\xi \sim_{R\to\infty} \sum_{\alpha\to-\infty} p_\alpha(\log R) R^\alpha, \tag{5.2}$$

where p_α is a polynomial of degree $k(\alpha)$. Then we define $⨍_{\mathbb{R}^p} f(x)dx$ to be the constant term in this asymptotic expansion, i.e.

$$⨍_{\mathbb{R}^p} f(x) dx := \mathrm{LIM}_{R\to\infty} \int_{|x|\leq R} f(x)dx := p_0(0), \tag{5.3}$$

(cf. [12, Sec. 5]). If $p = 1$, then $⨍$ coincides with the Hadamard partie finie.

Here and in the sequel we will use the common notation $\mathrm{LIM}_{R\to\infty}$ for the constant term in the log–polyhomogeneous expansion as $R \to \infty$.

The transformation behavior of $⨍_{\mathbb{R}^p}$ is more complicated than that of the usual integral. The following result gives the change of variables formula for $⨍$ under linear coordinate changes:

Proposition 5.3 (cf. [12, Prop. 5.2])**.** *Let $A \in \mathrm{GL}(n, \mathbb{R})$ be a regular matrix. Furthermore, let $f \in \mathrm{PS}^*(\mathbb{R}^p)$ with*

$$f \sim \sum_{\alpha\to-\infty} f_\alpha, \;\; f_\alpha(\xi) =: \sum_{l=0}^{k(\alpha)} f_{\alpha,l}, \;\; f_{\alpha,l}(\xi) = f_{\alpha,l}(\xi/|\xi|)|\xi|^\alpha \log^l(|\xi|), \qquad |\xi| \geq 1.$$

Then we have

$$\begin{aligned} &⨍_{\mathbb{R}^p} f(A\xi) d\xi \\ &\quad = |\det A|^{-1} \left(⨍_{\mathbb{R}^p} f(\xi)d\xi + \sum_{l=0}^{k} \frac{(-1)^{l+1}}{l+1} \int_{S^{n-1}} f_{-n,l}(\xi) \log^{l+1} |A^{-1}\xi| d\xi \right). \end{aligned}$$

Remark 5.4. Proposition 5.3 allows us to give an alternative existence proof for TR which in addition shows that TR is given by integration of a canonical density. Furthermore, this alternative construction reveals the analogy to the Kontsevich-Vishik trace.

We consider the local situation. For simplicity let $E = \mathbb{C}$, let U be a coordinate patch and $A \in \mathrm{CL}^m(U, \Gamma)$ as in (2.3). For fixed $x \in U, \mu \in \Gamma$ the symbol $\xi \mapsto \sigma_A(x, \xi, \mu)$ is polyhomogeneous. We define the density

$$\omega_A(x, \mu) := \fint_{\mathbb{R}^n} \sigma_A(x, \xi, \mu) đ\xi |dx| \mod C^\infty(U) \otimes \mathcal{P}. \tag{5.4}$$

Using Proposition 5.3 one shows similarly to [12, Lemma 5.3] that in this way we obtain a well–defined "density"

$$\omega_A \in \tilde{\mathrm{S}}^{m + \dim M}(M, |\Omega|, \Gamma) \Big/ C^\infty(M, |\Omega|) \otimes \mathcal{P} \tag{5.5}$$

fulfilling

$$\mathrm{TR}(A) = \int_M \omega_A. \tag{5.6}$$

Note that $\int_M \omega_A$ is a well–defined element of $\tilde{\mathrm{S}}^{m+\dim M}(M, \Gamma)/\mathcal{P}$. Thus we obtain the analogue of Theorem 2.3 for arbitrary m. The details are left to the reader.

In [12, Sec. 5] the first named author used this kind of argument to construct the Kontsevich-Vishik trace.

Let us mention that Stokes' theorem does not hold for $\fint$, or in other words $\fint$ is not a closed functional on $\Omega^*(\mathrm{PS}(\mathbb{R}^p))$. More precisely, we extend $\fint$ to $\Omega^*(\mathrm{PS}^*(\mathbb{R}^p))$ by putting

$$\fint : \omega \mapsto \begin{cases} 0, & \omega \in \Omega^k, k < p, \\ \fint_{\mathbb{R}^p} f(\xi) d\xi, & \omega = f(\xi) d\xi_1 \wedge \ldots \wedge d\xi_p. \end{cases} \tag{5.7}$$

In this way we obtain a graded trace on the complex $(\Omega^*(\mathrm{PS}^*(\mathbb{R}^p)), d)$. This would be a cycle in the sense of Connes [3, Sec. III.1.α] if $\fint$ were closed.

The next lemma shows that $d\fint$, which is defined by $(d\fint)\omega := \fint d\omega$, is nontrivial. However, it is local in the sense that it depends only on the log–polyhomogeneous expansion of ω.

Lemma 5.5. *Let* $f \in \mathrm{PS}^*(\mathbb{R}^p)$, $f \sim \sum_{\alpha \to -\infty} f_\alpha$, *where*

$$f_\alpha(\xi) = \sum_{l=0}^{k(\alpha)} f_{\alpha,l}(\xi/|\xi|) |\xi|^\alpha \log^l |\xi|, \quad |\xi| \geq 1, \quad f_{\alpha,l} \in C^\infty(S^{p-1}).$$

Then

$$\fint_{\mathbb{R}^p} \frac{\partial f}{\partial \xi_j} d\xi = \int_{S^{p-1}} f_{1-p,0}(\xi) \xi_j d\mathrm{vol}_S(\xi).$$

Proof. It suffices to prove this formula for $f \in C^\infty(\mathbb{R}^p)$ with

$$f(\xi) = f(\xi/|\xi|) |\xi|^\alpha \log^l |\xi|, \quad |\xi| \geq 1.$$

Then by Gauss' formula

$$\begin{aligned} \int_{|\xi|\le R} \frac{\partial f}{\partial \xi_j}\, d\xi &= \int_{|\xi|=R} f(\xi)\frac{\xi_j}{|\xi|}\, d\mathrm{vol}_S(\xi) \\ &= \int_{|\xi|=1} f(\xi)\xi_j\, d\mathrm{vol}_S(\xi)\, R^{p-1+\alpha}\log^l R \end{aligned} \tag{5.8}$$

and we reach the conclusion. □

Next we consider $f \in \mathcal{P}$. Since f is a sum of homogeneous polynomials

$$f = \sum_{k=0}^{m} f_k, \quad f_k(\lambda\xi) = \lambda^k f_k(\xi), \tag{5.9}$$

we have $\mathcal{P} \subset \mathrm{PS}^*(\mathbb{R}^p)$. Thus

$$\int_{|\xi|\le R} f(\xi) d\xi = \sum_{k=0}^{m} \int_{|\xi|\le 1} f_k(\xi) d\xi\, R^{k+p} \tag{5.10}$$

and hence

$$⨍_{\mathbb{R}^p} f(\xi) d\xi = 0. \tag{5.11}$$

As a consequence $⨍$ factorizes through $\mathcal{P}$ to a well–defined functional on $\mathrm{PS}^*(\mathbb{R}^p)/\mathcal{P}$.

For an arbitrary connected cone Γ with $H^1_{\mathrm{dR}}(\Gamma) = 0$ we can construct a complex from $\mathrm{CL}^*(M, E, \Gamma)$. Namely, similarly to (4.3) we put

$$\Omega^k \mathrm{CL}^*(M, E, \Gamma) := \Big\{ \sum A_I dx_I \,\Big|\, |I| = k, A_I \in \mathrm{CL}^*(M, E, \Gamma) \Big\}. \tag{5.12}$$

The exterior derivative maps $\Omega^*\mathrm{CL}^*(M, E, \Gamma)$ into itself and so we obtain a complex $(\Omega^*\mathrm{CL}^*(M, E, \mathbb{R}^p), d)$. The cup product makes $\Omega^*\mathrm{CL}^*(M, E, \mathbb{R}^p)$ into a graded algebra and TR extends naturally to a complex homomorphism

$$\begin{aligned} \mathrm{TR}: \quad (\Omega^*\mathrm{CL}^*(M, E, \Gamma), d) &\longrightarrow (\Omega^*\mathrm{PS}^*(\Gamma)/\mathcal{P}, d) \\ \sum A_I dx_I &\longmapsto \sum \mathrm{TR}(A_I) dx_I. \end{aligned} \tag{5.13}$$

That TR is indeed a complex homomorphism follows from Theorem 4.6.

If $\tau : \mathrm{PS}^*(\Gamma) \to \mathbb{C}$ is any linear functional with $\tau|\mathcal{P} = 0$, then τ factorizes through $\mathcal{P}$ to a linear functional on $\mathrm{PS}^*/\mathcal{P}$ and we obtain a graded trace on $(\Omega^*\mathrm{PS}^*(M, E, \Gamma)/\mathcal{P}, d)$ by putting

$$\bar\tau(\omega) := \begin{cases} 0, & \omega \in \Omega^k, k < p, \\ \tau(f), & \omega = f dx_1 \wedge \ldots \wedge dx_p. \end{cases} \tag{5.14}$$

τ induces a graded trace on $(\Omega^*\mathrm{CL}^*(M, E, \Gamma), d)$ by putting $\bar\tau := \tau \circ \mathrm{TR}$. In particular, we can apply this construction to $⨍$ and $d⨍$:

Definition 5.6. On $\Omega^*\mathrm{CL}^*(M, E, \mathbb{R}^p)$ we define the extended trace

$$\overline{\mathrm{Tr}} := ⨍ \circ \mathrm{TR},$$

and the formal trace

$$\widetilde{\mathrm{Tr}} := d\overline{\mathrm{Tr}} := \left(d ⨍\right) \circ \mathrm{TR} = ⨍ \circ d \circ \mathrm{TR} = ⨍ \circ \mathrm{TR} \circ d.$$

Remark 5.7.

1. The extended trace is graded, thus $((\Omega^*\mathrm{CL}^*(M,E,\mathbb{R}^p),d),\overline{\mathrm{Tr}})$ is almost a cycle in the sense of Connes [3, Sec. III.1.α], except that $\overline{\mathrm{Tr}}$ is not closed. Its derivative, the formal trace, $\widetilde{\mathrm{Tr}}$, is a closed graded trace on $\Omega^*\mathrm{CL}^*(M,E,\mathbb{R}^p)$. Furthermore, Lemma 5.5 shows that it is symbolic, i.e. depends only on finitely many terms of the symbol expansion of A (see the next proposition below).
2. If $p=1$, then via the isomorphism of Proposition 3.1 the traces $\overline{\mathrm{Tr}}, \frac{1}{2\pi}\widetilde{\mathrm{Tr}}$ coincide with the traces $\overline{\mathrm{Tr}}, \widetilde{\mathrm{Tr}}$ introduced by R. B. Melrose [15, Sec. 4 and 7]. Note that our normalization of the formal trace $\widetilde{\mathrm{Tr}}$ differs from the one of [15] by a factor of $\frac{1}{2\pi}$.

Proposition 5.8 (cf. [15, Prop. 6])**.** *Let $A\in\mathrm{CL}^m(M,E,\mathbb{R}^p)$ and put*

$$\omega := (-1)^{j-1} A\, d\mu_1\wedge \dots\wedge \widehat{d\mu_j}\wedge\dots\wedge d\mu_p. \tag{5.15}$$

Then

$$\widetilde{\mathrm{Tr}}(\omega) = \lim_{L\to\infty}\int_{|\xi|\le L}\int_{S^{p-1}} \mathrm{tr}(a_{1-p-n}(x,\xi,\mu))\,\mu_j\,d\mathrm{vol}(\mu)\,dx\,đ\xi. \tag{5.16}$$

Proof. It suffices to prove this for $E=\mathbb{C}$. Obviously, we have

$$\widetilde{\mathrm{Tr}}(\omega) = \overline{\mathrm{Tr}}(d\omega) = ⨍_{\mathbb{R}^p} \mathrm{TR}\left(\partial_{\mu_j}A\right)\,d\mu. \tag{5.17}$$

In view of Proposition 4.9 we introduce the abbreviation

$$b_N(x,\xi,\mu) := \sum_{|\alpha|\le N-1} \frac{(\partial_\mu^\alpha)a(x,\xi,0)}{\alpha!}\,\mu^\alpha. \tag{5.18}$$

Using Lemma 5.5 we find for N large enough

$$\begin{aligned}
&⨍_{\mathbb{R}^p} \mathrm{TR}\left(\partial_{\mu_j}A\right)(\mu)\,d\mu\\
&\quad = \mathrm{LIM}_{R\to\infty}\int_{|\mu|\le R}\int_U\int_{\mathbb{R}^n}\left[(\partial_{\mu_j}a)(x,\xi,\mu)-(\partial_{\mu_j}b_N)(x,\xi,\mu)\right]đ\xi\,dx\,d\mu\\
&\quad = \int_U\int_{S^{p-1}}\left\{\int_{\mathbb{R}^n} a(x,\xi,\mu)-b_N(x,\xi,\mu)\,đ\xi\right\}_{1-p}\mu_j\,d\mathrm{vol}_S(\mu)\,dx\\
&\quad = \lim_{L\to\infty}\int_{|\xi|\le L}\int_{S^{p-1}} a_{1-p-n}(x,\xi,\mu)\,\mu_j\,d\mathrm{vol}_S(\mu)\,đ\xi\,dx.
\end{aligned}$$

Here $\{\,.\,\}_{p-1}$ denotes the term of μ-homogeneity $1-p$. ∎

6. The eta–invariant

Now we have all prerequisites to define the higher eta–invariants:

Definition 6.1. If $p=2k-1$, then for elliptic and invertible $A\in\mathrm{CL}^*(M,E,\mathbb{R}^{2k-1})$ we put

$$\eta_k(A) := 2c_k\overline{\mathrm{Tr}}\left(\left(A^{-1}dA\right)^{2k-1}\right), \tag{6.1}$$

where $c_k=\frac{(-1)^{k-1}(k-1)!}{(2\pi i)^k(2k-1)!}$.

Remark 6.2.

1. If $k = 1$, then via the isomorphism of Proposition 3.1 $\eta_1(A) = \frac{1}{\pi i}\overline{\mathrm{Tr}}\left(A^{-1}dA\right)$ coincides with the η-invariant of Melrose [15, Sec. 5].
2. There are at least two motivations for the choice of the normalization constant c_k: It is well–known that for every smooth map $f : S^{2k-1} \to \mathrm{GL}(N, \mathbb{C})$ the number
$$w(f) := c_k \int_{S^{2k-1}} \mathrm{tr}((f^{-1}df)^{2k-1})$$
actually is an integer and w induces an isomorphism $\pi_{2k-1}(\mathrm{GL}(\infty, \mathbb{C})) \to \mathbb{Z}$. A map with $w(f) = 1$ can be constructed using Clifford matrices (cf. (6.25) below). In this sense η_k is a higher "winding number".

   The second motivation comes from the relation to the spectral eta–invariant (see Proposition 6.6 below).

Proposition 6.3 (cf. [15, Prop. 7])**.** *Let $A_s \in \mathrm{CL}^*(M, E, \mathbb{R}^{2k-1})$ be elliptic invertible and smoothly dependent on $s \in [0, 1]$. Then*

$$\frac{d}{ds}\eta_k(A_s) = 2(2k-1)c_k\widetilde{\mathrm{Tr}}\left((A_s^{-1}\partial_s A_s)(A_s^{-1}dA_s)^{2k-2}\right). \tag{6.2}$$

Proof. We introduce the 1–form $\omega := A^{-1}dA$ and note that since

$$d\omega = -\omega \wedge \omega \tag{6.3}$$

we have for $l \in \mathbb{Z}_+$

$$d\omega^l = \begin{cases} 0, & l \text{ even}, \\ -\omega^{l+1}, & l \text{ odd}. \end{cases} \tag{6.4}$$

Using this we find

$$\begin{aligned} \frac{d}{ds}\eta_k(A_s) &= 2c_k\,\overline{\mathrm{Tr}}\left(\frac{d}{ds}(A_s^{-1}dA_s)^{2k-1}\right) \\ &= 2(2k-1)\,c_k\,\overline{\mathrm{Tr}}\left((A_s^{-1}dA_s)^{2k-2}\frac{d}{ds}A_s^{-1}dA_s\right) \\ &= 2(2k-1)\,c_k\,\overline{\mathrm{Tr}}\left(-(A^{-1}dA)^{2k-1}\,A_s^{-1}\partial_s A\right. \\ &\qquad\qquad \left.+(A_s^{-1}dA_s)^{2k-2}\,A_s^{-1}\,d\partial_s A_s\right) \\ &= 2(2k-1)\,c_k\,\overline{\mathrm{Tr}}\left(d\left[(A_s^{-1}dA_s)^{2k-2}\,A_s^{-1}\,\partial_s A_s\right]\right) \\ &= 2(2k-1)\,c_k\,\widetilde{\mathrm{Tr}}\left((A_s^{-1}\partial_s A_s)(A_s^{-1}dA_s)^{2k-2}\right). \end{aligned} \tag{6.5}$$

□

Next we consider the complex Clifford algebra $\mathbb{C}\ell_p$, $p = 2k-1$ odd. $\mathbb{C}\ell_p$ is the universal C^*–algebra generated by p unitary elements $e_1, ..., e_p$ subject to the relations

$$e_i \cdot e_j + e_j \cdot e_i = -2\delta_{ij} \tag{6.6}$$

(cf. e.g. [11, Chap. I]). We choose a Clifford representation

$$c : \mathbb{R}^p \to \mathrm{M}(N, \mathbb{C}), \quad c(x) = \sum_{j=1}^{p} x_j E_j, \tag{6.7}$$

where the E_j are skew-adjoint Clifford matrices in $\mathbb{C}^N$. This means that $E_1, \dots, E_p$ are skew–adjoint matrices satisfying the Clifford relations (6.6). c induces a $*$–representation of $\mathbb{C}\ell_p$ in $\mathrm{M}(N,\mathbb{C})$.

Let us introduce the map

$$f : \mathbb{R} \times \mathbb{R}^p \to \mathrm{M}(N,\mathbb{C}), \quad x = (x_0, x') \mapsto x_0 + c(x') \tag{6.8}$$

and the p-form

$$\omega = \operatorname{tr}\left(f^{-1}\, df\right)^p \tag{6.9}$$

on $\mathbb{R}^{p+1} \setminus \{0\}$. That $f(x)$ is indeed invertible for $x \neq 0$ follows from (6.11) below.

Proposition 6.4. *The p-form ω is given by*

$$\omega = |x|^{-p-1}\, p!\, \operatorname{tr}(E_1 \cdot \ldots \cdot E_p) \sum_{j=0}^{p} (-1)^j\, x_j\, dx_0 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_p. \tag{6.10}$$

Proof. By definition of f we have $f(x)^* f(x) = |x|^2$, hence

$$f(x)^{-1} = |x|^{-2} f(x)^*, \tag{6.11}$$

and

$$f(x)^*\, df(x) = (x_0 - c(x')) \left(dx_0 + \sum_{j=1}^{p} E_j\, dx_j \right). \tag{6.12}$$

Setting $\omega_1 = (x_0 - c(x'))dx_0$ and $\omega_2 = (x_0 - c(x')) \sum_{j=1}^{p} E_j dx_j$ we get $\omega_1^2 = 0$ and

$$(f(x)^*\, df(x))^p = \omega_2^p + \sum_{j=0}^{p-1} \omega_2^j\, \omega_1\, \omega_2^{p-1-j}, \tag{6.13}$$

which implies

$$\omega = |x|^{-2p} \left(\operatorname{tr} \omega_2^p + p\, \operatorname{tr} \omega_1\, \omega_2^{p-1} \right), \tag{6.14}$$

as p is odd. We calculate the two terms on the right hand side separately. First note that both terms are invariant with respect to transformations of the form $x' \mapsto Ox'$ with $O \in SO(p)$, so we may assume $x' = (x_1, 0, \ldots, 0)$. Now

$$\begin{aligned} \operatorname{tr} \omega_2^p\big|_{(x_0,x')} &= \operatorname{tr} \left((x_0 - x_1 E_1) \sum_{j=1}^{p} E_j\, dx_j \right)^p \Big|_{(x_0,x')} \\ &= \sum_{\sigma \in \mathrm{S}_p} (\operatorname{sgn} \sigma)\, \operatorname{tr}\left((x_0 - x_1 E_1) E_{\sigma(1)} \cdot \ldots \cdot (x_0 - x_1 E_1) E_{\sigma(p)} \right) dx_1 \wedge \ldots \wedge dx_p. \end{aligned} \tag{6.15}$$

We fix a permutation $\sigma \in \mathrm{S}_p$ and put $j := \sigma^{-1}(1)$. Then for $l \neq j, j-1$ we have the relation

$$(x_0 - x_1 E_1) E_{\sigma(l)} (x_0 - x_1 E_1) E_{\sigma(l+1)} = |x|^2 E_{\sigma(l)}\, E_{\sigma(l+1)}. \tag{6.16}$$

Hence we obtain

$$\begin{aligned}
&\operatorname{tr}\big((x_0 - x_1E_1)E_{\sigma(1)} \cdot \ldots \cdot (x_0 - x_1E_1)E_{\sigma(p)}\big) \\
&= \operatorname{tr}\big((x_0 - x_1E_1)E_1\,(x_0 - x_1E_1)E_{\sigma(j+1)} \cdot \ldots \cdot (x_0 - x_1E_1)E_{\sigma(p)} \\
&\qquad\qquad \cdot (x_0 - x_1E_1)E_{\sigma(1)} \cdot \ldots \cdot (x_0 - x_1E_1)E_{\sigma(j-1)}\big) \\
&= |x|^{2(k-1)} \operatorname{tr}\,\big((x_0 - x_1E_{\sigma(j)}) \cdot E_{\sigma(j+1)} \cdot \ldots \cdot E_{\sigma(p)} \cdot E_{\sigma(1)} \cdot \ldots \cdot E_{\sigma(j)}\big) \\
&= x_0|x|^{p-1}\,\operatorname{tr}\big(E_{\sigma(1)} \cdot \ldots \cdot E_{\sigma(p)}\big)\,,
\end{aligned} \tag{6.17}$$

where we have used the Berezin Lemma (cf. [1, Prop. 3.21]). Summing up we have proved

$$\operatorname{tr}\omega_2^p = p!\,x_0\,|x|^{p-1}\,\operatorname{tr}\,(E_1 \cdot \ldots \cdot E_p)\;dx_1 \wedge \ldots \wedge dx_p \tag{6.18}$$

for all $x \in \mathbb{R}^{p+1} \setminus \{0\}$.

Next we calculate the second term in (6.14), where we again suppose $x' = (x_1, 0, \ldots, 0)$:

$$\begin{aligned}
\operatorname{tr}\omega_1\omega_2^{p-1}|_{(x_0,x')} = &\sum_{1\le i_1<\ldots<i_{p-1}\le p}\;\sum_{\sigma\in \mathrm{S}_{p-1}} \operatorname{sgn}\sigma \cdot \operatorname{tr}\big((x_0 - x_1E_1)^2E_{i_{\sigma(1)}} \\
&\cdot (x_0 - x_1E_1)E_{i_{\sigma(2)}} \cdot \ldots \cdot (x_0 - x_1E_1)E_{i_{\sigma(p-1)}}\big)\,dx_0 \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_p}.
\end{aligned} \tag{6.19}$$

Now consider the trace terms on the right hand side of the last equation with $i_1 = 1$ and $j = \sigma^{-1}(1)$:

$$\begin{aligned}
&\operatorname{tr}\big((x_0 - x_1E_1)^2E_{i_{\sigma(1)}} \cdot (x_0 - x_1E_1)E_{i_{\sigma(2)}} \cdot \ldots \cdot (x_0 - x_1E_1)E_{i_{\sigma(p-1)}}\big) \\
&= \operatorname{tr}\big((x_0 + (-1)^jx_1E_1)\,(x_0 - x_1E_1)E_{i_{\sigma(j)}} \cdot \ldots \cdot (x_0 - x_1E_1)E_{i_{\sigma(p-1)}} \\
&\qquad\qquad \cdot (x_0 - x_1E_1)E_{i_{\sigma(1)}} \cdot \ldots \cdot (x_0 - x_1E_1)E_{i_{\sigma(j-1)}}\big) \\
&= |x|^{p-1}\operatorname{tr}\big((x_0 + (-1)^jx_1E_1)E_1 \cdot E_{i_{\sigma(j+1)}} \cdot \ldots \\
&\qquad\qquad \cdot E_{i_{\sigma(p-1)}}E_{i_{\sigma(1)}} \cdot \ldots \cdot E_{i_{\sigma(j-1)}}\big) = 0.
\end{aligned} \tag{6.20}$$

In case $i_1 > 1$ an analogous argument proves

$$\begin{aligned}
&\operatorname{tr}\big((x_0 - x_1E_1)^2E_{i_{\sigma(1)}} \cdot (x_0 - x_1E_1)E_{i_{\sigma(2)}} \cdot \ldots \cdot (x_0 - x_1E_1)E_{i_{\sigma(p-1)}}\big) \\
&= |x|^{p-1}\,\operatorname{tr}\big((x_0 - x_1E_1) \cdot E_{i_{\sigma(1)}} \cdot \ldots \cdot E_{i_{\sigma(p-1)}}\big) \\
&= -x_1\,|x|^{p-1}\,\operatorname{tr}\big(E_1 \cdot E_{i_{\sigma(1)}} \cdot \ldots \cdot E_{i_{\sigma(p-1)}}\big)\,.
\end{aligned} \tag{6.21}$$

Hence by rotation symmetry

$$\begin{aligned}
\operatorname{tr}\omega_1\omega_2^{p-1} = &|x|^{p-1}\,(p-1)!\,\operatorname{tr}(E_1 \cdot \ldots \cdot E_p)dx_0 \\
&\wedge \sum_{j=1}^{p}(-1)^j\,x_jdx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_p
\end{aligned} \tag{6.22}$$

holds. The assertion follows from (6.14), (6.18) and (6.22). □

From now on we choose the standard representation of $\mathbb{C}\ell_{2k-1}$ in $\mathbb{C}^{2^{k-1}}$. For this representation one knows that

$$\operatorname{tr}(E_1 \cdot \ldots \cdot E_{2k-1}) = 2^{k-1}\,i^{-k}. \tag{6.23}$$

This is part of the Berezin Lemma (cf. [1, Prop. 3.21]), but can also easily seen as follows. In the standard representation the complex volume element $i^k\,e_1 \cdot \ldots \cdot e_{2k-1}$

acts as identity. Since the standard representation is of rank 2^{k-1} we obtain (6.23). Now, by Proposition 6.4

$$\omega = |x|^{-p-1}\, p!\, 2^{k-1}\, i^{-k} \sum_{j=0}^{p} (-1)^j\, x_j\, dx_0 \wedge \dots \wedge \widehat{dx_j} \wedge \dots \wedge dx_p, \tag{6.24}$$

and

$$\begin{aligned}\int_{S^{2k-1}} \omega &= \int_{B^{2k}} d(|x|^{p+1}\omega)\\ &= i^{-k}\, 2^{k-1}\, (2k-1)! \int_{B^{2k}} (p+1)\, d\mathrm{vol}\\ &= i^{-k}\, 2^{k}\, (2k-1)!\, \frac{\pi^k}{(k-1)!}\\ &= \frac{-1}{c_k},\end{aligned} \tag{6.25}$$

where c_k was defined in Definition 6.1.

We note another consequence of our calculations. Choose $a \in \mathbb{R}$ and let $f(x) = a + c(x)$, $x \in \mathbb{R}$. Then $(f^* df)^p = \omega_2^p$ and in view of (6.18) and (6.23)

$$\begin{aligned}\operatorname{tr}\big((f^{-1}df)^p\big) &= (a^2+|x|^2)^{-p}\, \operatorname{tr}\big((f^*df)^p\big)\\ &= p!\, a(a^2+|x|^2)^{-\frac{p+1}{2}}\, \operatorname{tr}(E_1 \cdot \dots \cdot E_p)\, dx_1 \wedge \dots \wedge dx_p\\ &= p!\, a(a^2+|x|^2)^{-\frac{p+1}{2}}\, 2^{k-1}\, i^{-k}\, dx_1 \wedge \dots \wedge dx_p.\end{aligned} \tag{6.26}$$

We are now able to calculate the integral of the p-form $\operatorname{tr}\big((f^{-1}df)^p\big)$:

(6.27)

$$\begin{aligned}\int_{|x|\le R} \operatorname{tr}\big((f^{-1}df)^p\big) &= 2^{k-1}\, i^{-k}\, p!\, a \int_{|x|\le R} (a^2+|x|^2)^{-k}\, dx\\ &= 2^{k-1}\, i^{-k}\, p!\, a\, \frac{(2k-1)\pi^{k-\frac12}}{\Gamma(k+\frac12)} \int_0^R (a^2+r^2)^{-k}\, r^{2k-2}\, dr\\ &\xrightarrow[R\to\infty]{} 2^{k-1}\, i^{-k}\, p!\, a\, \frac{(2k-1)\pi^{k-\frac12}}{\Gamma(k+\frac12)} \int_0^\infty (a^2+r^2)^{-k}\, r^{2k-2}\, dr.\end{aligned}$$

Using the formula

$$\int_0^\infty (1+u^2)^\alpha u^\beta\, du = \frac{1}{2}\frac{\Gamma(-\alpha-\frac{\beta}{2}-\frac12)\,\Gamma(\frac{\beta+1}{2})}{\Gamma(-\alpha)} \tag{6.28}$$

we find

$$\begin{aligned}\int_{\mathbb{R}^p} \operatorname{tr}\big((f^{-1}df)^p\big) &= 2^{k-1}\, i^{-k}\, p!\, (2k-1)\, (\operatorname{sgn} a)\, \frac{1}{2}\, \frac{\pi^{k-\frac12}}{\Gamma(k+\frac12)}\, \frac{\Gamma(\frac12)\,\Gamma(k-\frac12)}{\Gamma(k)}\\ &= 2^{k-1}\, i^{-k}\, p!\, (\operatorname{sgn} a)\, \frac{\pi^k}{(k-1)!} = \frac{-\operatorname{sgn} a}{2c_k}.\end{aligned} \tag{6.29}$$

Now we are ready to relate the η-invariant to the spectral η-invariant of an elliptic operator.

First we briefly recall the regularized integral for functions on $\mathbb{R}_+$ (cf. [13, Sec. 2.1], [14]). Let $f : (0,\infty) \to \mathbb{C}$ be a locally integrable function having log–polyhomogeneous asymptotic expansions as $x \to 0$ and as $x \to \infty$. Then one puts

$$\text{(6.30)} \qquad -\!\!\!\!\!\!\int_0^\infty f(x)dx := \mathrm{LIM}_{a\to 0}\int_a^1 f(x)dx + \mathrm{LIM}_{b\to\infty}\int_1^\infty f(x)dx.$$

For such a function its "Mellin transform"

$$\text{(6.31)} \qquad (\widetilde{\mathcal{M}}f)(s) := -\!\!\!\!\!\!\int_0^\infty x^{s-1}f(x)dx$$

is well–defined for $s\in\mathbb{C}$ and there is a discrete subset $A\subset\mathbb{C}$ such that $(\widetilde{\mathcal{M}}f)\restriction(\mathbb{C}\setminus A)$ extends to a meromorphic function $\mathcal{M}f$ on $\mathbb{C}$. For each s one has

$$\text{(6.32)} \qquad (\widetilde{\mathcal{M}}f)(s) = \mathrm{Res}_0(\mathcal{M}f)(s),$$

where Res_0 denotes the constant term in the Laurent expansion. In particular, if s is a regular point of $\mathcal{M}f$, then $(\widetilde{\mathcal{M}}f)(s) = (\mathcal{M}f)(s)$. We note that for $\alpha\in\mathbb{R}, k\in\mathbb{Z}_+$

$$\text{(6.33)} \qquad -\!\!\!\!\!\!\int_0^\infty x^\alpha \log^k x dx = 0.$$

(see [13, Sec. 2.1] for proofs of these facts). Of course, there is a simple relation between the integral (6.30) and the integral (5.3). Namely, if f is an even log–polyhomogeneous function on $\mathbb{R}$, then $2\,-\!\!\!\!\!\!\int\limits_0^\infty f(x)dx = -\!\!\!\!\!\!\int\limits_{\mathbb{R}} f(x)dx$.

Next let $D : C^\infty(E)\to C^\infty(E)$ be a first order invertible self-adjoint elliptic *differential* operator. Then for $k\in\mathbb{Z}_+, k>0$, the family

$$\text{(6.34)} \qquad \Phi_{k,p} : \mathbb{R}^p \ni x \mapsto D(D^2+|x|^2)^{-k}$$

lies in $\mathrm{CL}^{1-2k}(M,E,\mathbb{R}^p)$. Note that being a *differential* operator is really essential for this to be true. We apply Theorem 4.6 to $\Phi_{k,p}$. Thus for each $k\in\mathbb{Z}_+, k>0$, we have

$$\text{(6.35)} \qquad \mathrm{TR}(D(D^2+|\,\mathrm{id}_{\mathbb{R}^p}|^2)^{-k}) = \mathrm{TR}(\Phi_{k,p}) \in \tilde{\mathrm{S}}^{2k+\dim M}(\mathbb{R}^p)/\mathcal{P}.$$

We note for $p=1$ and $l>\frac{\dim M+1-2k}{2}$ the identity

$$\text{(6.36)} \qquad \left(\frac{1}{x}\frac{\partial}{\partial x}\right)^l \mathrm{TR}(\Phi_{k,p})(x) = (-2)^l\frac{(k+l-1)!}{(k-1)!}\,\mathrm{tr}_{L^2}(D(D^2+|x|^2)^{-k-l}).$$

Note that in view of (6.33) the Mellin transforms $\widetilde{\mathcal{M}}(\Phi_{k,1}), \mathcal{M}(\Phi_{k,1})$, are well–defined. For reasons of clarity we now write the various traces on $\mathrm{CL}(M,E,\mathbb{R}^p)$ with a subscript, i.e. $\mathrm{TR}_p, \overline{\mathrm{Tr}}_p, \widetilde{\mathrm{Tr}}_p$.

Proposition 6.5. *Let D be as before and let*

$$\text{(6.37)} \qquad \eta_D(s) := \sum_{\mu\in\operatorname{spec} D} \operatorname{sgn}\mu\,|\mu|^{-s} = \mathrm{tr}((D^2)^{-(s+1)/2})$$

be the spectral η-function of D. Then we have for $k\in\mathbb{Z}_+, k>0,$ and $\mathrm{Re}\, s$ *large*

(6.38)

$$\begin{aligned}\eta(s) &= \begin{pmatrix} k-1-\frac{s+1}{2} \\ k-1\end{pmatrix}^{-1} \frac{\sin\pi\frac{s+1}{2}}{\pi}\, 2 -\!\!\!\!\!\!\int_0^\infty x^{2k-2-s}\ \mathrm{TR}_1\left(D(D^2+\mathrm{id}_{\mathbb{R}}^2)^{-k}\right)(x)\,dx,\\ &= \begin{pmatrix} k-1-\frac{s+1}{2} \\ k-1\end{pmatrix}^{-1} \frac{\sin\pi\frac{s+1}{2}}{\pi}\, 2(\mathcal{M}\Phi_{k,1})(2k-1-s).\end{aligned}$$

Furthermore,

$$\begin{aligned}\eta(0) =&\frac{2\Gamma(k)}{\Gamma(k-\frac{1}{2})\sqrt{\pi}} \int\!\!\!\!\!\!-_0^\infty x^{2k-2}\,\mathrm{TR}_1\left(D(D^2+\mathrm{id}_{\mathbb{R}}^2)^{-k}\right)(x)\,dx\\ =&\frac{\Gamma(k)}{\Gamma(k-\frac{1}{2})\sqrt{\pi}}\overline{\mathrm{Tr}}_1(\mathrm{id}_{\mathbb{R}}^{2k-2}\,D(D^2+\mathrm{id}_{\mathbb{R}}^2)^{-k})\\ =&\frac{\Gamma(k)}{\pi^k}\overline{\mathrm{Tr}}_{2k-1}(D(D^2+|\,\mathrm{id}_{\mathbb{R}^{2k-1}}|^2)^{-k}).\end{aligned}\tag{6.39}$$

Proof. From the identities

$$\begin{aligned}\lambda^{-z} &= \frac{\sin\pi z}{\pi}\int_0^\infty x^{-z}\,(\lambda+x)^{-1}\,dx, \quad 0<\operatorname{Re} z<1,\\ \lambda^{-z} &= \binom{k-1-z}{k-1}^{-1}\frac{\sin\pi z}{\pi}\int_0^\infty x^{k-1-z}\,(\lambda+x)^{-k}\,dx, \quad 0<\operatorname{Re} z<k,\end{aligned}\tag{6.40}$$

and (6.37) we infer (6.38) for k large enough and $s_0(D)<\operatorname{Re} s<2k-1$. Integration by parts gives (6.38) for all $k\in\{1,2,3,...\}$ (cf. (6.36)). Since $\binom{k-1-\frac{s+1}{2}}{k-1}^{-1}\frac{\sin\pi\frac{s+1}{2}}{\pi}$ is regular and $\neq 0$ at $s=0$, and since $\eta(s)$ is regular at $s=0$ (cf. [5, Sec. 3.8]) we conclude from (6.38) that the meromorphic function $(\mathcal{M}\Phi_{k,1})(2k-1-s)$ is regular at $s=0$ and thus in view of (6.32) we arrive at the first equality of (6.39). The second equality of (6.39) is trivial. To prove the third one we note that from the uniqueness statement and (iii) of Theorem 4.6 we conclude the identity

$$\Phi_{k,p}(x)=\Phi_{k,1}(|x|)\tag{6.41}$$

and thus

$$\begin{aligned}&\frac{\Gamma(k)}{\pi^k}\overline{\mathrm{Tr}}_{2k-1}(D(D^2+|\,\mathrm{id}_{\mathbb{R}^{2k-1}}|^2)^{-k})\\ =&\frac{\Gamma(k)}{\pi^k}\int\!\!\!\!\!\!-_{\mathbb{R}^{2k-1}}\mathrm{TR}_{2k-1}(D(D^2+|\,\mathrm{id}_{\mathbb{R}^{2k-1}}|^2)^{-k})(x)dx\\ =&\frac{\Gamma(k)}{\pi^k}\,\mathrm{LIM}_{R\to\infty}\int_{|x|\le R}\mathrm{TR}_{2k-1}(D(D^2+|\,\mathrm{id}_{\mathbb{R}^{2k-1}}|^2)^{-k})(x)dx\\ =&\frac{\Gamma(k)}{\pi^k}\,\mathrm{LIM}_{R\to\infty}\frac{(2k-1)\pi^{k-1/2}}{\Gamma(k+\frac{1}{2})}\int_0^R\mathrm{TR}_1(D(D^2+\mathrm{id}_{\mathbb{R}}^2)^{-k})(x)x^{2k-2}dx\\ =&\frac{\Gamma(k)}{\Gamma(k-\frac{1}{2})\sqrt{\pi}}\overline{\mathrm{Tr}}_1(\mathrm{id}_{\mathbb{R}}^{2k-2}\,D(D^2+\mathrm{id}_{\mathbb{R}}^2)^{-k}).\end{aligned}\tag{6.42}$$

□

Proposition 6.6. *Let $D: C^\infty(E)\to C^\infty(E)$ be a first order invertible self-adjoint elliptic differential operator. Let $c:\mathbb{R}^{2k-1}\to \mathrm{M}(2^{k-1},\mathbb{C})$ be the standard Clifford representation (see (6.23)). Then the family $\mathcal{D}_\pm(\mu):=D\pm c(\mu)$ lies in $\mathrm{CL}^1(M,E,\mathbb{R}^{2k-1})$ and we have*

$$\eta(\mathcal{D}_\pm)=\mp\eta(D).\tag{6.43}$$

Remark 6.7.

1. This generalizes [15, Prop. 5]. Even for $k=1$ it is more general than there. In contrast to [15] our proof does not use the local index theorem, hence is not restricted to Dirac operators.

2. Since the spectral η–invariant is a regularization of the non–convergent sum

$$\sum_{\mu\in\operatorname{spec} D\setminus\{0\}} \operatorname{sgn}\mu,$$

this result is formally a consequence of the integral formula (6.29).

Proof. Since D is a differential operator, we have $\mathcal{D}\in \mathrm{CL}^1(M,E,\mathbb{R}^{2k-1})$, where being differential is really essential here. Note that the complete symbol $\sigma_{\mathcal{D}}(x,\xi,\mu)$ is (affine) linear in (ξ,μ). From (6.26) and the previous proposition we conclude

$$\begin{aligned}\eta(\mathcal{D}_\pm) &= 2\,c_k \;{-}\!\!\!\!\!\!\int_{\mathbb{R}^{2k-1}} \mathrm{TR}_{2k-1}\Big(\big((\pm D+c(\cdot))^{-1}\,dc(\cdot)\big)^{2k-1}\Big)(\mu)\,d\mu\\ &= \pm 2^k\,i^{-k}\,c_k\,(2k-1)!\;{-}\!\!\!\!\!\!\int_{\mathbb{R}^{2k-1}} \mathrm{TR}_{2k-1}(D(D^2+|\operatorname{id}_{\mathbb{R}^{2k-1}}|^2)^{-k})(x)dx\\ &= \;\pm 2^k\,i^{-k}\,c_k\,(2k-1)!\,\frac{\pi^k}{\Gamma(k)}\,\eta(D)\\ &= \mp\eta(D),\end{aligned}\tag{6.44}$$

and we are done. ∎

Remark 6.8. (i) η_1 is an additive homomorphism from the group of invertible elements of $\mathrm{CL}^*(M,E,\mathbb{R})$ into $\mathbb{C}$ [15, Prop. 4]. This follows immediately from the conjugation invariance of the trace $\overline{\mathrm{Tr}}$. Namely, given invertible $A,B\in\mathrm{CL}^*(M,E,\mathbb{R})$, then

$$\begin{aligned}(AB)^{-1}d(AB) &= B^{-1}(A^{-1}dA)B + B^{-1}dB\\ &=: \omega_1+\omega_2.\end{aligned}\tag{6.45}$$

Thus

$$\begin{aligned}\eta_1(AB) &= \frac{1}{\pi i}\overline{\mathrm{Tr}}(B^{-1}(A^{-1}dA)B+B^{-1}dB)\\ &= \eta_1(A)+\eta_1(B).\end{aligned}\tag{6.46}$$

However, η_k is not additive for $k\geq 2$. We illustrate this in the case $k=2$: the 1–forms ω_1,ω_2 of (6.45) have the following properties:

$$\begin{aligned}&d\omega_1 = -\omega_1^2-\omega_1\wedge\omega_2-\omega_2\wedge\omega_1, \quad d\omega_2=-\omega_2^2,\\ &d(\omega_1+\omega_2) = -(\omega_1+\omega_2)^2, \quad d(\omega_1\wedge\omega_2) = -(\omega_1+\omega_2)\wedge\omega_1\wedge\omega_2.\end{aligned}\tag{6.47}$$

Consequently,

$$\begin{aligned}\mathrm{TR}((\omega_1+\omega_2)^3)-\mathrm{TR}(\omega_1^3)-\mathrm{TR}(\omega_2^3) &= 3\,\mathrm{TR}((\omega_1+\omega_2)\wedge\omega_1\wedge\omega_2)\\ &= -3d\,\mathrm{TR}(\omega_1\wedge\omega_2)\end{aligned}\tag{6.48}$$

and

$$\eta_2(AB) = \eta_2(A)+\eta_2(B)-6c_2\widetilde{\mathrm{TR}}(\omega_1\wedge\omega_2).\tag{6.49}$$

So the defect of the additivity of η is a symbolic term.

(ii) Finally, we add some remarks concerning the divisor flow (cf. [15, Sec. 9]). Following [15, Sec. 8], the right hand side of the variation formula Proposition 6.3 can be defined if A_s is elliptic but not necessarily invertible. Namely, choose a smooth family of parametrices $Q_s\in\mathrm{CL}^{-m}(M,E,\mathbb{R}^{2k-1})$, i.e.

$$Q_sA_s-I,\quad A_sQ_s-I\in\mathrm{CL}^{-\infty}(M,E,\mathbb{R}^{2k-1}),$$

and put

$$\mathrm{v}\eta(A_s) := 2(2k-1)c_k \widetilde{\mathrm{Tr}}\left((Q_s\partial_s A_s)(Q_s dA_s)^{2k-2}\right). \tag{6.50}$$

In view of Proposition 5.8 $\mathrm{v}\eta(A_s)$ is independent of the choice of Q_s.

Now fix an elliptic and invertible $A \in \mathrm{CL}(M, E, \mathbb{R}^{2k-1})$. If $B \in \mathrm{Ell}_m(A)$, the component of A in the elliptic elements of order m, then according to [15, Def. 5] the divisor flow was defined to be

$$\mathrm{DF}(B, A) := \frac{1}{2}\left(\eta_k(B) - \eta_k(A) - \int_0^1 \mathrm{v}\eta(B_s) ds\right) \tag{6.51}$$

where B_s is any smooth family in $\mathrm{Ell}_m(A)$ with $B_0 = A, B_1 = B$.

However, it is not immediately clear, whether $\mathrm{DF}(B, A)$ is independent of the particular choice of a path B_s. There seems to be some evidence that this might not be the case. Let us give an example for M a point. Although this is quite an exceptional case it at least shows where the problems are:

Let $f \in C^\infty(\mathbb{R})$ with $f' \in C_0^\infty(\mathbb{R})$, and $\lim_{\lambda\to\pm\infty} f(\lambda) \neq 0$. If f is invertible, then

$$\eta(f) := \frac{1}{\pi i}\int_{\mathbb{R}} \frac{f'}{f}(\lambda) d\lambda. \tag{6.52}$$

If f_s is a family of such functions, the variation formula reads

$$\begin{aligned}\partial_s \eta(f_s) =& \frac{1}{\pi i}\int_{\mathbb{R}} \partial_\lambda \left(f_s^{-1}\partial_s f_s\right)\, d\lambda \\ =& \frac{1}{\pi i}\left(\frac{\partial_s f_s}{f_s}(+\infty) - \frac{\partial_s f_s}{f_s}(-\infty)\right) \\ =&: \mathrm{v}\eta(f_s).\end{aligned} \tag{6.53}$$

Now let

$$f_s(\lambda) := \begin{cases} 1, & \lambda \geq 1, \\ e^{2\pi i s}, & \lambda \leq s, \\ e^{2\pi i \lambda}, & s \leq \lambda \leq 1. \end{cases} \tag{6.54}$$

The two points at which f_s is just continuous but not smooth can easily be smoothed out.

Then one calculates

$$\int_0^1 \mathrm{v}\eta(f_s)\, ds = -2. \tag{6.55}$$

On the other hand,

$$g_s := (1-s)f_0 + s f_1, \quad 0 \leq s \leq 1, \tag{6.56}$$

also is an elliptic family joining f_0 and f_1. Here, elliptic means invertible outside a compact set. However,

$$\int_0^1 \mathrm{v}\eta(g_s)\, ds = 0 \neq \int_0^1 \mathrm{v}\eta(f_s)\, ds \tag{6.57}$$

proving the path dependence of the divisor flow.

References

[1] N. Berline, E. Getzler and M. Vergne: *Heat kernels and Dirac operators.* Grundlehren der Mathematischen Wissenschaften, vol. 298, Springer–Verlag, Berlin–Heidelberg–New York, 1992. MR **94e:**58130

[2] R. Bott and L. W. Tu: *Differential forms in algebraic topology.* Springer–Verlag, Berlin–Heidelberg–New York, 1982. MR **83i:**57016

[3] A. Connes: *Noncommutative geometry.* Academic Press, San Diego, 1994. MR **95j:**46063

[4] J. J. Duistermaat: *Fourier Integral Operators.* Birkhäuser, Basel, 1996. MR **96m:**58245

[5] P. B. Gilkey: *Invariance theory, the heat-equation, and the Atiyah-Singer index theorem.* second ed., CRC Press, Boca Raton, 1995. MR **98b:**58156

[6] A. Grigis and J. Sjøstrand: *Microlocal Analysis for Differential Operators.* London Mathematical Society Lecture Note Series, vol. 196, Cambridge University Press, 1994. MR **95d:**35009

[7] G. Grubb and R. T. Seeley: *Weakly parametric pseudodifferential operators and Atiyah-Patodi-Singer boundary problems.* Invent. Math. **121** (1995), 481–529. MR **95k:**58216

[8] V. Guillemin: *A new proof of Weyl's formula on the asymptotic distribution of eigenvalues.* Adv. in Math. **55** (1985), 131–160. MR **86i:**58135

[9] M. Kontsevich and S. Vishik: *Determinants of elliptic pseudo–differential operators.* Preprint, 1994.

[10] M. Kontsevich and S. Vishik: *Geometry of determinants of elliptic operators.* In: *Functional analysis on the eve of the 21st century.* Volume I (S. Gindikin et al., editor). In honor of the eightieth birthday of I. M. Gelfand. Proceedings of the conference, held at Rutgers University, New Brunswick, NJ, October 24-27, 1993. Boston: Birkhäuser, Prog. Math. 131, 173-197 (1995). MR **96m:**58264

[11] H. B. Lawson and M. L. Michelsohn: *Spin geometry.* Princeton University Press, Princeton, NJ, 1989. MR **91g:**53001

[12] M. Lesch: *On the noncommutative residue for pseudodifferential operators with* log*–polyhomogeneous symbols.* Ann. Global Anal. Geom **17** (1999), 151–187. CMP 99:09

[13] M. Lesch: *Operators of Fuchs type, conical singularities, and asymptotic methods.* Teubner Texte zur Mathematik, vol. 136, B. G. Teubner, Leipzig, 1997. MR **98d:**58174

[14] M. Lesch and J. Tolksdorf: *On the determinant of one–dimensional elliptic boundary value problems.* SFB Commun. Math. Phys. **193** (1998), 643–660. CMP 98:13

[15] R. B. Melrose: *The eta invariant and families of pseudodifferential operators.* Math. Res. Lett. **2** (1995), 541–561. MR **96h:**58169

[16] R.B. Melrose and V. Nistor: *C^*–algebras of B–pseudodifferential operators and an $\mathbb{R}^k$–equivariant index theorem.* Preprint, 1996; funct-an/9610003 .

[17] R.B. Melrose and V. Nistor: *Homology of pseudodifferential operators I. Manifolds with boundary.* Preprint, 1996; funct-an/9606005.

[18] R.B. Melrose and V. Nistor: In preparation.

[19] M. A. Shubin: *Pseudodifferential Operators and Spectral Theory.* Springer-Verlag, Berlin–Heidelberg–New York, 1987. MR **88c:**47105

[20] M. Wodzicki: *Noncommutative residue I: Fundamentals, K-Theory.* LNM 1289, Springer-Verlag, Berlin–Heidelberg–New York, 1987. MR **90a:**58175

Institut für Mathematik, Humboldt-Universität, Unter den Linden 6, 10099 Berlin, Germany

Current address: Department of Mathematics, The University of Arizona, Tucson, Arizona 85721-0089

E-mail address: `lesch@math.arizona.edu`

URL: `http://www.math.arizona.edu/~lesch`

Institut für Mathematik, Humboldt-Universität, Unter den Linden 6, 10099 Berlin, Germany

E-mail address: `pflaum@mathematik.hu-berlin.de`

URL: `http://spectrum.mathematik.hu-berlin.de/~pflaum`