TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 352, Number 11, Pages 5293–5316
S 0002-9947(00)02648-9
Article electronically published on July 18, 2000

HYPERCYCLIC OPERATORS THAT COMMUTE WITH THE BERGMAN BACKWARD SHIFT

PAUL S. BOURDON AND JOEL H. SHAPIRO

ABSTRACT. The backward shift B on the Bergman space of the unit disc is known to be *hypercyclic* (meaning: it has a dense orbit). Here we ask: "Which operators that commute with B inherit its hypercyclicity?" We show that the problem reduces to the study of operators of the form $\varphi(B)$ where φ is a holomorphic self-map of the unit disc that multiplies the Dirichlet space into itself, and that the question of hypercyclicity for such an operator depends on how freely $\varphi(z)$ is allowed to approach the unit circle as $|z| \to 1-$.

INTRODUCTION

A *hypercyclic* operator on a Banach space is a linear operator that has a dense orbit. Surprisingly many concrete operators have been shown to have this property, and from this abundance of examples has arisen a lively literature on the subject. For an excellent guide to both this body of work and its historical background in classical analysis, see Grosse-Erdmann's recent survey article [17].

One much-studied operator that has recently been identified as hypercyclic is the backward shift B on the Bergman space A^2 of the unit disc (see §1 below for the definitions of B and A^2, and [15] or [25, §7.4, Exercise 2] for the proof of hypercyclicity). We initiate here the study of the "Commutant Hypercyclicity Problem" for B:

Which operators that commute with B are also hypercyclic?

It is known that each operator on A^2 that commutes with B has a natural representation of the form $\varphi(B)$ where φ is a *multiplier of the Dirichlet space* (Theorem 1.7 below). As we will explain in §1.10–§1.11, the problem of understanding the hypercyclic behavior of $\varphi(B)$ reduces to that of understanding the special case where $\varphi(\mathbb{U})$ is a subset of $\mathbb{U}$ whose closure intersects the unit circle. After that we consider only this case, for which our results indicate that whether $\varphi(B)$ is hypercyclic or not depends on how freely the points $\varphi(z)$ are allowed to approach the unit circle as $|z| \to 1-$.

For example, we will show that $\varphi(B)$ is hypercyclic whenever φ has radial limits of modulus one on a set of positive measure (Theorem 2.8). Although sufficient, this positive-measure condition is not necessary; we show this in §2.12 by constructing a Dirichlet multiplier $\varphi : \mathbb{U} \to \mathbb{U}$ for which $\varphi(B)$ is hypercyclic on A^2, yet φ has radial limit of modulus one at just a single point of $\partial\mathbb{U}$.

Received by the editors January 28, 1999 and, in revised form, September 13, 1999.
2000 *Mathematics Subject Classification.* Primary 47B38.
Key words and phrases. Hypercyclic operator, Bergman space, backward shift.
Both authors were supported in part by the National Science Foundation.

In the other direction we show in Corollary 3.3 that if the degree of contact between $\varphi(\mathbb{U})$ and $\partial\mathbb{U}$ is "limited" in a certain sense, then $\varphi(B)$ cannot be hypercyclic; in particular, this happens if $\varphi(\mathbb{U})$ lies in a disk internally tangent to the unit circle, so for example the operator $(I+B)/2$ is not hypercyclic on A^2 (see Corollary 3.4).

In §4 we show that there is some precision to our "positive-measure" sufficient condition for hypercyclicity by giving examples of Dirichlet multipliers φ that map the unit disc into itself such that $\varphi(B)$ is not hypercyclic, yet for which φ has radial limits of modulus one on a set of *Hausdorff dimension one*.

All of this contrasts sharply with what is known for the Hardy space H^2, where the backward shift is not hypercyclic (it is a contraction), and the hypercyclic operators that commute with it are easily described (see §§1.2-1.3 below). In further contrast with what happens in the H^2 setting our present work leads into diverse issues concerning multipliers of the Dirichlet space, Carleson sets, and regularity of outer functions.

Our results bear some similarity with those obtained for the isolation problem for composition operators. We comment briefly on this in §5.

1. Fundamentals

In this section we introduce the spaces of functions analytic on the unit disc that form the infrastructure of our work. These are the Bergman space A^2, the Dirichlet space $\mathcal{D}$ and its pointwise multipliers, and the Hardy space H^2. We indicate why the commutant hypercyclicity problem is interesting for the backward shift on the Bergman space, and show how it reduces to the consideration of geometric properties of multipliers of the Dirichlet space that map the unit disc into itself.

1.1. **The Bergman space.** Our primary setting is the Bergman space A^2 of the open unit disc $\mathbb{U}$. This is the space of functions f that are holomorphic on $\mathbb{U}$ and whose moduli are square integrable with respect to Lebesgue area measure on $\mathbb{U}$. A^2 is a closed subspace of $L^2(d\lambda)$, where $d\lambda$ is Lebesgue area measure on $\mathbb{U}$, normalized so as to have unit mass. Therefore A^2 is a Hilbert space in the $L^2(d\lambda)$-norm $\|\cdot\|$ defined by

$$\|f\|^2 = \int_{\mathbb{U}} |f|^2\, d\lambda \qquad (f \in A^2). \tag{1}$$

A^2 and its norm can be described as well by Taylor coefficients. A straightforward computation shows that if $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ is holomorphic on $\mathbb{U}$, then

$$\int_{\mathbb{U}} |f|^2\, d\lambda = \sum_{n=0}^{\infty} \frac{|\hat{f}(n)|^2}{n+1}, \tag{2}$$

where now the value ∞ is allowed. Thus f belongs to A^2 if and only if the series on the right converges, in which case the sum of this series is equal to $\|f\|^2$.

We study bounded linear operators on A^2 that commute with the *backward shift* B. This is the operator on A^2 defined by

$$Bf(z) = \frac{f(z)-f(0)}{z} = \sum_{n=0}^{\infty} \hat{f}(n+1) z^n \qquad (f \in A^2, z \in \mathbb{U}).$$

B gets its name from the fact that it shifts the Taylor coefficient sequence of f one unit to the left (and drops off the constant term). An easy calculation using the Taylor coefficient description of the Bergman norm shows that B is a bounded operator on A^2 with $\|B\| = \sqrt{2}$.

1.2. **Commutant hypercyclicity.** Work of Godefroy and Shapiro [16] suggests that operators behaving like backward shifts tend to transfer hypercyclicity (if they have it) to appropriate operators in their commutants. The word "appropriate" here must be interpreted properly, since the commutant will always contain operators that are not hypercyclic (e.g. contractions, and scalar multiples of the identity).

Carol Kitai proved in her 1982 Toronto dissertation [12] that a necessary condition for an operator on a Banach space to be hypercyclic is that every component of its spectrum must intersect the unit circle. While clearly not sufficient for hypercyclicity (e.g. the identity operator, whose spectrum is the singleton $\{1\}$, is not hypercyclic), this spectral condition has a stronger version which, for some classes of operators, *does* suffice. A striking instance of this occurs in the following result due to Godefroy and Shapiro about the backward shift on the Hardy space H^2 [16, Theorem 4.9]:

1.3. Theorem. *If T is a bounded operator on H^2 that commutes with the backward shift, then the following statements are equivalent:*

(a) *T is hypercyclic on H^2.*
(b) *The interior of the spectrum of T intersects $\partial\mathbb{U}$.*

The sufficiency part "(b) $\rightarrow$ (a)" of this result holds for very general spaces of analytic functions, in particular for the Bergman space [16, Theorem 4.5]. However, the converse "(a) $\rightarrow$ (b)" *fails* for the Bergman space, as is shown by the backward shift itself, which is hypercyclic, but whose spectrum is well known (and easily seen) to be the closed unit disc.

Note that, according to Theorem 1.3, operators in the commutant of the H^2-backward shift having the same spectrum also display the same hypercyclic behavior. For the Bergman backward shift the commutant hypercyclicity problem is much more delicate. We just mentioned that B itself is hypercyclic on A^2 and that its spectrum is the closed unit disc, but in §3.6 below we will present an example of an operator that commutes with B and has spectrum equal to the closed disk, but is *not* hypercyclic. Thus in the Bergman setting the spectrum alone does not provide sufficient information to resolve the issue of hypercyclicity.

As a further complicating factor, the commutant of the Bergman backward shift is a far more subtle object than the corresponding Hardy space commutant. It is known that any operator commuting with B has the form $\varphi(B)$, where φ is a Dirichlet space multiplier (see §1.7–§1.9 for the details). By contrast, the corresponding representation for the Hardy space commutant involves the full algebra H^∞ of bounded analytic functions (see, for example, [18, Problem 147, page 79] for the dual version of this involving the forward shift).

1.4. **The Hardy and Dirichlet spaces.** Two Hardy spaces of analytic functions arise during the course of our work. First there is H^2, the space of functions f holomorphic on $\mathbb{U}$ for which

$$\|f\|_2^2 \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty.$$

The norm $\|\cdot\|_2$ makes H^2 a Hilbert space. Next there is the collection H^∞ of bounded analytic functions on $\mathbb{U}$, which is a Banach algebra in the "supremum norm"

$$\|f\|_\infty \stackrel{\text{def}}{=} \sup\{|f(z)| : z \in \mathbb{U}\} \qquad (f \in H^\infty).$$

The commutant of the Bergman backward shift is intimately connected, via duality, with yet a third space: the *Dirichlet space.* This is the collection $\mathcal{D}$ of functions holomorphic on $\mathbb{U}$ whose *first derivatives* have square integrable modulus over $\mathbb{U}$. The norm $\|\cdot\|_{\mathcal{D}}$ defined by

$$\|f\|_{\mathcal{D}}^2 = \|f\|_2^2 + \int_{\mathbb{U}} |f'|^2 \, d\lambda \tag{3}$$

makes $\mathcal{D}$ into a Hilbert space. The calculation used to establish (2) shows that for each f holomorphic on $\mathbb{U}$,

$$\|f\|_{\mathcal{D}}^2 = \sum_{n=0}^{\infty} (n+1)\, |\hat{f}(n)|^2, \tag{4}$$

where again the value ∞ is allowed. Thus $\mathcal{D}$ emerges as the space of functions holomorphic on $\mathbb{U}$ whose power series coefficients make the sum on the right-hand side of (4) finite.

Neither of the spaces $\mathcal{D}$ nor H^∞ contains the other, but letting X denote either space, and letting $H(\mathbb{U})$ denote the space of all functions holomorphic on U, endowed with the topology of uniform convergence on compact subsets of $\mathbb{U}$, we have the inclusions

$$X \subset H^2 \subset A^2 \subset H(\mathbb{U}),$$

where all the embedding maps are continuous. In particular, a sequence that converges in any of these spaces also converges uniformly on compact subsets of $\mathbb{U}$.

1.5. **Duality.** For $f \in A^2$ and $g \in \mathcal{D}$, define

$$\langle f, \, g\rangle \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \hat{f}(n)\, \hat{g}(n). \tag{5}$$

The Cauchy-Schwarz inequality teams up with the coefficient descriptions (2) and (4) of the norms in A^2 and $\mathcal{D}$ to show that the sum on the right-hand side of (5) converges absolutely. The result is a bilinear pairing between the two spaces with respect to which each is isometrically the dual of the other. For example, a linear functional Λ on A^2 is continuous if and only if there is a function $g \in \mathcal{D}$ such that $\Lambda(f) = \langle f, \, g\rangle$ for each $f \in A^2$. Moreover, the norm of Λ is precisely the $\mathcal{D}$-norm of g.

This way of representing the dual space of A^2 is more natural for studying the backward shift than is the usual self-dual Hilbert space representation. In the representation above the adjoint of $B : A^2 \to A^2$, is easily seen to be the forward shift $M_z : \mathcal{D} \to \mathcal{D}$ defined by $(M_z f)(z) = zf(z)$ for $z \in \mathbb{U}$ and $f \in \mathcal{D}$ (the notation "M_z" employs a standard abuse of functional notation which will show up again later on). More precisely,

$$\langle Bf, \, g\rangle = \langle f, \, M_z g\rangle \qquad (f \in A^2, \; g \in \mathcal{D}).$$

In the same way B is the adjoint of M_z. By contrast, if we represent the dual of A^2 in the standard way, as A^2 itself acting through the Bergman space inner product

$$\langle f, g\rangle_{A^2} \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{\hat{f}(n)\overline{\hat{g}(n)}}{n+1},$$

then the adjoint of B on A^2 becomes the operator

$$f \to \sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right) \hat{f}(n-1) z^n;$$

i.e., M_z followed by a coefficient multiplier.

For the rest of this paper we adopt the convention that if S is a bounded linear operator on $\mathcal{D}$, then S^* denotes the adjoint of S, *computed with respect to the bilinear form* (5). More precisely, S^* is that bounded operator on A^2 defined by

$$\langle S^* f, g\rangle = \langle f, Sg\rangle \qquad (f \in A^2,\, g \in \mathcal{D}). \tag{6}$$

Thus our previous discussion of the duality between the backward shift on A^2 and the forward shift on $\mathcal{D}$ can be summarized as $(M_z)^* = B$.

1.6. **Multipliers and commutants.** The answer to the question: "What is the commutant of the Bergman backward shift?" emerges, not in terms of the Dirichlet space itself, but in terms of its multipliers. A complex valued function φ on $\mathbb{U}$ is said to be a *multiplier* of $\mathcal{D}$ if the pointwise product φf is in $\mathcal{D}$ for every $f \in \mathcal{D}$. We use $\mathcal{M}(\mathcal{D})$ to denote the collection of multipliers of $\mathcal{D}$.

If $\varphi \in \mathcal{M}(\mathcal{D})$ then, because $\varphi = \varphi \cdot 1$ and the constant function 1 belongs to $\mathcal{D}$, we see that that $\varphi \in \mathcal{D}$. Moreover, Banach algebra considerations show that each multiplier is bounded on $\mathbb{U}$ [27, Theorem 10(iii), page 74], but it is known that there are bounded functions in $\mathcal{D}$ that are *not* multipliers of $\mathcal{D}$ [30, Theorem 9]. Thus $\mathcal{M}(\mathcal{D})$ is a proper subset of $\mathcal{D} \cap H^\infty$. We will say more about membership in $\mathcal{M}(\mathcal{D})$ in §1.8.

Each $\varphi \in \mathcal{M}(\mathcal{D})$ induces a linear transformation $M_\varphi : \mathcal{D} \to \mathcal{D}$ defined in the obvious way:

$$M_\varphi f = \varphi f \qquad (f \in \mathcal{D}).$$

A standard argument using the closed graph theorem, along with the fact that convergence in $\mathcal{D}$ implies uniform convergence on compact subsets of $\mathbb{U}$, shows that M_φ is a bounded operator on $\mathcal{D}$. In the resulting operator norm, $\mathcal{M}(\mathcal{D})$ is a commutative Banach algebra.

The following result characterizes the commutant of the Bergman backward shift in terms of Dirichlet multipliers. It is well known, but in order to keep our exposition reasonably self-contained we give a proof. Recall once again our convention that adjoints are to be computed relative to the duality described in §1.5.

1.7. Theorem. *A bounded operator T on A^2 commutes with the backward shift B if and only if $T = M_\varphi^*$ for some $\varphi \in \mathcal{M}(\mathcal{D})$.*

Proof. We prove the equivalent dual statement:

> *A bounded operator T on $\mathcal{D}$ commutes with the forward shift M_z if and only if $T = M_\varphi$ for some $\varphi \in \mathcal{M}(\mathcal{D})$.*

Only one direction deserves attention. Suppose T commutes with M_z; we claim that $T = M_\varphi$ where $\varphi = T(1)$. An induction shows that T also commutes with $(M_z)^n = M_{z^n}$ for each positive integer n, from which it follows that $T(z^n) = z^n\,\varphi$, and then by linearity that $Tf = \varphi f$ for any holomorphic polynomial f. Now if $f \in \mathcal{D}$, then its Taylor polynomials $\{f_n\}$ (center at the origin) converge in $\mathcal{D}$ to f, hence by the continuity of T and our observation about the polynomial case, $\varphi f_n = Tf_n \to Tf$ in $\mathcal{D}$, and therefore uniformly on compact subsets of $\mathbb{U}$. Since $\varphi f_n \to \varphi f$ uniformly on compact subsets of $\mathbb{U}$, we see that $Tf = \varphi f$, hence φ is a multiplier of $\mathcal{D}$ and $T = M_\varphi$. □

The result above has more general formulations; see [27, Theorem 3(b), page 62] for one that deals with weighted shift operators.

1.8. **Sufficient conditions for multipliers.** The previous result underscores the importance of knowing just when a function holomorphic on $\mathbb{U}$ is a Dirichlet multiplier. Characterization of these functions is a significant problem to which much effort has been devoted. To illustrate the difficulty involved we note that Cochran, Shapiro and Ullrich [9] have shown that for each $f \in \mathcal{D}$ the power series $\sum_{n=0}^{\infty} \pm \hat{f}(n) z^n$ is a Dirichlet multiplier for "almost every choice of sign $\pm$." Thus Dirichlet multipliers cannot be characterized by any condition that involves only the moduli of Taylor coefficients.

In 1980 Stegenga [29] gave a Carleson-type capacitary condition characterizing the multipliers of $\mathcal{D}$. Subsequently, Brown and Shields studied the connection between Dirichlet multipliers and cyclic vectors of M_z acting on $\mathcal{D}$. Among their results is this one ([2, Corollary 7, page 70] and [6, Proposition 19, page 300]):

If φ is holomorphic on $\mathbb{U}$ with $\varphi' \in H^{1+\varepsilon}$ for some $\varepsilon > 0$, then $\varphi \in \mathcal{M}(\mathcal{D})$.

It is interesting to note that this result fails if $\varepsilon = 0$; in [2] Axler and Shields give an example of a Jordan domain with rectifiable boundary for which the Riemann map (which necessarily has derivative in H^1) is not a Dirichlet multiplier.

Axler and Shields [2, Theorem 3] gave further interesting geometric results about univalent multipliers of $\mathcal{D}$. One of the most useful for our purposes is this:

Every univalent mapping taking $\mathbb{U}$ onto a bounded starlike domain is a Dirichlet multiplier.

(A domain G is called "starlike" if there is a point $w_0 \in G$ such that for any $w \in G$ the entire line segment joining w_0 to w lies in G.)

1.9. **A functional calculus for B.** It follows quickly from the equation $B = M_z^*$ and the "non-conjugate" bilinear nature of our duality (5) between A^2 and $\mathcal{D}$ that $p(B) = M_p^*$ for any holomorphic polynomial p. More generally, it is easy to check that $\|B^n\| = \sqrt{n+1}$, from which it follows that if $\sum_n \sqrt{n}\,|\hat{\varphi}(n)| < \infty$ (a condition that is fulfilled if, for example, φ has C^2-smoothness on the closed unit disc), then the series $\sum_n \hat{\varphi}(n) B^n$ converges in the operator norm of A^2 to a bounded linear operator which deserves to be called $\varphi(B)$. The operator-norm convergence just noted insures that the MacLaurin series of φ converges in the norm of $\mathcal{M}(\mathcal{D})$, so $\varphi \in \mathcal{M}(\mathcal{D})$, and all this norm convergence makes it easy to check that once again $\varphi(B) = M_\varphi^*$.

Something like this argument works in full generality, but with the *weak operator topology* replacing the norm topology. If φ is any function in $\mathcal{M}(\mathcal{D})$, let φ_n denote the n-th arithmetic mean of the sequence of Taylor polynomials of φ (center at the origin). It is known that $M_{\varphi_n} f \to M_\varphi f$ for every $f \in \mathcal{D}$ [27, Theorem 12, page 90]. Thus for every $f \in A^2$ and $g \in \mathcal{D}$,

$$\langle \varphi_n(B)f,\, g\rangle = \langle f,\, M_{\varphi_n} g\rangle \to \langle f,\, M_\varphi g\rangle = \langle M_\varphi^* f,\, g\rangle$$

as $n \to \infty$. In other words, $\varphi_n(B) \to M_\varphi^*$ in the weak operator topology of A^2. This justifies the following all-encompassing definition of our functional calculus for B:

$$\varphi(B) \stackrel{\text{def}}{=} M_\varphi^* \qquad \forall \varphi \in \mathcal{M}(\mathcal{D}). \tag{7}$$

The next result, which is well known, asserts that the functional calculus defined by (7) behaves as it should relative to spectra. For the reader's convenience we sketch a proof.

1.10. Spectral Mapping Theorem. *If $\varphi \in \mathcal{M}(\mathcal{D})$, then the spectrum of $\varphi(B)$ is $\overline{\varphi(\mathbb{U})}$, the closure of $\varphi(\mathbb{U})$ in $\mathbb{C}$.*

Proof. The spectrum of $\varphi(B) = M_\varphi^* : A^2 \to A^2$ coincides with the spectrum of $M_\varphi : \mathcal{D} \to \mathcal{D}$. Thus we have only to prove that the spectrum of M_φ is $\overline{\varphi(\mathbb{U})}$, and for this it is enough to prove that M_φ is invertible on $\mathcal{D}$ if and only if φ is bounded away from zero on $\mathbb{U}$.

For this we note an easy consequence of the product rule for differentiation: *A holomorphic function on $\mathbb{U}$ is a Dirichlet multiplier if and only if its derivative multiplies $\mathcal{D}$ into A^2.* Suppose, then, that $\varphi \in \mathcal{M}(\mathcal{D})$ is bounded away from zero on $\mathbb{U}$. Then $(1/\varphi)'$ is bounded by a constant multiple of φ', and since φ' multiplies $\mathcal{D}$ into A^2, so does $(1/\varphi)'$. Thus $1/\varphi$ is a Dirichlet multiplier, so M_φ is invertible on $\mathcal{D}$, with inverse $M_{1/\varphi}$.

Conversely, suppose M_φ is invertible on $\mathcal{D}$. Let T be its inverse. Then for every $f \in \mathcal{D}$,

$$f = M_\varphi T f = \varphi T f$$

so $1/\varphi$ is a Dirichlet multiplier, and $T = M_{1/\varphi}$. In particular, $1/\varphi$ is bounded on $\mathbb{U}$; i.e., φ is bounded away from zero. $\square$

The previous results transform our commutant hypercyclicity problem for the Bergman backward shift into a study of holomorphic functions φ that are multipliers of the Dirichlet space. Our spectral mapping theorem and Kitai's necessary condition for hypercyclicity (§1.2) show that if $\varphi(B)$ is to be hypercyclic, then $\overline{\varphi(\mathbb{U})}$ has to intersect the unit circle. If $\varphi(\mathbb{U})$ itself intersects $\partial\mathbb{U}$, then the work of Godefroy and Shapiro mentioned after Theorem 1.3 shows that $\varphi(B)$ is hypercyclic on the Bergman space. Thus we need only consider multipliers φ for which $\varphi(\mathbb{U})$ lies either inside $\mathbb{U}$ or outside $\overline{\mathbb{U}}$, and for which $\overline{\varphi(\mathbb{U})} \cap \partial\mathbb{U} \neq \emptyset$.

One further reduction: if $\varphi(\mathbb{U})$ lies outside $\overline{\mathbb{U}}$, then $\varphi(B)$ is invertible (its spectrum $\overline{\varphi(\mathbb{U})}$ does not contain the origin) and the spectrum of its inverse, namely the collection of reciprocals of points in the original spectrum, lies in $\overline{\mathbb{U}}$. Since an invertible operator is hypercyclic if and only if its inverse is hypercyclic (see Corollary 2.2 below) this reduces the formulation of our problem to the following:

1.11. **Reduced commutant hypercyclicity problem.** *For which multipliers φ of $\mathcal{D}$, with $\|\varphi\|_\infty = 1$, is $\varphi(B)$ hypercyclic on A^2?*

We note that G. Herzog and C. Schmoeger [19] have considered the question of hypercyclicity for $f(T)$ where T is a bounded operator on a Banach space, f is holomorphic on a neighborhood of the spectrum of T, and T generalizes the notion of backward shift in that it is surjective and the union of the null spaces of its powers is dense. Herzog and Schmoeger show that in this case, if f has no zero on the spectrum of T and $|f(0)| = 1$, then $f(T)$ is hypercyclic. In the special case where T is the backward shift on A^2 these hypotheses imply that $f \in \mathcal{M}(\mathcal{D})$ and $f(\mathbb{U})$ intersect the unit circle, so the hypercyclicity of $f(T)$ follows from the above-mentioned results in [16]. This emphasizes the difference between the work of [19], where the point is the generality of the operator T, and our work here, which aims for precise results about functions of a very special operator.

2. Hypercyclicity for $\varphi(B)$

Since $\mathcal{M}(\mathcal{D}) \subset \mathcal{D} \subset H^2$, every Dirichlet multiplier φ has a radial limit function φ^* defined for a.e. ζ on $\partial\mathbb{U}$ by

$$\varphi^*(\zeta) \stackrel{\text{def}}{=} \lim_{r\to 1-} \varphi(r\zeta).$$

To avoid trivialities we will always assume our multipliers φ are nonconstant. Here and throughout the rest of our work, "almost every" refers to Lebesgue measure m on the unit circle. We normalize m to have unit mass.

In view of our previous reduction of the commutant hypercyclicity problem for B, we are concerned with multipliers φ of $\mathcal{D}$ for which $\|\varphi\|_\infty = 1$. In this section we explore the connection between hypercyclicity for $\varphi(B)$ and the size of the *precontact set*

$$E_\varphi \stackrel{\text{def}}{=} \{\zeta \in \partial\mathbb{U} : |\varphi^*(\zeta)| = 1\}$$

of φ. We show that the condition $m(E_\varphi) > 0$ is sufficient, but not necessary, for $\varphi(B)$ to be hypercyclic on A^2.

The hypercyclicity of B itself is the special case $\varphi(z) \equiv z$ of our sufficient condition. More generally, $\varphi(B)$ is hypercyclic whenever φ is any finite Blaschke product (these are the only inner functions that belong to $\mathcal{D}$; see [21, page 250] or [28, Theorem 3.4]).

In a more geometric vein, suppose φ maps $\mathbb{U}$ univalently onto a starlike Jordan domain $G \subset U$ whose boundary is rectifiable and contacts $\partial\mathbb{U}$ in a set of positive measure (for example G could be the top half of $\mathbb{U}$). By the Axler-Shields "starlike" theorem mentioned in §1.8, $\varphi \in \mathcal{M}(\mathcal{D})$. By Carathéodory's extension theorem, φ extends to a homeomorphism of $\overline{\mathbb{U}}$ onto $\overline{G}$. The rectifiability of ∂G insures that $\varphi' \in H^1$, hence:

(a) The boundary function φ^* is absolutely continuous on $\partial\mathbb{U}$, with derivative $ie^{i\theta}\varphi'(e^{i\theta})$ (see, for example, [11, Theorem 3.11, page 42]), and
(b) $\varphi'(e^{i\theta})$ cannot vanish on a set of positive measure.

Thus

$$0 < m(\varphi(E_\varphi)) = \int_{E_\varphi} |\varphi'(\zeta)|\, dm(\zeta),$$

which guarantees that $m(E_\varphi) > 0$, hence $\varphi(B)$ is hypercyclic on A^2.

Our proof of sufficiency will require a number of preliminary lemmas and constructions, all heading toward application of the following characterization of hypercyclicity (see [16, Theorem 1.2, page 233]).

2.1. Proposition. *A bounded linear operator T on a Banach space X is hypercyclic if and only if for every pair V, W of nonempty open subsets of X there is a non-negative integer n such that $T^n(V) \cap W \neq \emptyset$.*

Actually no linearity is required for this result: it applies equally well to continuous self-maps of complete metric spaces, in which it is known as *Birkhoff's Transitivity Theorem* (see [22, §7.2, Theorem 2.1, page 245]). The Proposition says that there is a point in X whose orbit is dense precisely when the orbit of every nonvoid open set is dense. The transition between orbits of points and orbits of open sets is negotiated by the Baire Category Theorem.

Note that $T^n(V) \cap W$ is nonempty if and only if the same is true of $V \cap T^{-n}(W)$. Thus Proposition 2.1 has the following corollary, which played an important role in the reduction argument that preceded §1.11:

2.2. Corollary. *If T is* invertible *on X, then T is hypercyclic if and only if T^{-1} is hypercyclic.*

Our proof that $m(E_\varphi) > 0$ is sufficient for hypercyclicity depends critically on the properties of an operator that intertwines $\varphi(B)$ with a certain multiplication operator acting on L^2. Here is the notation required for the discussion.

2.3. *Notation.* We write L^2 for $L^2(m)$, and L^∞ for $L^\infty(m)$. For $f \in L^2$ and $n \in \mathbb{Z}$ we let $\hat{f}(n)$ denote the n-th Fourier coefficient of f:

$$\hat{f}(n) \stackrel{\text{def}}{=} \int_{\partial\mathbb{U}} f(\zeta)\,\overline{\zeta}^{\,n}\,dm(\zeta).$$

Previously, when f denoted a function holomorphic in $\mathbb{U}$, we used $\hat{f}(n)$ to denote the n-th *Taylor coefficient* of f in its expansion about the origin. In what follows we will use both conventions, allowing the context to determine the meaning. In case f belongs to H^2 and n is a non-negative integer, then $\hat{f}(n)$ can be correctly interpreted either as the n-th Taylor coefficient of f or the n-th Fourier coefficient of the radial limit function f^*.

In keeping with our setup for the Bergman-Dirichlet duality, we represent the self-dual nature of L^2, not in the usual conjugate-linear fashion involving the Hilbert space inner product, but instead through the bilinear form

$$\langle f,\, g\rangle \stackrel{\text{def}}{=} \sum_{n=-\infty}^{\infty} \hat{f}(n)\hat{g}(n) = \int_{\partial\mathbb{U}} f(\zeta)g(\overline{\zeta})\,dm(\zeta) \quad (f, g \in L^2) \tag{8}$$

(note that we use the same notation as for the pairing (5) between A^2 and $\mathcal{D}$, relying upon the context to determine the meaning).

In what follows, subsets of $\partial\mathbb{U}$ are always assumed to be measurable. For $E \subset \partial\mathbb{U}$, we let $L^2(E)$ denote the subspace of L^2 consisting of functions that vanish almost everywhere off of E. Relative to the duality pairing (8) the dual space of $L^2(E)$ is $L^2(\overline{E})$, where $\overline{E}$ denotes the set of complex conjugates of points in E.

It is easy to check that if $\psi \in L^\infty$ then, relative to the pairing (8), the adjoint of the multiplication operator $M_\psi : L^2 \to L^2$ is the multiplication operator induced by the function $\zeta \to \psi(\overline{\zeta})$. In the spirit of conserving notation we simply refer to this reflected function as $\psi(\overline{\zeta})$, letting the context determine whether we are discussing the function or one of its values. Thus, $(M_\psi)^* = M_{\psi(\overline{\zeta})}$.

Finally, we will no longer use a special notation for radial limits of functions in the Hardy or Dirichlet spaces. Thus for such a function f, the notation $f(z)$ will denote the value of f at z if $z \in \mathbb{U}$, and the radial limit of f at z if $z \in \partial\mathbb{U}$. In other words, we regard f to be extended to almost every point of the unit circle via radial limits. If there is any danger of confusion we will write "$f|_{\partial\mathbb{U}}$" to denote this radial limit function.

2.4. **The complex Riesz projection.** This is the operator $Q : L^2 \to A^2$ defined by

$$Q[f](z) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \hat{f}(n) z^n \quad (f \in L^2,\ z \in \mathbb{U}), \tag{9}$$

so that $\widehat{Q[f]}(n) = \hat{f}(n)$ for all integers $n \geq 0$. While not itself a projection, Q is related in an obvious way to the usual Riesz projection which takes L^2 orthogonally onto the subspace of boundary restrictions of H^2-functions.

2.5. Lemma: Properties of Q. (a) *Q is a* compact *operator $L^2 \to A^2$.*

(b) *$Q^* : \mathcal{D} \to L^2$ is the map $g \to g|_{\partial\mathbb{U}}$.*

(c) *For $E \subset \partial\mathbb{U}$ the adjoint of $Q : L^2(E) \to A^2$ is the operator $\mathcal{D} \to L^2(\overline{E})$ given by*

$$Q^* g = (g|_{\partial\mathbb{U}})\chi_{\overline{E}} \qquad (g \in \mathcal{D}).$$

Proof. (a) Q is the composition of itself, viewed as an operator from L^2 into H^2 (clearly a bounded operator—in fact, a contraction) and the identity map from H^2 into A^2, which is easily seen to be compact.

(c) Suppose $f \in L^2(E)$ and $g \in \mathcal{D}$. From (8) and the fact that f vanishes a.e. off E we have

$$\begin{aligned}
\langle f, Q^* g \rangle &\stackrel{\text{def}}{=} \langle Qf, g \rangle = \sum_{n=0}^{\infty} \hat{f}(n)\hat{g}(n) \\
&= \int_{\partial\mathbb{U}} f(\zeta) g(\overline{\zeta})\, dm(\zeta) \\
&= \int_{\partial\mathbb{U}} \chi_E(\zeta) f(\zeta) g(\overline{\zeta})\, dm(\zeta) \\
&= \int_{\partial\mathbb{U}} f(\zeta)(g\chi_{\overline{E}})(\overline{\zeta})\, dm(\zeta) \\
&= \langle f, (g|_{\partial\mathbb{U}})\chi_{\overline{E}} \rangle,
\end{aligned}$$

which is the desired result.

(b) This is the special case $E = \partial\mathbb{U}$ of (c). $\square$

2.6. Corollary. *If $E \subset \partial\mathbb{U}$ has positive measure, then the image of $L^2(E)$ under Q is a dense subspace of A^2.*

Proof. It is enough to prove that the adjoint of $Q : L^2(E) \to A^2$ is one-to-one. By Lemma 2.5 this is the operator that takes $g \in \mathcal{D}$ to $\chi_{\overline{E}}(g|_{\partial\mathbb{U}})$. If this latter function is identically zero, then g must vanish identically on $\overline{E}$. Since $g \in \mathcal{D} \subset H^2$ and $\overline{E}$ has positive measure, g must vanish identically on $\mathbb{U}$. Thus the operator in question is one-to-one. $\square$

The next result shows that for each Dirichlet multiplier φ, the Riesz projection Q intertwines $\varphi(B)$ with the multiplication operator $M_{\varphi(\overline{\zeta})} : L^2 \to L^2$ (a bounded operator on L^2 because $\varphi \in H^\infty$). The special case $\varphi(z) \equiv z$ is particularly easy to understand since the operator in question is now $M_{\overline{\zeta}}$, which simply performs a leftward shift on Fourier coefficients of L^2 functions.

2.7. Proposition. *$\varphi(B)Q = QM_{\varphi(\overline{\zeta})}$ for each $\varphi \in \mathcal{M}(\mathcal{D})$.*

Proof. For each $g \in \mathcal{D}$:

$$(Q^*M_\varphi)(g) = Q^*(\varphi g) = (\varphi g)|_{\partial\mathbb{U}} = M_{\varphi(\zeta)}\,(g|_{\partial\mathbb{U}}) = (M_{\varphi(\zeta)}Q^*)(g),$$

where the second and the last equalities follow from part (c) of Lemma 2.5, and the symbol $M_{\varphi(\zeta)}$ denotes the operator of multiplication by $\varphi|_{\partial\mathbb{U}}$, acting on L^2. Thus $Q^*M_\varphi = M_{\varphi(\zeta)}Q^*$, from which the desired result follows upon taking adjoints (recalling that the adjoint of $M_{\varphi(\zeta)}$ is $M_{\varphi(\overline{\zeta})}$, and that M_φ^* is, by definition, $\varphi(B)$). $\square$

We can now complete the proof of our sufficient condition for hypercyclicity. For reference we restate it as:

2.8. Theorem. *Suppose $\varphi \in \mathcal{M}(\mathcal{D})$ and $\|\varphi\|_\infty = 1$. If E_φ has positive measure, then $\varphi(B)$ is hypercyclic.*

Proof. We are assuming that φ is a Dirichlet multiplier mapping $\mathbb{U}$ into itself whose precontact set E_φ has positive measure. To simplify notation for the rest of this proof, let $T = \varphi(B)$. To prove that T is hypercyclic we will use Proposition 2.1; i.e., we will show that for each pair V, W of nonvoid open subsets of A^2 there is a non-negative integer n such that $T^n(V) \cap W \neq \emptyset$.

Fix such a pair of open sets. Let $\overline{E_\varphi}$ denote the set of complex conjugates of points in E_φ—also a subset of $\partial\mathbb{U}$ having positive measure. By Corollary 2.6 there exist functions F and G in $L^2(\overline{E_\varphi})$ such that $Q[F] \in V$ and $Q[G] \in W$. Since the bounded functions in $L^2(\overline{E_\varphi})$ are dense and the operator Q is continuous, we may assume further that F and G are bounded. For n a non-negative integer let

$$f_n \stackrel{\text{def}}{=} Q[\varphi(\overline{\zeta})^n F(\zeta)]$$

and note that $f_0 = Q[F] \in V$. Our intertwining relationship (Proposition 2.7) now shows that $Tf_n = f_{n+1}$ for each n; i.e., *$\{f_n\}$ is the T-orbit of f_0.*

We claim that $\|f_n\| \to 0$ as $n \to \infty$. For this observe that, since φ is a self-map of the unit disc, $\varphi^n \to 0$ uniformly on compact subsets of $\mathbb{U}$ as $n \to +\infty$. Since the sequence $\{\varphi^n : n \geq 0\}$ is uniformly bounded on $\mathbb{U}$ it is bounded in H^2. Because of this and the uniform convergence on compact sets, $\varphi^n \to 0$ weakly in H^2, and therefore the corresponding sequence of boundary functions converges weakly to zero in L^2. Because $F \in L^\infty$ the same holds for the sequence $\{\varphi^n(\zeta)F(\overline{\zeta}) : n \geq 0\}$, and therefore for the reflected sequence $\{\varphi^n(\overline{\zeta})F(\zeta) : n \geq 0\}$. This reveals the sequence $\{f_n\}$ as the Q-image of a weakly null sequence in L^2, and since $Q : L^2 \to A^2$ is compact (Lemma 2.5), $f_n \to 0$ in the A^2-norm as $n \to \infty$.

Informally speaking, we have produced a "forward null-orbit" $\{f_n\}$, with initial point in V. A similar argument yields a "backward null-orbit" with initial point in W. Let

$$g_n \stackrel{\text{def}}{=} Q[\varphi(\overline{\zeta})^{-n} G(\zeta)]$$

(so that, in particular, $g_0 = Q[G] \in W$), and note that, since $\varphi(\overline{\zeta})$ has modulus one on $\overline{E_\varphi}$, the function $\varphi(\overline{\zeta})^{-n}$ is, on $\overline{E_\varphi}$, just the complex conjugate of $\varphi(\overline{\zeta})^n$. By the same arguments we used above, $Tg_n = g_{n-1}$ for each $n > 0$, and $\|g_n\| \to 0$ as $n \to \infty$.

To complete the proof, for each non-negative integer n let $h_n = f_0 + g_n$. Recalling that $g_n \to 0$ we see that $h_n \to f_0$, hence $h_n \in V$ for all sufficiently large n. Now the "orbit" properties of $\{f_n\}$ and $\{g_n\}$, along with the fact that $f_n \to 0$, imply that

$$T^n h_n = f_n + g_0 \to g_0 \quad (n \to \infty),$$

hence $T^n h_n \in W$ for all sufficiently large n. So if n is large enough then $T^n h_n$ is in both $T^n(V)$ and W, and our proof is complete. □

The converse of Theorem 2.8 is not true. This is a consequence of Theorem 2.12 below, which produces a Dirichlet multiplier $\varphi : \mathbb{U} \to \mathbb{U}$ with E_φ a single point, yet for which $\varphi(B)$ is hypercyclic on A^2. Once a few prerequisites have been set out, the construction is simple and intuitive; it was suggested to us by Fedor Nazarov.

2.9. **Smoothness classes.** Suppose n is a non-negative integer. We say a holomorphic function f on $\mathbb{U}$ is of *class* $C^{(n)}$ if its n-th complex derivative $f^{(n)}$ has a continuous extension to $\overline{\mathbb{U}}$ (in this context we use the notation $f^{(0)}$ for f itself). We let $H^{(n)}(\overline{\mathbb{U}})$ denote the collection of all such functions. It is easy to check that the classes $H^{(n)}(\overline{\mathbb{U}})$ decrease as n increases, and that $H^{(n)}(\overline{\mathbb{U}})$ is the collection of functions f holomorphic on $\mathbb{U}$ and continuous on $\overline{\mathbb{U}}$ for which $f(e^{it})$ has n continuous derivatives with respect to t. We denote the intersection of all the classes $H^{(n)}(\overline{\mathbb{U}})$ by $H^{(\infty)}(\overline{\mathbb{U}})$ (not to be confused with the space H^∞ of bounded holomorphic functions on $\mathbb{U}$).

There is a natural metric topology on $H^{(\infty)}(\overline{\mathbb{U}})$ in which a sequence of functions converges if and only if each derivative converges uniformly on $\overline{\mathbb{U}}$ (or equivalently, on $\mathbb{U}$). A metric that does the job is

$$d(f,g) = \sum_{n=0}^{\infty} 2^{-n} \frac{\|f^{(n)} - g^{(n)}\|_\infty}{1 + \|f^{(n)} - g^{(n)}\|_\infty} \qquad (f, g \in H^{(\infty)}(\overline{\mathbb{U}})).$$

Similarly, one can define a metric on the space $C^{(\infty)}([-1,1])$ of infinitely differentiable functions $\gamma : [-1,1] \to \mathbb{C}$ (where differentiability at the endpoints is defined in terms of one-sided limits); we leave the details to the reader. From now on we take it for granted that the spaces $H^{(\infty)}(\overline{\mathbb{U}})$ and $C^{(\infty)}([-1,1])$ are topologized by these metrics.

2.10. **Jordan domains.** The plane region interior to a Jordan curve is called a *Jordan domain.* For definiteness we will always parameterize Jordan curves by functions defined on $[-1,1]$. For $0 < n \le \infty$ we say a Jordan domain is *of class* $C^{(n)}$ if its boundary is parameterized by a function in $C^{(n)}([-1,1])$. Such a parameterizing function is characterized within $C^{(n)}([-1,1])$ by the fact that it is one-to-one on $(-1,1]$ and both the function and its derivatives through order n take the same values at the endpoints -1 and 1.

Suppose G is a Jordan domain and φ a univalent (holomorphic) map of $\mathbb{U}$ onto G. A famous result of Carathéodory asserts that φ extends to a homeomorphism of $\overline{\mathbb{U}}$ onto the closure of G (see [23, §§14.18–14.20] for example). Earlier Painlevé, proved the existence of this extension for Jordan domains G of class C^∞, in which case he showed that φ belongs to $H^{(\infty)}(\overline{\mathbb{U}})$. For more on the history of this result, see the interesting expository paper [4] of Bell and Krantz.

We are going to consider the class $\mathcal{J}^{(\infty)}$ of C^∞ Jordan curves $\gamma : [-1,1] \to \mathbb{C}$ that surround the origin. Let G_γ denote the Jordan domain with boundary γ (so $0 \in G_\gamma$), and let φ_γ be the Riemann map of $\mathbb{U}$ onto G_γ ($\varphi_\gamma(0) = 0$ and $\varphi'_\gamma(0) > 0$). We assume without change of notation that φ_γ is extended by the theorem of Painlevé-Carathéodory to $\overline{\mathbb{U}}$. Thus the map $\gamma \to \varphi_\gamma$ takes $\mathcal{J}^{(\infty)}$ into $H^{(\infty)}(\overline{\mathbb{U}})$. It will be important for our purposes to know that this map is continuous. This is the content of the following:

Stability Theorem ([3, Theorem 26.1, page 112])**.** *The map $\gamma \to \varphi_\gamma$ is continuous when both $\mathcal{J}^{(\infty)}$ and $H^{(\infty)}(\overline{\mathbb{U}})$ have their natural C^∞ topologies.*

2.11. **The connection with multipliers.** The work of §1.9 shows that the identity map embeds the smoothness class $H^{(2)}(\overline{\mathbb{U}})$ into the space $\mathcal{M}(\mathcal{D})$ of Dirichlet multipliers, and that this embedding is continuous if each space is given its natural norm. Recall that the map $\varphi \to M_\varphi$ is an isometry of $\mathcal{M}(\mathcal{D})$ into $\mathcal{L}(\mathcal{D})$, the space of bounded operators on $\mathcal{D}$, and that the adjoint map is an isometry on $\mathcal{L}(H)$ for any Hilbert space H. Taken together, these observations show that the map $\gamma \to \varphi_\gamma(B)$ takes the space $\mathcal{J}^{(\infty)}$ of C^∞ Jordan curves that surround the origin continuously into the (normed) space of bounded operators on A^2.

With these preliminary results in hand we proceed to the construction of our example. Here is the official statement of our result.

2.12. Theorem. *There is a holomorphic map φ that takes $\mathbb{U}$ univalently onto a C^∞ starlike Jordan subdomain of $\mathbb{U}$ such that E_φ is a single point, yet $\varphi(B)$ is hypercyclic on A^2.*

Proof. The idea of the proof is that if $\gamma \in \mathcal{J}^{(\infty)}$ impacts the unit circle in an arc, so that $\varphi_\gamma(B)$ is hypercyclic (by Theorem 2.8), then a very small perturbation in the C^∞ topology can produce a new $\gamma \in \mathcal{J}^{(\infty)}$ that impacts the circle in a *much* smaller arc. The new $\varphi_\gamma(B)$ is still hypercyclic, and the process can be repeated. The argument below shows that this procedure can be carried out so that in the limit we arrive at a hypercyclic operator $\varphi_\gamma(B)$ where $\gamma \in \mathcal{J}^{(\infty)}$ intersects the unit circle in a single point. We break the proof into several steps.

Step I: *Choose a basis.* Fix a countable basis of open subsets for the topology of A^2, and enumerate the *pairs* of these basis elements as $\{(V_j, W_j)\}_0^\infty$ (so, in this list of pairs, each of the original basis elements will show up infinitely often in both the first and second positions). For each index j choose an open subset $\widetilde{W}_j$ of W_j that has its *closure* contained in W_j.

Step II. *Some starlike Jordan domains.* Suppose $\rho : [-1,1] \to [\frac{1}{2}, 1]$ is a C^∞ function with $\rho^{(n)}(-1) = \rho^{(n)}(1)$ for each $n = 0, 1, 2, \ldots$. We associate to ρ the $C^{(\infty)}$ Jordan curve γ_ρ defined by

$$\gamma_\rho(t) = \rho(t)e^{i\pi t} \qquad (t \in [-1,1]).$$

Then γ_ρ bounds a C^∞ Jordan domain

$$G_\rho = \{re^{i\theta} : 0 \le r < \gamma_\rho(\theta)\}$$

that contains the origin, with respect to which it is starlike. The example we are going to construct will be the Riemann map onto just such a domain.

Step III. *An induction.* Fix a strictly decreasing sequence $\{\theta_n\}_0^\infty$ of positive numbers with $0 < \theta_n \leq 1$, $\theta_0 = 1$, and $\theta_n \to 0$.

We are going to produce:

(a) A sequence $\{\rho_n\}_0^\infty$ of functions as in Step II such that for each n:
 (i) $\rho_n(t) = 1 \iff t \in [-\theta_n, \theta_n]$,
 (ii) $\rho_{n+1} \leq \rho_n$ pointwise on $[-1, 1]$,
 (iii) $d(\rho_n, \rho_{n-1}) < 1/2^n$, where d is the metric on $C^{(\infty)}([-1, 1])$ as defined in §2.9.
 (iv) $\rho_n > 1/2$ at each point of $[1, 1]$ (so that ρ_n is actually bounded away from $1/2$ on $[-1, 1]$).

(b) A sequence of positive integers $\{\nu_n\}_0^\infty$ and a sequence of vectors $\{f_n\}_0^\infty$ in A^2 such that for each index n we have $f_n \in V_n$ and

$$\varphi_n(B)^{\nu_j} f_j \in \widetilde{W}_j \quad \forall\ 0 \leq j \leq n,$$

where $\varphi_j = \varphi_{\rho_j}$ is the Riemann map of $\mathbb{U}$ onto the C^∞ starlike Jordan domain $G_j = G_{\rho_j}$ defined as in Step II.

The argument is by induction. For $n = 0$ let $\rho_0(t) \equiv 1$, so that $G_0 = \mathbb{U}$ and φ_0 is the identity map of $\mathbb{U}$. Then $\varphi_0(B) = B$ is hypercyclic on A^2, so by Proposition 2.1 there exists a non-negative integer ν_0 such that $\varphi_0(B)^{\nu_0}(V_0) \cap \widetilde{W}_0 \neq \emptyset$; i.e., there exists $f_0 \in V_0$ such that $\varphi_0(B)^{\nu_0} f_0 \in \widetilde{W}_0$.

Suppose $n \geq 0$ and that we have produced the appropriate $C^{(\infty)}$ functions $\rho_0, \ldots, \rho_n$, the positive integers $\nu_0, \ldots, \nu_n$, and the A^2-functions $f_0, \ldots, f_n$. To get to the next stage, fix a non-negative $C^{(\infty)}$ function h on $[-1, 1]$, whose values and all of whose derivatives coincide at both $+1$ and -1, and whose zero-set is the interval $[-\theta_{n+1}, \theta_{n+1}]$. For $\varepsilon > 0$ let $\rho_{n+1} = \rho_n - \varepsilon h$, where ε remains to be chosen. For ε sufficiently small, ρ_{n+1} has the four properties of (a) listed above, with $n+1$ in place of n (the third of these comes from the fact that scalar multiplication is continuous in the "$C^{(\infty)}$ topology").

Let $\varphi_{n+1} = \varphi_{\rho_{n+1}}$ (a map which also depends on the still-to-be-chosen parameter ε). By the discussion of §2.10 the map $\rho \to \varphi_\rho(B)$ is continuous from $C^{(\infty)}([-1, 1])$ into $\mathcal{L}(A^2)$, hence by choosing ε sufficiently smaller we may insure that φ_{n+1} is sufficiently close to φ_n so that $\varphi_{n+1}(B)^{\nu_j} f_j \in \widetilde{W}_j$ for $0 \leq j \leq n$. Now $\varphi_{n+1}(\overline{\mathbb{U}}) \cap \partial\mathbb{U}$ is the arc $\{e^{it} : |t| \leq \theta_{n+1}\}$, hence (because φ_{n+1} is a homeomorphism on $\overline{\mathbb{U}}$), the precontact set $E(\varphi_{n+1})$ is also an arc of $\partial\mathbb{U}$. Thus $\varphi_{n+1}(B)$ is hypercyclic on A^2 by Theorem 2.8, so there exists a vector $f_{n+1} \in V_{n+1}$ and a positive integer ν_{n+1} such that $\varphi_{n+1}(B)^{\nu_{n+1}} f_{n+1} \in \widetilde{W}_{n+1}$. This completes the induction.

Step IV. *Passing to the limit.* We have arranged matters so that the sequence $\{\rho_n\}$ converges in $C^{(\infty)}([-1, 1])$ to a function $\rho \in C^{(\infty)}([-1, 1])$ with values in the interval $[\frac{1}{2}, 1]$, and which takes the value 1 only at the origin. Let $G = G_\rho$, a $C^{(\infty)}$ Jordan sub-domain of $\mathbb{U}$ that contains the disc $\{|z| < \frac{1}{2}\}$, is starlike with respect to the origin, and whose closure touches $\partial\mathbb{U}$ only at the point 1. Let φ be the Riemann map taking $\mathbb{U}$ onto G, so φ is non-constant and extends to a $C^{(\infty)}$ homeomorphism taking $\overline{\mathbb{U}}$ onto the closure of G. Thus $E_\varphi = \varphi^{-1}(G \cap \partial\mathbb{U}) = \varphi^{-1}(1)$ is a single point.

The stability results of §2.10 show that $\varphi_n(B) \to \varphi(B)$ in the norm of $\mathcal{L}(A^2)$, so by (b) of Step III, for each non-negative integer j the vector $\varphi(B)^{\nu_j} f_j$ belongs to the closure of $\widetilde{W}_j$, and therefore to W_j. Thus for each j we have $f_j \in V_j$ and

$\varphi(B)^{\nu_j} f_j \in W_j$, so $\varphi(B)$ is hypercyclic, by Proposition 2.1. This completes the construction of our example. □

We close this section with a subordination theorem that reinforces the connection between geometric properties of φ and hypercyclic behavior for $\varphi(B)$. It shows, for example, that if G is a simply connected subdomain of $\mathbb{U}$ that contains the one promised by Theorem 2.12, and if the Riemann map ψ of $\mathbb{U}$ onto G is a Dirichlet multiplier (e.g. if ∂G is sufficiently smooth, or G is starlike), then $\psi(B)$ will be hypercyclic on A^2.

2.13. Theorem. *Suppose φ and ψ belong to $\mathcal{M}(\mathcal{D})$, both are univalent self-mappings of $\mathbb{U}$, and $\varphi(\mathbb{U}) \subset \psi(\mathbb{U})$. If $\varphi(B)$ is hypercyclic on A^2, then so is $\psi(B)$.*

Proof. $\omega = \psi^{-1} \circ \varphi$ is a univalent self-map of $\mathbb{U}$, so it induces a bounded composition operator $C_\omega : \mathcal{D} \to \mathcal{D}$ defined by

$$C_\omega f = f \circ \omega \qquad (f \in \mathcal{D}).$$

A little calculation shows that $C_\omega M_\psi^n = M_\varphi^n C_\omega$ for each non-negative integer n hence, upon taking adjoints,

$$\psi(B)^n C_\omega^* = C_\omega^* \varphi(B)^n \qquad (n = 0, 1, 2, \dots). \tag{10}$$

Now C_ω is one-to-one on $\mathcal{D}$ so its adjoint, viewed as an operator on A^2, has dense range. Thus if $f \in A^2$ is hypercyclic for $\varphi(B)$, then equation (10) shows that $C_\omega^* f$ is hypercyclic for $\psi(B)$. □

3. Non-hypercyclicity and degree of contact

In this section we give a criterion for $\varphi(B)$ to be *non-hypercyclic*, and we apply it to show that if the closure of $\varphi(\mathbb{U})$ touches the boundary of the unit circle at just finitely many points, and approaches those points in a certain "exponentially limited" way, then $\varphi(B)$ is not hypercyclic. This limitation holds if, for example, $\varphi(\mathbb{U})$ lies in a subdisc of U that is tangent to $\partial\mathbb{U}$ at a single point; hence our result shows, in particular, that the operator $(I + B)/2$ is not hypercyclic on A^2. Note that we have already seen an extreme case of this phenomenon: if $\varphi(\mathbb{U})$ does not approach the unit circle at all; i.e., if $\|\varphi\|_\infty < 1$, then $\varphi(B)$ is not hypercyclic because its spectrum (the closure of $\varphi(\mathbb{U})$) does not intersect the unit circle.

Our argument hinges on the following simple observation:

3.1. Lemma. *Suppose X is a Banach space and T a bounded linear operator on X. If there exists $\Lambda \neq 0$ in X^* such that the orbit $\{T^{*\,n}\Lambda\}_0^\infty$ is bounded in X^*, then T is not hypercyclic.*

Proof. Our assumption is that there is a positive number M such that $\|T^{*\,n}\Lambda\| \leq M$ for every non-negative integer n. Let x be any vector in X. Then

$$|\Lambda(T^n x)| = |(T^{*\,n}\Lambda)(x)| \leq \|T^{*\,n}\Lambda\| \|x\| \leq M \|x\|;$$

i.e., the sequence of complex numbers $\{\Lambda(T^n x)\}_0^\infty$ is bounded. Thus the orbit $\{T^n x\}_0^\infty$ is not dense in X, so x cannot be a hypercyclic vector for T. Since x is arbitrary, T is not hypercyclic. □

This lemma leads to a useful sufficient condition for non-hypercyclicity of operators in the commutant of B.

3.2. Theorem. *Suppose $\varphi \in \mathcal{M}(\mathcal{D})$ with $\|\varphi\|_\infty = 1$, and that there exists a function $f \in \mathcal{D}\backslash\{0\}$ and a positive number β such that*

$$|f(z)| \leq \beta(1 - |\varphi(z)|) \qquad \forall z \in \mathbb{U}. \tag{11}$$

Then $\varphi(B)$ is not hypercyclic on A^2.

Proof. We will show that the orbit $\{M_\varphi^n f\}_0^\infty$ is bounded in $\mathcal{D}$, from which the non-hypercyclicity of $\varphi(B) = M_\varphi^*$ on A^2 will follow from Lemma 3.1. The argument begins with a simple estimate that is easily derived from the chain rule, the Cauchy-Schwarz inequality, and the fact that $\|\varphi\|_\infty = 1$:

$$\|\varphi^n f\|_{\mathcal{D}} \leq 2\|f\|_{\mathcal{D}} + n\left(\int_{\mathbb{U}} |\varphi|^{2(n-1)}|f|^2|\varphi'|^2\, d\lambda\right)^{1/2}.$$

This, along with condition (11), yields

$$\begin{aligned}\|\varphi^n f\|_{\mathcal{D}} &\leq 2\|f\|_{\mathcal{D}} + \beta\, n\left(\int_{\mathbb{U}} \left[|\varphi|^{n-1}(1-|\varphi|)\right]^2 |\varphi'|^2\, d\lambda\right)^{1/2} \\ &\leq 2\|f\|_{\mathcal{D}} + \beta\, \|\varphi\|_{\mathcal{D}},\end{aligned}$$

where the last inequality follows from the fact that $x^{n-1}(1-x) < 1/n$ for $0 \leq x \leq 1$. Thus the M_φ-orbit of f is bounded in $\mathcal{D}$, as promised. ∎

Our first application of Theorem 3.2 requires some descriptive terminology. Suppose G is a subset of $\mathbb{U}$ and $\eta \in \partial\mathbb{U}$ lies in the closure (in $\mathbb{C}$) of G. Then we say G *contacts* the unit circle at η. If there exist an open disc Δ centered at η and positive constants α, β, and γ such that

$$\exp\left\{\frac{-\alpha}{|\eta - w|^\gamma}\right\} \leq \beta(1 - |w|) \qquad (w \in G \cap \Delta),$$

then we say G has *exponential contact with $\partial\mathbb{U}$ of order $\leq \gamma$ at η.*

To get some intuition for what this definition is saying, suppose more generally that $h : [0,2] \to [0,\infty)$ is a non-negative, continuous, strictly increasing function with $h(0) = 0$, and let us say that G *approaches η h-tangentially* if there exists an open disc Δ with center at η such that

$$h(|\eta - w|) \leq 1 - |w| \qquad \forall w \in G \cap \Delta.$$

The faster h approaches 0 as $x \to 0+$, the more closely G is allowed to contact the unit circle at η. In the case of exponential contact, $h(x) = \beta^{-1}\exp(-\alpha/x^\gamma)$, whereas if G were confined to a subdisc of $\mathbb{U}$ with boundary tangent to the unit circle at η, then we could do no better than $h(x) = \beta x^2$ for some $\beta > 0$.

More generally, we say G has *finite order contact* with $\partial\mathbb{U}$ at η if its approach to η is h-tangential with $h(x) = \beta x^\alpha$ for some $\alpha, \beta > 0$. If, for example, G were confined to a triangle in $\mathbb{U}$ with a vertex at η, then the definition of finite order contact at η would be satisfied with $\alpha = 1$. The point here is that "exponential contact" allows significantly closer approach to the boundary than does finite order contact.

3.3. Corollary. *Suppose $\varphi \in \mathcal{M}(\mathcal{D})$ with $\|\varphi\|_\infty = 1$ and that $\varphi(\mathbb{U})$ contacts the unit circle at only a finite number of points. If, at each of these points, $\varphi(\mathbb{U})$ has exponential contact with the circle of order < 1, then $\varphi(B)$ is not hypercyclic.*

Proof. Suppose first that $\varphi(\mathbb{U})$ contacts $\partial\mathbb{U}$ at just one point, which without loss of generality we may assume is the point 1. Then our hypothesis on φ is that there exist positive numbers α and β, and $0 < \gamma < 1$, such that

$$\exp\left\{\frac{-\alpha}{|1-\varphi(z)|^\gamma}\right\} \le \beta(1-|\varphi(z)|) \qquad \forall z \in \mathbb{U}. \tag{12}$$

We claim that for a suitable $a > 0$ the function f defined below belongs to $\mathcal{D}$ and satisfies inequality (11):

$$f \stackrel{\text{def}}{=} \exp\left\{-\frac{a}{(1-\varphi)^\gamma}\right\}.$$

The key here is that $\operatorname{Re}(1-z)^{-1} > 0$ (in fact it is $> 1/2$) for each $z \in U$. Thus the same is true of $\operatorname{Re}(1-\varphi(z))^{-1}$, and so for the argument of $(1-\varphi(z))^{-1}$ we may choose a unique value $t(z)$ in the open interval $(-\pi/2, \pi/2)$. Consequently, every $z \in U$,

$$\operatorname{Re}\frac{1}{(1-\varphi(z))^\gamma} = \frac{\cos(\gamma t(z))}{|1-\varphi(z)|^\gamma} \ge \frac{\cos(\gamma\pi/2)}{|1-\varphi(z)|^\gamma},$$

whereupon

$$|f(z)| = \exp\left\{\operatorname{Re}\frac{-a}{|1-\varphi(z)|^\gamma}\right\} \le \exp\left\{\frac{-a\cos(\gamma\pi/2)}{|1-\varphi(z)|^\gamma}\right\}. \tag{13}$$

Upon using the chain rule to compute f', taking absolute values, and then substituting inequality (13) into the result, we obtain

$$|f'(z)| \le |\varphi'(z)|\frac{a}{|1-\varphi(z)|^{\gamma+1}}\exp\left\{\frac{-a\cos(\gamma\pi/2)}{|1-\varphi(z)|^\gamma}\right\}$$

for each $z \in \mathbb{U}$. Since $\varphi \in \mathcal{D}$ we have $\varphi' \in A^2$. Note that on the right-hand side of the last inequality, the term that multiplies $|\varphi'(z)|$ is bounded on $\mathbb{U}$. Thus also $f' \in A^2$; i.e., $f \in \mathcal{D}$ for every $a > 0$.

Finally, set $a = \alpha/\cos(\gamma\pi/2)$ and observe that, thanks to (12) and (13), the function f now satisfies condition (11). Thus all the hypotheses of Theorem 3.2 are satisfied, and therefore $\varphi(B)$ is not hypercyclic.

Suppose now that $\varphi(\mathbb{U})$ contacts $\partial\mathbb{U}$ at just the n points $\eta_1, \eta_2, \ldots, \eta_n$. Then we can choose $\alpha, \beta > 0$, $\gamma < 1$, and open discs $\Delta_1, \ldots, \Delta_n$, with Δ_j centered at η_j, so that if $h(x) = \beta^{-1}\exp(-\alpha/x^\gamma)$, then for each j,

$$h(|\eta_j - \varphi(z)|) \le 1 - |\varphi(z)| \qquad \forall z \in \bigcup_{j=1}^n \Delta_j.$$

Let

$$f_j = \exp\left\{\frac{-a}{(\eta_j - \varphi)^\gamma}\right\},$$

where, as before, $a = \alpha/\cos(\gamma\pi/2)$. Then by the previous argument, each f_j has derivative with modulus that is bounded on $\mathbb{U}$ by a constant multiple of $|\varphi'|$, so the same is true of

$$f \stackrel{\text{def}}{=} f_1 f_2 \cdots f_n$$

(because each f_j is bounded on $\mathbb{U}$). Thus $f \in \mathcal{D}$. Finally, for each index j we know that $|f_j| < 1$ on $\mathbb{U}$, and that f_j satisfies (11) whenever $\varphi(z) \in \Delta_j$. Since $\varphi(z)$ is bounded away from the unit circle for z in the complement of $\varphi^{-1}\left(\bigcup_{j=1}^{n}\Delta_j\right)$, it follows that f satisfies (11) on all of $\mathbb{U}$, possibly with different constants. Thus once again f and φ satisfy the hypotheses of Theorem 3.2, so $\varphi(B)$ is not hypercyclic. □

In case φ is analytic in a neighborhood of a point ζ_0 of its precontact set, then there is this dichotomy: *either* $\varphi(\mathbb{U})$ *has finite order contact with* $\partial\mathbb{U}$ *at* $\varphi(\zeta_0)$*, or* $|\varphi| \equiv 1$ *on some arc centered at* ζ_0*.*

To see why this is so, suppose (without loss of generality) that $\zeta_0 = 1$, and that $\varphi(\mathbb{U})$ does not have finite order contact with $\partial\mathbb{U}$ at $\varphi(1)$. We are assuming that φ is analytic in a disc Δ centered at 1. Let $I = \log(\Delta \cap \partial\mathbb{U})$, where we use the principal branch of the logarithm. For $t \in I$ (so that $e^{it} \in \Delta \cap \partial\mathbb{U}$) set $g(t) = 1 - |\varphi(e^{it})|^2$. Then g is real-analytic on I, and our contact hypothesis guarantees that for each fixed positive integer n there exists a real sequence $t_j \to 0$ such that $|g(t_j)| = o(|\varphi(e^{it_j}) - \varphi(1)|^n)$ as $j \to \infty$. Since φ is analytic at 1 we know in addition that $|\varphi(e^{it}) - \varphi(1)| = \mathrm{O}(|t|)$ for $e^{it} \in I$ with $t \to 0$, hence $|g(t_j)| = o(|t_j|^n)$ as $j \to \infty$. Thus the n-th derivative of g vanishes at 0. Since n is an arbitrary positive integer and g is real-analytic on I, this shows that g is constant on I. But also $g(0) = 0$, so $g \equiv 0$ on I, as promised.

3.4. Corollary. *Suppose* $\varphi \in \mathcal{M}(\mathcal{D})$ *is a holomorphic self-map of* $\mathbb{U}$ *for which* E_φ *is a finite set at each point of which* φ *is analytic. Then* $\varphi(U)$ *makes finite order contact with* $\partial\mathbb{U}$ *at each point of* $\varphi(E_\varphi)$*, and therefore* $\varphi(B)$ *is not hypercyclic.*

Proof. If $\varphi(U)$ does not make finite order contact with $\partial\mathbb{U}$ at $\varphi(\zeta_0)$ for some $\zeta_0 \in E_\varphi$ then we saw above that $|\varphi| \equiv 1$ on an arc of $\partial\mathbb{U}$ about ζ_0, contradicting the hypothesis that E_φ is finite. □

We have an even stronger dichotomy in case φ is analytic across *every* point of the unit circle.

3.5. Corollary. *If* $\varphi \in \mathcal{M}(\mathcal{D})$ *is a self-map of* $\mathbb{U}$ *that is analytic in a neighborhood of the closed unit disc, then* $\varphi(B)$ *is hypercyclic on* A^2 *if and only if* φ *is a finite Blaschke product.*

Proof. If φ is a finite Blaschke product, then it is analytic in a neighborhood of $\overline{\mathbb{U}}$ and therefore a multiplier of $\mathcal{D}$. Since $|\varphi| \equiv 1$ on $\partial\mathbb{U}$, it follows from Theorem 2.8 that $\varphi(B)$ is hypercyclic.

Conversely, if $\varphi(B)$ is hypercyclic, then by Corollary 3.4 E_φ must have infinitely many points, hence the function $g(t) = 1 - |\varphi(e^{it})|^2$, which is now real-analytic on the whole real line, vanishes on a set having a finite limit point, and therefore on all of $\mathbb{R}$. Thus $|\varphi| \equiv 1$ on $\partial\mathbb{U}$, so in view of its analyticity across the entire unit circle, φ must be a finite Blaschke product. □

To this point we have shown that limited geometric contact between $\varphi(\mathbb{U})$ and $\partial\mathbb{U}$ leads to non-hypercyclicity. Thus limited contact between the spectrum of $\varphi(B)$ and $\partial\mathbb{U}$ leads to non-hypercyclicity. The next result shows that, even if φ is univalent, the geometry of the spectrum of $\varphi(B)$ cannot tell the whole story.

3.6. **Example.** There exists a univalent Dirichlet multiplier $\varphi : \mathbb{U} \to \mathbb{U}$ such that $\varphi(\mathbb{U})$ is dense in $\mathbb{U}$ (so that the spectrum of $\varphi(B)$ is $\overline{\mathbb{U}}$), yet for which $\varphi(B)$ is not hypercyclic on A^2.

Proof. First we need another sufficient condition for non-hypercyclicity. Suppose that $\varphi \in \mathcal{M}(\mathcal{D})$ maps $\mathbb{U}$ into itself, and that

$$\int_{\mathbb{U}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\, d\lambda(z) < \infty. \tag{14}$$

We claim that $\varphi(B)$ is not hypercyclic on A^2.

For this it is enough to show, by Lemma 3.1, that the orbit $\{(M_\varphi)^n 1\} = \{\varphi^n\}$ is bounded in $\mathcal{D}$. The calculation proceeds along the lines of the proof of Theorem 3.2. For each positive integer n we have from the definition (3) of the norm in $\mathcal{D}$ and the chain rule

$$\begin{aligned}\|\varphi^n\|_{\mathcal{D}}^2 &= \|\varphi^n\|_2^2 + n^2 \int_{\mathbb{U}} |\varphi|^{2(n-1)} |\varphi'|^2\, d\lambda \\ &= \|\varphi^n\|_2^2 + n^2 \int_{\mathbb{U}} \left[|\varphi|^{(n-1)}(1-|\varphi|^2)\right]^2 \frac{|\varphi'|^2}{(1-|\varphi|^2)^2}\, d\lambda \\ &\le 1 + 4\int_{\mathbb{U}} \frac{|\varphi'|^2}{(1-|\varphi|^2)^2}\, d\lambda,\end{aligned}$$

where in the last line we have used the fact that $x^{n-1}(1-x^2) \le 2/n$ for $0 \le x \le 1$. Thus the orbit $\{M_\varphi^n 1\}$ is a bounded subset of $\mathcal{D}$, as promised.

Now we can give our example; this one comes directly from [5], where it is used to construct a compact composition operator on the "little Bloch space" for which the image of the inducing map is dense in $\mathbb{U}$. Let $\{\omega_k : k = 1, 2, \ldots\}$ be a countable dense subset of $\partial\mathbb{U}$, and let $\{h_n\}$ be a sequence of positive numbers less than (say) $1/2$, such that $\sum_{k=1}^{\infty} h_k < \infty$. For each positive integer k let E_k denote the open region in the right-half disk bounded between the curve $y = h_k(x-1)^2$ and its reflection in the x-axis. An easy estimate using polar coordinates based at the point 1 shows that

$$\int_{E_k} \frac{d\lambda(w)}{(1-|w|^2)^2} = \mathrm{O}(h_k) \quad \text{as } k \to \infty. \tag{15}$$

Set $G = \left(\frac{1}{2}\mathbb{U}\right) \cup \left(\bigcup_{k=1}^{\infty} \omega_k E_k\right)$, and observe that G is star-like with respect to the origin. Thus G is simply connected, and upon letting φ denote a univalent mapping of $\mathbb{U}$ onto G we see from [2, Theorem 3] that $\varphi \in \mathcal{M}(\mathcal{D})$. Now G contains the ray $\{r\omega_k : 0 \le r < 1\}$ for each k, and since $\{\omega_k\}$ is dense in $\partial\mathbb{U}$ it follows that G is dense in $\mathbb{U}$.

Nevertheless, we claim that φ satisfies the integrability condition (14) above, so that $\varphi(B)$ is not hypercyclic on A^2. To see this, use the univalence of φ to effect a change of variable that begins the following chain of estimates:

$$\begin{aligned}\int_{\mathbb{U}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\, d\lambda(z) &= \int_G \frac{d\lambda(w)}{(1-|w|^2)^2} \\ &\le \left(\int_{\frac{1}{2}\mathbb{U}} + \sum_k \int_{\omega_k E_k}\right) \frac{d\lambda(w)}{(1-|w|^2)^2} \\ &\le \frac{1}{3} + \text{const.} \sum_k h_k \\ &< \infty,\end{aligned}$$

where the next-to-last line follows from (15), and the last one from the choice of the sequence $\{h_k\}$. $\square$

We remark that condition (14) asserts that $\varphi(\mathbb{U})$ has *finite hyperbolic area*, where the multiplicity of the mapping is figured into the calculation. This same condition is easily seen to characterize the Hilbert-Schmidt composition operators on the Dirichlet space.

4. Non-hypercyclicity with large precontact sets

In this section we construct a class of non-hypercyclic $\varphi(B)$'s where the precontact set of φ is, in the sense of Hausdorff dimension, as large as possible.

Recall that in Theorem 2.8 we saw that if $\varphi \in \mathcal{M}(\mathcal{D})$ with $\|\varphi\|_\infty = 1$, and if the precontact set E_φ has positive measure, then $\varphi(B)$ is hypercyclic on A^2. We will show below (Theorem 4.3) that in this result the condition "$m(E_\varphi) > 0$" cannot be replaced by "E_φ has Hausdorff dimension one." Our construction depends on Carleson's characterization of the boundary zeros of analytic functions in $\mathbb{U}$ that extend smoothly to the boundary, and on the following corollary of Theorem 3.2.

4.1. Proposition. *Suppose $\varphi \in \mathcal{M}(\mathcal{D})$ and*

$$\int_{\partial\mathbb{U}} \log(1 - |\varphi^*|)\, dm > -\infty. \tag{16}$$

If there exists $f \in \mathcal{D}$ with $|f^*| = 1 - |\varphi^*|$ a.e. on $\partial\mathbb{U}$, then $\varphi(B)$ is not hypercyclic on A^2.

Proof. Suppose f is a function that satisfies the hypotheses of the proposition. In view of Theorem 3.2 it will be enough to show that

$$|f(z)| + |\varphi(z)| \le 1 \qquad \forall z \in \mathbb{U}. \tag{17}$$

Fix $z \in \mathbb{U}$ and choose $\gamma \in \partial\mathbb{U}$ such that

$$|f(z)| + |\gamma\varphi(z)| = |f(z) + \gamma\varphi(z)|. \tag{18}$$

Now f and φ are both bounded analytic functions, so $f + \gamma\varphi$ is a bounded analytic function which, because its radial limit function has modulus ≤ 1 a.e. on $\partial\mathbb{U}$, is itself ≤ 1 at every point of $\mathbb{U}$. The desired inequality (17) follows from this and (18). □

Remark. The referee suggested an idea that led to this proof. One might be tempted to draw the desired conclusion about $|f|+|\varphi|$ simply from the fact that it is a positive subharmonic function on $\mathbb{U}$ whose radial limit function is 1 a.e. on the boundary. But the unbounded function $z \to \operatorname{Re}\{2/(1-z)\}$ also has these properties, so something more—in this case boundedness on $\mathbb{U}$—is needed. A careful treatment of such "generalized maximum principles" for subharmonic functions can be found in the short paper [14] of Gårding and Hörmander.

The question of how to determine the regularity of an analytic function from the regularity of its boundary-modulus has drawn much attention. For outer functions F, Carleson [7] has given a condition on $|F^*|$ that is necessary and sufficient for $F \in \mathcal{D}$. Although Carleson's condition is often difficult to verify, Aleksandrov, Džrbašjan, and Havin [1] succeeded in using it to show that if $h : \partial\mathbb{U} \to [0, \infty]$ has integrable logarithm and is absolutely continuous on $\partial\mathbb{U}$ with derivative in $L^2(m)$, then the outer function with boundary-modulus equal to h lies in $\mathcal{D}$. This result, along with Proposition 4.1, yields the following:

4.2. Corollary. *Suppose φ is a holomorphic self-map of $\mathbb{U}$ that obeys the logarithmic integrability condition* (16). *If, in addition, $\varphi' \in H^2$, then $\varphi \in \mathcal{M}(\mathcal{D})$ and $\varphi(B)$ is not hypercyclic on A^2.*

Proof. Recall from §1.8 that the condition $\varphi' \in H^2$ guarantees that φ is a Dirichlet multiplier. It also renders the radial limit function φ^* absolutely continuous on $\partial\mathbb{U}$, with derivative in L^2. Thus the same is true of $1 - |\varphi^*|^2$, so the result of Aleksandrov, Džrbašjan, and Havin guarantees that the outer function F with boundary-modulus $1 - |\varphi^*|^2$ lies in $\mathcal{D}$. The argument we gave to prove Proposition 4.1 goes through almost word-for-word to show that $|F| \leq 1 - |\varphi|^2 \leq 2(1 - |\varphi|)$ at every point of $\mathbb{U}$. Thus the non-hypercyclicity of $\varphi(B)$ follows once again from Theorem 3.2. □

We remark in passing that condition (16) characterizes those functions φ on the unit sphere of H^∞ that are *not* extreme points of the closed unit ball (see [11, Theorem 7.9, page 125], for example).

With these preliminaries out of the way we can finally move on to the main result of this section:

4.3. Theorem. *There exists φ holomorphic on $\mathbb{U}$ with $\|\varphi\|_\infty = 1$ and $\varphi' \in H^2$ such that E_φ has Hausdorff dimension one, yet $\varphi(B)$ is not hypercyclic on A^2.*

The proof of this result depends critically on the structure of zero sets of holomorphic functions possessing significant boundary smoothness. Such sets were characterized by Carleson in [8]. Suppose E is a closed subset of $\partial\mathbb{U}$ and that E has Lebesgue measure zero. Then the complement of E is a disjoint union of at most countably many open subarcs $\{I_n\}$. If

$$\sum_n m(I_n) \log m(I_n) > -\infty,$$

then E is called a *Carleson set.* Not every set of measure zero has this property; non-Carleson *sequences* can be easily constructed. Nevertheless, the Cantor middle-thirds set is Carleson, and by varying the ratio of dissection properly one can produce Carleson sets of Hausdorff dimension d for any $0 \leq d \leq 1$.

Carleson showed that the sets bearing his name are precisely the boundary zero-sets of functions that are analytic on $\mathbb{U}$ and extend to be Lipschitz on $\overline{\mathbb{U}}$, or even C^n-differentiable there ($n = 1, 2, \dots$) [8, Theorem 1]. Most important for our purposes is this part of his argument:

> *Given any Carleson set E there is an* outer function *F that extends C^2 to $\overline{\mathbb{U}}$, and vanishes precisely on E.*

Other investigators later refined Carleson's construction to produce outer functions with infinite differentiability on $\overline{\mathbb{U}}$ having E as zero-set, but we will not need this extra precision. Carleson's outer function provides the crucial step in the following result, from which Theorem 4.3 follows immediately.

4.4. Theorem. *Suppose $E \subset \partial\mathbb{U}$ is a Carleson set. Then there exists φ holomorphic on $\mathbb{U}$ with $\|\varphi\|_\infty = 1$ and $\varphi' \in H^2$ such that $E_\varphi = E$ and $\varphi(B)$ is not hypercyclic on A^2.*

Proof. Let F denote a "Carleson" outer function with C^2-smoothness on $\overline{\mathbb{U}}$ that vanishes precisely on E. Upon multiplying by an appropriate constant, if necessary, we may additionally assume that

$$|F(z)| \leq 1/\sqrt{2} \qquad \forall z \in \overline{\mathbb{U}}. \tag{19}$$

In what follows it will be convenient to retain the notation F^* for the restriction of F to $\partial\mathbb{U}$. The boundary-smoothness of F guarantees that $|F^*|^2 \in C^2(\partial\mathbb{U})$, and because of (19) this smoothness transfers to

$$w \stackrel{\text{def}}{=} \log(1 - |F^*|^2).$$

In particular, w is integrable on $\partial\mathbb{U}$, so we may form the outer function φ with boundary-modulus $e^w = 1 - |F^*|^2$. We claim that φ furnishes the desired example.

For this, note that $\varphi = e^h$ where h is the holomorphic completion of the Poisson integral of w; i.e., for each $z \in \mathbb{U}$:

$$h(z) = \int_{\partial\mathbb{U}} \frac{\zeta + z}{\zeta - z} w(\zeta)\, dm(\zeta) = \hat{w}(0) + 2\sum_{n=1}^{\infty} \hat{w}(n) z^n, \tag{20}$$

where $\hat{w}(n)$ is the n-th Fourier coefficient of w. Now the values of the function $1 - |F^*|^2$ all lie in the interval $(0, 1]$, so its logarithm w is ≤ 0 on $\partial\mathbb{U}$. It follows that

$$|\varphi| = \exp(\operatorname{Re} h) = \exp(P[w]) \leq 1 \text{ on } \overline{\mathbb{U}},$$

with equality precisely when $w = 0$, i.e. on E. Thus we have established that φ is a holomorphic self-map of $\mathbb{U}$ with precontact set E_φ equal to E.

The next order of business is to show that $\varphi' \in H^2$. For this recall that since $w \in C^2(\partial\mathbb{U})$ we know that $\sum_{-\infty}^{\infty} |n\, \hat{w}(n)|^2 < \infty$, so by the last equality of (20) we also have $\sum_0^\infty |n\, \hat{h}(n)|^2 < \infty$, (where now $\hat{h}(n)$ is a Taylor coefficient); i.e., $h' \in H^2$. Thus $|\varphi'| = |h'\varphi| \leq |h'|$ on $\mathbb{U}$, so $\varphi' \in H^2$.

It remains to prove that $\varphi(B)$ is not hypercyclic; for this we will verify that φ satisfies the hypotheses of Proposition 4.1. To check logarithmic integrability, recall that φ is the outer function with boundary-modulus $1 - |F^*|^2$, so $1 - |\varphi^*| = |F^*|^2$ on $\partial\mathbb{U}$. Thus

$$\int_{\partial\mathbb{U}} \log(1 - |\varphi^*|)\, dm = 2\int_{\partial\mathbb{U}} \log|F^*|\, dm > -\infty,$$

the integrability of $\log|F^*|$ being a standard fact about analytic functions with some boundary regularity (in fact, for this it suffices merely to have F belong to some Hardy space, or even to the Nevanlinna class [11, Theorem 2.2, page 17]).

At this point we could quote Corollary 4.2 to finish the proof, but in order to keep the exposition as self-contained as possible we prefer to use Proposition 4.1. For this it remains only to show that there is a function in $\mathcal{D}$ with boundary-modulus $1 - |\varphi^*|$. Now the definition of φ has been arranged so that $1 - |\varphi^*|$ is the boundary-modulus of F^2, so we need only know that F^2 belongs to $\mathcal{D}$. This too is obvious: F has C^2-regularity on $\overline{\mathbb{U}}$, hence so does F^2, and this is more than enough to guarantee that $F^2 \in \mathcal{D}$. $\square$

5. Final remarks

The results we have obtained here—especially Theorem 2.8, Corollaries 3.3–3.5 (14), and the examples of §2.12 and §3.6—indicate that there is a theorem waiting to be proved giving a function-theoretic characterization of hypercyclicity for $\varphi(B)$ on A^2 in terms of how freely the point $\varphi(z)$ is allowed to approach the unit circle as $|z| \to 1-$ (φ is, as usual, a holomorphic self-map of $\mathbb{U}$ that is also a Dirichlet multiplier). A similar question arises for composition operators on the Hardy and Bergman spaces, both when one tries to characterize which of these operators are *non-compact* (see [10, §3.2], [24], [25]), and when one tries to characterize which ones are *isolated* from the other composition operators in the operator-norm topology (see [10, §9.3] and [26]). Our results on the commutant hypercyclicity problem resemble most closely those obtained in [26] for the isolation problem, although why there should be such a connection remains mysterious.

Particularly striking is the association with extreme points of the H^∞ unit ball, which we recall are characterized for all bounded analytic functions φ with $\|\varphi\|_\infty = 1$ by failure of the logarithmic integrability condition (16). In [26] it is proved that if C_φ is isolated from other composition operators on H^2, then φ must be an extreme point (but not conversely). We do not know if the analogous result holds for our present problem:

> *If $\varphi \in \mathcal{M}(\mathcal{D})$ is a holomorphic self-map of $\mathbb{U}$ and $\varphi(B)$ is hypercyclic on A^2, is φ an extreme point of the unit ball of H^∞?*

Corollary 3.3 can be regarded as providing evidence in favor of an affirmative answer to this question: For the class of mappings considered there, "exponential contact of order 1" can be thought of as a sort of dividing line between extreme points and non-extreme points. Does it also divide hypercyclic from non-hypercyclic? In this regard it would be especially interesting to see if the construction of §2.12 could be refined to produce a univalently induced hypercyclic example where $\varphi(\mathbb{U})$ has exponential order of contact 1 with the unit circle.

Acknowledgment

We wish to thank Professor Fedor Nazarov of Michigan State University for his contributions to our work. His observations greatly improved both the statement and the proof of Theorem 2.8, and—as we mentioned above—he suggested the proof of Theorem 2.12. We also thank both the referee of this paper and Dr. Ullrich Boettger of the University of Karlsruhe for their constructive comments on our original preprint.

References

1. A. B. Aleksandrov, A. É. Džrbašjan, and V. P. Havin, *On the Carleson formula for the Dirichlet integral of an analytic function,* (Russian), Vestnik Leningrad. University. Mat. Mekh. Astronom 4 (1979), 8–14. *English translation:* Vestnik Leningrad Univ. Math. 12 (1980), 237–245. MR **81d:**30051
2. S. Axler and A. L. Shields, *Univalent multipliers of the Dirichlet space*, Michigan Math. J. 32 (1985), 65–80. MR **86c:**30043
3. S. R. Bell, *The Cauchy Transform, Potential Theory, and Conformal Mapping*, CRC Press, Boca Raton, 1992. MR **94k:**30013
4. S. R. Bell and S. G. Krantz, *Smoothness to the boundary of conformal maps*, Rocky Mountain J. Math. 17 (1987) 24–40. MR **88e:**30023
5. P. S. Bourdon, J. A. Cima and A. L. Matheson, *Compact composition operators on BMOA*, Trans. Amer. Math. Soc. 351 (1999), 2183–2196. MR **99i:**47052
6. L. Brown and A. L. Shields, *Cyclic vectors in the Dirichlet space,* Trans. Amer. Math. Soc. 285 (1984), 269–303. MR **86d:**30079
7. L. Carleson, *A representation formula for the Dirichlet integral,* Math. Z. 73 (1960), 190–196. MR **22:**3803
8. L. Carleson, *Sets of uniqueness for functions regular in the unit circle,* Acta Math. 87 (1952), 325–345. MR **14:**261a

9. W. G. Cochran, J. H. Shapiro and D. C. Ullrich, *Random Dirichlet functions: multipliers and smoothness,* Canadian J. Math. 45 (2) (1993), 255–268. MR **94f:**30001
10. C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, 1995. MR **97i:**47056
11. P. L. Duren, *Theory of H^p Spaces*, Pure Appl. Math., 38, Academic Press, 1970. MR **42:**3552
12. C. Kitai, *Invariant closed sets for linear operators*, Thesis, Univ. of Toronto, 1982.
13. E. Flytzanis, *Unimodular eigenvalues and linear chaos in Hilbert spaces,* Geometric and Functional Analysis 5 (1995), 1–13. MR **95k:**28034
14. L. Gårding and L. Hörmander, *Strongly subharmonic functions,* Math. Scand. 15 (1964), 93–96. MR **31:**3621
15. R. M. Gethner and J. H. Shapiro, *Universal vectors for operators on spaces of holomorphic functions*, Proc. Amer. Math. Soc. 100 (1987), 281–288. MR **88g:**47060
16. G. Godefroy and J. H. Shapiro, *Operators with dense, invariant, cyclic vector manifolds*, J. Funct. Anal. 98 (1991), 229–269. MR **92d:**47029
17. K.-G. Grosse-Erdmann, *Universal families and hypercyclic operators*, Bull. Amer. Math. Soc. 36 (1999) 345–381. MR **2000c:**47001
18. P. R. Halmos, *A Hilbert Space Problem Book*, Second Edition, Springer-Verlag, 1982. MR **84e:**47001
19. G. Herzog and C. Schmoeger, *On operators T such that $f(T)$ is hypercyclic,* Studia Math. 108 (3) (1994), 209–216. MR **95f:**47031
20. Y. Katznelson, *An Introduction to Harmonic Analysis,* Second ed., Dover, New York, 1976. MR **54:**10976
21. D. J. Newman and H. S. Shapiro, *The Taylor coefficients of inner functions,* Michigan Math. J. 9 (1962) 249–255. MR **26:**6371
22. C. Robinson, *Dynamical Systems. Stability, Symbolic Dynamics, and Chaos*, CRC Press, 1995. MR **97e:**58064
23. W. Rudin, *Real and Complex Analysis*, Third Edition, McGraw-Hill, New York, 1987. MR **88k:**00002
24. J. H. Shapiro, *The essential norm of a composition operator,* Annals of Math. 125 (1987), 375–404. MR **88c:**47058
25. J. H. Shapiro, *Composition Operators and Classical Function Theory,* Springer-Verlag, 1993. MR **94k:**47049
26. J. H. Shapiro and C. Sundberg, *Isolation amongst the composition operators,* Pacific J. Math. 145 (1990), 117–152. MR **92g:**47041
27. A. L. Shields, *Weighted shift operators and analytic function theory,* Math. Surveys 13: *Topics in Operator Theory,* C. Pearcy, ed., American Math. Society 1974, 49–128. MR **50:**14341
28. D. A. Stegenga, *Bounded Toeplitz operators on H^1 and applications of the duality between H^1 and the functions of bounded mean oscillation,* American J. Math. 98 (1973), 573–589. MR **54:**8340
29. D. A. Stegenga, *Multipliers of the Dirichlet space,* Illinois J. Math 24 (1980), 113–139. MR **81a:**30027
30. G. D. Taylor, *Multipliers on D_α,* Trans. Amer. Math. Soc. 123 (1966), 229–240. MR **34:**6514

Department of Mathematics, Washington and Lee University, Lexington, Virginia 24450

E-mail address: `pbourdon@wlu.edu`

Department of Mathematics, Michigan State University, East Lansing, Michigan 48824-1027

E-mail address: `shapiro@math.msu.edu`