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INFINITELY RENORMALIZABLE QUADRATIC POLYNOMIALS

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ABSTRACT. We prove that the Julia set of a quadratic polynomial which admits an infinite sequence of unbranched, simple renormalizations with complex bounds is locally connected. The method in this study is three-dimensional puzzles.

1. INTRODUCTION

Let $P(z) = z^2 + c$ be a quadratic polynomial where z is a complex variable and c is a complex parameter. The *filled-in Julia set* K of P is, by definition, the set of points z which remain bounded under iterations of P. The *Julia set* J of P is the boundary of K. A central problem in the study of the dynamical system generated by P is to understand the topology of a Julia set J, in particular, the local connectivity for a connected Julia set. A connected set J in the complex plane is said to be *locally connected* if for every point p in J and every neighborhood U of p there is a neighborhood $V \subseteq U$ such that $V \cap J$ is connected.

We first give the definition of renormalizability. A quadratic-like map $F : U \to V$ is a holomorphic, proper, degree two branched cover map, where U and V are two domains isomorphic to a disc and $\overline{U} \subset V$. Then $K_F = \bigcap_{n=0}^{\infty} F^{-n}(U)$ and $J_F = \partial K_F$ are the *filled-in Julia set* and *the Julia set* of F, respectively. We only consider those quadratic-like maps whose Julia sets are connected. Let us assume the only branch point of F is 0. A quadratic-like map $F : U \to V$ is said to be (once) *renormalizable* if there are an integer $n' > 1$ and an open subdomain U' containing 0 such that $U' \subset U$ and such that $F_1 = F^{\circ n'} : U' \to V' \subset V$ is a quadratic-like map with connected Julia set $J_{F_1} = J(n', U', V')$. The choice of (U', V') is called an n'-renormalization of (U, V). An annulus A is a double connected domain. The definition of the modulus $\mathrm{mod}(A)$ of an annulus A is defined in many books in complex analysis (see, for example, [AL]). It is $\log r$ if A is holomorphically diffeomorphic to the annulus $A_r = D_r \setminus \overline{D}_1$, where D_r is the open disk centered at 0 with radius $r > 1$. The sets $U \setminus \overline{U'}$ and $V \setminus \overline{U}$ are annuli. In §3, we prove a *modulus inequality* in renormalization as follows.

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Theorem 1. *Suppose $F : U \to V$ is a renormalizable quadratic-like map. Consider any n'-renormalization (U', V'), $n' > 1$. Then*

$$\operatorname{mod}(U \setminus \overline{U'}) \geq \frac{1}{2} \operatorname{mod}(V \setminus \overline{U}).$$

A quadratic-like map $F : U \to V$ is said to be twice renormalizable if F is once renormalizable, and there is an m_1-renormalization (U', V') of (U, V) such that $F_1 = F^{\circ m_1} : U' \to V'$ is once renormalizable. Consequently, we have renormalizations $F_1 = F^{\circ m_1} : U_1 \to V_1$ and $F_2 = F^{\circ m_2} : U_2 \to V_2$ and their Julia sets (J_{F_1}, J_{F_2}). Similarly, we can definite a *k-times* renormalizable quadratic-like map $F : U \to V$ and renormalizations $\{F_i = F^{\circ m_i} : U_i \to V_i\}_{i=1}^{k}$, where $m_1 < m_2 < \cdots < m_k$. A quadratic-like map $F : U \to V$ is *infinitely renormalizable* if it is *k-times* renormalizable for every $k > 0$.

For an n'-renormalization (U', V') of a renormalizable quadratic-like map $F : U \to V$, suppose that $F_1 = F^{\circ n'} : U' \to V'$ has two repelling fixed points α and β in the filled-in Julia set K_{F_1}. One fixed point β does not disconnect K_{F_1}, i.e., $K_{F_1} \setminus \{\beta\}$ is still connected, and the other fixed point α disconnects K_{F_1}, i.e., $K_{F_1} \setminus \{\alpha\}$ is disconnected. McMullen [MC1] discovered that different types of renormalizations can occur in the renormalization theory of quadratic-like maps. Let $K(i) = F^{\circ i}(K_{F_1})$, $1 \leq i < n'$. An n'-renormalization (U', V') is

α-type if $K(i) \cap K(j) = \{\alpha\}$ for some $i \neq j$, $0 \leq i, j < n'$;

β-type if $K(i) \cap K(j) = \{\beta\}$ for some $i \neq j$, $0 \leq i, j < n'$;

disjoint type if $K(i) \cap K(j) = \emptyset$ for all $i \neq j$, $0 \leq i, j < n'$.

The β-type and the disjoint type are also called *simple*. If $F : U \to V$ is infinitely renormalizable, i.e., it has an infinite sequence of renormalizations, then it will have an infinite sequence of simple renormalizations. We will show a construction of a most natural infinite sequence

$$\{F_i = F^{\circ m_i} : U_i \to V_i\}_{i=1}^{\infty},$$

where $m_1 < m_2 < \cdots < m_k < \cdots$, of simple renormalizations by using two-dimensional puzzles in §2. Henceforth, we will always use this sequence as a sequence of renormalizations for an infinitely renormalizable quadratic-like map.

Now let $F : U \to V$ be an infinitely renormalizable quadratic-like map and let $\{F_i = F^{\circ m_i} : U_i \to V_i\}_{i=1}^{\infty}$ be the most natural infinite sequence of simple renormalizations in the previous paragraph. Suppose $\{J_{F_i}\}_{i=1}^{\infty}$ is the corresponding infinite sequence of Julia sets. (For an infinitely renormalizable quadratic-like map, its filled-in Julia set equals its Julia set.) We prove in §2 that J_{F_i} is *independent* of the choice of m_i-renormalizations (U_i, V_i) (Theorem 2). Thus J_{F_i} can be denoted as J_{m_i} and called a renormalization of J_F. The map $F : U \to V$ is said to have *complex bounds* if there are an infinite subsequence of simple renormalizations

$$\{F_{i_s} = F^{\circ m_{i_s}} : U_{i_s} \to V_{i_s}\}_{s=1}^{\infty}$$

and a constant $\lambda > 0$ such that the modulus of the annulus $V_{i_s} \setminus U_{i_s}$ is greater than λ for every $s > 0$ (see Definition 1). It is said to be *unbranched* if there are an infinite subsequence of renormalizations $\{J_{m_{i_l}}\}_{l=1}^{\infty}$ of the Julia set J_F, neighborhoods W_l of $J_{m_{i_l}}$, and a constant $\mu > 0$, such that the modulus of the annulus $W_l \setminus J_{m_{i_l}}$ is greater than μ and $W_l \setminus J_{m_{i_l}}$ is disjoint with the critical orbit $\{F^{\circ n}(0)\}_{n=0}^{\infty}$ for every $l > 0$ (see Definition 2).

For a quadratic polynomial $P(z) = z^2 + c$, let U be a fixed domain bounded by an equipotential curve of P (see §2) and let $V = P(U)$. Then $P : U \to V$ is a quadratic-like map whose Julia set is always J. We say P is infinitely renormalizable if $P : U \to V$ is infinitely renormalizable. The main result in this paper is

Theorem 3. *The Julia set of an unbranched infinitely renormalizable quadratic polynomial having complex bounds is locally connected.*

Two quadratic-like maps $F : U \to V$ and $G : W \to X$ are hybrid equivalent if there is a quasiconformal homeomorphism $H : V \to X$ such that $H \circ F = G \circ H$ and $H|K_F$, the restriction of H on the filled-in Julia set K_F of F, is conformal. The reader may refer to Ahlfors' book [AL] about definitions of quasiconformal and conformal maps. Any quadratic-like map with connected Julia set is hybrid equivalent to a unique quadratic polynomial, as shown in [DH]. Therefore, Theorem 3 applies to quadratic-like maps too. An example of an unbranched infinitely renormalizable quadratic polynomial having complex bounds is a real quadratic polynomial of bounded type (for example, the Feigenbaum polynomial) (see [SU], [MV], [MC2], [JI]). Before going into the next section, we mention some background information about renormalization and local connectivity in the study of the dynamics of quadratic polynomials. The renormalization technique was introduced into the study of dynamical systems by physicists Feigenbaum [FE1], [FE2] and Coullet and Tresser [CT] when the period doubling bifurcations following a universal law in any family of one-dimensional maps like the family of quadratic polynomials was observed about two decades ago. The technique is extensively used in the recent study of one-dimensional and complex dynamical systems (see, for example, [SU], [MV], [MC1], [MC2], [JI]). During this same period, computer-generated pictures of Julia sets of quadratic polynomials and the Mandelbrot set, which is the set of parameters c such that the Julia sets J_c of quadratic polynomials $P_c(z) = z^2 + c$ are connected, showed a fascinating world of fractal geometry (see [MA]). Douady and Hubbard (see [CG]) proved that the Mandelbrot set is connected. The next important problem in this direction is to show that the Mandelbrot set is locally connected. This has been a long-standing conjecture. The study of the local connectivity of the connected Julia set of a quadratic polynomial will give some important information on this conjecture. Moreover, if the Julia set of a quadratic polynomial is locally connected, then the *combinatorics* of this polynomial, that is the landing pattern of external rays (see §2 for the definition), determines completely the *topology* of the Julia set. Recently Yoccoz constructed a puzzle for the connected Julia set of a quadratic polynomial. Using these puzzles, he showed that the connected Julia set of a quadratic polynomial having no indifferent periodic points (see §2 for the definition) is locally connected if it is not infinitely renormalizable (see [HU]). Further, he translated these puzzles into a puzzle on the parameter space and showed that the Mandelbrot set is locally connected at all non-infinitely-renormalizable points (see [HU]). The remaining points to be verified are all infinitely renormalizable ones in the Mandelbrot set. We study these infinitely renormalizable quadratic polynomials. We construct a three-dimensional puzzle for the Julia set of an infinitely renormalizable quadratic polynomial. By using these three-dimensional puzzles we prove in this paper that the Julia set of a quadratic polynomial which admits an infinite sequence of unbranched, simple renormalizations with complex bounds is locally connected. The local connectivity of the Julia set of an infinitely renormalizable quadratic polynomial is not always guaranteed. Actually, Douady and Hubbard have constructed an infinitely renormalizable quadratic polynomial whose Julia set is not locally connected by a method called tuning (see [MI2]). Combining

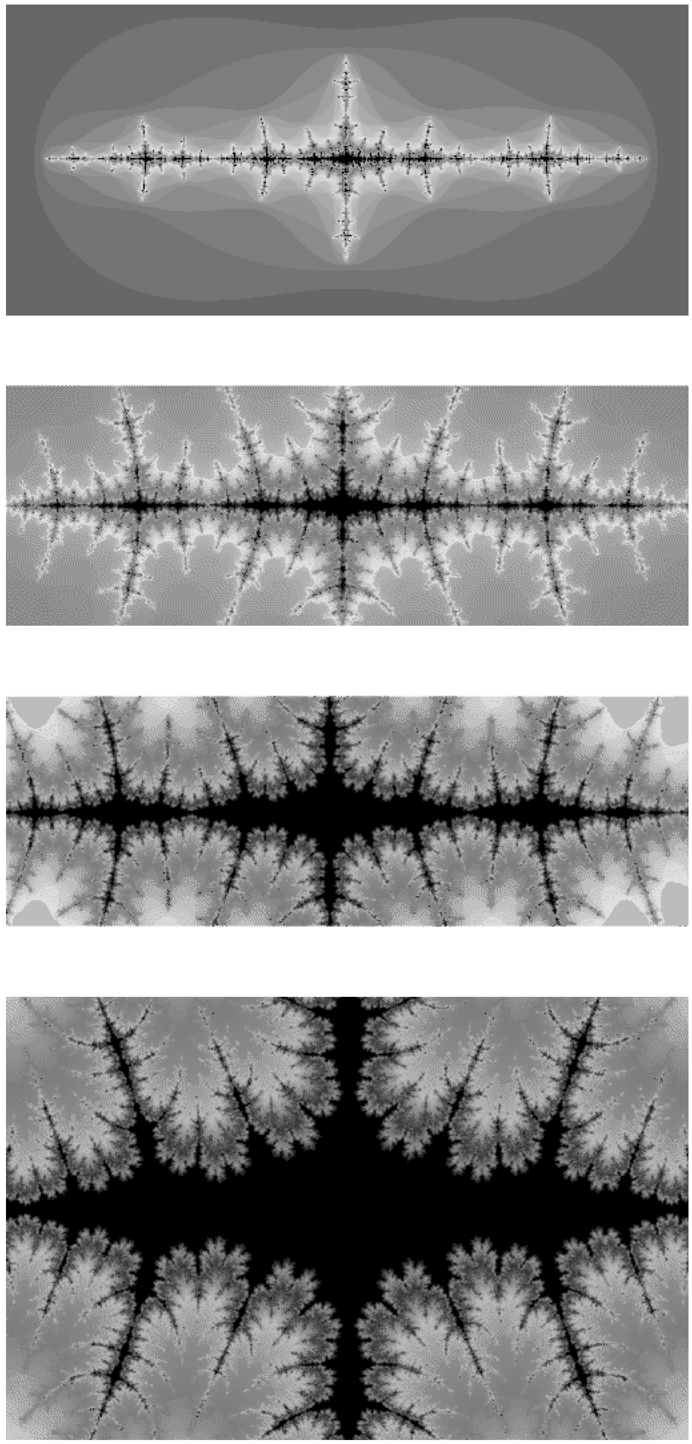

FIGURE 1. A computer picture of the Julia set of the Feigenbaum polynomial and three enlargements around 0.

this method and the Yoccoz inequality (see [HU]), Douady constructed a generic set of infinitely renormalizable points on the boundary of the Mandelbrot set such that the Mandelbrot set is locally connected at this set of points and such that the corresponding Julia sets are not locally connected (see [PM]). Douady's construction can be summarized as follows: In any copy of the Mandelbrot set, the rational limb p/q with $q >> p$ are small due to the Yoccoz inequality. Considering a nested sequence of copies of the Mandelbrot set which belong to an infinite number of limbs as described above, one sees that the intersection is a singular point at which the Mandelbrot set is locally connected. By further choosing orders of tunings higher and higher in the construction, one can make such a set such that the corresponding Julia sets are not locally connected. On the other hand, by translating the three-dimensional puzzles in this paper into a three-dimensional puzzle on the parameter space, we proved in [JIM] that there is a subset of infinitely renormalizable points in the Mandelbrot set such that the subset is dense on the boundary of the Mandelbrot set, and the Mandelbrot set is locally connected at this set of points, and the corresponding Julia sets are also locally connected. The reader may also refer to McMullen's recent book [MC2] for the latest developments in this direction and for an excellent dictionary between the study of complex dynamical systems and the study of Kleinian groups and hyperbolic geometry. The series of computer pictures in Figure 1 shows us how complicated the Julia set of an infinitely renormalizable quadratic-like map can be.

This paper is organized as follows: We study some properties of renormalizable quadratic polynomials in §2, and prove Theorem 2. In §3, we prove Theorem 1 and Theorem 3.

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2. Renormalization on quadratic polynomials

2.1. Equipotential curves and external rays. Let $P(z) = z^2+c$ be a quadratic polynomial and let J be its Julia set. A point p in $\mathbb{C}$ is called a periodic point of P of period n, where $n \geq 1$ is an integer, if $P^{\circ i}(p) \neq p$ for $1 \leq i < n$ and $P^{\circ n}(p) = p$. The number $E_p = (P^{\circ n})'(p)$ is called the multiplier of P at a periodic point p of period n. Then p is called attractive if $|E_p| < 1$; repelling if $|E_p| > 1$; and indifferent if $|E_p| = 1$. The point 0 is the critical point of P in the complex plane $\mathbb{C}$. Let $c_i = P^{\circ i}(0)$, $i \geq 1$, be the i^{th} critical value of P. Then $PCO = \{c_i\}_{i=1}^{\infty}$ is the post-critical orbit and $CO = PCO \cup \{0\}$ is the critical orbit. The critical point 0 is said to be recurrent if for any neighborhood W of 0 there is a critical value c_i, $i \geq 1$, in W. Henceforth we will only consider those quadratic polynomials whose critical points are recurrent. We will also assume that P has no attractive and indifferent periodic points. These are assumed for a quadratic-like map too. Therefore all periodic points of P will be repelling, and the Julia set J is connected and equal to its filled-in Julia set K.

Let D_1 be the open unit disk in the complex plane $\mathbb{C}$, and let $P_0(z) = z^2$. There is a Riemann mapping h from $\overline{\mathbb{C}} \setminus \overline{D}_1$ onto $\overline{\mathbb{C}} \setminus J$ such that $h(z)/z \to 1$ as z tends to infinity and

$$h \circ P_0 = P \circ h$$

on $\mathbb{C} \setminus \overline{D}_1$ (see [MC1], [MI1]). The image S_t of a circle $s_t = \{te^{2\pi i\theta} \mid 0 \leq \theta < 1\}$ for $1 < t < \infty$ under h is called an equipotential curve of P. If we consider the Green's function $G(z) = \log |h^{-1}(z)| = \lim_{n\to\infty}(\log^+ |P^{\circ n}(z)|)/2^n$ defined on $\mathbb{C} \setminus J$, then $G(P(z)) = 2G(z)$, and it takes the constant value $\log t$ when it is restricted on S_t, where $\log^+ x = \max\{\log x, 0\}$. This implies that

$$P(S_t) = S_{t^2}.$$

Let U_t be the open domain bounded by S_t. Then $P : U_t \to U_{t^2}$ is a quadratic-like map. The image R_θ of a ray $r_\theta = \{te^{2\pi i\theta} \mid 1 < t < \infty\}$ under h is called an external ray of angle θ. Thus,

$$P(R_\theta) = R_{2\theta\,(\mathrm{mod}\,1)}.$$

An external ray R_θ is called periodic if $P^{\circ k}(R_\theta) = R_\theta$ for some k, and the smallest such k is called the period of R_θ. Douady and Yoccoz proved that every repelling periodic point of P is a landing point of finitely many periodic external rays of the same period, and furthermore, these landing rays are in the same orbit because P is a quadratic polynomial (see [HU], [MI2]).

2.2. Construction of Yoccoz puzzles. Take a quadratic polynomial $P(z) = z^2 + c$ as in §2.1. Let us fix an equipotential curve S_t and the domain $U = U_t$ bounded by S_t. Then we have a quadratic-like map $F = P : U \to V = P(U)$ whose Julia set is J. Then F has two fixed points. One of them, say β, is non-separating, and the other, say α, is separating, i.e., $J \setminus \{\beta\}$ is still connected and $J \setminus \{\alpha\}$ is not. There are at least two, but a finite number, external rays of P landing at α. Let Γ_0^0 be the union of a cycle of external rays landing at α. Then Γ_0^0 cuts $U_0^0 = U$ into finitely many domains. Let η_0 be the collection of the closure of these domains. Let $\Gamma_n^0 = F^{-n}(\Gamma_0^0)$ for any $n > 0$. Then Γ_n^0 cuts $U_n^0 = F^{-n}(U_0^0)$ into finitely many domains. Let η_n be the collection of the closures of these domains. The sequence $\xi^0 = \{\eta_n\}_{n=0}^\infty$ is called the *Yoccoz puzzle* for J (refer to [BH], [HU], [MI2], [JI]). The domain C_n in η_n containing 0 is called the critical piece in η_n. It is clear that P restricted to all domains but C_n is bijective to domains in η_{n-1}, and $P|C_n$ is a degree two branched cover map onto a domain in η_{n-1}. Let

$$J_1 = \bigcap_{n=0}^\infty C_n.$$

The following result follows directly from the result of Yoccoz about local connectivity of nonrenormalizable quadratic polynomials (refer to [HU], [MI2], [JI]), and gives an equivalent definition of renormalizability:

Theorem A (Yoccoz)**.** *Suppose $P(z) = z^2 + c$ has recurrent critical orbit. Then P is renormalizable if and only if J_1 consists of more than one point.*

We will use $N(X, \epsilon) = \{x \in \mathbb{C} \mid d(x, X) < \epsilon\}$ to denote the ϵ-neighborhood of X in the complex plane in this paper. Suppose P is renormalizable. We have two integers $n_1 \geq 0$, $m_1 > 1$ such that $F_1 = F^{m_1} : C_{m_1+n_1} \to C_{n_1}$ is a degree two branched cover map and such that $C_{m_1+n_1} \subset N(J_1, 1)$ (refer to [HU], [MI2], [JI]). We can further take domains $C_{n_1+m_1} \subseteq U_1 \subset U$ and $C_{n_1} \subseteq V_1 \subset V$ such that

$$F_1 = F^{\circ m_1} : U_1 \to V_1$$

is a quadratic-like map. Then its Julia set is J_1. (Note that $F_1 = F^{\circ m_1} : U_1 \to V_1$ is a simple renormalization of $F : U \to V$.)

Theorem 2. *Suppose $G = P^{\circ m_1} : U' \to V'$ is any m_1-renormalization of P. Then its filled-in Julia set is always J_1.*

Proof. The point 0 is in the intersection $U' \cap U_1$. Suppose U'' is the connected component of $U' \cap U_1$ containing 0. Then $H = P^{\circ m_1} : U'' \to V'' \subset V' \cap V_1$ is also a renormalization of P, and its Julia set J_H is connected. It is easy to check that $J_H \subseteq J_G \cap J_{F_1}$. Let β_{F_1} and α_{F_1} be the non-separating and separating fixed points of F_1, and let β_G and α_G be the non-separating and separating fixed points of G. All β_{F_1}, α_{F_1}, β_G, and α_G are in J_H, since each of G, F and H has exactly two fixed points in its domain, and every fixed point of H has to be a common fixed point of G and F. Since $F_1 = P^{\circ m_1} : U_1 \to V_1$ and $G = P^{\circ m_1} : U' \to V'$ are both degree two branched cover maps and β_{F_1} is in both U' and U_1, all preimages of β_{F_1} under iterates of G and F_1 are in both U' and U_1. But each of J_H, J_1, and J_G is the closure of the set of all preimages of β_{F_1} under iterates of H, G, and F_1. Therefore, $J_H = J_G = J_1$. □

Remark 1. From Theorem 2, for any m_1-renormalization (U', V') of $F : U \to V$, there is $C_{m_1+n} \subset U'$ such that $F_1 = F^{m_1} : C_{m_1+n} \to C_n \subset V'$ is a degree two branched cover map.

Remark 2. From the Douady and Hubbard theorem [DH] that every quadratic-like map $F : U \to V$ whose Julia set is connected is hybrid equivalent to a unique quadratic polynomial, all arguments in this section can be applied to quadratic-like maps (by considering induced external rays and equipotential curves from the hybrid equivalent quadratic polynomial).

2.3. Construction of three-dimensional puzzles. Now we assume $P(z) = z^2 + c$ is an infinitely renormalizable quadratic polynomial. Let $k_1 = m_1$, $\alpha_1 = \alpha$, $\beta_1 = \beta$, $\eta_n^0 = \eta_n$, $C_n^0 = C_n$, and let n_1, J_1, Γ_n^0, U_n^0, ξ^0, $F : U \to V$ be as in the previous subsection. Suppose β_2 and α_2 are the non-separating and separating fixed points of F_1, i.e., $J_1 \setminus \{\beta_2\}$ is still connected and $J_1 \setminus \{\alpha_2\}$ is not. The points β_2 and α_2 are also repelling periodic points of P. There are at least two, but finitely many, external rays of P landing at α_2. Let Γ_0^1 be the union of a cycle of external rays landing at α_2. Then Γ_0^1 cuts $U_0^1 = C_{n_1+m_1}^0$ into finitely many domains. Let η_0^1 be the collection of the closures of these domains. Let $\Gamma_n^1 = F_1^{-n}(\Gamma_0^1)$ for any $n > 0$. Then Γ_n^1 cuts $U_n^1 = F_1^{-n}(U_0^1)$ into finitely many domains. Let η_n^1 be the collection of the closures of these domains. The sequence $\xi^1 = \{\eta_n^1\}_{n=0}^{\infty}$ is the two-dimensional puzzle for J_1. We call it the first puzzle. (We also call ξ^0 the 0^{th} puzzle.)

The domain C_n^1 in η_n^1 containing 0 is called the critical piece in η_n^1. It is clear that F_1 restricted to all domains but C_n^1 is bijective to domains in η_{n-1}^1, and $P|C_n^1$ is a degree two branched cover map onto a domain in η_{n-1}^1. Let

$$J_2 = \bigcap_{n=0}^{\infty} C_n^1.$$

There are two integers $n_2 \geq 0$, $k_2 > 1$ such that

$$F_2 = F_1^{\circ k_2} : C_{n_2+k_2}^1 \to C_{n_2}^1$$

is a degree two branched cover map and such that $C_{n_2+k_2}^1 \subset N(J_2, 1/2)$. We take domains $C^1_{n_2+k_2} \subseteq U_2 \subset U_1$ and $C^1_{n_2} \subseteq V_2 \subset V_1$ such that

$$F_2 = F_1^{\circ k_2} : U_2 \to V_2$$

is a quadratic-like map. Then its Julia set is J_2.

Inductively, for every $i \geq 2$, suppose we have constructed

$$F_i = F_{i-1}^{\circ k_i} : C^{i-1}_{n_i+k_i} \to C^{i-1}_{n_i} \qquad \text{and} \qquad F_i = F_{i-1}^{\circ k_i} : U_i \to V_i,$$

whose Julia set is J_i. Let β_{i+1} and α_{i+1} be the non-separating and separating fixed points of F_i; i.e., $J_i \setminus \{\beta_{i+1}\}$ is still connected and $J_i \setminus \{\alpha_{i+1}\}$ is not. The points β_{i+1} and α_{i+1} are also repelling periodic points of P. There are at least two, but a finite number of, external rays of P landing at α_{i+1}. Let Γ^i_0 be the union of a cycle of external rays landing at α_{i+1}. Then Γ^i_0 cuts $U^i_0 = C^{i-1}_{n_i+k_i}$ into finitely many domains. Let η^i_0 be the collection of the closures of these domains. Let $\Gamma^i_n = F_i^{-n}(\Gamma^i_0)$ for any $n > 0$. Then Γ^i_n cuts $U^i_n = F_i^{-n}(U^i_0)$ into finitely many domains. Let η^i_n be the collection of the closures of these domains. The domain C^i_n in η^i_n containing 0 is called the critical piece in η^i_n. It is clear that F_i restricted to all domains but C^i_n is bijective to domains in η^i_{n-1}, and $P|C^i_n$ is a degree two branched cover map onto a domain in η^i_{n-1}. Let

$$J_{i+1} = \bigcap_{n=0}^{\infty} C^i_n.$$

There are two integers $n_{i+1} \geq 0$, $k_{i+1} > 1$ such that

$$F_{i+1} = F_i^{\circ k_{i+1}} : C^i_{n_{i+1}+k_{i+1}} \to C^i_{n_{i+1}}$$

is a degree two branched cover map and such that $C^i_{n_{i+1}+k_{i+1}} \subset N(J_{i+1}, 1/(i+1))$. We take domains $C^i_{n_{i+1}+k_{i+1}} \subseteq U_{i+1} \subset U_i$ and $C^i_{n_{i+1}} \subseteq V_{i+1} \subset V_i$ such that

$$F_{i+1} = F_i^{\circ k_{i+1}} : U_{i+1} \to V_{i+1}$$

is a quadratic-like map. Then its Julia set is J_{i+1}. Let $\xi^i = \{\eta^i_n\}_{n=0}^{\infty}$. It is the two-dimensioanl puzzle for J_i. We call it the i^{th} partition.

Remark 3. For any k_{i+1}-renormalization (U', V') of $F_i : U_i \to V_i$, we have an integer $n > 0$ such that $C^i_{n+k_{i+1}} \subset U' \cap N(J_{i+1}, 1/(i+1))$ and $C^i_n \subset V'$, and such that $F_{i+1} = F_i^{\circ k_{i+1}} : C^i_{n+k_{i+1}} \to C^i_n$ is a degree two branched cover map. We will still use ξ^i to mean $\xi^i \cap C^i_{n+k_{i+1}}$. Therefore, (U_{i+1}, V_{i+1}) can be an arbitrary k_{i+1}-renormalization of $F_i : U_i \to V_i$.

Let $m_i = \prod_{j=1}^{i} k_i$, $1 \leq i < \infty$. We have thus constructed a most natural infinite sequence of simple renormalizations of $F : U \to V$,

$$\{F_i = F^{\circ m_i} : U_i \to V_i\}_{i=1}^{\infty},$$

and the nested-nested sequence $\{\xi^i\}_{i=0}^{\infty}$ of partitions for $\{J_i\}_{i=0}^{\infty}$ (where $J_0 = J$), which we call a *three-dimensional puzzle*. Henceforth, we will fix all notations in this subsection.

Definition 1. We say an infinitely renormalizable quadratic polynomial $P(z) = z^2 + c$ has *complex bounds* if there are a constant $\lambda > 0$ and an infinite sequence of simple renormalizations $\{F_{i_s} = F^{\circ m_{i_s}} : U_{i_s} \to V_{i_s}\}_{s=1}^{\infty}$ such that the modulus $\mathrm{mod}(V_{i_s} \setminus \overline{U}_{i_s})$ is greater than λ for every $s \geq 1$.

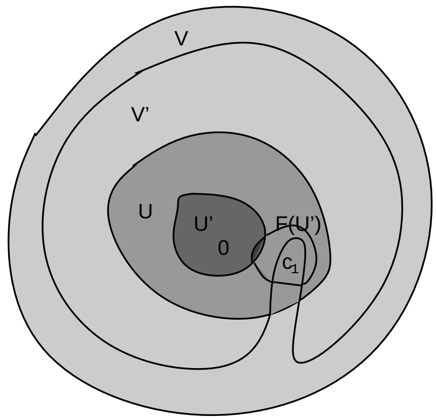

FIGURE 2. Modulus inequality in renormalization

Definition 2. We say an infinitely renormalizable quadratic polynomial $P(z) = z^2+c$ is *unbranched* if there are an infinite subsequence of renormalizations $\{J_{i_l}\}_{l=1}^{\infty}$ of the Julia set J of P, neighborhoods W_l of J_{i_l} for every $l > 0$, and a constant $\mu > 0$ such that the modulus $\mathrm{mod}(W_l \setminus J_{i_l})$ is greater than μ and $W_l \setminus J_{i_l}$ contains no point in the critical orbit CO of P.

Remark 4. For the same reason as that in Remark 2, all arguments in this section can be applied to an infinitely renormalizable quadratic-like map.

3. THREE-DIMENSIONAL PUZZLES AND LOCAL CONNECTIVITY

Before we use the three-dimensional puzzle for the Julia set J of an infinitely renormalizable quadratic polynomial $P(z) = z^2 + c$ to study the local connectivity of J, we first prove the following result, which we call a *modulus inequality in renormalization* (see Figure 2). Remember that an n'-renormalization (U', V') of a quadratic-like map $F : U \to V$ means a pair of domains $U' \subset U$ and $V' \subset V$ such that $F_1 = F^{\circ n'} : U' \to V'$ is a quadratic-like map with connected filled-in Julia set.

Theorem 1. *Suppose $F : U \to V$ is a renormalizable quadratic-like map. Consider any n'-renormalization (U', V'), $n' > 1$ (which may or may not be simple). Then*

$$\mathrm{mod}(U \setminus \overline{U'}) \geq \frac{1}{2}\,\mathrm{mod}(V \setminus \overline{U}).$$

Proof. Since $F_1 = F^{\circ n'} : U' \to V'$ is quadratic-like, its only critical point is 0. So the first critical value $c_1 = F(0)$ of F is not in V'. (Otherwise, there will be a point $x \neq 0$ in U' such that $F^{\circ(n'-1)}(x) = 0$, since c_1 has only one preimage 0 under F and 0 is not a periodic point of F as we assumed in §2.1. Then x will be a critical point of F_1.) Since V' is simply connected, F has two analytic inverse branches,

$$g_0 : V' \to g_0(V') \subset U \qquad \text{and} \qquad g_1 : V' \to g_1(V') \subset U.$$

One of them is $F^{\circ(n'-1)}(U')$. Therefore $F(U')$ is a domain inside U and containing c_1. Consider the annuli $U \setminus \overline{U'}$ and $V \setminus \overline{F(U')}$. Then

$$F : U \setminus \overline{U'} \to V \setminus \overline{F(U')}$$

is a degree two cover map. This implies that

$$\mathrm{mod}(U \setminus \overline{U'}) = \frac{1}{2}\,\mathrm{mod}(V \setminus \overline{F(U')}).$$

Since $F(U')$ is a subset of U, the annulus $V \setminus \overline{U}$ is a sub-annulus of the annulus $V \setminus \overline{F(U')}$, i.e., $V \setminus \overline{U} \subseteq V \setminus \overline{F(U')}$. Thus we have that

$$\mathrm{mod}(U \setminus \overline{U'}) \geq \frac{1}{2}\mathrm{mod}(V \setminus \overline{U}).$$

□

Now we prove the main result about local connectivity by using three-dimensional puzzles.

Theorem 3. *The Julia set of an unbranched infinitely renormalizable quadratic polynomial having complex bounds is locally connected.*

We break the proof of the theorem into four lemmas. We use the same notations as in the previous section.

Suppose $P(z) = z^2 + c$ is an infinitely renormalizable quadratic polynomial. Let J be its Julia set and $\{\xi^i = \{\eta_n^i\}_{n=0}^{\infty}\}_{i=0}^{\infty}$ be the three-dimensional puzzle. We first classify points in J into two categories. For a point x in J, let $O(x) = \{P^{\circ n}(x)\}_{n=0}^{\infty}$ be its orbit and $\overline{O(x)}$ the closure of its orbit. Then x is *non-recurrent* to 0 in the three-dimensional puzzle if $\overline{O(x)} \cap J_i = \emptyset$ for some $i \geq 1$; otherwise it is *recurrent* to 0 in the three-dimensional puzzle. For examples, all non-separating and separating fixed points α_{j+1} and β_{j+1} of F_j, $1 \leq j < \infty$, and their preimages under iterations of P are non-recurrent, and the critical orbit and its preimages under iterations of P are recurrent. There are many other non-recurrent and recurrent points.

Lemma 1. *For any domain D in η_n^i, $i, n \geq 0$, $D \cap J$ is connected.*

Proof. Since the domain D is bounded by finitely many external rays $\Pi = \{r_{\theta_j}\}_{j=1}^{m}$ and by some equipotential curve, then $\partial D \cap J$ consists of a finite number of points $\{p_i\}_{j=1}^{m'}$. Every p_i is a landing point of two external rays in Π. Suppose $D \cap J$ is not connected for some D in η_n^i. Then there are two disjoint open sets X and Y such that $D \cap J = (D \cap J \cap X) \cup (D \cap J \cap Y)$. Suppose that $p_1, \ldots, p_{m''}$ are in X and that $p_{m''+1}, \ldots, p_{m'}$ are in Y. The two external rays in Π landing at p_j cut $\mathbb{C}$ into two domains. One of them, denoted by Z_j, is disjoint with D, i.e., $D \cap Z_j = \emptyset$. Then $U' = \cup \bigcup_{j=1}^{m''} Z_j$ and $V' = V \cup \bigcup_{j=m''+1}^{m'} Z_j$ are two disjoint open sets, and $J = (U' \cap J) \cup (V' \cap J)$. This contradicts the fact that J is connected. □

Lemma 2. *The Julia set J is locally connected at any non-recurrent point.*

Proof. Suppose x in J is non-current. Take $n_0 = 0$, $k_0 = 1$, and $C_{n_0+k_0}^{-1} = \overline{U}$. We also have $C_{n_j+k_j}^{j-1}$ for all $j \geq 1$. Then there is the smallest integer $i \geq 0$ such that $O(x) \cap C_{n_i+k_i}^{i-1} \neq \emptyset$ and $\overline{O(x)} \cap J_{i+1} = \emptyset$. Consider the i^{th} puzzle $\xi^i = \{\eta_n^i\}_{n=0}^{\infty}$ inside $C_{n_i+k_i}^{i-1}$. Since $J_{i+1} = \bigcap_{n=0}^{\infty} C_n^i$, there is an integer $N \geq 0$ such that $\overline{O(x)} \cap C_N^i = \emptyset$. First let us assume that x is in $C_{n_i+k_i}^{i-1}$. Consider

$$\eta_N^i = \{C_N^i, B_{N,1}, B_{N,2}, \ldots, B_{N,q}\}$$

in the i^{th} puzzle ξ^i. Assume that $B_{N,1}$ contains $P(0)$. (Then $P(0)$ is an interior point of $B_{N,1}$.) The orbit $P(O(x))$ is disjoint with $B_{N,1}$, since $O(x)$ is disjoint with $C^i_{N+1} \subseteq C^i_N$ and $P(C^i_{N+1}) = B_{N,1}$. Suppose $x \in D$ is a domain in η^i_{N+1} in the i^{th} puzzle ξ^i. Suppose $D \subseteq B_{N,j}$ and $P(D) = B_{N,i}$ for $2 \leq i, j \leq q$. Then $P : D \to B_{N,i}$ has the inverse $g_{ij} : B_{N,i} \to D$. Therefore for each pair $(B_{N,i}, B_{N,j})$, $2 \leq i, j \leq q$, satisfying $g_{ij}(B_{N,i}) \subseteq B_{N,j}$, we can thicken $B_{N,i}$ and $B_{N,j}$ to simply connected domains $\tilde{B}_{N,i,ij}$ and $\tilde{B}_{N,j,ij}$ such that $B_{N,i} \subset \tilde{B}_{N,i,ij}$ and $B_{N,j} \subset \tilde{B}_{N,j,ij}$ and such that g_{ij} can be extended to a schlicht function from $\tilde{B}_{N,i,ij}$ into $\tilde{B}_{N,j,ij}$. We still use g_{ij} to denote this extended function. Now consider $\tilde{B}_{N,i,ij}$ and $\tilde{B}_{N,j,ij}$ as hyperbolic Riemann surfaces with hyperbolic distances $d_{H,i,ij}$ and $d_{H,j,ij}$. Then g_{ij} on $B_{N,i}$ strictly contracts these hyperbolic distances; more precisely, there is a constant $0 < \lambda_{ij} < 1$ such that $d_{H,i,ij}(g_{ij}(x), g_{ij}(y)) < \lambda_{ij} d_{H,j,ij}(x, y)$ for x and y in $B_{N,i}$. Since there are only a finite number of such pairs, we have a constant $0 < \lambda < 1$ such that $d_{H,i,ij}(g_{ij}(x), g_{ij}(y)) < \lambda d_{H,j,ij}(x, y)$ for all such pairs. Let

$$x \in \cdots \subseteq D^i_n(x) \subseteq D^i_{n-1}(x) \subseteq \cdots \subseteq D^i_1(x) \subseteq D^i_0(x)$$

be any x-end in the i^{th} puzzle ξ^i, where $D^i_n(x) \in \eta^i_n$. Then $P^{\circ m}(D^i_n(x))$ for $n - m > N$ is in $B_{N,j}$ for some $2 \leq j \leq q$. Therefore, there is a constant $C > 0$ such that for any $D^i_n(x)$ and for any $n > N$,

$$d(D^i_n(x)) = \max_{y,z \in D^i_n(x)} |y - z| \leq C\lambda^{n-N}.$$

Thus $d(D^i_n(x))$ tends to zero as n goes to infinity. From Lemma 1, $J \cap D^i_n(x)$ is connected. If x is an interior point of $D^i_n(x)$ for all n, then $\{D^i_n(x)\}_{i=1}^{\infty}$ is a basis of connected neighborhoods of J at x. Thus J is locally connected at x.

If x is on the boundary of $D^i_n(x)$ but in the interior of $C^{i-1}_{n_i+k_i}$, then we consider all x-ends

$$x \in \cdots \subseteq D^i_{s,n}(x) \subseteq D^i_{s,n-1}(x) \subseteq \cdots \subseteq D^i_{s,1}(x) \subseteq D^i_{s,0}(x)$$

in the i^{th} puzzle ξ^i. The number of these ends is finite. Then x is an interior point of the domain $\bigcup_s D^i_{s,n}(x)$ for all $n \geq 1$. Also, since $\bigcup_s D^i_{s,n}(x)$ is bounded by external rays and equipotential curves of P, $J \cap (\bigcup_s D^i_{s,n}(x))$ is connected, following a similar proof for Lemma 1. Thus $\{\bigcup_s D^i_{s,n}(x)\}_{i=1}^{\infty}$ is a basis of connected neighborhoods of J at x, and J is locally connected at x.

If x is on the boundary of $C^{i-1}_{n_i+k_i}$, then $i \geq 1$, and x is a preimage of the separating fixed point α_i of $F_{i-1} : U_{i-1} \to V_{i-1}$ under some iterates of F_{i-1}. (Denote $F_0 = F$, $U_0 = U$, $V_0 = V$.) Suppose $P^{\circ q}(x) = \alpha_i$. Then $P^{\circ q}$ restricted on a small neighborhood of x is homeomorphic. Thus we have sequences of nested domains

$$\alpha_i \in \cdots \subseteq P^{\circ q}(D^i_{s,n}(x)) \subseteq P^{\circ q}(D^i_{s,n-1}(x)) \subseteq \cdots \subseteq P^{\circ q}(D^i_{s,N_0}(x))$$

for some big $N_0 > 0$ such that the domains $E_n = \bigcup_s P^{\circ q}(D^i_{s,n}(x))$ are bounded by external rays and equipotential curves of P and the diameter $\mathrm{diam}(E_n)$ tends to 0 as n goes to ∞. Since α_i is the repelling periodic point of P of period m_i, we have a finite number of domains $E_{n,i} = P^{\circ i m_i}(E_n)$ cyclically around α_i such that the domain $\bigcup_i E_{n,i}$ has x as an interior point and is bounded by external rays and equipotential curves. The diameter $\mathrm{diam}(\bigcup_i E_{n,i})$ tends to 0 as n goes to ∞. Thus, similarly to Lemma 1, $J \cap (\bigcup_i E_{n,i})$ is connected and $\{\bigcup_i E_{n,i}\}_{n=N_1}^{\infty}$ for some large $N_1 \geq N_0$ is a basis of connected neighborhoods of J at α_i. This basis can be pulled back by the local homeomorphism $P^{\circ q}$ on a small neighborhood of x to get a basis of connected neighborhoods of J at x. Therefore, J is locally connected at x.

If x is not in $C^{i-1}_{n_i+k_i}$, let $r \geq 1$ be the smallest integer such that $y = P^{\circ r}(x)$ is in $C^{i-1}_{n_i+k_i}$. The above argument says J is locally connected at y. Since x is not a critical point of $P^{\circ r}$, $P^{\circ r}$ restricted on a small neighborhood of x is homeomorphic. Thus J is also locally connected at x. $\square$

Lemma 3. *If P has complex bounds, then the Julia set J is locally connected at* 0.

Proof. To make notations simple, we assume $\{F_i = F^{\circ m_i} : U_i \to V_i\}_{i=1}^{\infty}$ is the infinite sequence of simple renormalizations in Definition 1. Let $\lambda > 0$ be the constant in Definition 1. Then $\{U_i\}_{i=1}^{\infty}$ is a sequence of nested domains containing 0. From the modulus inequality, Theorem 1, we have

$$\mathrm{mod}(U_i \setminus \overline{U}_{i+1}) \geq \frac{1}{2}\,\mathrm{mod}(V_i \setminus \overline{U}_i) > \frac{\lambda}{2}.$$

Let $A_i = U_i \setminus \overline{U}_{i+1}$ for $i \geq 1$ and $X = \bigcap_{i=1}^{\infty} U_i$. Since $U_1 \setminus X = \bigcup_{i=1}^{\infty} A_i$,

$$\mathrm{mod}(U_1 \setminus X) \geq \sum_{i=1}^{\infty} \mathrm{mod}(A_i) = \infty.$$

Thus, $X = \{0\}$. This implies that the diameter $d(U_i)$ tends to 0 as i goes to infinity.

Consider the i^{th} puzzle $\xi^i = \{\eta^i_n\}_{n=0}^{\infty}$ for every $i \geq 0$. Remember that C^i_n is the member in η^i_n containing 0. Consider the corresponding critical end

$$0 \in \cdots \subseteq C^i_n \subseteq C^i_{n-1} \subseteq \cdots \subseteq C^i_1 \subseteq C^i_0$$

in the i^{th}-puzzle ξ^i. Since $J_{i+1} = \bigcap_{j=0}^{\infty} C^i_j$ is the Julia set of $F_{i+1} : U_{i+1} \to V_{i+1}$, there is a $C^i_{n(i)}$ contained in U_{i+1}. The diameter $d(C^i_{n(i)})$ of $C^i_{n(i)}$ tends to zero as i goes to infinity. But $C^i_{n(i)} \cap J$ is connected from Lemma 1. So $\{C^i_{n(i)}\}_{i=1}^{\infty}$ is a basis of connected neighborhoods of J at 0, and J is locally connected at 0. $\square$

Corollary 1. *Suppose $\{J_i\}_{i=1}^{\infty}$ is the infinite sequence of renormalizations of the Julia set J. If P has complex bounds, then $\bigcap_{i=1}^{\infty} J_i = \{0\}$.*

Proof. It follows that $J_i \subseteq C^i_{n(i)}$ and $\bigcap_{i=1}^{\infty} C^i_{n(i)} = \{0\}$. $\square$

Lemma 4. *If $\bigcap_{i=1}^{\infty} J_i = \{0\}$ and if P is unbranched, then J is locally connected at all recurrent points.*

Proof. Consider the i^{th} puzzle $\xi^i = \{\eta^i_n\}_{n=0}^{\infty}$ for every $i \geq 0$, and the corresponding critical end

$$0 \in \cdots \subseteq C^i_n \subseteq C^i_{n-1} \subseteq \cdots \subseteq C^i_1 \subseteq C^i_0.$$

Consider $F_{i+1} = F_i^{k_{i+1}} : C^i_{n_{i+1}+k_{i+1}} \to C^i_{n_{i+1}}$. Let $k(i) = n_{i+1} + k_{i+1}$. Since $\bigcap_{i=1}^{\infty} J_i = \{0\}$ and $C^i_{k(i)} \subset N(J_{i+1}, 1/(i+1))$, $\{C^i_{k(i)}\}_{i=0}^{\infty}$ is a basis of connected neighborhoods of J at 0.

Let $\mu > 0$ be a constant and let $\{W_l\}_{l=1}^{\infty}$ be domains satisfying Definition 2. Without loss of generality, we assume in Definition 2 that $l = i$ and $i_l = i + 1$. Since $\mathrm{mod}(W_i \setminus J_{i+1}) \geq \mu$ and $J_{i+1} = \bigcap_{n=0}^{\infty} C^i_n$, by choosing $k(i)$ large enough, we can assume that

$$\mathrm{mod}(W_i \setminus C^i_{k(i)}) \geq \frac{\mu}{2}.$$

for all $i \geq 1$. Also, by modifying W_i, we can assume the diameter $\mathrm{diam}(W_i)$ tends to zero as i goes to ∞.

We first construct a sequence of partitions for the Julia set J from the three-dimensional puzzle $\{\xi^i\}_{i=0}^{\infty}$. Denote by τ_1 the first partition, which will be constructed as follows: Consider the 0^{th} puzzle $\xi^0 = \{\eta_n^0\}_{n=0}^{\infty}$. Take $C_{k(0)}^0 \in \eta_{k(0)}^0 \in \xi^0$. Put all domains in $\eta_{k(0)+1}$ which are the preimages of $C_{k(0)}^0$ under F in τ_1, and let $\eta_{k(0)+1}^c$ be the rest of the domains. Consider $\eta_{k(0)+2} \cap \eta_{k(0)+1}^c$, consisting of all domains in $\eta_{k(0)+2}$ which are subdomains of the domains in $\eta_{k(0)+1}^c$. Put all domains in $\eta_{k(0)+2} \cap \eta_{k(0)+1}^c$ which are the preimages of $C_{k(0)}^0$ under $F^{\circ 2}$ in τ_1, and let $\eta_{k(0)+2}^c$ be the rest of the domains. Suppose we already have $\eta_{k(0)+s}^c$ for $s \geq 2$. Consider $\eta_{k(0)+s+1} \cap \eta_{k(0)+s}^c$, consisting of all domains in $\eta_{k(0)+s+1}$ which are subdomains of the domains in $\eta_{k(0)+s}^c$. Put all domains in $\eta_{k(0)+s+1} \cap \eta_{k(0)+s}^c$ which are the preimages of $C_{k(0)}^0$ under $F^{\circ(s+1)}$ in τ_1, and let $\eta_{k(0)+s+1}^c$ be the rest of the domains. Thus we can construct the partition τ_1 inductively. This partition covers the Julia set J minus all points not entering the interior of $C_{k(0)}^0$ under all iterations of F.

Consider the first puzzle $\xi^1 = \{\eta_n^1\}_{n=0}^{\infty}$. Take $C_{k(1)}^1 \in \eta_{k(1)}^1 \in \xi^1$. We can use arguments similar to those in the previous paragraph by considering $F_1 : C_{k(0)}^0 \to C_{k(0)-m_1}^0$ to get a partition $\tau_{1,1}$ in $C_{k(0)}^0$. Then we use all iterations of F to pull back this partition following τ_1 to get a partition τ_2. It is a sub-partition of τ_1, and covers the Julia set J minus all points not entering the interior of $C_{k(1)}^1$ under iterations of F.

Suppose we have already constructed the $(j-1)^{th}$ partition τ_{j-1} for $j > 2$. Consider the puzzle $\xi^j = \{\eta_n^j\}_{n=0}^{\infty}$. Take $C_{k(j)}^j \in \eta_{k(j)}^j \in \xi^j$. Similarly, by considering $F_j : C_{k(j-1)}^{j-1} \to C_{k(j-1)-m_j}^{j-1}$, we get a partition $\tau_{j,1}$ in $C_{k(j-1)}^{j-1}$. Then we use all iterations of F_{j-1} to pull back this partition following τ_{j-1} to get a partition $\tau_{j,2}$ in $C_{k(j-2)}^{j-2}$, and all iterations of F_{j-2} to pull back this partition following τ_{j-1} to get a partition $\tau_{j,3}$ in $C_{k(j-3)}^{j-3}$, and so on, to obtain a partition $\tau_j = \tau_{j,j}$ in U. It is a sub-partition of τ_{j-1}, and covers the Julia set minus all points not entering the interior of $C_{k(j)}^j$ under iterations of F. By induction, we have a sequence of nested partitions $\{\tau_j\}_{j=1}^{\infty}$, which covers the Julia set J minus all non-recurrent points. We call $\{\tau_j\}_{j=1}^{\infty}$ the *(extended) three-dimensional puzzle for the Julia set J*.

Suppose $x \neq 0$ in J is recurrent. Then the orbit $O(x) = \{P^{\circ n}(x)\}_{n=0}^{\infty}$ enters every $C_{k(i)}^i$ infinitely many times. Consider the (extended) three-dimensional puzzle $\{\tau_j\}_{j=1}^{\infty}$ and the x-end in this puzzle,

$$x \in \cdots \subseteq D_j(x) \subseteq D_{j-1}(x) \subseteq \cdots \subseteq D_1(x),$$

where $D_j(x) \in \tau_j$. Let $q_j(x) \geq 0$ be the unique integer such that

$$F^{\circ q_j(x)} : D_{j+1}(x) \to C_{k(j)}^j$$

is a proper holomorphic diffeomorphism (see Figure 3). Let

$$g_{j,x} : C_{k(j)}^j \to D_{j+1}(x)$$

be its inverse. *At this point we use the unbranched condition.* Since there are no critical values $\{c_r = P^{\circ r}(0)\}_{r=1}^{\infty}$ in $W_j \setminus C_{k(j)}^j$, $g_{j,x}$ can be extended to a proper holomorphic diffeomorphism on W_j which we still denote by $g_{j,x}$.

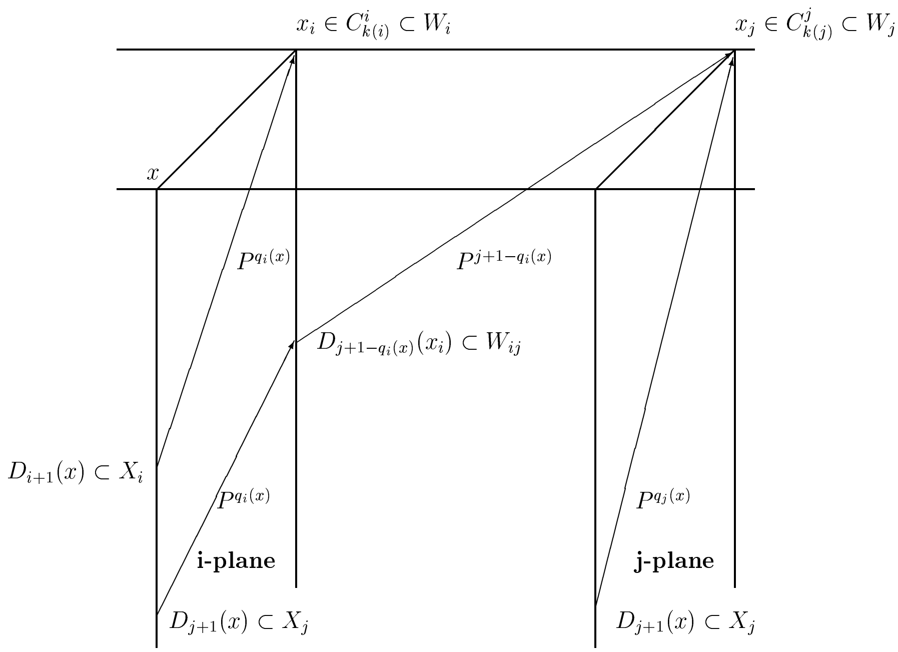

FIGURE 3. The (extended) three-dimensional puzzle

For each $i \geq 1$, since the diameter $d(W_j)$ tends to 0 as j goes to ∞, we can find an integer $j = j(i) > i$ such that $W_j \subset C^i_{k(i)}$. Let $x_i = P^{\circ q_i(x)}(x) \in C^i_{k(i)}$ and $x_j = P^{\circ q_j(x)}(x) = P^{\circ q_j(x_i)}(x_i) \in C^j_{k(j)}$.

Consider the x_i-end in the (extended) three-dimensional puzzle,

$$x_i \in \cdots \subseteq D_l(x_i) \subseteq D_{l-1}(x_i) \subseteq \cdots \subseteq D_1(x_i),$$

where $D_l(x_i) \in \tau_l$. Then $P^{\circ q_{j+1-q_i(x)}(x_i)} : D_{j+1-q_i(x)}(x_i) \to C^j_{k(j)}$ is a proper holomorphic diffeomorphism (see Figure 3). Let g_{ij} be its inverse. Then g_{ij} can be extended to W_j because of the unbranched condition. We still use g_{ij} to denote this extension. Since $C^i_{k(i)}$ is bounded by external rays landing at some pre-images of α_i under iterations of P and by equipotential curves of P, it follows that $W_{ij} = g_{ij}(W_j)$ is contained in $C^i_{k(i)}$. Thus

$$\mathrm{mod}(W_i \setminus W_{ij}) \geq \frac{\mu}{2}.$$

Consider $X_i = g_{i,x}(W_i)$ and $X_j = g_{j,x}(W_j) = g_{i,x}(W_{ij})$. Then

$$\mathrm{mod}(X_i \setminus X_j) \geq \frac{\mu}{2},$$

since $g_{i,x}$ is conformal.

Therefore, inductively, we find an infinite sequence of nested domains $\{X_{i_t}\}_{t=1}^{\infty}$ such that

$$\mathrm{mod}(X_{i_t} \setminus X_{i_{t+1}}) \geq \frac{\mu}{2}$$

for $t \geq 1$. Thus the diameter of X_{i_t} tends to zero as t goes to infinity. Since $D_{i_t+1}(x) = g_{i,x}(C^{i_t}_{k(i_t)})$, we have that

$$D_{i_t+1}(x) \subseteq X_{i_t}.$$

So the diameter $d(D_{i_t+1}(x))$ tends to zero as t goes to infinity. Since each $D_{i_t+1}(x)$ is bounded by external rays and equipotential curves of P, similarly to Lemma 1, $D_{i_t+1}(x) \cap J$ is connected. Therefore, $\{D_{i_t+1}(x)\}_{t=1}^{\infty}$ forms a basis of connected neighborhoods of J at x, and J is locally connected at x. $\square$

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