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A CLASSIFICATION OF ONE DIMENSIONAL ALMOST PERIODIC TILINGS ARISING FROM THE PROJECTION METHOD

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ABSTRACT. For each irrational number α, with continued fraction expansion $[0; a_1, a_2, a_3, \dots]$, we classify, up to translation, the one dimensional almost periodic tilings which can be constructed by the projection method starting with a line of slope α. The invariant is a sequence of integers in the space $X_\alpha = \{(x_i)_{i=1}^\infty \mid x_i \in \{0, 1, 2, \dots, a_i\}$ and $x_{i+1} = 0$ whenever $x_i = a_i\}$ modulo the equivalence relation generated by tail equivalence and $(a_1, 0, a_3, 0, \dots) \sim (0, a_2, 0, a_4, \dots) \sim (a_1 - 1, a_2 - 1, a_3 - 1, \dots)$. Each tile in a tiling T, of slope α, is coded by an integer $0 \le x \le [\alpha]$. Using a composition operation, we produce a sequence of tilings $\mathsf{T}_1 = \mathsf{T}, \mathsf{T}_2, \mathsf{T}_3, \dots$. Each tile in T_i gets absorbed into a tile in T_{i+1}. A choice of a starting tile in T_1 will thus produce a sequence in X_α. This is the invariant.

The subject of this paper is the classification one dimensional almost periodic tilings obtained by the projection method. These sequences were extensively studied by Morse and Hedlund under the name of Sturmian trajectories; but there are earlier works by J. Bernoulli [JB], H. J. S. Smith [HJSS], and E. Christoffel [EBC] among others. For a survey of the recent literature see the references in the papers of T. Brown [TCB] and Lunnon and Pleasants [LP].

There are a number of axiomatic characterizations of these sequences. We have found it convenient to use one found by C. Series [CS] (which is essentially that of Morse and Hedlund). The equivalence of this with other characterizations is presented in [LP]. We shall review below the axioms of Series.

With each tiling there is an associated real number α, which we shall call the *slope* of the tiling; the continued fraction expansion of α is reflected in the structure of the tiling [CS]. The tilings can be grouped into families, $\mathcal{T}_\alpha$, in which all members have the same slope. Moreover there is a simple geometric construction for obtaining members of $\mathcal{T}_\alpha$ using the projection method with lines of slope α.

We shall be concerned with the case when $0 < \alpha < 1$ and α is irrational. In this case the members of $\mathcal{T}_\alpha$ have a property characteristic of almost periodic tilings: if T_1 and T_2 are in the family $\mathcal{T}_\alpha$, then every segment of T_1 appears in T_2 and appears infinitely often. Thus on a local level all elements of $\mathcal{T}_\alpha$ appear similar.

To distinguish the elements of $\mathcal{T}_\alpha$ we use a space X_α constructed from the continued fraction expansion of $\alpha = [0; a_1, a_2, a_3, \dots]$. Let $X_\alpha = \{(x_i)_{i=1}^\infty \mid x_i \in \{0, 1, \dots, a_i\}$ and $x_{i+1} = 0$ whenever $x_i = a_i\}$. From a tiling $\mathsf{T} \in \mathcal{T}_\alpha$ and a tile $\mathsf{t} \in \mathsf{T}$

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we construct $x \in X_\alpha$, and we show that these classify the tilings up to translation. For certain irrationals this expansion is related to the β-expansions of Rényi and Parry ([AR], [WP]).

When $\alpha = (\sqrt{5}-1)/2$ we have $X_\alpha = \{(x_i) \mid x_i \in \{0,1\}$ and $x_i = 1 \Rightarrow x_{i+1} = 0\}$, and X_α, modulo tail equivalence, was called by Connes [AC] (section II.3) the *Penrose universe* because it classifies Penrose tilings (see [GS], section 10.6; also [AP] and [EAR], where the dynamical properties of these and other tilings are explored). In [NDB$_1$] and [NDB$_2$] a classification of Beatty sequences (sequences of the form $\{[n\alpha+\beta]\}_{n=1}^\infty$) for $\alpha = (\sqrt{5}-1)/2$ or $\alpha = \sqrt{2}-1$ was discussed. However our method is simpler, more general, and works for every irrational.

1. Introduction

1.1. Definition. A two sided infinite sequence $\{\mathsf{t}_i\}_{i=1}^\infty$ is *composable* if it satisfies the following axioms:

A$_1$: The letter a is isolated: i.e. if $\mathsf{t}_i = \mathsf{a}$ then $\mathsf{t}_{i\pm 1} = \mathsf{b}$.
A$_2$: There is an integer n such that between a's there are either n or $n+1$ b's.

If T is a composable sequence, we can produce a new sequence T' by *composition*: each segment beginning with an a and followed by n b's gets replaced by a b, and any remaining b's are replaced by an a. Thus

$$\mathsf{a}\underbrace{\mathsf{bbb}\ldots\mathsf{b}}_{n} \mapsto \mathsf{b} \quad \text{and} \quad \mathsf{a}\underbrace{\mathsf{bbb}\ldots\mathsf{b}}_{n+1} \mapsto \mathsf{ba}$$

The third axiom asserts that the sequence can be composed infinitely many times.

A$_3$: Each composition satisfies A$_1$ and A$_2$.

If $\mathsf{T} = \{\mathsf{t}_i\}_{i=1}^\infty$ satisfies axioms A$_1$, A$_2$, and A$_3$, then we call T a *cutting sequence* (following Series [CS]).

1.2. *Remark.* As explained in Series [CS], cutting sequences can be constructed as follows. Let α be an irrational number between 0 and 1 and let β be any real number. Let L be the line with equation $y = \alpha x + \beta$. Each intersection of L with a horizontal line will be marked with an a, and each intersection of L with a vertical line will be marked with a b. Thus along L we obtain a two sided infinite sequence T of a's and b's. These are the one dimensional almost periodic sequences obtained by the projection method (see [MS], section 4.3). In [LP], Lemmas 2.2 and 3.3, it is shown that T is a cutting sequence (see also [GS], 10.6.2).

When a cutting sequence for the line $y = \alpha x + \beta$ is constructed by the projection method, a tiling of the line is produced with a-tiles intervals of length $\sin\vartheta$ and b-tiles intervals of length $\cos\vartheta$, where $\alpha = \tan\vartheta$.

It will be convenient later on to have a notation for labelling the symbols (tiles). The intersection of L with the horizontal line $y = m$ produces an a tile; let us label it a_m. The intersection of L with the vertical line $x = n$ produces a b tile; let us label it b_n. The intersection of L with the y-axis produces b_0. By following the path of b_0 through successive compositions we shall produce our invariant.

If L passes through a point of $\mathbb{Z}^2$ then an a and a b will coincide. We will in this case obtain two sequences T^+ and T^-. T^+ is constructed by writing all coinciding pairs $\{\mathsf{a},\mathsf{b}\}$ with the a's preceding the b's (i.e. to the left), and T^- with the a's following the b's.

1.3. Definition. Given a cutting sequence T, let $\mathsf{T}_1, \mathsf{T}_2, \dots$ be the sequence obtained from T by composition: i.e. T_2 is obtained from T_1 by composition, and in general T_{i+1} is obtained from T_i by composition. Let a_i be the integer for T_i postulated by axiom $\text{A}2$. Then α, the real number with continued fraction expansion $[0;, a_1, a_2, a_3, \dots]$, is the *slope* of T.

1.4. Definition. Let T be a cutting sequence with slope $\alpha = [0; a_1, a_2, a_3, \dots]$ and let $\{\mathsf{T}_i\}_{i=1}^{\infty}$ be the sequence of cutting sequences obtained from T by composition (setting $\mathsf{T}_1 = \mathsf{T}$). Let t be a letter in T and let t_2 be the letter in T_2 into which t is absorbed by composition. In general let t_{i+1} be the letter of T_{i+1} into which t_i gets absorbed by composition. Letting $\mathsf{t}_1 = \mathsf{t}$ we obtain a sequence $\{\mathsf{t}_i\}_{i=1}^{\infty}$ with $\mathsf{t}_i \in \mathsf{T}_i$. We construct $x \in X_\alpha$ as follows. If $\mathsf{t}_i = \mathsf{a}$ then $x_i = 0$; if $\mathsf{t}_i = \mathsf{b}$ then x_i is the number of b's between t and the first a to the left. Then $x = (x_i) \in X_\alpha$ is the *coding sequence* of the pair (T, t):

$$\begin{array}{ccc} \underbrace{\mathsf{ab}\boxed{\mathsf{b}}\mathsf{b}\dots\mathsf{b}} & \mathsf{t}_i & \in & \mathsf{T}_i \\ & \downarrow & & \\ \mathsf{ba}\boxed{\mathsf{b}}\mathsf{ab} & \mathsf{t}_{i+1} & \in & \mathsf{T}_{i+1} \end{array}$$

In the example above, $x_i = 1$ and $x_{i+1} = 0$.

We next wish to describe the process by which, given $x \in X_\alpha$, one can construct $\mathsf{T} \in \mathcal{T}_\alpha$ and $\mathsf{t} \in \mathsf{T}$ such that the coding sequence of (T, t) is x. T will be written as a limit of words $\{T_1, T_2, T_3, \dots\}$ in $\{\mathsf{a}, \mathsf{b}\}$, where each T_{i-1} is a subword of T_i. A special case of this (when $x = (a_1 - 1, a_2 - 1, a_3 - 1, \dots)$) was given by Smith [HJSS], and so we shall call $\{T_i\}_{i=0}^{\infty}$ the *Smith sequence* of $x \in X_\alpha$.

Suppose $\alpha = [0; a_1, a_1, a_3, \dots]$ and $(x_1, x_2, x_3, \dots) \in X_\alpha$. We shall construct sequences $\{S_{-1}, S_0, S_1, \dots\}$ and $\{T_0, T_1, T_2, \dots\}$ of words in the letters $\{\mathsf{a}, \mathsf{b}\}$. We can think of these words as elements in the free group with generators $\{\mathsf{a}, \mathsf{b}\}$. Let

$$\begin{aligned} S_{-1} &= \mathsf{a} \\ S_0 &= \mathsf{b} \\ S_1 &= S_{-1}\underbrace{S_0 \dots S_0}_{a_1} = \mathsf{a}\underbrace{\mathsf{b}\dots\mathsf{b}}_{a_1} \\ &\vdots \\ S_{i+1} &= S_{i-1}\underbrace{S_i \dots S_i}_{a_{i+1}} \end{aligned}$$

Define $T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots$ by

$$T_0 = \boxed{\mathsf{b}}$$

$$T_1 = \begin{cases} S_{-1}\underbrace{S_0 \cdots S_0}_{x_1}\boxed{T_0}\underbrace{S_0 \cdots S_0}_{a_1 - x_1 - 1} = \mathsf{a}\underbrace{\mathsf{b}\cdots\mathsf{b}}_{x_1}\boxed{\mathsf{b}}\ \underbrace{\mathsf{b}\cdots\mathsf{b}}_{a_1 - x_1 - 1} & \text{if } x_1 \neq a_1 \\ \boxed{\mathsf{b}} & \text{if } x_1 = a_1 \end{cases}$$

$$T_{i+1} = \begin{cases} \boxed{T_i} & \text{if } x_{i+1} = a_{i+1} \\ S_{i-1}\underbrace{S_i \cdots S_i}_{x_{i+1}}\boxed{T_i}\ \underbrace{S_i \cdots S_i}_{a_{i+1} - x_{i+1} - 1} & \text{if } x_{i+1} < a_{i+1} \text{ and } x_i \neq a_i \\ \boxed{T_{i-1}}\underbrace{S_i \cdots S_i}_{a_{i+1}} & \text{if } x_i = a_i \end{cases}$$

Note that when $x_i = a_{i+i}$ we have $T_i = T_{i-1}$, so $T_{i+1} = T_i \overbrace{S_i \cdots S_i}^{a_{i+1}}$; and so in all cases T_i is a subword of T_{i+1}, the inclusions being indicated by the boxes.

1.5. Definition. We shall call $\{T_i\}$ the *Smith sequence* of $x \in X_\alpha$.

By taking the union T of the T_i's and $\mathsf{t} = T_0$ we have a sequence T and a tile t in T. We shall show that T is a cutting sequence and that the coding sequence of (T, t) is x (Theorem C).

1.6. Examples. Two sequences x and y in X_α are if there is an n such that $x_k = y_k$ for $k \geq n$. If $x \in X_\alpha$ is tail equivalent to one of $\{-\alpha, 0^+, 0^-\}$, then the limit sequence $T_0 \to T_1 \to T_2 \to \cdots \to \mathsf{T}$ will not be two sided. In fact, when $x = -\alpha = (a_1 - 1, a_2 - 1, a_3 - 1, \dots)$ the sequence produced is the limit of the T_i's:

$$\begin{array}{rcl} \boxed{\mathsf{b}} & = & T_0 \\ \mathsf{a}\,\underbrace{\mathsf{b}\cdots\boxed{\mathsf{b}}}_{a_1} & = & T_1 \\ \underbrace{\mathsf{a}\,\underbrace{\mathsf{b}\cdots\mathsf{b}}_{a_1}\cdots\mathsf{a}\,\underbrace{\mathsf{b}\cdots\boxed{\mathsf{b}}}_{a_1}}_{a_2} & = & T_2 \\ & \vdots & \end{array}$$

Let us call this sequence $\mathsf{T}_{-\alpha}$. This is an infinite sequence extending off to the left, so it it not a cutting sequence, although one can compose it an infinite number of times. If $x = 0^+ = (0, a_2, 0, a_4, \dots)$, then the sequence produced is the limit of the T_i's:

$$\begin{array}{rcl} T_0 & = & \boxed{\mathsf{b}} \\ T_1 = T_2 & = & \mathsf{a}\,\underbrace{\boxed{\mathsf{b}}\,\mathsf{b}\cdots\mathsf{b}}_{a_1} \\ T_3 = T_4 & = & T_1\,\underbrace{S_2 \dots S_2}_{a_3} \\ & = & \mathsf{a}\,\underbrace{\boxed{\mathsf{b}}\,\mathsf{b}\cdots\mathsf{b}}_{a_1}\,\underbrace{\overbrace{\mathsf{ba}\,\underbrace{\mathsf{b}\cdots\mathsf{b}}_{a_1}\cdots\mathsf{ba}\,\underbrace{\mathsf{b}\cdots\mathsf{b}}_{a_1}}^{a_2}\cdots\overbrace{\mathsf{ba}\,\underbrace{\mathsf{b}\cdots\mathsf{b}}_{a_1}\cdots\mathsf{ba}\,\underbrace{\mathsf{b}\cdots\mathsf{b}}_{a_1}}^{a_2}}_{a_3} \\ & \vdots & \end{array}$$

Let us denote this sequence T_{0^+}. This infinite sequence, which is also not a cutting sequence, extends off to the right. Finally, suppose $x = 0^- = (a_1, 0, a_3, 0,$

...). Then the sequence obtained is

$$\begin{aligned} T_0 = T_1 &= \boxed{\mathsf{b}} \\ T_2 = T_3 &= T_0\,\underbrace{S_1 \cdots S_1}_{a_2} \\ &= \boxed{\mathsf{b}}\,\overbrace{\mathsf{a}\underbrace{\mathsf{b}\cdots\mathsf{b}}_{a_1}\cdots\mathsf{a}\underbrace{\mathsf{b}\cdots\mathsf{b}}_{a_1}}^{a_2} \\ &\vdots \end{aligned}$$

Let us denote this sequence T_{0^-}. This infinite sequence, which is also not a cutting sequence, extends off to the right.

We can assemble $\mathsf{T}_{-\alpha}$, T_{0^+}, and T_{0^-} into cutting sequences; however, $\mathsf{T}_{-\alpha}|\mathsf{T}_{0^+}$ and $\mathsf{T}_{-\alpha}|\mathsf{T}_{0^-}$ are both cutting sequences (here $\cdot \mid \cdot$ denotes concatenation). The first produces the cutting sequence T^+ for $\beta = 0^+$ and the second produces T^- for $\beta = 0^-$. The only difference between T_{0^+} and T_{0^-} is in the first two letters: the former begins with $\boxed{\mathsf{ab}}$ and the latter with $\boxed{\mathsf{ba}}$. If we let $\mathsf{T}^{\mathrm{op}}_{-\alpha}$ denote the reflection of $\mathsf{T}_{-\alpha}$ so as to produce an infinite sequence extending to the right, then $\mathsf{T}_{0^+} = \mathsf{ab}\mathsf{T}^{\mathrm{op}}_{-\alpha}$ and $\mathsf{T}_{0^-} = \mathsf{ba}\mathsf{T}^{\mathrm{op}}_{-\alpha}$, and thus $\mathsf{T}^+ = \mathsf{T}_{-\alpha}|\mathsf{ab}|\mathsf{T}^{\mathrm{op}}_{-\alpha}$ and $\mathsf{T}^- = \mathsf{T}_{-\alpha}|\mathsf{ba}|\mathsf{T}^{\mathrm{op}}_{-\alpha}$. We shall call such a pair of cutting sequences *conjugate* (cf. [LP], section 8). For example, when $\alpha = [0; 3, 1, 3, 1, \dots]$ we have

$$\begin{aligned} \mathsf{T}^+ &= \dots\mathsf{abbbbabbbbabbb}\,\boxed{\mathsf{ab}}\,\mathsf{bbbabbbbabbbabbbb}\dots \\ \mathsf{T}^- &= \dots\mathsf{abbbbabbbbabbb}\,\boxed{\mathsf{ba}}\,\mathsf{bbbabbbbabbbabbbb}\dots \end{aligned}$$

1.7. Definition. Let T be a cutting sequence and t a tile in T. Let $\mathsf{T}_1 = \mathsf{T}$, T_2 the cutting sequence obtained from T_1 by composition, and let t_2 be the tile of T_2 which absorbs $\mathsf{t}_1 = \mathsf{t}$. Iterating this construction, we obtain a sequence $\{(\mathsf{T}_1, \mathsf{t}_1), (\mathsf{T}_2, \mathsf{t}_2), \dots\}$ of cutting sequences with a specified letter t_i in the cutting sequence T_i. From this we may obtain an increasing sequence of words $W_1 \subseteq W_2 \subseteq W_3 \subseteq \cdots \subseteq \mathsf{T}$ as follows. We let $W_0 = \mathsf{t}_1$, and we let W_1 be all the letters in T which under composition get absorbed into t_2. Let W_2 be all the letters in T which get absorbed, under two compositions, into t_3. Similarly let W_k be all the letters of T which get absorbed, under k compositions, into t_{k+1}. The sequence $\{W_i\}_{i=1}^{\infty}$ is called the *inflation sequence* of (T, t).

Now either $\bigcup_{1=1}^{\infty} W_i = \mathsf{T}$, and we shall say that T is *non-singular*, or $\bigcup_{i=1}^{\infty} W_i \neq \mathsf{T}$, in which case we shall say that T is *singular*. By construction $\bigcup_{1=1}^{\infty} W_i$ is infinite and contains no gaps, so when T is singular we can write it as the union of $\bigcup_{1=1}^{\infty} W_i$ and $\bigcup_{1=1}^{\infty} W_i'$, where the W_s''s are constructed by choosing another starting tile not in $\bigcup_{1=1}^{\infty} W_i$. We shall show in Theorem C that the Smith sequence reconstructs the inflation sequence. From this it follows that when T is singular $\bigcup_{i=1}^{\infty} W_i$ is either $\mathsf{T}_{-\alpha}$, T_{0^+}, or T_{0^-}

The last construction we need tells us which coding sequence is obtained from the line L: $y = \alpha x + \beta$, when we choose as our initial tile b_0 (the intersection of L with the y-axis). To take account of the singular tiles we need to introduce some notation.

1.8. Notation. Let $\mathbb{R}_\alpha$ be the Cantor set obtained by cutting the real line $\mathbb{R}$ at each of the points $\mathbb{Z} + \alpha\mathbb{N} = \{m + \alpha n \mid m, n \in \mathbb{Z}, \text{ and } n \geq 0\}$. Each of the points $m + \alpha n$ of $\mathbb{Z} + \alpha\mathbb{N}$ will be split into two points $(m + \alpha n)^+$ and $(m + \alpha n)^-$.

We can make $\mathbb{R}_\alpha$ a locally compact Hausdorff space by writing it as an inverse limit. Let $\{\gamma_n\}_{n=1}^\infty$ be an enumeration of $\mathbb{Z} + \alpha\mathbb{N}$. Then $\mathbb{R}_\alpha = \lim_\leftarrow R_n$, where R_n is obtained from R_{n-1} by splitting at the point γ_n. The map $R_{n+1} \to R_n$ forgets about the splitting at γ_n.

Let $\pi : \mathbb{R}_\alpha \to \mathbb{R}$ be the map that 'forgets' the '+' or '−'. For each $x \in \mathbb{R}_\alpha$, let $[x]$ be the greatest integer less than or equal to x — with the convention that $[n^+] = n$ and $[n^-] = n - 1$ for $n \in \mathbb{Z}$. Also $\{x\}$ will denote the fractional part of x: $\{x\} = \pi(x) - [\pi(x)]$.

1.9. Definition. For $\alpha = [0; a_1, a_2, a_3, \ldots]$ let

$$\begin{aligned} &\alpha_0 = 1 && \alpha_1 = \alpha \\ &\alpha_2 = 1 - a_1\alpha = \alpha\{\alpha^{-1}\} && \alpha_3 = \alpha_1 - a_2\alpha_2 = \alpha_2\{\alpha_1\alpha_2^{-1}\} \\ &\quad\vdots && \\ &\alpha_{n+1} = \alpha_{n-1} - a_n\alpha_n = \alpha_n\{\alpha_{n-1}\alpha_n^{-1}\} && \end{aligned}$$

and for $\beta \in \mathbb{R}_\alpha$, let $\beta_1 = \beta - [\beta]$ and

$$x_1 = [\beta_1/\alpha_1] \quad \text{and} \quad \beta_2 = \begin{cases} (1 + x_1)\alpha_1 - \beta_1 & \text{if } x_1 < a_1 \\ \alpha_0 - \beta_1 & \text{if } x_1 = a_1. \end{cases}$$

If $\beta_1, \beta_2, \ldots, \beta_n$ and $x_1, x_2, x_3, \ldots, x_{n-1}$ have been constructed, let

$$x_n = [\beta_n/\alpha_n] \qquad \text{and} \quad \beta_{n+1} = \begin{cases} (1 + x_n)\alpha_n - \beta_n & \text{if } x_n < a_n \\ \alpha_{n-1} - \beta_n & \text{if } x_n = a_n. \end{cases}$$

Let $\varphi : \mathbb{R}_\alpha \to X_\alpha$ be the map $\varphi(\beta) = (x_i)_{i=1}^\infty$.

We shall show in Theorem B that if we start with a line $\mathsf{L} : y = \alpha x + \beta$, and let T be the corresponding coding sequence, and $\mathsf{t}_1 = \mathsf{b}_0$, the intersection of L with the y-axis, then the coding sequence associated with (T, t) is $\varphi(\beta) = (x_1, x_2, x_3, \ldots) \in X_\alpha$.

1.10. *Remark.* Let us recall two results from [JAM]. Let $S^1_{\mathbb{N}\alpha}$ be the Cantor set obtained by cutting the circle S^1 along the forward orbit of 0 under the action of rotation by the angle $2\pi\alpha$. Writing S^1 as [0,1] with 0 and 1 identified and the group law as addition mod 1, $S^1_{\mathbb{N}\alpha}$ becomes the interval $[0^+, 1^-] \subseteq \mathbb{R}_\alpha$. Theorem 3.6 of [JAM] showed that $\varphi : [0^+, 1^-] \to X_\alpha$ is a homeomorphism and that (by Theorem 3.8 and Corollary 3.9) $\varphi(x)$ is tail equivalent to $-\alpha$, 0^+, or 0^- if and only if $x = -m\alpha$, $x = n\alpha^+$, or $x = n\alpha^-$ respectively for $m > 0$ and $n \geq 0$. This implies that, given a coding sequence (x_i), we can use the formulas of 1.9 to construct a line which gives the coding sequence (x_i). In the paragraph below we give a graphical interpretation of this.

1.11. *Remark.* In Figure 1 we illustrate the expansion of β, i.e. the map $[0,1] \ni \beta \mapsto (x_i) \in X_\alpha$ when $\alpha = [0; 3, 1, 3, 1, \ldots] \approx 0.26$. In the figure the interval [0,1] has been cut at the points $n\alpha$ (mod 1) for integers $n \geq 0$. In the first row we have made a partition of [0,1] using the points $\{0, \alpha, 2\alpha, \ldots, a_1\alpha\}$. In the second row we have refined the partition by adding the points $\{(1 + a_1)\alpha, \ldots, a_1(1 + a_2)\alpha\}$. The rule for the i-th row is to add the points $\{(q_{i-2} + q_{i-1})\alpha, \ldots, (q_i + q_{i-1} - 1)\alpha\}$,

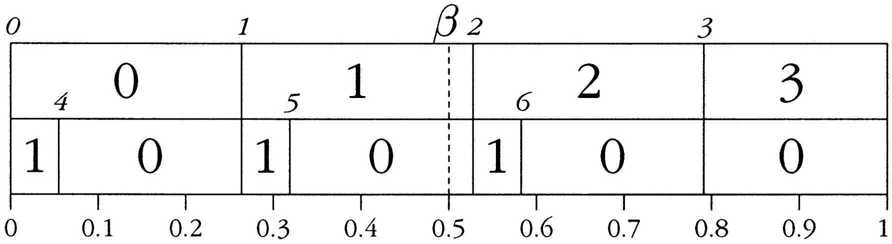

Figure 1.

where $q_0 = 1$, $q_1 = a_1$, and $q_{i+1} = a_{i+1}q_i + q_{i-1}$ are the usual denominators of the convergents in the continued fraction expansion of α.

We can also view the partition as being constructed by a 'paving'. We begin by 'paving' the interval [0,1] from left to right by interval of length α. We can get in a_1 of these intervals, and there is a remainder of length α_2. So in the top row the intervals are of length $\alpha_1 = \alpha$, except for the last which is of length $\alpha_2 = 1 - a_1\alpha_1$. To obtain the second row we 'pave' from right to left each of the intervals on the first row with intervals of length α_2. We can fit in a_2 intervals of length α_2 in each interval of length α. Thus on the second row there are intervals of length α_2 and remainders of length α_3. To construct the i-th row we 'pave' each interval of the $(i-1)$-st row with intervals of length α_i. When i is odd we pave from left to right and when i is even we pave from right to left.

The numbers $\mathit{1}, \mathit{2}, \mathit{3}, \ldots, \mathit{6}$ indicate the multiples of α (modulo 1). In this example $\alpha \approx 0.26$, so $\mathit{4}\alpha$ (mod 1) is a little to the right of 0. On the top row we calculate x_1. Thus if $\beta \in [i\alpha, (i+1)\alpha]$ then $x_i = i$. On the bottom row we calculate x_2. In this example $x_2 \in \{0, 1\}$. If $x_1 = a_1$ then x_2 can only be 0. Otherwise we count the number of intervals from the right wall of the containing interval on the first row. Thus when $\beta = 0.5$ we have $x_1 = 1$ and $x_2 = 0$.

1.12. Definition. Given two cutting sequences T and T' in $\mathcal{T}_\alpha$ and chosen letters t and t' in T and T' respectively, we say that (T, t) and $(\mathsf{T}', \mathsf{t}')$ are *translation equivalent* if there is $\tilde{\mathsf{t}}'$ in T' such that $(\mathsf{T}, \mathsf{t}) = (\mathsf{T}', \tilde{\mathsf{t}}')$.

Recall from 1.2 that for each α there is a pair of singular cutting sequences T^+ and T^-:

$$\begin{aligned} \mathsf{T}^+ &= \ldots \mathsf{bbb}\boxed{\mathsf{ab}}\mathsf{bbb} \\ \mathsf{T}^- &= \ldots \mathsf{bbb}\boxed{\mathsf{ba}}\mathsf{bbb}\ldots \end{aligned}$$

These sequences were called conjugate because reflection about the centre of the box takes one to the other. Our invariant cannot separate these two sequences. If we choose a letter to the left of the box in either sequence we get a coding sequence tail equivalent to $-\alpha$; if we choose a letter to the right of the box we get a coding sequence tail equivalent to 0^+ or 0^- respectively. Therefore on our space $\mathcal{T}_\alpha^\bullet$ of pointed cutting sequence with slope α we put the equivalence relation $\approx$ generated by translation and conjugation.

To conclude this introduction we present the main results of this paper. The numbers following the lettering of the theorems will direct the reader to the section containing the proof. Our results were inspired by and are analogous to the statements 10.5.9–10.5.14 in [GS] for Penrose tilings.

Theorem A (3.1)**.** *Let T be a non-singular cutting sequence and t and t' be tiles in T. Let x and x' in X_α be the corresponding coding sequences. Then x and x' are tail equivalent.*

Let T be a singular cutting sequence and t a tile in T. Then the coding sequence of t is tail equivalent to one of $-\alpha$, 0^+, or 0^-.

Theorem A shows that except for the singular cutting sequences, the tail equivalence in X_α of any coding sequence is an invariant of T under translation. For the singular cutting sequences there are three possible tail equivalence classes.

Theorem B (3.2)**.** *Let $\beta \in \mathbb{R}_\alpha$, and let L be the line $y = \alpha x + \beta$. Let T be the corresponding cutting sequence, let t be the tile b_0 (as described in 1.2), and let $x \in X_\alpha$ be the corresponding coding sequence. Then $x = \varphi(\beta)$, where φ is constructed in* 1.9*. In particular, the cutting sequence T is singular if and only if L passes through a point of $\mathbb{Z}^2$.*

Theorem C (3.4)**.** *Let T be a cutting sequence, t a tile in T, and x the coding sequence of (T, t). Then the Smith sequence $\{T_i\}_{i=1}^\infty$ of x equals the inflation sequence $\{W_i\}_{i=1}^\infty$ of (T, t) i.e. $W_i = T_i$ for $i = 1, 2, 3, \ldots$.*

Theorem D (3.5)**.** *Let $\sim$ be the equivalence relation on X_α generated by tail equivalence, and let $-\alpha \sim 0^+ \sim 0^-$. Let $(\mathsf{T}_1, \mathsf{t}_1)$ and $(\mathsf{T}_2, \mathsf{t}_2)$ be in $\mathcal{T}_\alpha^\bullet$, and x_1 and x_2 in X_α the corresponding coding sequences. Then $x_1 \sim x_2$ if and only if $\mathsf{T}_1 \approx \mathsf{T}_2$.*

2. Preliminaries

Before proving our main results we recall some elementary lemmas. Let $0 < \alpha < 1$ be irrational and $\beta \in \mathbb{R}_\alpha$. Let L be the line $y = \alpha x + \beta$. Let T be the sequence obtained by the cutting method (1.2).

Lemma 2.1. *Between adjacent a's there are either $[\alpha^{-1}]$ or $1 + [\alpha^{-1}]$ b's. There are $1 + [\alpha^{-1}]$ b's between a_i and a_{i+1} if and only if $\left\{i + 1 - \frac{\beta}{\alpha}\right\} \le \left\{\frac{1}{2}\right\}$, unless $\mathsf{T} = \mathsf{T}^-$ and $\left\{i - \frac{\beta}{\alpha}\right\} \in \mathbb{Z}$, in which case there are $[\alpha^{-1}]$ b's between a_i and a_{i+1}, or $\mathsf{T} = \mathsf{T}^+$ and $i + 1 - \frac{\beta}{\alpha} \in \mathbb{Z}$, in which case there are $[\alpha^{-1}]$ b'a between a_i and a_{i+1}.*

Proof. Suppose neither $i - \frac{\beta}{\alpha}$ nor $i + 1 - \frac{\beta}{\alpha}$ is an integer. Then the number of b's between a_i and a_{i+1} is the number of integers in the interval $\left[i - \frac{\beta}{\alpha}, i + 1 - \frac{\beta}{\alpha}\right]$. This interval contains $\left[i + 1 - \frac{\beta}{\alpha}\right] - \left[i - \frac{\beta}{\alpha}\right] = [\alpha^{-1}] + \left[\left\{i - \frac{\beta}{\alpha}\right\} + \{\alpha^{-1}\}\right]$ integers. Thus there are either $[\alpha^{-1}]$ or $1 + [\alpha^{-1}]$ b's between a_i and a_{i+1}. The interval $\left[i - \frac{\beta}{\alpha}, i + 1 - \frac{\beta}{\alpha}\right]$ contains $1 + [\alpha^{-1}]$ integers $\iff$ $\left[\left\{i - \frac{\beta}{\alpha}\right\} + \{\alpha^{-1}\}\right] = 1$ $\iff$ $\left\{i - \frac{\beta}{\alpha}\right\} + \{\alpha^{-1}\} > 1$ $\iff$ $\left\{i + 1 - \frac{\beta}{\alpha}\right\} = \left\{\left\{i - \frac{\beta}{\alpha}\right\} + \{\alpha^{-1}\}\right\} < \{\alpha^{-1}\}$.

Suppose $i - \frac{\beta}{\alpha} \in \mathbb{Z}$. Then $\mathsf{b}_{i - \frac{\beta}{\alpha}}$ is between a_i and a_{i+1} only when $\mathsf{T} = \mathsf{T}^+$. So when $\mathsf{T} = \mathsf{T}^+$, we have $\left\{i + 1 - \frac{\beta}{\alpha}\right\} = \left\{\frac{1}{\alpha}\right\}$, and there are $1 + [\alpha^{-1}]$ b's between a_i and a_{i+1}, and when $\mathsf{T} = \mathsf{T}^-$ we have $[\alpha^{-1}]$ b's between a_i and a_{i+1}.

Suppose $i + 1 - \frac{\beta}{\alpha} \in \mathbb{Z}$. Then $\mathsf{b}_{i+1-\frac{\beta}{\alpha}}$ is between a_i and a_{i+1} only when $\mathsf{T} = \mathsf{T}^-$. So when $\mathsf{T} = \mathsf{T}^-$ and $i + 1 - \frac{\beta}{\alpha} \in \mathbb{Z}$ we have $\left\{i + 1 - \frac{\beta}{\alpha}\right\} = 0 < \left\{\frac{1}{\alpha}\right\}$, and there are $1 + [\alpha^{-1}]$ b's between a_i and a_{i+1}, and when $\mathsf{T} = \mathsf{T}^+$, we have $[\alpha^{-1}]$ b's between a_i and a_{i+1}.

Lemma 2.2. *Let $k(\mathsf{b}_i)$ be the number of b's between b_i and the first a to the left of b_i. Then $k(\mathsf{b}_i) = [\{i\alpha+\beta\}\alpha^{-1}]$ except in the cases*

(i) *$\mathsf{T} = \mathsf{T}^-$ and $i\alpha+\beta \in \mathbb{Z}$, in which case $k(\mathsf{b}_i) = [\alpha^{-1}]$, and*
(ii) *$\mathsf{T} = \mathsf{T}^-$, $[i\alpha+\beta] - \frac{\beta}{\alpha} \in \mathbb{Z}$ but $i\alpha+\beta \notin \mathbb{Z}$, in which case*

$$k(\mathsf{b}_i) = \{i\alpha+\beta\}\alpha^{-1} - 1.$$

Proof. Suppose neither $i\alpha+\beta$ nor $[i\alpha+\beta] - \frac{\beta}{\alpha}$ is an integer. The first a to the left of b_i is $\mathsf{a}_{[i\alpha+\beta]}$. The first b following $\mathsf{a}_{[i\alpha+\beta]}$ is $\mathsf{b}_{1+\left[[i\alpha+\beta]-\frac{\beta}{\alpha}\right]}$. Thus the number of b's between b_i and $\mathsf{a}_{[i\alpha+\beta]}$ is $i - \left(1 + \left[[i\alpha+\beta] - \frac{\beta}{\alpha}\right]\right) = \left[\{i\alpha+\beta\}\alpha^{-1}\right]$.

Suppose $i\alpha+\beta \in \mathbb{Z}$. When $\mathsf{T} = \mathsf{T}^+$ there are 0 b's between b_i and $\mathsf{a}_{i\alpha+\beta}$, and the first a to the left of b_i. When $\mathsf{T} = \mathsf{T}^-$ the first a to the left is $\mathsf{a}_{i\alpha+\beta-1}$, the first b following $\mathsf{a}_{i\alpha+\beta-1}$ is $\mathsf{b}_{1+\left[i\alpha+\beta-1-\frac{\beta}{\alpha}\right]} = \mathsf{b}_{i-[\alpha^{-1}]}$. Thus there are $i-(i-[\alpha^{-1}])$ b's between b_i and the first a to its left.

Suppose $[i\alpha+\beta] - \frac{\beta}{\alpha} \in \mathbb{Z}$ but $i\alpha+\beta \notin \mathbb{Z}$. When $\mathsf{T} = \mathsf{T}^+$, the first a to the left of b_i is $\mathsf{a}_{[i\alpha+\beta]}$ and the first b following $\mathsf{a}_{[i\alpha+\beta]}$ is $\mathsf{b}_{[i\alpha+\beta]-\frac{\beta}{\alpha}}$; thus the number of b's between b_i and $\mathsf{a}_{[i\alpha+\beta]}$ is $i - [i\alpha+\beta] - \frac{\beta}{\alpha} = \{i\alpha+\beta\}\alpha^{-1}$. When $\mathsf{T} = \mathsf{T}^-$, the first a to the left of b_i is again $\mathsf{a}_{[i\alpha+\beta]}$, but the first b following $\mathsf{a}_{[i\alpha+\beta]}$ is now $\mathsf{b}_{1+[i\alpha+\beta]-\frac{\beta}{\alpha}}$; thus the number of b's between $\mathsf{a}_{[i\alpha+\beta]}$ and b_i is $\{i\alpha+\beta\}\alpha^{-1} - 1$.

The next proposition shows how to keep track of b_0 under composition.

Proposition 2.3. *Let L be the line $y = \alpha x + \beta$, T the sequence obtained from L by the cutting method, and let T' be the sequence obtained from the the line $y = \{\alpha^{-1}\}x - \beta\alpha^{-1}$ by the cutting method. If L passes through a point of $\mathbb{Z}^2$ and $\mathsf{T} = \mathsf{T}^+$ let $\mathsf{T}' = \mathsf{T}'^-$, and if $\mathsf{T} = \mathsf{T}^-$ let $\mathsf{T}' = \mathsf{T}'^+$.*

For $\mathsf{T} \neq \mathsf{T}^-$ we define a map from T to T' by sending

$$\mathsf{T} \ni \mathsf{a}_i \mapsto \mathsf{b}_i \in \mathsf{T}'$$
$$\mathsf{T} \ni \mathsf{b}_i \mapsto \begin{cases} \mathsf{b}_{[i\alpha+\beta]} \in \mathsf{T}' & \text{if } \{i\alpha+\beta\}\alpha^{-1} < [\alpha^{-1}] \\ \mathsf{a}_{i-[\alpha^{-1}](1+[i\alpha+\beta])} \in \mathsf{T}' & \text{if } \{i\alpha+\beta\}\alpha^{-1} \geq [\alpha^{-1}]. \end{cases}$$

For $\mathsf{T} = \mathsf{T}^-$ the map is given by

$$\mathsf{T} \ni \mathsf{a}_i \mapsto \mathsf{b}_i \in \mathsf{T}'$$
$$\mathsf{T} \ni \mathsf{b}_i \mapsto \begin{cases} \mathsf{b}_{[i\alpha+\beta]} \in \mathsf{T}' & \text{if } \{i\alpha+\beta\}\alpha^{-1} < [\alpha^{-1}] \\ & \text{and } i\alpha+\beta \notin \mathbb{Z} \\ \mathsf{a}_{i-[\alpha^{-1}](1+[i\alpha+\beta])} \in \mathsf{T}' & \text{if } \{i\alpha+\beta\}\alpha^{-1} \geq [\alpha^{-1}] \\ \mathsf{a}_{i-[\alpha^{-1}](i\alpha+\beta)} \in \mathsf{T}' & \text{if } i\alpha+\beta \in \mathbb{Z}. \end{cases}$$

Then T' is the sequence obtained from T by composition, and the map above from T to T' is the composition map.

Proof. We must show that each word $\mathsf{ab}\cdots\mathsf{b}$ in T with $[\alpha^{-1}]$ b's gets mapped to a b in T', and each word $\mathsf{ab}\cdots\mathsf{b}$ in T with $1+[\alpha^{-1}]$ b's gets mapped to ba in T'.

To simplify the calculations let L denote the original line $y = \alpha x+\beta$ and $\widetilde{\mathsf{L}}$ denote the line $y = \{\alpha^{-1}\}^{-1}x + \beta\alpha^{-1}\{\alpha^{-1}\}^{-1}$. Note that $\widetilde{\mathsf{L}}$ is the reflection in $y = x$ of the line $y = \{\alpha^{-1}\}x - \beta\alpha^{-1}$. Let us denote by A_i the intersection of $\widetilde{\mathsf{L}}$ with $y = i$

and B_i the intersection of $\widetilde{\mathsf{L}}$ with $x = i$. Under reflection in the line $y = x$, A_i gets mapped to b_i and B_i to a_i. Under this transformation our map then is

$$\begin{aligned} \mathsf{T} \ni \mathsf{a}_i \quad &\mapsto \quad \mathsf{A}_i \\ \mathsf{T} \ni \mathsf{b}_i \quad &\mapsto \quad \begin{cases} \mathsf{A}_{[i\alpha+\beta]} & \text{if } \{i\alpha+\beta\}\alpha^{-1} < [\alpha^{-1}] \\ \mathsf{B}_{i-[\alpha^{-1}](1+[i\alpha+\beta])} & \text{if } \{i\alpha+\beta\}\alpha^{-1} \geq [\alpha^{-1}] \end{cases} \end{aligned}$$

and when $\mathsf{T} = \mathsf{T}^-$

$$\begin{aligned} \mathsf{T} \ni \mathsf{a}_i \quad &\mapsto \quad \mathsf{A}_i \\ \mathsf{T} \ni \mathsf{b}_i \quad &\mapsto \quad \begin{cases} \mathsf{A}_{[i\alpha+\beta]} & \text{if } \{i\alpha+\beta\}\alpha^{-1} < [\alpha^{-1}] \\ & \text{and } i\alpha+\beta \notin \mathbb{Z} \\ \mathsf{B}_{i-[\alpha^{-1}](1+[i\alpha+\beta])} & \text{if } \{i\alpha+\beta\}\alpha^{-1} \geq [\alpha^{-1}] \\ \mathsf{B}_{i-[\alpha^{-1}](i\alpha+\beta)} & \text{if } i\alpha+\beta \in \mathbb{Z}. \end{cases} \end{aligned}$$

To start with, let us suppose that $\mathsf{T} \neq \mathsf{T}^-$. For each b_i the first a to the left is $\mathsf{a}_{[i\alpha+\beta]}$, and thus b_i and $\mathsf{a}_{[i\alpha+\beta]}$ get mapped to the same letter: $\mathsf{A}_{[i\alpha+\beta]}$, unless $k(\mathsf{b}_i) = [\alpha^{-1}]$, i.e. $\{i\alpha+\beta\}\alpha^{-1} \geq [\alpha^{-1}]$. So we must check that

(†) *There are* $1 + [\alpha^{-1}]$ *b's between* $\mathsf{a}_{[i\alpha+\beta]}$ *and* $\mathsf{a}_{1+[i\alpha+\beta]}$ *if and only if there is a B between* $\mathsf{A}_{[i\alpha+\beta]}$ *and* $\mathsf{A}_{1+[i\alpha+\beta]}$.

Now for any j

$$\begin{aligned} &\text{there are } 1+[\alpha^{-1}]\mathsf{b}'s \text{ between } \mathsf{a}_j \text{ and } \mathsf{a}_{j+1} \\ &\iff \left\{j+1-\frac{\beta}{\alpha}\right\} < \left\{\frac{1}{\alpha}\right\} \\ &\iff \left\{\{\alpha^{-1}\}(j+1)-\beta\alpha^{-1}\right\} = \left\{\alpha^{-1}(j+1)-\beta\alpha^{-1}-[\alpha^{-1}](j+1)\right\} \\ &\qquad = \left\{j+1-\frac{\beta}{\alpha}\right\} < \left\{\frac{1}{\alpha}\right\} \\ &\iff \{\alpha^{-1}\}(j+1)-\beta\alpha^{-1}-\left\{\{\alpha^{-1}\}(j+1)-\beta\alpha^{-1}\right\} \\ &\qquad > \{\alpha^{-1}\}j-\beta\alpha^{-1} \\ &\iff \left[\{\alpha^{-1}\}(j+1)-\beta\alpha^{-1}\right] > \{\alpha^{-1}\}j-\beta\alpha^{-1} \\ &\iff \text{there is a } \mathsf{B} \text{ between } \mathsf{A}_j \text{ and } \mathsf{A}j+1. \end{aligned}$$

Letting $j = [i\alpha+\beta]$, we have (†). This equivalence holds even when $j+1-\frac{\beta}{\alpha} \in \mathbb{Z}$ (and $\mathsf{T} = \mathsf{T}^+$), for then a_{j+1} is at a point of $\mathbb{Z}^2$ and there will be only $[\alpha^{-1}]$ b's between a_j and a_{j+1}, while there will not be a B between A_j and A_{j+1} in this case because A_{j+1} will also be at a point of $\mathbb{Z}^2$ and so A_{j+1} will be to the left of the coinciding B (since $\mathsf{T}' = \mathsf{T}'^-$).

When there is a B between $\mathsf{A}_{[i\alpha+\beta]}$ and $\mathsf{A}_{1+[i\alpha+\beta]}$ it will be

$$\mathsf{B}_{1+\left[\frac{[i\alpha+\beta]-\beta\alpha^{-1}\{\alpha^{-1}\}^{-1}}{\{\alpha^{-1}\}^{-1}}\right]} = \mathsf{B}_{1+[\{\alpha^{-1}\}[i\alpha+\beta]-\beta\alpha^{-1}]} = \mathsf{B}_{i-[\alpha^{-1}](1+[i\alpha+\beta])}.$$

Now suppose $\mathsf{T} = \mathsf{T}^-$. The only additional complication is when b_i is at a point of $\mathbb{Z}^2$. Since $\mathsf{T} = \mathsf{T}^-$, this can only occur when b_i is the last b in a block of $1+[\alpha^{-1}]$. So suppose $i\alpha+\beta \in \mathbb{Z}$; then b_i and $\mathsf{a}_{i\alpha+\beta}$ coincide at $(i, i\alpha+\beta) \in \mathbb{Z}^2$ and b_i is to the left of $\mathsf{a}_{i\alpha+\beta}$. $\mathsf{a}_{i\alpha+\beta}$ gets sent to $\mathsf{A}_{i\alpha+\beta}$, which will be at the point $(i-[\alpha^{-1}](i\alpha+\beta), i\alpha+\beta)$ of $\mathbb{Z}^2$ on the line $\widetilde{\mathsf{L}}$. So $\mathsf{B}_{i-[\alpha^{-1}][i\alpha+\beta]}$ will be to the left of

$\mathsf{A}_{i\alpha+\beta}$, because $\mathsf{a}_{i-[\alpha^{-1}](i\alpha+\beta)}$ will be to the left of $\mathsf{b}_{i\alpha+\beta}$ in $\mathsf{T}' = \mathsf{T}'^{+}$. Hence there will be a B between $\mathsf{A}_{i\alpha+\beta-1}$, and let $\mathsf{A}_{i\alpha+\beta}$.

3. The Proof of Theorems A, B, C, and D

Theorem 3.1 (Theorem A)**.** *Let T be a non-singular cutting sequence and t and t' be tiles in T. Let x and x' in X_α be the corresponding coding sequences. Then x and x' are tail equivalent.*

Let T be a singular cutting sequence, and let t a tile in T. Then the coding sequence of t is tail equivalent to one of $-\alpha$, 0^+, or 0^-.

Proof. Suppose that T is non-singular. Then $\mathsf{T} = \bigcup_{i=0}^{\infty} W_i$, where $\{W_i\}$ is the inflation sequence of some letter $\widetilde{\mathsf{t}}$ in T (cf. section 1.7). Then for all i larger than some i_0, t and t' are in the same segment W_i. For each letter s in W_i the i-th element of the coding sequence of s is the same as the i-th element of the coding sequence of $\widetilde{\mathsf{t}}$. Hence from the i-th element on the coding sequences of t and t' agree.

Suppose that T is singular and $\mathsf{t} \in \mathsf{T}$. Let $W_0 = \{\mathsf{t}\} \subseteq W_1 \subseteq W_2 \cdots$ be the inflation sequence of t. Then $\bigcup_{i=0}^{\infty} W_i$ is a connected infinite proper subset of T and is thus a right or left half line.

Case 1: $\bigcup_{i=0}^{\infty} W_i$ is a left half line. Let t^- be the rightmost letter of $\bigcup_{i=0}^{\infty} W_i$. We claim that t^- is a b and that to the left of t^- is the segment $\mathsf{a}\underbrace{\mathsf{b}\cdots\mathsf{b}}_{a_1-1}$, where $\alpha = [0; a_1, a_2, a_3, \ldots]$ is the slope of T. Indeed, t^- cannot be an a, as it would be grouped with a b to the right (and thus outside of $\bigcup_{i=0}^{\infty} W_i$) on the first composition. Thus t^- is a b. If there are fewer than $a_1 - 2$ b's between t^- and the first a to the left, then a b to the right of t^- would be grouped with t^- on the first composition, which is again impossible.

When we perform composition on T, t^- gets grouped into a letter which is also the rightmost letter in a half line, and by the same argument as above this letter is a b and to its left is the segment $\mathsf{a}\underbrace{\mathsf{b}\cdots\mathsf{b}}_{a_2-1}$. This is repeated under successive compositions, and hence the coding sequence of t^- is $(a_1-1, a_2-1, a_3-1, \ldots) = -\alpha$. If t is any letter in $\bigcup_{i=0}^{\infty} W_i$, then eventually t and t^- are in some W_i. From this point on the coding sequence of t and t^- are the same, and thus the coding sequence of t is tail equivalent to $-\alpha$, and $\bigcup_{i=1}^{\infty} W_i = \mathsf{T}_{-\alpha}$ (see 1.6).

Case 2: Now let us suppose that $\bigcup_{i=0}^{\infty} W_i$ is a right half line. This time, let t^+ be the leftmost letter of $\bigcup_{i=0}^{\infty} W_i$, and let $\mathsf{t}_1 = \mathsf{t}^+, \mathsf{t}_2, \ldots$ be the sequence of letters obtained by following t^+ through successive compositions. From Case 1 we see that t_i will always have $\mathsf{a}\underbrace{\mathsf{b}\cdots\mathsf{b}}_{a_i}$ to its immediate left. Thus, if t^+ is an a, then $x_1 = 0$ and t_2 is a b. Then $x_2 = a_2$ and t_3 is an a again, and the process repeats. Hence $(x_i) = (0, a_2, 0, a_4, \ldots) = 0^+$, and $\bigcup_{i=1}^{\infty} W_i = \mathsf{T}_{0^+}$ (see 1.6). Similarly, if t^+ is a b, then t_2 is an a and t_3 is a b. In this case the coding sequence $(x_i) = (a_1, 0, a_3, 0, \ldots) = 0^-$, and $\bigcup_{i=1}^{\infty} W_i = \mathsf{T}_{0^-}$ (see 1.6).

If t is any letter in $\bigcup_{i=0}^{\infty} W_i$, then t and t^+ are eventually in some W_i, and thus from i onwards their coding sequences will be equal. Hence the coding sequence for t will be tail equivalent to either 0^+ or 0^-.

Theorem 3.2 (Theorem B). *Let $\beta \in \mathbb{R}_\alpha$, and let L be the line $y = \alpha x + \beta$. Let T be the corresponding cutting sequence, let t be the tile b_0 (as described in 1.2), and let $x \in X_\alpha$ be the corresponding coding sequence. Then $x = \varphi(\beta)$, where φ is constructed in 1.9. In particular, the cutting sequence T is singular if and only if L passes through a point of $\mathbb{Z}^2$.*

Proof. Let $\varphi(\beta) = (x_1, x_2, x_3, \dots)$. Both $\varphi(\beta)$ and the coding sequence for b_0 are unchanged by replacing β by $\beta - [\beta]$, so we shall suppose that $0 \le \beta \le 1$.

We shall show that if L'' is the line $y = \{\alpha^{-1}\}x + \beta_2$, where $\beta_2 = 1 - \frac{\beta}{\alpha}$ if $\beta\alpha^{-1} \ge [\alpha^{-1}]$ or $\beta_2 = (1 + x_1)\alpha - \frac{\beta}{\alpha}$ if $\beta\alpha^{-1} < [\alpha^{-1}]$, and T'' is the corresponding cutting sequence, then T'' is the composition of T, and b_0 in T gets sent to either b_0 in T'' when $\beta\alpha^{-1} < [\alpha^{-1}]$, or to a_0 in T'' when $\beta\alpha^{-1} \ge [\alpha^{-1}]$.

When $\beta\alpha^{-1} \ge [\alpha^{-1}]$, b_0 is the first b in T'' to the right of a_0. So a_0, and b_0 get grouped under composition and have the same coding sequence. Thus the theorem follows by induction if we can prove the claim made above about L'' and T''.

Let us recall some notation from Proposition 2.3. Let L' be the line $y = \{\alpha^{-1}\}x - \beta\alpha^{-1}$ and T' its cutting sequence. Applying Proposition 2.3 to the case $i = 0$ we see that b_0 in T gets sent either to b_0 in T' when $\beta\alpha^{-1} < [\alpha^{-1}]$, or to $\mathsf{a}_{-[\alpha^{-1}]}$ in T' when $\beta\alpha^{-1} \ge [\alpha^{-1}]$.

Case 1: $\beta\alpha^{-1} < [\alpha^{-1}]$. Then $\beta_2 = 1 + x_1 - \{\beta\alpha^{-1}\} = -\beta\alpha^{-1} + 1 + x_1 + [\beta\alpha^{-1}]$. Thus L'' is just L' raised by $1 + x_1 + [\beta\alpha^{-1}]$. This means that T' and T'' are the same cutting sequence up to translation, and under this translation b_0 in T' gets sent to b_0 in T''.

Case 2: $\beta\alpha^{-1} \ge [\alpha^{-1}]$. Then $\beta_2 = \{\alpha^{-1}\} - \{\beta\alpha^{-1}\} = \{\alpha^{-1}\} + [\alpha^{-1}] - \beta\alpha^{-1}$. So we may write L'' as $y = \{\alpha^{-1}\}(x+1) - \beta\alpha^{-1} + [\alpha^{-1}]$. Thus L'' is L' raised by $[\alpha^{-1}]$ units and shifted one unit to the left. So T' and T'' are same cutting sequences up to translation, and $\mathsf{a}_{-[\alpha^{-1}]}$ in T' is sent to a_0 in T''.

To prove the last assertion, note that

$$\begin{aligned} \mathsf{T} \text{ is singular} \quad &\iff \quad X \text{ is tail equivalent to one of } \{-\alpha, 0^+, 0^-\} \\ &\iff \quad \beta \in \mathbb{Z} + \alpha\mathbb{Z} \\ &\iff \quad \mathsf{L} \text{ passes through a point of } \mathbb{Z}^2. \end{aligned}$$

Lemma 3.3. *When $x_i < a_i$ we have $T_i = S_i$, and when $x_i = a_i$ we have $T_i = S_{i-1}$.*

Proof. We will prove this by induction on i. When $i = 1$, T_1 is either $\mathsf{a}\underbrace{\mathsf{b}\cdots\mathsf{b}}_{a_1} = S_1$ (if $x_i < a_i$), or $\mathsf{b} = S_0$ (if $x_i = a_i$). Suppose the result is true for $k - 1$.

Case (i): $x_k < a_k$. Then

$$T_k = \begin{cases} S_{k-2}\underbrace{S_{k-1}\cdots S_{k-1}}_{x_k}T_{k-1}S_{k-1}\cdots S_{k-1} & \text{if } x_{k-1} < a_{k-1} \\ T_{k-2}\underbrace{S_{k-1}\cdots S_{k-1}}_{a_k} & \text{if } x_{k-1} = a_{k-1}. \end{cases}$$

In the first case $T_{k-1} = S_{k-1}$, so $T_k = S_k$; and in the second case $T_{k-2} = S_{k-2}$, since x_{k-2} must be different from a_{k-2}. Thus $T_k = S_k$.

Case (ii): $x_k = a_k$. In this case $T_k = T_{k-1} = S_{k-1}$, since $x_{k-1} \ne a_{k-1}$.

Theorem 3.4 (Theorem C). *Let T be a cutting sequence, t a tile in T, and x the coding sequence of (T, t). Then the Smith sequence $\{T_i\}_{i=1}^\infty$ of x equals the inflation sequence $\{W_i\}_{i=1}^\infty$ of (T, t), i.e. $W_i = T_i$ for $i = 1, 2, 3, \dots$.*

Proof. Given a cutting sequence T, let $\mathsf{T}_1 = \mathsf{T}, \mathsf{T}_2, \mathsf{T}_3, \dots$ be the cutting sequences obtained from T_1 by composition. We shall show that T_k can be obtained by rewriting T_1 using the words S_{k-2} and S_{k-1}.

We begin by replacing all a's in T_1 by S_{-1} and all b's in T_1 by S_0. We have now rewritten T_1 in the words S_{-1} and S_0 (see 1.4).

Now replace all words $S_{-1}\underbrace{S_0 \cdots S_0}_{a_1}$ with S_1. We now have a sequence of S_1's and S_0's. By making the substitution $S_1 \mapsto \mathsf{b}$ and $S_0 \mapsto \mathsf{a}$ we have the cutting sequence T_2.

If we now take our sequence of S_1's and S_0's and replace each word $S_0\underbrace{S_1 \cdots S_1}_{a_2}$ by S_2, we have a sequence of S_2's and S_1's. The map that takes $S_2 \mapsto \mathsf{b}$ and $S_1 \mapsto \mathsf{a}$ transforms T into T_3.

Suppose we have performed this substitution $k-1$ times and have rewritten T as a sequence of S_{k-1}'s and S_{k-2}'s which becomes T_k under the map $S_{k-1} \mapsto \mathsf{b}$ and $S_{k-2} \mapsto \mathsf{a}$. Let us show that if we make one more iteration we obtain a rewriting of T as a sequence of S_{k-1}'s and S_k's which becomes T_{k+1} under the map $S_k \mapsto \mathsf{b}$ and $S_{k-1} \mapsto \mathsf{a}$.

First notice that because of the bijection with T_k there are in T either a_k or $1 + a_k$ S_{k-1}'s between adjacent S_{k-2}'s. Thus the rewrite rule $S_{k-2}\underbrace{S_{k-1} \cdots S_{k-1}}_{a_k} \mapsto S_k$ can be performed. Moreover, by transforming this rewrite rule by $S_{k-2} \mapsto \mathsf{a}$ and $S_{k-1} \mapsto \mathsf{b}$ we get exactly the composition rule for passing from T_k to T_{k+1}; i.e. the following diagrams commute:

$$\begin{array}{ccc}
\mathsf{T} \ni S_{k-2}\underbrace{S_{k-1} \cdots S_{k-1}}_{a_k} & \longrightarrow & S_k \in \mathsf{T} \\
\big\downarrow & & \big\downarrow \\
\mathsf{T}_k \ni \mathsf{a}\underbrace{\mathsf{b} \cdots \mathsf{b}}_{a_k} & \longrightarrow & \mathsf{b} \in \mathsf{T}_{k+1}
\end{array}$$

$$\begin{array}{ccc}
\mathsf{T} \ni S_{k-2}\underbrace{S_{k-1} \cdots S_{k-1}}_{1+a_k} & \longrightarrow & S_k S_{k-1} \in \mathsf{T} \\
\big\downarrow & & \big\downarrow \\
\mathsf{T}_k \ni \mathsf{a}\underbrace{\mathsf{b} \cdots \mathsf{b}}_{1+a_k} & \longrightarrow & \mathsf{ba} \in \mathsf{T}_{k+1}
\end{array}$$

where in the left hand columns we are using the rule $S_{k-2} \mapsto \mathsf{a}$ and $S_{k-1} \mapsto \mathsf{b}$, and in the right hand columns we are using the rule $S_{k-1} \mapsto \mathsf{a}$ and $S_k \mapsto \mathsf{b}$. This establishes our claim.

We shall prove that $W_i = T_i$ by induction on i. For $i = 1$ we have either $W_1 = \mathsf{a}\underbrace{\mathsf{b} \cdots \mathsf{b}}_{a_1} = T_1$ when $x_1 < a_1$, or $W_1 = \mathsf{b} = T_1$ when $x_1 = a_1$. So let us suppose that $W_i = T_i$ for $1 \le i \le k$, and show that $W_{k+1} = T_{k+1}$. We have three cases to consider:

(i) $x_{k+1} = a_{k+1}$.
(ii) $x_{k+1} < a_{k+1}$ and $x_k < a_k$.
(iii) $x_k = a_k$.

Case (i): In this case $\mathsf{t}_{k+1} = \mathsf{b}$ and appears in T_{k+1} as $\cdots \mathsf{a} \underbrace{\mathsf{b} \cdots \mathsf{b}}_{a_{k+1}} \boxed{\mathsf{b}} \mathsf{a} \cdots$. This $\boxed{\mathsf{b}}$ gets sent to an a in T_{k+2}, so no additional letters are picked up; i.e. $W_{k+1} = W_k = T_k = T_{k+1}$.

Case (ii): In this case $x_{k+1} < a_{k+1}$ and $x_k < a_k$, so $\mathsf{t}_{k+1} = \boxed{\mathsf{b}}$ in $\cdots \mathsf{a} \underbrace{\mathsf{b} \cdots \mathsf{b}}_{x_{k+1}} \boxed{\mathsf{b}}\ \mathsf{b} \cdots \mathsf{b}$, which gets mapped to $\boxed{\mathsf{b}}$ in T_{k+2}. W_{k+1} is the inverse image in T_1 of this b. Under the rewrite rule $S_k \mapsto \mathsf{b}$, $S_{k-1} \mapsto \mathsf{a} : \mathsf{T}_1 \to \mathsf{T}_{k+1}$, the corresponding interval in T_1 is $\cdots S_{k-1} \underbrace{S_k \cdots S_k}_{x_{k+1}} \boxed{T_k} \underbrace{S_k \cdots S_k}_{a_{k+1} - x_{k+1} - 1} \cdots$ (where we know that $T_k = S_k$ because $x_k < a_k$). Thus

$$W_{k+1} = S_{k-1} \underbrace{S_k \cdots S_k}_{x_{k+1}} \boxed{T_k} \underbrace{S_k \cdots S_k}_{a_{k+1} - x_{k+1} - 1} = T_{k+1}$$

Case (iii): In this case $x_k = a_k$ and $x_{k+1} = 0$. Thus $\mathsf{t}_{k+1} = \boxed{\mathsf{a}}$ in $\cdots \boxed{\mathsf{a}} \underbrace{\mathsf{b} \cdots \mathsf{b}}_{a_{k+1}} \cdots \subseteq \mathsf{T}_{k+1}$. This segment of letters gets mapped to a b in T_{k+2}, so W_{k+1} is the inverse image in T_1 of $\boxed{\mathsf{a}} \underbrace{\mathsf{b} \cdots \mathsf{b}}_{a_{k+1}}$ in T_{k+1}. Using the rewrite rule $S_{k-1} \mapsto \mathsf{a}$ and $S_k \mapsto \mathsf{b}$, the corresponding interval in T_1 is

$$W_{k+1} = \boxed{S_{k-1}} \underbrace{S_k \cdots S_k}_{a_{k+1}} = \boxed{T_{k-1}} \underbrace{S_k \cdots S_k}_{a_{k+1}} = T_{k+1}$$

since $x_{k-1} < a_{k-1}$ (as $x_k = a_k$).

Theorem 3.5 (Theorem D)**.** *Let $\sim$ be the equivalence relation on X_α generated by tail equivalence, and let $-\alpha \sim 0^+ \sim 0^-$. Let $(\mathcal{T}_1, \mathfrak{t}_1)$ and $(\mathcal{T}_2, \mathfrak{t}_2)$ be in $\mathcal{T}_\alpha^\bullet$, and x_1 and x_2 in X_α the corresponding coding sequences. Then $x_1 \sim x_2$ if and only if $\mathcal{T}_1 \approx \mathcal{T}_2$.*

Proof. Suppose (T, t) and $(\mathsf{T}', \mathsf{t}')$ are in $\mathcal{T}_\alpha^\bullet$ with corresponding coding sequences (x_i) and (x_i') in X_α. Suppose that (T, t) and $(\mathsf{T}', \mathsf{t}')$ are equivalent. Then either they are both singular, in which case (x_i) and (x_i') are in the tail equivalence class of $\{-\alpha, 0^+, 0^-\}$ by Theorem B, or neither is singular, in which case (x_i) and (x_i') are tail equivalent, again by Theorem B.

Suppose that (x_i) and (x_i') are equivalent. Then there are two cases.

Case (i): (x_i) and (x_i') are tail equivalent. We shall show that that there is $\tilde{\mathsf{t}}'$ in T' such that $(\mathsf{T}, \mathsf{t}) = (\mathsf{T}', \tilde{\mathsf{t}}')$. Suppose that $x_i = x_i'$ for $i \geq k$. Let T_i for $i = 1, 2, 3, \ldots$ and T_i' for $i = 1, 2, 3, \ldots$ be the sequences obtained from T and T' respectively by composition. Also let $\mathsf{t}_k \in \mathsf{T}_k$ and $\mathsf{t}_k' \in \mathsf{T}_k'$ be the letters which after $k-1$ compositions absorb t and t' respectively. The coding sequences of $(\mathsf{T}_k, \mathsf{t}_k)$ and $(\mathsf{T}_k', \mathsf{t}_k')$ are respectively $(x_k, x_{k+1}, x_{k+2}, \ldots)$ and $(x_k', x_{k+1}', x_{k+2}', \ldots)$, and hence are equal. Thus by Theorem C the inflation sequence of t_k in T_k is equal to the inflation sequence of t_k' in T_k'. Since these cutting sequences are non-singular, they are the union of the corresponding inflation sequences. Hence $(\mathsf{T}_k, \mathsf{t}_k) = (\mathsf{T}'_k, \mathsf{t}'_k)$. Now by the argument presented at the beginning of the proof of Theorem C let us rewrite T as a sequence of S_{k-1}'s and S_{k-2}'s so that, under the transformation $S_{k-1} \mapsto \mathsf{b}$ and $S_{k-2} \mapsto \mathsf{a}$, T is transformed into T_k. Let us do the same for T' and T'_k. Since $(\mathsf{T}_k, \mathsf{t}_k) = (\mathsf{T}'_k, \mathsf{t}'_k)$, we see that, when written as a sequence of S_{k-1}'s and S_{k-2}'s, T and T' are equal, and that t and t' lie in the same word (either a S_{k-1} or a S_{k-2}). Hence $(\mathsf{T}, \mathsf{t}) = (\mathsf{T}', \tilde{\mathsf{t}}')$, for some letter $\tilde{\mathsf{t}}'$ in T' no more distant from t than $|S_{k-1}|$.

Case (ii) : (x_i) and (x'_i) are in the tail equivalence class of $\{-\alpha, 0^+, 0^-\}$. By Theorem B we know that T and T' come from lines which pass through a point of $\mathbb{Z}^2$, and thus both T and T' are singular. By Theorem C, we know that up to a translation there are only the two singular cutting sequences T^+ and T^- discussed in 1.6, and these are conjugate. Hence $\mathsf{T} \approx \mathsf{T}'$.

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