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EFFECTIVELY DENSE BOOLEAN ALGEBRAS AND THEIR APPLICATIONS

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ABSTRACT. A computably enumerable Boolean algebra $\mathcal{B}$ is *effectively dense* if for each $x \in \mathcal{B}$ we can effectively determine an $F(x) \leq x$ such that $x \neq 0$ implies $0 < F(x) < x$. We give an interpretation of true arithmetic in the theory of the lattice of computably enumerable ideals of such a Boolean algebra. As an application, we also obtain an interpretation of true arithmetic in all theories of intervals of $\mathcal{E}$ (the lattice of computably enumerable sets under inclusion) which are not Boolean algebras. We derive a similar result for theories of certain initial intervals $[\mathbf{o}, \boldsymbol{a}]$ of subrecursive degree structures, where $\boldsymbol{a}$ is the degree of a set of relatively small complexity, for instance a set in exponential time.

1. INTRODUCTION

We describe a uniform method to interpret $\mathrm{Th}(\mathbb{N}, +, \times)$ in the theories of a wide variety of seemingly well-behaved structures. These structures stem from formal logic, complexity theory and computability theory. In many cases, they are closely related to dense distributive lattices. In spite of the structure's apparent well-behavedness, the theory turns out to be as complex as possible, namely it has the same Turing degree as $\emptyset^{(\omega)}$.

An *interpretation* of a theory T_1 in T_2 is a many-one reduction from T_1 to T_2 which is defined in some natural way on the sentences of the language of T_1. A good first step towards understanding a theory is to find out which well-understood theories can be interpreted. Interpretations of structures are defined e.g. in [11]. Our method to interpret $\mathrm{Th}(\mathbb{N}, +, \times)$ uses concepts from effective algebra. First we investigate a lattice of ideals of certain effective Boolean algebras, with the goal of showing that its theory interprets $\mathrm{Th}(\mathbb{N}, +, \times)$. Then we interpret such lattices in all structures under consideration. A precursor of this method was derived in [19], where it is proved that such lattices of ideals have an undecidable theory.

In the following we will discuss the applications.

Application I: Formal logic. We consider lattices of computably enumerable (c.e.) theories under inclusion. In the first-order language based on the symbol set $\{0, 1, +, \times\}$, let Q denote Robinson arithmetic, and let T be a recursively axiomatizable, consistent theory containing Q. (Thus T is a theory where Gödel's theorems apply.) Now, let $\mathcal{L}_T$ be the lattice of c.e. extensions of T closed under inference.

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Theorem 1.1. $\mathrm{Th}(\mathbb{N}, +, \times)$ *can be interpreted in* $\mathrm{Th}(\mathcal{L}_T)$.

Application II: Complexity theory. In complexity theory, one considers subsets X, Y of e.g. $\{0,1\}^{<\omega}$. Polynomial time bounded analogs of the recursion theoretic reducibilities are introduced. For instance, polynomial time many-one reducibility is defined by $X \le_m^p Y \Leftrightarrow \exists f \in \textsc{Ptime}[X = f^{-1}(Y)]$, and polynomial time Turing reducibility by $X \le_T^p Y \Leftrightarrow$ there is a polynomial time bounded oracle Turing machine taking inputs in $\{0,1\}^{<\omega}$ which computes X if the oracle is Y. Analogs of other reducibilities, like truth-table reducibility, can be defined in a similar way. We let $(\mathrm{Rec}_r^p, \le)$ be the p.o. of polynomial time r–degrees of computable sets, where $\le_r^p$ is a polynomial time reducibility in between (and possibly equal to) $\le_m^p$ and $\le_T^p$. Ladner [13] proved that Rec_r^p is dense, thereby introducing the method of delayed diagonalization (see also [4]). Slaman and Shinoda [20] gave an interpretation of $\mathrm{Th}(\mathbb{N}, +, \times)$ in $\mathrm{Th}(\mathrm{Rec}_T^p)$, but left open the case of polynomial time many-one degrees. Three years later, Ambos Spies and the author [3] proved that $\mathrm{Th}(\mathrm{Rec}_m^p)$ is undecidable. However, the two latter results use computable sets of very high complexity (usually nonelementary sets), and therefore don't allow us to obtain information about degree structures based on complexity classes low down.

Recall that $\textsc{Dtime}(h) := \{X \subseteq \{0,1\}^{<\omega} : X$ can be computed in time $O(h)\}$.

Based on the general method developed in [19], R. Downey and the author proved that $\mathrm{Th}(\textsc{Dtime}(2^n), \le_r^p)$ is undecidable [7]. Recall that a function $h : \mathbb{N} \mapsto \mathbb{N}$ is *time constructible* if $h(n)$ can be computed in time $O(h(n))$. In [7] we prove in fact that the result above holds for any time constructible hyperpolynomial function $h(n)$ in place of 2^n (e.g., $n^{\log n}$), where h is *hyperpolynomial* if h eventually dominates all polynomials. Here we prove a related result. First recall that A is *super sparse* [2] if there is a strictly increasing, time constructible $f : \mathbb{N} \mapsto \mathbb{N}$ such that $A \subseteq \{0^{f(k)} : k \in \mathbb{N}\}$ and "$0^{f(k)} \in A$?" can be determined in time $O(f(k+1))$ Here we require that, in addition, $\forall p$ (a.e. n) $[f(n)^p < f(n+1)]$. Given a reducibility $\le_r^p$, we denote the degree of a set X by $\boldsymbol{x}$ and also write $\deg_r^p(X)$ for $\boldsymbol{x}$. $\mathrm{Rec}_r^p(\le \boldsymbol{a})$ denotes the initial segment of r-degrees $\le \boldsymbol{a}$.

Theorem 1.2. *If* $A \subseteq \{0\}^*$ *is super sparse and* $A \notin \textsc{Ptime}$*, then* $\mathrm{Th}(\mathrm{Rec}_r^p(\le \boldsymbol{a}))$ *interprets* $\mathrm{Th}(\mathbb{N}, +, \times)$.

It was proved essentially in [2] that each class $\textsc{Dtime}(h)$, where h is hyperpolynomial and time constructible, contains such a strongly super sparse set A. Because $\boldsymbol{a} = \deg_r^p(A)$ can be used as a parameter and sufficiently many degrees in $\mathrm{Rec}_r^p(\le \boldsymbol{a})$ are in $\textsc{Dtime}(h)$, Theorem 1.2 implies the result in [7].

Notice that there is actually only one type of structure even if $\le_r^p$ varies: in [2] it is proved that the p-T-degree of a set $X \le_T^p A$ collapses to a single 1-tt-degree, and that $\mathrm{Rec}_m^p(\le \boldsymbol{a})$ is computably isomorphic to $\mathrm{Rec}_{1\text{-tt}}^p(\le \boldsymbol{a})$.

Application III: Computability theory. Recall that $\mathcal{E}$ is the lattice of c.e. sets under inclusion. We will consider intervals of $\mathcal{E}$ and of $\mathcal{E}^* := \mathcal{E}/_{=^*}$.

If an interval is a Boolean algebra, then by a result of Tarski (see [6]), its theory is decidable. We show that otherwise the theory has the maximum possible complexity. Maass and Stob [14] had asked whether the theory of $[D, A]$, for $D \subseteq_m A$, is undecidable. This was answered to the affirmative in Nies [19].

Theorem 1.3. *Suppose* $D \subseteq A$*, where* $D, A \in \mathcal{E}$*. If* $[D, A]_{\mathcal{E}}$ *is not a Boolean algebra, then* $\mathrm{Th}(\mathbb{N}, +, \times)$ *can be interpreted in* $\mathrm{Th}([D, A])$*. A similar statement holds for* $\mathcal{E}^*$.

2. Effectively dense Boolean algebras

As in [19], a c.e. Boolean algebras is given by a model $(\mathbb{N}, \preceq, \vee, \wedge)$ such that $\preceq$ is a c.e. relation which is a pre-ordering, $\vee, \wedge$ are total computable binary functions, and the quotient structure $\mathcal{B} = (\mathbb{N}, \preceq, \vee, \wedge)/_{\approx}$ is a Boolean algebra (where $n \approx m \;\Leftrightarrow\; n \preceq m \;\&\; m \preceq n$.) More generally, for Σ^0_k-Boolean algebras, one requires that $\preceq$ be Σ^0_k and that $\wedge, \vee$ be recursive in $\emptyset^{(k-1)}$. Note that the complement of $b \in \mathcal{B}$ can be computed using a $\emptyset^{(k-1)}$-oracle.

While developing some theory of Σ^0_k-Boolean algebras, we will introduce and refine three examples which will lead us towards the applications. First we introduce the various Boolean algebras we need.

Examples 2.1. Σ^0_1: Let T be a consistent recursively axiomatizable theory, and let $\mathcal{B}_T$ be the Lindenbaum algebra of sentences over T.

Σ^0_2: the Boolean algebra of complemented elements in $\mathrm{Rec}^p_r(\leq \boldsymbol{a})$, where $\boldsymbol{a}$ is the degree of a super sparse set (see [7]).

Σ^0_3: the Boolean algebra of complemented elements in $[D^*, A^*]_{\mathcal{E}^*}$, where $D, A \in \mathcal{E}$, $D \subset_\infty A$.

As in [19], for a Σ^0_k–Boolean algebra $\mathcal{B}$, let

$$\mathcal{I}(\mathcal{B}) \;:=\; \text{the lattice of } \Sigma^0_k\text{–ideals of } \mathcal{B}.$$

In the following we will use the terminology of c.e. Boolean algebras. It should be clear how to relativize the notions to the Σ^0_k-cases. We list some properties of $\mathcal{I}(\mathcal{B})$ which show that, in a sense, $\mathcal{I}(\mathcal{B})$ is similar to $\mathcal{E}$. $\mathcal{I}(\mathcal{B})$ is a distributive lattice with least and greatest elements (the infimum of $A, B \in \mathcal{I}(\mathcal{B})$ is $A \cap B$, and the supremum is $\{a \vee b : a \in A \;\&\; b \in B\}$). It is easy to prove that $\mathcal{I}(\mathcal{B})$ also has the reduction property (see [21]), namely each supremum of two elements is the disjoint supremum of two smaller elements. All principal ideals $[0, b]_{\mathcal{B}}$ of $\mathcal{B}$ are in $\mathcal{I}(\mathcal{B})$. The class of principal ideals is definable: an ideal is principal iff it is complemented in $\mathcal{I}(\mathcal{B})$.

It can occur that $\mathcal{I}(\mathcal{B}) \cong \mathcal{B}$, even for a dense c.e. $\mathcal{B}$: one can construct a dense $\mathcal{B}$ such that every c.e. ideal is principal [15]. However, the c.e. Boolean algebras we consider now have a very complex lattice of c.e. ideals. A c.e. Boolean algebra $\mathcal{B}$ is called *effectively dense* [19] if there is a computable F such that $\forall x\, [F(x) \preceq x]$ and

$$\forall x \not\approx 0 \;\; [0 \prec F(x) \prec x]. \tag{1}$$

More generally, a Σ^0_k Boolean algebra $\mathcal{B}$ is effectively dense if the above holds with some $F \leq_T \emptyset^{(k-1)}$. All effectively dense Boolean algebras are isomorphic to $\mathcal{D}$, but not necessarily effectively isomorphic. Thus our study of Boolean algebras is in the spirit of recursive model theory, and not along the lines of [8], where (classical) isomorphism types of c.e. Boolean algebras are investigated.

Examples, continued 2.2. Σ^0_1: If T is a recursively axiomatizable consistent theory containing Robinson's Q, then $\mathcal{B}_T$ is effectively dense.

Σ^0_2: As, before, suppose $\boldsymbol{a}$ is the degree of a super sparse set. Then the complemented elements in $\mathrm{Rec}^p_r(\leq \boldsymbol{a})$ form an effectively dense Σ^0_2 Boolean algebra.

Σ^0_3: If D is a major subset of A (denoted by $D \subset_m A$, see [21]), then the complemented elements in $[D^*, A^*]_{\mathcal{E}^*}$ form an effectively dense Σ^0_3 Boolean algebra.

Proofs. To prove the first assertion, we use Rosser's Theorem (see e.g. [9]), which asserts that from an index of a c.e. theory $S \supseteq Q$ one can effectively obtain a sentence α such that

$$S \text{ consistent } \Rightarrow S \nvdash \alpha \text{ and } S \nvdash \neg\alpha.$$

Given $\varphi \in \mathcal{B}_T$, to determine $F(\varphi)$ let $S = T \cup \{\varphi\}$. If $\varphi \not\approx 0$ (in $\mathcal{B}_T$), then S is consistent, so $\varphi \not\preceq \alpha$ and $\varphi \not\preceq \neg\alpha$. Thus let $F(\varphi) = \varphi \wedge \alpha$. Notice that, by a result of Montagna and Sorbi [16], the Boolean algebras for all such theories are effectively isomorphic. (In that paper the notion of effective density for general c.e. preorderings is introduced.)

The effective density of $\mathrm{Rec}_r^p(\leq \boldsymbol{a})$ will be proved in 3.2. Finally, the effective density of $[D^*, A^*]_{\mathcal{E}^*}$ follows from the Owings splitting theorem (see [19]). $\square$

3. The Main Theorem

Main Theorem 3.1. *Suppose $\mathcal{B}$ is effectively dense. Then* $\mathrm{Th}(\mathbb{N}, +, \times)$ *can be interpreted in* $\mathrm{Th}(\mathcal{I}(\mathcal{B}))$.

This seems to be the first time that an interpretation of arithmetic can be given for any structure satisfying a fairly general set of conditions.

The main theorem can also be viewed as a theorem about Π_1^0-classes under inclusion. C.e. ideals of $\mathcal{D}$ correspond to Π_1^0 classes $\subseteq 2^\omega$ via the usual Stone duality. For a Π_1^0-class $P \subseteq 2^\omega$, let $[\emptyset, P]$ be the set of Π_1^0-classes $Q \subseteq P$. The interval $[\emptyset, P]$ is called *effectively dense* if, for each basic open set U in 2^ω, one can effectively find disjoint basic open sets $V, W \subseteq U$ such that $U \cap P \neq \emptyset \;\Rightarrow\; V \cap P, W \cap P \neq \emptyset$. Thus, $U \cap P$ is not a singleton, in an effective way. The Main Theorem asserts in this context that $\mathrm{Th}([\emptyset, P])$ interprets true arithmetic.

A further application of the Main Theorem to a quite different type of interval has been given in [5]. A Π_1^0-class P is *decidable* if there is a decision procedure to tell whether $U \cap P \neq \emptyset$ for a basic open U. In a partial analogy to Theorem 1.3, it is shown that if P is decidable and $[\emptyset, P]$ is not a Boolean algebra, then $\mathrm{Th}([\emptyset, P])$ interprets true arithmetic. However, this fails for P in general. As in the proof of Theorem 1.3, one applies the Main Theorem relativized to $\emptyset''$.

The Main Theorem will be proved in Sections 5 and 6. In this section we apply it to give proofs for the Theorems 1.1–1.3.

3.1. **Proving Theorems 1.1 and 1.3.**

To show that $\mathrm{Th}(\mathbb{N}, +, \times) \leq_m \mathrm{Th}(\mathcal{L}_T)$, just notice that elements of $\mathcal{L}_T$ are the c.e. filters in $\mathcal{B} = \mathcal{B}_T$. So $\mathcal{L}_T \cong \mathcal{I}(\mathcal{B})$ via complementation (which is effective in a c.e. Boolean algebra).

To prove Theorem 1.3 we rely on some auxiliary results from [19], where it is shown that intervals of $\mathcal{E}^*$ and $\mathcal{E}$ which are not Boolean algebras have an undecidable theory. We will obtain the following.

Claim. Suppose $\widetilde{D} \subset_m \widetilde{A}$. Then there is a fixed interpretation G^* of $\mathrm{Th}(\mathbb{N}, +, \times)$ *in each theory* $\mathrm{Th}([\widetilde{D}^*, \widetilde{A}^*])$ *and an interpretation G of* $\mathrm{Th}(\mathbb{N}, +, \times)$ *in* $\mathrm{Th}([\widetilde{D}, \widetilde{A}])$.

(By [14], all these intervals are isomorphic. But we don't make use of this fact, since in both cases the interpretation is independent of the particular choices of D and A.)

The claim suffices for the following reason. First, we can assume that $A = \mathbb{N}$, since each closed interval of $\mathcal{E}$ is isomorphic to an end interval of $\mathcal{E}$. Now, as explained in [19], since $[D, \mathbb{N}]$ is not a Boolean algebra, there is a subinterval $[\widetilde{D}, \widetilde{A}]$ of $[D, \mathbb{N}]$ such that $\widetilde{D} \subset_m \widetilde{A}$. Let $\mu(x, y)$ be the formula describing the major subset relation in $\mathcal{E}^*$, namely, $\mu(x, y) \;=\; x < y \;\&\; \forall w \;\; (y \vee w = 1 \Rightarrow x \vee w = 1)$. If x, y

denote elements of $\mathcal{L}^*(D) = [D^*, \mathbb{N}^*]$, then we can restrict the quantifier as well to $\mathcal{L}^*(D)$. Thus we obtain an interpretation F^* of $\mathrm{Th}(\mathbb{N}, +, \times)$ in $\mathrm{Th}(\mathcal{L}^*(D))$ from G^* as follows:

$$F^*(\varphi) = \exists x \exists y[\mu(x, y) \ \&\ G^*(\varphi)^{[x,y]}].$$

In a similar way we obtain an interpretation F of $\mathrm{Th}(\mathbb{N}, +, \times)$ in $\mathrm{Th}(\mathcal{L}(D))$ from G.

The claim is proved in [19]: first one considers the case of $\mathcal{E}^*$. By 2.2, the complemented elements in $[\widetilde{D}^*, \widetilde{A}^*]$ form an effectively dense Σ^0_3 Boolean algebra $\mathcal{B}$. As in [19], one gives an interpretation (of structures) without parameters of $\mathcal{I}(\mathcal{B})$ in $[\widetilde{D}^*, \widetilde{A}^*]$. By Lemma 3.2 in [19], if I is a Σ^0_3–ideal of $\mathcal{B}$, then there is C_I, $\widetilde{D} \subseteq^* C_I \subseteq^* \widetilde{A}$, such that

$$I = \{j : W_{f(j)} \cap C_I \subseteq^* \widetilde{D}\}. \tag{2}$$

Conversely, an ideal I satisfying (2) for some C must be Σ^0_3. Now, for the desired interpretation, one represents Σ^0_3-ideals I of $\mathcal{B}$ ambiguously by any element $c = C^*_I$. Inclusion of Σ^0_3–ideals can be defined within $[\widetilde{D}^*, \widetilde{A}^*]$ using the formula

$$\varphi_{\subseteq}(c_1, c_2) \equiv \forall x(x \text{ complemented in } [\tilde{d}, \tilde{a}] \Rightarrow (x \wedge c_1 = \tilde{d} \Rightarrow x \wedge c_2 = \tilde{d})), \tag{3}$$

where $\tilde{d} = \widetilde{D}^*$, etc. Composing the interpretation $\mathrm{Th}(\mathbb{N}, +, \times) \mapsto \mathrm{Th}(\mathcal{I}(\mathcal{B}))$ given by the Main Theorem with this interpretation, we obtain G^*. Now G can easily be obtained from G^*, since $\mathrm{Th}(\widetilde{D}^*, \widetilde{A}^*)$ can be interpreted in $\mathrm{Th}(\widetilde{D}, \widetilde{A})$: for $\widetilde{D} \subseteq X$ and $Y \subseteq \widetilde{A}$ we have $X =^* Y \Leftrightarrow [X \cap Y, X \cup Y]$ is a Boolean algebra (see [19]).

3.2. **Proving Theorem 1.2.** In [7], it is shown that the Boolean algebra $\mathcal{B} = \mathcal{B}(\boldsymbol{a})$ of complemented degrees in $\mathrm{Rec}^p_r(\leq \boldsymbol{a})$ equals $\{\deg^p_r(A \cap P_e) : e \in \mathbb{N}\}$, where (P_e) is a uniformly recursive list of sets in PTIME. The complement of $\deg^p_r(A \cap P_e)$ is $\deg^p_r(A \cap \overline{P}_e)$, and inclusion of sets $A \cap P_e$ corresponds to the ordering of the degrees. Thus $\mathcal{B}(\boldsymbol{a})$ is a Σ^0_2-Boolean algebra, and Ladner's technique shows that it is effectively dense [7]. In view of the Main Theorem, it now suffices to give an interpretation without parameters of the lattice $\mathcal{I}(\mathcal{B})$ in $\mathrm{Rec}^p_r(\leq \boldsymbol{a})$. Since the p-T-degree of a set $X \leq^p_T A$ collapses to a single 1-tt-degree, we can assume that $\leq^p_r$ is $\leq^p_m$ or $\leq^p_{\text{1-tt}}$. In [7] a weaker version of the next lemma is proved, where C_I is merely in an appropriate time class. Here we improve this to $C_I \leq^p_m A$.

Lemma 3.2. *Suppose that A is strongly super sparse via f. Then for each Σ^0_2 ideal $I \lhd \mathcal{B}(\boldsymbol{a})$ there is $\boldsymbol{c}_I \leq \boldsymbol{a}$ such that $\forall x \in \mathcal{B}(\boldsymbol{a})$ $(x \in I \Leftrightarrow x \leq c_I)$.*

Conversely, each ideal defined in this way must be Σ^0_2. Thus we obtain a parameterless interpretation of $\mathrm{Th}(\mathcal{I}(\mathcal{B}))$ in $\mathrm{Th}(\mathrm{Rec}^p_r(\leq \boldsymbol{a}))$ using the same framework as we did for intervals $[D^*, A^*]$ of $\mathcal{E}^*$ where $D \subset_m A$: we represent Σ^0_2-ideals I by elements $\boldsymbol{c}_I$. Inclusion of Σ^0_2–ideals can be defined within $\mathrm{Rec}^p_r(\leq \boldsymbol{a})$ using the formula

$$\varphi_{\subseteq}(c_1, c_2) \equiv \forall x(x \text{ complemented in } \mathrm{Rec}^p_r(\leq \boldsymbol{a}) \Rightarrow (x \leq c_1 \Rightarrow x \leq c_2)).$$

Proof of Lemma 3.2. By the remarks after Theorem 1.2, it suffices to consider the case of many-one reducibility. We say that s is *relevant* if $s = 0^{f(k)}$ for some k. We will build $C_I \leq^p_m A$ via a g which is computable in polynomial time. Since I is in Σ^0_2, there is a function $q \leq_T \emptyset'$ such that $\mathrm{range}(q) = \{e : \deg^p_r(P_e \cap A) \in I\}$. By the Limit

Lemma [21], there is a computable function $q(e,t)$ such that $q(e) = \lim_t q(e,t)$. Let (h_j) be a list of p-m-reductions. We meet the coding requirements

$$K_e : A \cap P_{q(e)} \leq_m^p C_I$$

by specifying polynomial time m-reductions to C_I. To do so, we assign K_e-*coding locations* to certain relevant 0^s. If $s = f(m)$, a K_e-coding location for 0^s will have the form 0^n, $n = \langle e, r\rangle$, where $r \geq e$ and $f(m) \leq n < f(m+1)$. We will ensure that K_e-coding locations exists for all sufficiently long relevant 0^s. We require that in n steps one can determine that $0^s \in P_u$, where u is the current guess at $q(e) = \lim_t q(e,t)$. We define C_I by specifying a polynomial time computable g such that $C_I = g^{-1}(A)$, mapping coding locations for relevant strings 0^s to 0^s. Thus, eventually just the relevant $0^s \in P_{q(e)}$ are assigned a K_e-coding location, which is in C_I just if 0^s is in A. An appropriate choice of the K_e-coding locations will ensure that the requirements

$$H_{\langle i,j\rangle} : A \cap P_i \leq_r^p C_I \text{ via } h_j \Rightarrow A \cap P_i \leq_r^p \bigoplus_{m \leq k} A \cap P_{q(m)} \ (k = \langle i,j\rangle)$$

are met. We can suppose that computing $h_j(x)$ takes at most $p_j(|x|)$ steps, where

$$p_j(n) = (n+2)^j.$$

The main idea of the proof is how to ensure that the coding of K_e does not interfere with the requirements H_i, $i < e$: We make the length of any K_e-coding location for 0^s exceed $p_{e-1}(s)$.

The algorithm for g.

Given an input x, $n = |x|$, first determine in quadratic time the maximal $s \leq n$ such that 0^s is relevant. This is possible by the time constructibility of f. Now proceed as follows.

1. See if there are e, r such that $x = 0^{\langle e,r\rangle}$.
2. Perform computations $q(e,0), q(e,1), \ldots$ till n steps have passed, and let u be the last value (or $u = 0$ if there was no value so far).
3. See if $0^s \in P_u$ in n steps.
4. Check if $p_{e-1}(s) \leq n$.

If (1) and (3) are answered affirmatively and the computation in (4) stops, then let $g(x) = 0^s$ (so x is a K_e-coding location for 0^s). Else let $g(x)$ be the string $(1) \notin A$. This completes the algorithm. Clearly the algorithm takes at most $O(n^2)$ steps.

Let $C_I = g^{-1}(A)$. We verify that C_I has the required properties.

Claim 1. Let $q(e) = \lim_t q(e,t)$. Then $A \cap P_{q(e)} \leq_m^p C_I$.

Proof. Let $p(s)$ be a polynomial which dominates $p_{e-1}(s)$ and the number of steps it takes to compute $P_{q(e)}$ on the input 0^s. Pick an $s_0 = f(m)$ such that the value returned in (2.) of the algorithm is $q(e)$ for all $s \geq s_0$ and also that, by super sparseness, $\langle e, p(f(k))\rangle < f(k+1)$ for all $k \geq m$. Then for all $s \geq s_0$, 0^s relevant,

$$0^s \in A \cap P_{q(e)} \ \Leftrightarrow \ 0^{\langle e, p(s)\rangle} \in C_I.$$

Claim 2. The requirements $H_{\langle i,j\rangle}$ are met.

Proof. Suppose that $A \cap P_i \leq_r^p C_I$ via h_j. We obtain an m-reduction of $A \cap P_i$ to $\bigoplus_{m<k} A \cap P_{q(m)}$ ($k = \langle i, j \rangle$) as follows. Given a relevant string 0^s, first compute $x = h_j(0^s)$. Since $0^s \in A \cap P_i \iff x \in C_I$, it is sufficient to determine if $x \in C_I$. Run the algorithm for g on input x. If $g(x) = (1)$ then $x \notin C_I$. Otherwise x is a coding location.

Case 1: $|x| < s$. Then give $A(g(x))$ as an answer. Since A is super sparse and $|g(x)| < s$, this answer can be found in time $O(s)$.

Case 2: $n = |x| \geq s$. We can suppose that $s \geq s_0$, where s_0 is so large that, for all relevant $t \geq s_0$, $|h_j(0^t)|$ is less than the least relevant number bigger than t (by the last condition in the definition of super sparse), and also the computation in Step 2 of the algorithm for g with input 0^t gives the final value $q(e)$ for each $e \leq k$. By the main idea, if $x \in C_I$, then x must be a coding location for a requirement K_e, $e \leq k$. Since $s \geq s_0$, $x \in C_I \Leftrightarrow g(x) \in A \cap P_{q(e)}$. □

Corollary 3.3 ([7])**.** *Suppose that h is time constructible and hyperpolynomial. Then the degrees of (1) all sets and (2) all tally sets in* $\textsc{Dtime}(h)$ *have an undecidable theory.*

Proof. Choose a strongly super sparse $A \in \textsc{Dtime}(h) - \textsc{Ptime}$. (1) was proved in [7]. We obtain it here because all sets $A \cap P_e$, as well as the sets C_I, are in $\textsc{Dtime}(h)$ (since h is hyperpolynomial). Thus, we get an interpretation of the structure $(\mathbb{N}, +, \times)$ in $(\textsc{Dtime}(h), \leq_r^p)$ with parameter $\boldsymbol{a}$, and $\mathrm{Th}(\textsc{Dtime}(h), \leq_r^p)$ is undecidable. For (2), observe that all sets involved are tally sets. □

4. Preliminary facts about c.e. ideals

We first introduce a useful alternative representation of a Σ_k^0 Boolean algebra $\mathcal{B}$. Choose a computable sequence (d_i) of free generators for the recursive dense Boolean algebra $\mathcal{D}$. The map $f : d_i \mapsto i$ extends to a recursive map $\overline{f} : \mathcal{D} \mapsto \mathbb{N}$. Thus, if $t(x_0, \ldots, x_{n-1})$ is some Boolean term, the element $t(d_0, \ldots, d_{n-1})$ is mapped to $t^{\mathcal{B}}(0, \ldots, n-1)$ (recall that $\mathcal{B}$ is defined as a model with domain $\mathbb{N}$; we use the fact that $\wedge, \vee$ and complementation in $\mathcal{B}$ are effective). We write $\mathrm{Cpl}(d)$ for the complement of $d \in \mathcal{D}$ and say that $d, e \in \mathcal{D}$ are *disjoint* if $d \wedge e = 0$. If we are considering Σ_k^0–Boolean algebras, the map $\overline{f}$ is recursive in $\emptyset^{(k-1)}$. The kernel of $\overline{f}$,

$$H = \{d_0, \ldots, d_{n-1}) : t^{\mathcal{B}}(0, \ldots, n-1) \preceq 0\},$$

is a Σ_k^0-ideal.

Now suppose that $\mathcal{B}$ is effectively dense via $F_{\mathcal{B}} : \mathbb{N} \to \mathbb{N}$. Let F be a "preimage" of $F_{\mathcal{B}}$ in $\mathcal{D}$, namely, if $d \in \mathcal{D}$, let $F(d) = e$ if $e \leq d$ is the first element of $\mathcal{D}$ (with respect to some effective listing) such that we discover $\overline{f}(e) \approx F_{\mathcal{B}}(\overline{f}(d))$. Then F can be chosen recursive in $\emptyset^{(k-1)}$. We have obtained the following.

Fact 4.1. *Suppose $\mathcal{B}$ is a Σ_k^0-Boolean algebra.*

(i) *$\mathcal{B}$ is $\emptyset^{(k-1)}$–isomorphic to $\mathcal{D}/_H$ (with the canonical presentation), for some Σ_k^0–ideal H of $\mathcal{D}$.*
(ii) *If $\mathcal{B}$ is effectively dense, then there is $F : \mathcal{D} \mapsto \mathcal{D}$, $F \leq_T \emptyset^{(k-1)}$, such that*

$$\forall d \in \mathcal{D}(d \notin H \Rightarrow F(d) \notin H \ \&\ d - F(d) \notin H.$$

Definition 4.2. Throughout, $\mathcal{B}$ will be an effectively dense Boolean algebra and H will denote the ideal of elements of $\mathcal{D}$ representing $0^{\mathcal{B}}$. An ideal X of $\mathcal{D}$ is principal if $X = H \vee [0, t]$ for some $t \in \mathcal{D}$. Given $t \in \mathcal{D}$, we denote by "t" as well the principal ideal $H \vee [0, t]$. For any ideal D, we write $D - t$ for $D \cap \mathrm{Cpl}(t)$.

Thus the trivial ideal $\mathcal{B}$ is denoted by 1.

We begin with some simple constructions of c.e. ideals which will be needed later. First we prove that $\mathcal{I}(\mathcal{B})$ is not a Boolean algebra if $\mathcal{B}$ is effectively dense. See the discussion before (1).

Fact 4.3. *There is a nonprincipal ideal E.*

Proof. Let $e_0 = 0$. If e_n has been defined, let $\hat{e}_n = e_0 \vee \ldots \vee e_n$ and

$$e_{n+1} = F(\mathrm{Cpl}(\hat{e}_n)).$$

Let E be the ideal of $\mathcal{D}$ generated by H and $\{e_i : i \in \mathbb{N}\}$. We claim that E is nonprincipal. First we show that $\hat{e}_n \prec \hat{e}_{n+1}$ for each n: $e_0 = 0$, and if $\hat{e}_n \prec 1$, then by (1), $0 \prec e_{n+1} \prec \mathrm{Cpl}(\hat{e}_n)$. Moreover, $e_{n+1} \wedge \hat{e}_n = 0$ (in $\mathcal{D}$), so $\hat{e}_n \prec \hat{e}_{n+1} \prec 1$.

If $E = H \vee [0, t]$ for some t, then there is an n such that $E = H \vee [0, \hat{e}_n]$. So $e_{n+1} = h \vee \hat{e}_n$ for some $h \in H$, which implies that $e_{n+1} \leq h$, contrary to $e_{n+1} \notin H$. ☐

Before we proceed we introduce some notation. We use the language of the unrelativized case.

Definition 4.4.

1. A c.e. ideal X of $\mathcal{B} = \mathcal{D}/_H$ is given by a c.e. subset $\widetilde{X}$ of $\mathcal{D}$ such that X is the ideal of $\mathcal{D}$ generated by $\widetilde{X} \cup H$. We let (V_e) be a uniform enumeration of all c.e. ideals containing H.
2. For a c.e. ideal X, we let

$$\hat{x}_s = \sup_{\mathcal{D}}(\widetilde{X}_s), \qquad x_0 = 0, \qquad x_s = \hat{x}_s - \hat{x}_{s-1} \quad (s > 0). \tag{4}$$

   Thus, $(\hat{x}_n)_{n \in \mathbb{N}}$ is an effective ascending sequence in $\mathcal{D}$ generating X, and $(x_n)_{n \in \mathbb{N}}$ is an effective "partition" generating X.
3. Capital letters $A, \ldots, E, G, X, Y, W$ range over c.e. ideals of $\mathcal{D}$ containing H.
4. *(Splittings of ideals)* We write $B \sqcup C = A$ if $B \cap C = H$ and $B \vee C = A$. In this case we denote C by $\mathrm{Cpl}_A(B)$. Note that if $a \in A$ and $b = \mathrm{Cpl}(a)$ (in $\mathcal{D}$), then $\mathrm{Cpl}_A(a) = A \cap b$. We write $B \sqsubset A$ if $\exists C\ B \sqcup C = A$, and $D - a$ for $D \cap \mathrm{Cpl}(a)$. Then $D \sqsubset 1$ means that D is principal.

When building a c.e. ideal Y we speak of enumerating an element $z \in \mathcal{D}$ into Y when we actually mean to enumerate z into $\widetilde{Y}$.

The following concept is of central importance for the proof of the Main Theorem.

Definition 4.5. We say that A is *locally principal* in E if $A \subseteq E$ and

$$\forall e \in E[e \cap A \text{ is principal}].$$

Note that this property of A, E can be expressed in $\mathcal{I}(\mathcal{B})$ in a first-order way, since the principal ideals are just the complemented elements. The motivation is that the situation $A \subseteq E$ is in a sense similar to an inclusion of sets. Whenever $e \in E$, the the intersection $A \cap e$ has only a finite amount of information. In what follows, given a nonprincipal ideal E, we construct $A \subseteq E$ which is not a component of a split of E, but A is locally principal in E. Note that the hypothesis $E \not\sqsubset 1$ is necessary. As required in (1.) of Definition 4.4, we will enumerate elements of $\mathcal{D}$ into a c.e. set $\widetilde{A}$ to determine A.

Lemma 4.6. *Given E, we can effectively obtain an $A \subseteq E$ such that A is locally principal in E and*

$$E \not\sqsubset 1 \;\Rightarrow\; A \not\sqsubset E. \tag{5}$$

Proof. We meet the requirements

$$R_i : E \not\sqsubset 1 \;\Rightarrow\; \neg\, A \sqcup (V_i \cap E) = E,$$

while enumerating A in such a way that A is locally principal in E. Let (e_n) be the "partition" corresponding to E given by (4) in Definition 4.4. We put movable markers γ_i associated with R_i on the elements e_n. If $E \not\sqsubset 1$, then γ_i will come to rest on the i-th element e_j which is not in H. Thus let

$$\gamma_{i,s} = \text{ the } i\text{-th element } \notin H_s \text{ in the sequence } e_0, e_1, \ldots .$$

After γ_i has settled down, R_i enumerates an element $\leq \gamma_i$ into A at most once. This implies that A is locally principal in E: given $e \in E$, we want to show that $e \cap A \sqsubset 1$. Since E is the ideal generated by $\{e_n : n \in \mathbb{N}\}$, we can assume that $e = e_n$ for some n and $e \notin H$. Then $e = \lim_s \gamma_{i,s}$ for some i, and after γ_i has stabilized, since we enumerate at most once into $A \cap [0, e]$, $A \cap e \sqsubset 1$.

For (5), if actually $A \sqcup (V_i \cap E) = E$, then at some stage s when γ_i has settled down, we will discover that $\gamma_i \leq \hat{a}_{s-1} \vee \hat{v}_{i,s}$. In that case we put $F(\gamma_i - \hat{a}_{s-1})$ into A, thereby causing $A \cap E \cap V_i \not\subseteq H$.

Construction.

Stage $s > 0$. If $i \leq s$ is minimal such that $\gamma_{i,s-1} \neq \gamma_{i,s}$, then declare all requirements R_j, $j \geq i$, unsatisfied.

For each $i \leq s$ do the following: if R_i is unsatisfied and now $\gamma_{i,s} \leq \hat{a}_{s-1} \vee \hat{v}_{i,s}$, then put $F(\gamma_{i,s} - \hat{a}_{s-1})$ into $\widetilde{A}_s$ and declare R_i satisfied.

Verification. Clearly R_i enumerates into A at most once after γ_i has settled down. Moreover, if $e_n \notin H$, then $e_n \notin A$, since the elements $\leq e_n$ we enumerate at finitely many stages t have the form $F(e_n - \hat{a}_{t-1})$. Finally, it is not the case that $A \sqcup (V_i \cap E) = E$. For choose s minimal such that γ_i is stable from s on, and let e be its limit. Since $e \notin A$, if R_i is never active at stages $\geq s$, then $A \cap e = \hat{a}_s$ (as ideals), $\hat{a}_s \wedge e \prec e$ and $e \notin A \vee (V_i \cap E)$. Now suppose R_i is active at $t \geq s$. Then $e \wedge \hat{a}_{t-1} \notin H$, and we put $F(e - \hat{a}_{t-1})$ into $\widetilde{A}_t$. But $F(e - \hat{a}_{t-1}) \notin H$ by (1), and $F(e - \hat{a}_{t-1}) \in A \cap E \cap V_i$. ◻

We next prove the analog of the Friedberg Splitting Theorem (see [21]). First some more notation. Recall that $(d_i)_{i \in \mathbb{N}}$ is an effective free generating sequence of $\mathcal{D}$. It is very useful to make some restrictions on the way ideals can be enumerated, which are embodied in the following convention. We continue to use the language of the unrelativized case.

Convention 4.7. 1. ($\mathcal{D}_s$) At stage s of our constructions, we will work only with elements in $\mathcal{D}_s$, the finite Boolean algebra of Boolean combinations of $d_0, \ldots, d_{s-1}$. In particular, we assume an enumeration of H such that $\widetilde{H}_s$ is an ideal of $\mathcal{D}_s$.

2. *(Enumerating ideals).* We require that a c.e. set $\widetilde{X}$ determining an ideal X of $\mathcal{B} = \mathcal{D}/_H$ in the sense of Definition 4.4, (1.) satisfies $\widetilde{X}_s \subseteq \mathcal{D}_s$. We let

$$X_s = \text{the ideal of } \mathcal{D}_s \text{ generated by } \widetilde{X}_s \cup H_s. \tag{6}$$

Note that $\hat{x}_s, x_s \in X_s$.

The point in (2.) is that ideals of $\mathcal{D}_s$ are finite approximations of ideals of $\mathcal{D}$, but they behave like sets with respect to the atoms d_σ ($|\sigma| = s$) of $\mathcal{D}_s$. For instance, if $d_\sigma \notin H_s$ and $X, Y \supseteq H$ are c.e. ideals, then

$$d_\sigma \in X_s \vee Y_s \;\Leftrightarrow\; d_\sigma \in X_s \text{ or } d_\sigma \in Y_s. \tag{7}$$

Another useful property is that, if $t \leq s$, then

$$d_\sigma \notin X_t \Rightarrow d_\sigma \wedge \sup(X_t) = 0. \tag{8}$$

Suppose that at some stage s of a construction we have $x \in \mathcal{D}_s$ and want to enumerate $F(x)$ into an ideal G. The problem is that $F(x)$ may not be in $\mathcal{D}_s$. The solution is to replace $F(x)$ by $y = x \wedge F^*_s$ and enumerate y at stage $h(s)$, where F^*_s and h are defined below.

Definition 4.8. 1. If $\sigma \in 2^{<\omega}, |\sigma| = s$, then let

$$d_\sigma = \bigwedge_{i<s \;\&\; \sigma(i)=1} d_i \wedge \bigwedge_{i<s \;\&\; \sigma(i)=0} \mathrm{Cpl}(d_i)$$

(so that $d_\sigma \in \mathcal{D}_s$).

2. We let

$$F^*_s = \sup_{|\sigma|=s} F(d_\sigma) \tag{9}$$

and define an increasing computable function h by

$$h(s) = \mu t \geq s \forall s' \leq s \; (F^*_{s'} \in \mathcal{D}_t). \tag{10}$$

This "finer" choice $x \wedge F^*_s$ instead of $F(x)$ makes the combinatorics of our main construction, the proof of the Trace Lemma below, much easier, roughly speaking because we never completely put a $d_\sigma \notin H, |d_\sigma| = s$ (an element some other requirement could rely on) into G at stage s. The reason we wanted $F(x)$ in G was that $0 \prec x \Rightarrow 0 \prec F(x) \prec x$. But this holds for $x \wedge F^*_s$ as well: if $0 \prec x$, pick σ of length s such that $0 \prec d_\sigma \leq x$. Then $0 \prec F(d_\sigma) \leq x \wedge F^*_s \prec x$.

Theorem 4.9. *If $A \not\subseteq E$, then, effectively in indices for A, E, one can obtain ideals B, C such that $A = B \sqcup C$ and $B, C \not\subseteq E$. In particular, each non-principal A can be split into two non-principal ideals.*

We begin with a lemma. For a c.e. ideal W let $W \searrow B$ be the ideal X given by enumerating (into a set $\widetilde{X}$) at stage s those x such that

$$x \in W_{s-1} \;\&\; x \notin B_{s-1} \;\&\; x \in B_s,$$

(and, as always, letting X_s be the ideal of $\mathcal{D}_s$ generated by $\widetilde{X}_s \cup H_s$).

Lemma 4.10. *For each A there is a splitting $A = B \sqcup C$ such that*

$$\forall e[V_e \searrow A \not\subseteq H \;\Rightarrow\; V_e \searrow B \not\subseteq H \;\&\; V_e \searrow C \not\subseteq H]. \tag{11}$$

Proof. The construction of B, C is simple: if $t = h(s)+1$, then at stage t enumerate $a_s \wedge F_s^*$ into B and $a_s - F_s^*$ into C.

Clearly, $A = B \sqcup C$. Now suppose that $V_e \searrow A \not\subseteq H$. Then there exist $r < s$ and a string σ of length r such that $d_\sigma \notin H$, $d_\sigma \in V_{e,r}$ and $d_\sigma \in A_s - A_{s-1}$. Pick $\rho \supset \sigma$ of length s such that also $d_\rho \notin H$. Then $d_\rho \leq \hat{a}_s$, and $d_\rho \wedge \hat{a}_{s-1} = 0$ using the fact (8). So $d_\rho \leq a_s (= \hat{a}_s - \hat{a}_{s-1})$, and $F(d_\rho), d_\rho - F(d_\rho) \notin H$. But $a_s \wedge F_s^* \geq F(d_\rho)$ and $a_s - F_s^* \geq d_\rho - F(d_\rho)$. Now, if $t = h(s) + 1$, then $a_s \wedge F_s^*$ is enumerated into B at stage t and $a_s - F_s^*$ into C at t. Moreover, $\sup B_{t-1} \leq \hat{a}_{s-1}$ by the monotonicity of h, whence $F(d_\rho) \notin B_{t-1}$, and similarly $d_\rho - F(d_\rho) \notin C$. Therefore, $F(d_\rho) \in B_t - B_{t-1}, d_\rho - F(d_\rho) \in C_t - C_{t-1}$. Both elements are already in $V_{e,t-1}$. □

Proof of the Theorem. Obtain B, C as in the preceding lemma. Assume for a contradiction that $B \sqcup W = E$ for some W. We claim that $W \searrow A \not\subseteq H$. Then also $W \searrow B \not\subseteq H$, contrary to $W \cap B = H$.

Let

$$\widetilde{D}_s = \{x \in \mathcal{D}_s : x \leq \hat{w}_s - \hat{a}_s\}.$$

We show that $W \searrow A \subseteq H$ implies $A \sqcup D = E$. For $A \vee D = E$, given $e \in E$, choose an s so that $e \leq \hat{b}_s \vee \hat{w}_s$. Then

$$e \leq [(\hat{b}_s \vee \hat{w}_s) \wedge \hat{a}_s] \ \vee \ [(\hat{b}_s \vee \hat{w}_s) - \hat{a}_s].$$

The first term in this supremum is in A; the second is in D, since, by the definition of the enumeration of B, $\hat{b}_s \leq \hat{a}_s$. To see that $A \cap D = H$, assume that there are $u \notin H$ and s such that $u \leq \hat{w}_s - \hat{a}_s$, and $u \in A$. Then $u \leq \hat{a}_t$ for some t. Since $t > s$, $u \in W \searrow A$; hence $u \in H$. We can conclude that $A \sqcup D = E$. Since our assumption was that $A \not\sqsubset E$, in fact $W \searrow A \not\subseteq H$. □

5. Proof of the Main Theorem

To prove that $\mathrm{Th}(\mathbb{N}, +, \times)$ can be interpreted in $\mathrm{Th}(\mathcal{I}(\mathcal{B}))$ for any c.e. effectively dense $\mathcal{B}$, we will give an interpretation of $\mathrm{Th}(\Sigma^0_7, \subseteq)$ in $\mathrm{Th}(\mathcal{I}(\mathcal{B}))$. (More generally, if $\mathcal{B}$ is a Σ^0_k-Boolean algebra, we will give an interpretation of $(\Sigma^0_{k+6}, \subseteq)$ in $\mathcal{I}(\mathcal{B})$.) By a result of Harrington (see [10]), $\mathrm{Th}(\mathbb{N}, +, \times)$ can be interpreted in $\mathrm{Th}(\mathcal{E})$. This result relativizes to $\emptyset^{(6)}$, so the same interpretation works for $\mathrm{Th}(\mathbb{N}, +, \times)$ and $\mathrm{Th}(\Sigma^0_7, \subseteq)$. Altogether we will have the following interpretations:

$$\mathrm{Th}(\mathbb{N}, +, \times) \Rightarrow \mathrm{Th}(\Sigma^0_7, \subseteq) \Rightarrow \mathrm{Th}(\mathcal{I}(\mathcal{B})).$$

In analogy to Harrington and Nies [10], we consider the Boolean algebra of splittings of a non-principal ideal A. Let

$$\mathcal{B}(A) = \{X : X \sqsubset A\}.$$

Since $\mathcal{I}(\mathcal{B})$ is a distributive lattice, $(\mathcal{B}(A), \cap, \vee, \mathrm{Cpl}_A, H, A)$ is a Σ^0_3-Boolean algebra. We consider ideals of $\mathcal{B}(A)$. To avoid confusion, we will write "IDEAL" if we mean such a level 2 ideal. Boldface letters $\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{M}$ will denote IDEALS of some Boolean algebra $\mathcal{B}(A)$. For certain A, E such that $A \subseteq E$, we will view

$$\mathcal{R}_E(A) = \{X \sqsubset E : X \subseteq A\}$$

as the IDEAL of negligible splittings of A. They play the same role as the recursive sets in [10]. The reason why we cannot take the principal ideals instead is that, in a construction of ideals, a principal ideal is done after a finite number of steps, while building a splitting of E may be distributed over the whole construction. In this way, we allow for a more flexible notion of negligibility. This idea only works because A is locally principal in E, else all the complexity of A could be concentrated on $e \cap A$ for some $e \in E$, and splittings of E would be no more useful than splittings of e.

In (ii) and (iii) below we make key definitions: we introduce classes of complex IDEALS with the goal of being able to quantify over them in the first-order language of $\mathcal{I}(\mathcal{B})$.

Definition 5.1. Suppose that $A \subseteq E$.

(i) The index set for an IDEAL $\boldsymbol{I} \lhd \mathcal{B}(A)$ is the set $\{e : V_e \in \boldsymbol{I}\}$.
(ii) Let $k \geq 3$. An IDEAL $\boldsymbol{I} \lhd \mathcal{B}(A)$ is k–acceptable$_E$ if $\boldsymbol{I}$ has a Σ^0_k index set and $\mathcal{R}_E(A) \subseteq \boldsymbol{I}$.
(iii) A class $\mathcal{C}$ of IDEALS of $\mathcal{B}(A)$ containing $\mathcal{R}_E(A)$ is uniformly definable if, for some formula $\psi(X; P_1, \dots, P_n, A, E)$ in the language of lattices with $0, 1$,
$$\mathcal{C} = \{\{X \sqsubset A : \mathcal{I}(\mathcal{B}) \models \psi(X; P_1, \dots, P_n, A, E)\} : \ P_1, \dots, P_n \in \mathcal{I}(\mathcal{B})\}.$$

Recall that A is locally principal in E if $A \subseteq E$ and $\forall e \in E[e \cap A \sqsubset 1]$.

Definability Lemma 5.2. *Suppose that A is locally principal in E. Then for each odd $N \geq 3$, the class of N-acceptable$_E$ IDEALS of $\mathcal{B}(A)$ is uniformly definable.*

The Definability Lemma is trivial if $A \sqsubset E$, since in that case $\mathcal{R}_E(A) = \mathcal{B}(A)$. In our applications it will be the case that $A \not\sqsubset E$, and hence $E \not\sqsubset 1$.

We first show that the Definability Lemma is sufficient to give the desired interpretation. By Fact 4.3, choose any nonprincipal E, and let $A \not\sqsubset E$ be an ideal locally principal in E obtained by the Subideal Lemma 4.6. Recall that a Boolean algebra is *atomic* if each nonzero element bounds an atom.

Lemma 5.3. *There is a 5-acceptable $\boldsymbol{I} \lhd \mathcal{B}(A)$ such that $\mathcal{B}(A)/_{\boldsymbol{I}}$ is an infinite atomic Boolean algebra.*

Proof. By iterating applications of Theorem 4.9, we obtain a uniformly c.e. sequence (A_k) such that $A_k \sqsubset A$, $A_k \not\sqsubset A$ and $A_i \cap A_j = H$ for $i \neq j$. Next, let $\boldsymbol{I}_k$ be the pre-image in $\mathcal{B}(A_k)$ of a maximal IDEAL of $\mathcal{B}(A_k)/_{\mathcal{R}_E(A_k)}$. Clearly we can ensure that $\boldsymbol{I}_k$ is recursive in $\emptyset^{(3)}$, uniformly in k. Now let $\boldsymbol{I} = \{X \sqsubset A : \forall k \ X \cap A_k \in \boldsymbol{I}_k\}$. $\boldsymbol{I}$ is 5-acceptable, and, for each k, $\boldsymbol{I} \cap \mathcal{B}(A_k) = \boldsymbol{I}_k$. Therefore $A_k/_{\boldsymbol{I}}$ is an atom in $\mathcal{B}(A)/_{\boldsymbol{I}}$. Clearly, $A_k/_{\boldsymbol{I}} \neq A_r/_{\boldsymbol{I}}$ for $k \neq r$. Finally, if $X \notin \boldsymbol{I}$, then $X \cap A_k \notin \boldsymbol{I}_k$ for some k, so $A_k - X \in \boldsymbol{I}_k$ and therefore $A_k/_{\boldsymbol{I}} \leq X/_{\boldsymbol{I}}$. □

The IDEAL $\boldsymbol{J}$ generated by $\{A_k\}$ and $\boldsymbol{I}$ is 5-acceptable and is the pre-image in $\mathcal{B}(A)$ of the IDEAL generated by the atoms of $\mathcal{B}(A)/_{\boldsymbol{I}}$. Let $\mathcal{L}$ be the lattice of 7-acceptable IDEALS of $\mathcal{B}(A)$. Whenever we are in the above situation, namely 5-acceptable IDEALS $\boldsymbol{I}, \boldsymbol{J}$ are given such that $\mathcal{B}(A)/_{\boldsymbol{I}}$ is infinite atomic and $\boldsymbol{J}$ is the pre-image in $\mathcal{B}(A)$ of the IDEAL generated by atoms in $\mathcal{B}(A)/_{\boldsymbol{I}}$, then the lattice $[\boldsymbol{I}, \boldsymbol{J}]_{\mathcal{L}}$ of 7-acceptable IDEALS between $\boldsymbol{I}$ and $\boldsymbol{J}$ is isomorphic to $(\Sigma^0_7, \subseteq)$: Notice that "$V_e/_{\boldsymbol{I}}$ is an atom of $\mathcal{B}(A)/_{\boldsymbol{I}}$" is a Π^0_6-property of indices, so there is a function $f \leq_T \emptyset^{(6)}$ such that $(V_{f(n)}/_{\boldsymbol{I}})_{n \in \mathbb{N}}$ is an enumeration of the atoms of $\mathcal{B}(A)/_{\boldsymbol{I}}$ without repetition. This implies that
$$S \mapsto \{n \in \mathbb{N} : f(n) \in S\}$$
is an isomorphism between $[\boldsymbol{I}, \boldsymbol{J}]_{\mathcal{L}}$ and $(\Sigma^0_7, \subseteq)$.

Finally, to obtain the desired interpretation, we have to express in a first-order way that parameters code the above situation. First we can express that A is locally principal in E and $A \not\subseteq E$. By the Definability Lemma 5.2, we can quantify over 5-acceptable and 7-acceptable IDEALS of $\mathcal{B}(A)$. We express that our parameters code 5-acceptable $\boldsymbol{I} \subseteq \boldsymbol{J}$ such that

(a) $\mathcal{B}(A)/_{\boldsymbol{I}}$ is atomic,
(b) $\boldsymbol{J}/_{\boldsymbol{I}}$ is non-principal in $\mathcal{B}(A)/_{\boldsymbol{I}}$, and
(c) $\boldsymbol{J}$ is the pre-image of the IDEAL generated by the atoms of $\mathcal{B}(A)/_{\boldsymbol{I}}$, i.e.
  - $\boldsymbol{J}/_{\boldsymbol{I}}$ contains all the atoms, and
  - for each 7-acceptable $\boldsymbol{K} \supseteq \boldsymbol{I}$, if also $\boldsymbol{K}/_{\boldsymbol{I}}$ contains all the atoms, then $\boldsymbol{J} \subseteq \boldsymbol{K}$.

The interpretation is given by

$$(\Sigma^0_7, \subseteq) \models \varphi \;\Leftrightarrow\; \mathcal{I}(\mathcal{B}) \models \exists A, E(A \not\subseteq E \;\&\; A \text{ locally principal in } E$$
$$\& \;\exists \boldsymbol{I}, \boldsymbol{J} \text{ satisfying (a)-(c) } [\boldsymbol{I}, \boldsymbol{J}]_{\mathcal{L}} \models \varphi).$$

"$[\boldsymbol{I}, \boldsymbol{J}]_{\mathcal{L}} \models \varphi$" can be expressed by a formula involving the parameters for the 5-acceptable $\boldsymbol{I}, \boldsymbol{J}$ and quantifying over parameters coding 7-acceptable IDEALS.

6. Proof of the Definability Lemma

We proceed by induction over odd $N \geq 3$. First we prove that, whenever A is locally principal in E, then the class of 3-acceptable$_E$ IDEALS of $\mathcal{B}(A)$ is uniformly definable. Then we show that, if A is locally principal in E, there are ideals $C \subseteq G \subseteq A$ such that C is locally principal in G, and there is a 3-acceptable$_G$ IDEAL $\boldsymbol{M} \lhd \mathcal{B}(C)$ such that any $N+2$-acceptable$_E$ IDEAL $\boldsymbol{I} \lhd \mathcal{B}(A)$ can be defined from a N-acceptable$_G$ IDEAL $\boldsymbol{J} \lhd \mathcal{B}(C)$ and $\boldsymbol{M}$, i.e. the formula to define $\boldsymbol{I}$ contains statements of the form "$X \in \boldsymbol{M}$" and "$X \in \boldsymbol{J}$". Since C is locally principal in G, these statements can then be eliminated by the inductive hypothesis. On the other hand, the first-order formula obtained in this way only allows us to define $N+2$-acceptable$_E$ IDEALS .

We need some more preliminaries. Several times we will show that ideals are splittings using the following fact.

Fact 6.1. *Suppose $B \subseteq E$ is an ideal such that $\forall m \; B \cap [0, e_m] = [0, b_m]$, where b_m is obtained effectively in m. Then $B \sqsubset E$.*

Proof. Let C be the ideal generated by $H \cup \{e_m - b_m\}_{m \in \mathbb{N}}$. Then $B \sqcup C = E$. □

Next we introduce some more notation for splittings of ideals. It is our goal to define a u.c.e. sequence (X_e) of ideals in $\mathcal{B}(A)$ such that each element of $\mathcal{B}(A)$ is represented. Also, we will define a uniformly c.e. sequence $(\overline{X}_e)$ of ideals such that $X_e \cap \overline{X}_e = H$ and $X_e \vee \overline{X}_e$ is principal or equals A.

Definition 6.2. Given $e = \langle i, j \rangle$, consider the pair $U_i = V_i \cap A, U_j = V_j \cap A$ with the canonical enumerations. We define enumerations

$$(\widetilde{X}_{e,s})_{s \in \mathbb{N}}, \qquad (\widetilde{\widetilde{X}}_{e,s})_{s \in \mathbb{N}}$$

by enumerating at certain active stages. We declare $s = 0$ active and define $\widetilde{X}_{e,0} = \widetilde{\widetilde{X}}_{e,0} = H_0$. At stage $s > 0$, if $t < s$ is the last active stage, see if $U_{i,t} \cap U_{j,t} \subseteq H_s \;\&\; \hat{a}_t \in U_{i,s} \vee U_{j,s}$. If so, declare s active and let $\widetilde{X}_{e,s} = U_{i,s}$, $\widetilde{\widetilde{X}}_{e,s} = U_{j,s}$. Else $\widetilde{X}_{e,s} = \widetilde{X}_{e,s-1}$ and $\widetilde{\widetilde{X}}_{e,s} = \widetilde{\widetilde{X}}_{e,s-1}$.

We are now ready to begin our proof by induction.

Lemma 6.3. *Suppose that $A \subseteq E$ is locally principal in E. Then the class of 3-acceptable$_E$* IDEALS *of $\mathcal{B}(A)$ is uniformly definable.*

Proof. In this proof, if $C, D \subseteq A$, we use the notation $C \leq_E D$ for $\exists S \in \mathcal{R}_E(A)\ C \leq D \vee S$. Let

$$\psi_3(X; C, A, E) = \exists R(R \subseteq A \ \&\ R \sqsubset E \ \&\ \ C \cap X \subseteq R). \tag{12}$$

Thus, $\psi_3(X)$ expresses that $C \cap X$ is negligible. Clearly, each subset of $\mathcal{B}(A)$ defined via ψ_3 is an IDEAL which is 3-acceptable$_E$. Conversely, we now show that each 3-acceptable$_E$ IDEAL $\boldsymbol{I}$ equals $\{X : \psi_3(X; C_{\boldsymbol{I}}, A, E)\}$ for some $C_{\boldsymbol{I}} \subseteq A$. We use some ideas from [19], where a similar fact is proven for splittings of sets in $\mathcal{E}$. First we find a good representation of $\boldsymbol{I}$.

Fact 6.4. *If $g \leq_T \emptyset''$, there is a uniformly c.e. sequence (Y_i) of elements of $\mathcal{B}(A)$ such that $\forall i \exists S \in \mathcal{R}_E(A)\ X_{g(i)} \vee S = Y_i \vee S$.*

Proof. We make use of the hypothesis that A is locally principal in E in an essential way. Since "$p = g(i)$" is Σ^0_3, we can choose a u.c.e. sequence (V_k) of initial segments of $\mathbb{N}$ such that

$$X_p = X_{g(i)} \Leftrightarrow \exists n\ V_{\langle i,p,n\rangle} = \mathbb{N}.$$

Now, for each i, in a uniform way define an ideal $Y_i = X_{h(i)}$ by determining $Y_i \cap [0, e_m]$ for each m. At stage s, for each $r = \langle i, p, n\rangle < s$, if

$$\max \bigcup_{\langle i,p',n'\rangle < r} V_{\langle i,p',n'\rangle,s} < m \leq \max(V_{r,s})$$

and $V_{r,s} \neq V_{r,s-1}$, then put $b = \sup(X_{p,s}) \wedge e_m$ into Y_i (e_m was defined in 4.4). We say that b is *enumerated via* V_r.

We will now determine $S \sqsubset E, S \subseteq A$, such that $Y_i \vee S = X_{g(i)} \vee S$. Let q be the least number of the form $\langle i, p, n\rangle$ such that $V_q = \mathbb{N}$. Choose a $\tilde{t}$ such that $V_{q',\tilde{t}} = V_{q'}$ for each $q' < q$ of the form $\langle i, p', n'\rangle$. Let k be the maximum of all elements of such $V_{q'}$. Since A is locally principal in E, $A \cap \hat{e}_k = a$ for some $a \in A$. Let S be the ideal generated by a and all the elements enumerated via V_r, where $r > q$ and r is of the form $\langle i, p', n'\rangle$. Then $S \subseteq A$. Since $V_q = \mathbb{N}$, given m we can determine b_m as needed in Fact 6.1 in order to show that $S \sqsubset E$: let $b_m = e_m \wedge \sup(S_t)$, where $t \geq \tilde{t}$ is least such that $\max(V_{q,t}) \geq m$.

We verify that $Y_i \vee S = X_{g(i)} \vee S$ or, equivalently, $Y_i \cap \mathrm{Cpl}_A(S) = X_{g(i)} \cap \mathrm{Cpl}_A(S)$. Suppose $u \in \mathrm{Cpl}_A(S)$ and $u \notin H$; then $u \wedge a \in H$ and hence $u \wedge \hat{e}_k \in H$, so $u \preceq e_{k+1} \vee \ldots \vee e_h$ for some $h > k$. Let s be least such that $u \in X_{p,s}$ and $h \leq \max(V_{q,s})$; then $u \in Y_{i,s}$. Conversely, if $u \in Y_i$, then, because $u \in \mathrm{Cpl}_A(S)$ and $u \wedge \hat{e}_k \in H$, for some $v_1, \ldots, v_l$ enumerated via V_q, $u \preceq v_1 \vee \ldots \vee v_l$. Since $q = \langle i, p, n\rangle$, we obtain $u \in X_p = X_{g(i)}$. ∎

We have to find $C = C_{\boldsymbol{I}}$ such that

$$\boldsymbol{I} = \{X : X \cap C_{\boldsymbol{I}} \leq_E H\}. \tag{13}$$

Since I is Σ^0_3 and complementation in $\mathcal{B}(A)$ is recursive in $\emptyset''$, there is $g \leq_T \emptyset''$ such that $\{X_{g(i)}\}$ is the filter of complements in $\mathcal{B}(A)$ of elements of $\boldsymbol{I}$. Applying the preceding fact to the sequence $(X_{g(i)})$, we obtain a uniformly c.e. sequence (Y_i). Let $Z_n = \bigcap_{m \le n} Y_m$. We satisfy the requirements

$$P_n : X_n \cap Z_n \not\subseteq H \Rightarrow X_n \cap C \not\subseteq H,$$

while ensuring that $\forall n\ C \le_E Z_n$. This suffices to show (13):

For the inclusion "$\subseteq$", if $X_j \in \boldsymbol{I}$, then $Z_n \le_E \mathrm{Cpl}_A(X_j)$ for some n, so $Z_n \cap X_j \le_E H$ and hence $C \cap X_j \le_E H$.

For the converse inclusion, if $X_j \notin \boldsymbol{I}$, then for each $R \in \mathcal{R}_E(A)$, $X_j \cap \mathrm{Cpl}_A(R) \notin \boldsymbol{I}$, so $\mathrm{Cpl}_A(X_j) \vee R$ is not in the filter. Thus, if $X_n = X_j \cap \mathrm{Cpl}_A(R)$, $Z_n \not\le_E \mathrm{Cpl}_A(X_n)$, so $X_n \cap Z_n \not\subseteq H$. By the requirements P_n, $X_n \cap C \not\subseteq H$, i.e. $X_j \cap \mathrm{Cpl}_A(R) \cap C \not\subseteq H$. This implies that $X_j \cap C \not\subseteq R$.

Construction of C. At stage $s > 0$, for each $n < s$ do the following. Let $t < s$ be the greatest stage such that $t = 0$ or P_n acted at stage t. If $X_{n,t} \cap C_t \not\subseteq H_s$, then declare P_n *satisfied* at stage s. Else P_n *acts* by enumerating $\sup(X_{n,s} \cap Z_{n,s})$ into C.

Claim 1. $\forall n \exists u \in A[C \subseteq Z_n \vee u]$. *Thus* $C \le_E Z_n$.

Proof. For each $m < n$, if P_m is permanently satisfied from some stage on, let u_m be the last element P_m enumerates into C. Let u be the supremum of all such u_m. Then $u \in A$ and $C \subseteq Z_n \vee u$, because a requirement P_m, $m \ge n$, only enumerates elements which are in Z_n and those $P_m, m < n$, which will never be permanently satisfied only enumerate elements which later go into H.

Claim 2. For each n, the requirement P_n is met.

Proof. Suppose $z \in X_n \cap Z_n$ and $z \notin H$. Let s be a stage such that $z \in X_{n,s} \cap Z_{n,s}$. If P_n never acts at a stage $t \ge s$, then it is satisfied from s on and therefore met. If it acts at $t \ge s$ by enumerating y into C, then $z \le y$. So $y \notin H$, and P_n is satisfied from t on. ☐

Next we carry out the inductive step in the proof of the Definability Lemma. Suppose that $N \ge 3$ and A is locally principal in E. As described above, we want to define a given $N+2$-acceptable$_E$ IDEAL $\boldsymbol{I} \lhd \mathcal{B}(A)$ from an N-acceptable$_G$ $\boldsymbol{J} \lhd \mathcal{B}(C)$, for some fixed $C \subseteq G \subseteq A$ to be constructed. The main idea is to use a "tracing" procedure. Through a *Trace Lemma*, we construct G such that each $X \sqsubset A$ is assigned a trace $p_X \in X$ in a $\emptyset''$-way, with the property that $X \not\sqsubseteq E \;\Rightarrow\; p_X \cap G \not\sqsubseteq E$. In the following, let p, q be traces. The construction ensures that $G \cap p \cap q = H$ for $p \ne q$. We apply the Subideal Lemma 4.6 to each ideal $G \cap p$ inside $[0,p]$, and obtain an ideal $C_p \subseteq p$ which is locally principal in $G \cap p$. Let $C \subseteq G$ be the ideal generated by all the ideals C_p. Since $G \cap p \cap q = H$ for $p \ne q$, we see that $C_p = C \cap p$ for each p, and C is locally principal in G. If $X \not\sqsubseteq E$, then $p_X \cap G \not\sqsubseteq 1$; hence by the Subideal Lemma 4.6 $C \cap p_X \not\sqsubseteq G \cap p_X$. Thus the trace for X carries some non-negligible information about X into $\mathcal{B}(C)$, where we view the splits in $\mathcal{R}_G(C)$ as negligible. We let $\boldsymbol{J} \lhd \mathcal{B}(C)$ be the IDEAL generated by $\mathcal{R}_G(C)$ and all intersections of the form $X \cap u_n \cap C$, where $u_0 = 0$ and, for $n > 0$, u_n is the supremum of traces for finitely many ideals $Y - u_{n-1} \sqsubset A$, and one of the ideals is X; moreover X is guessed to be in $\boldsymbol{I}$ according to the n-th guess of an approximation procedure to the index set of $\boldsymbol{I}$. Similarly to [10], such a procedure consists of an array of finite Σ^0_N sets $Z_n \subseteq \{0, \dots, n\}$, where an index for Z_n can be obtained using a $\emptyset^{(N-1)}$ oracle, and $e \in Z_n$ means that X_e is guessed to be in

$\boldsymbol{I}$. This implies that $\boldsymbol{J}$ is N-acceptable$_G$. The auxiliary 3-acceptable$_G$ IDEAL $\boldsymbol{M}$ is generated by $\mathcal{R}_G(C)$ and all the intersections $u_n \cap C$. We will verify that

$$X \in \boldsymbol{I} \Leftrightarrow \exists R \in \boldsymbol{M} \forall S \in \boldsymbol{M}[S \cap R = H \Rightarrow X \cap S \in \boldsymbol{J}],$$

so $\boldsymbol{I}$ can be defined in a first-order way from $\boldsymbol{J}$ and $\boldsymbol{M}$. See Table 1 near the end of the paper for a summary of all the objects introduced.

We now focus on a main recursion theoretic ingredient, which corresponds to Lemma 3 in the Appendix to [10].

Trace Lemma 6.5. *Suppose that A is locally principal in E. Then there is a $G \subseteq A$ such that*

(i) *$\forall i\ \exists p_i \in X_i[X_i \not\sqsubseteq E \ \Rightarrow\ p_i \cap G \not\sqsubseteq E]$, and the function $i \mapsto p_i$ is recursive in $\emptyset''$, and*

(ii) *there are a recursive subset $\mathcal{P} \subseteq \mathcal{D}$ and a u.c.e. sequence of ideals $(G_p)_{p \in \mathcal{P}}$ such that*

   (a) *$\forall p \in \mathcal{P}\ G_p \subseteq p$, and G is the ideal generated by $\bigcup_{p \in \mathcal{P}} G_p$,*

   (b) *$\forall i\ p_i \in \mathcal{P} \cup \{0\}$, and*

   (c) *if $p, q \in \mathcal{P}$ are distinct, then $G_p \cap G_q = H$. In particular $G_p = G \cap p$.*

If $X = X_i$, we will also use the notation p_X instead of p_i. Thus, p_X actually depends on an index for X.

Proof. We first concentrate on (i). The additional properties (ii) will follow easily. Since $p_i \in X_i \subseteq E$, for $p_i \cap G \not\sqsubseteq E$ in (i) it is sufficient to ensure that $p_i \cap G \not\sqsubseteq 1$. For each i, a strategy R_i attempts to define a trace $p_X \in X$ (where $X = X_i$). The main task for the R_i strategy is to find a value for p_X which is not in H. If $X \not\sqsubseteq E$, such a value will eventually appear in X. So if R_i abandons the old value once it has been enumerated into H, it will eventually make a good choice. However, if $X_i \subseteq H$, R_i may go through an infinite list of possible values, and all of them are useless because they turn out to be in H.

To make $p_X \cap G$ non-principal, similarly to the proof of Fact 4.3, the R_i-strategy will, at each stage s where p_X is defined, enumerate $p_X - \hat{g}_s \wedge F_s^*$ into G (where $\hat{g}_s = \sup(G_s)$). To conform with the convention that elements enumerated at a stage t be in $\mathcal{D}_t$, the element $(p_X - \hat{g}_s) \wedge F_s^*$ will in fact be enumerated only at stage $h(s)$ (see Definition 4.8), and the construction resumes at stage $h(s)+1$. Thus we proceed in stages $s_0 < s_1 < \dots$, where $s_0 = 0$ and $s_{n+1} = h(s_n) + 1$.

When considering the interaction of different strategies, the problem is to avoid that $p_X \in G$ due to the action of other strategies (in which case $p_X \cap G = p_X$ would be principal). To ensure $p_X \notin G$ is difficult, because the R_i-strategy may go for a long while with some useless p_X, which eventually appears in H. Or, a good choice for p_X may appear very late in X_i, but at an early stage in some other X_j. In these cases, R_i has to prevent other strategies from putting a possible future choice for p_X into G completely.

1. The G-enumeration of a higher priority strategy R_e is not too hard to deal with, since the R_i strategy may assume (a) its parameter p_e has already reached its limit, which means a finitary restriction, unless (b) all values of p_e are in H anyway, in which case R_e cannot enumerate an element $\notin H$ into G. In case (a), since $X_i \sqsubset A$, if $X_i \not\sqsubseteq E$, then also $X_i - p_e \not\sqsubseteq E$, so R_i can trust that a good choice for p_i which is disjoint from p_e exists. The condition (15) in the construction takes into account the higher priority strategies.

2. We must force a strategy for R_j of priority lower than R_i not to assign its parameter p_j until p_j is in $\sup(X_i) \vee \sup(\overline{X}_i)$. Then, by the properties of F_s^* discussed after Definition 4.8, R_j will never put a possible choice of the R_i-strategy for p_X into G completely, but rather certain elements $\prec p_X$.

The strategy R_j now needs a guess whether $X_i \vee \overline{X}_i = A$. If so, as explained above, only those elements are candidates for p_{X_j} which have already appeared in $X_i \vee \overline{X}_i$. In that case, R_j also needs to know whether the limit of R_i's p parameter exists or not (i.e. whether Case (a) or (b) above with R_i in place of R_e holds). If $X_i \vee \overline{X}_i \neq A$, then the R_i-strategy will be ignored by R_j.

To equip the strategies R_i with such a guess, we organize them on the tree $3^{<\omega}$. We use the usual terminology for tree constructions (see [21]). In particular, α, β denote strings on the tree and $\alpha <_L \beta$ means that α is to the left of β, where the tree is pictured as growing downward. Moreover, $\alpha \leq \beta \Leftrightarrow \alpha <_L \beta \ \vee \ \alpha \subseteq \beta$. At stage s_n of the construction, we define a string $\delta_n \in 3^{<\omega}$ of length n. We let the *true path* f be the leftmost path visited infinitely often. If $\alpha \in 3^{<\omega}$ and $|\alpha| = i$, then α is a version of the R_i-strategy. If $\alpha = f|i$, then $f(i) = 2$ means that $X_i \vee \overline{X}_i \neq A$. Otherwise $f(i) = 1$ signifies that the limit of the parameter p_α exists, $f(i) = 0$ that it fails to exist. The guessing procedure is as follows. At stage s_n, if we have determined $\alpha = \delta_n|i$, let $\delta_n(i) = 2$ if no new active stage for X_i in the sense of Definition 6.2 appeared. Else let $\delta_n(i) = 0$ if at this stage we cancelled p_α (since we discovered that it is in H), and $\delta_n(i) = 1$ otherwise. Note that we do *not* guess at whether $p_\alpha \in G$. Now a strategy on the true path can assume that, for $\beta \subseteq \alpha$, $\lim_s p_{\beta,s}$ has been reached if it exists at all, else α is initialized another time. For the rest of the discussion, assume that $\alpha 1 \subseteq f$.

An extra problem is now that we have a new type of lower priority strategy: the strategies $\beta >_L \alpha 1$, whose G-enumeration also may cause $p_\alpha \in G$. We will initialize such a β whenever $\alpha 1 \subseteq \delta_n$. This means that, at stages $> s_n$, β can only enumerate elements which are disjoint from $\hat{e}_{s_n}$. By Fact 6.1, all the β-strategies $>_L \alpha 1$ together will then only contribute a negligible set $S \in \mathcal{R}_E(A)$ to G. But if $X_i \not\subseteq E$, then also $X_i \cap \mathrm{Cpl}_A(S) \not\subseteq E$. Thus a good choice for the strategy α will appear. In the construction we implement this idea in (16).

Of course, strategies $\gamma <_L \alpha$ will do the same to α, so a further problem is that a good choice p for p_α may be forbidden because $p \leq \hat{e}_t$, where $t = s_m$ is the last stage such that α is initialized (namely $\delta_m <_L \alpha$). Here the hypothesis that A is locally principal in E saves us: for some $a \in A$, a (viewed as an ideal) $= \hat{e}_t \cap A$, so if $X_i \cap \mathrm{Cpl}_A(S) \not\subseteq E$, then also $X_i \cap \mathrm{Cpl}_A(S \vee a) \not\subseteq E$. Hence some good choice $p \in X_i \cap \mathrm{Cpl}_A(S)$ must appear such that $p \wedge a = 0$, and hence $p \wedge \hat{e}_t = 0$. See (17) in the construction below.

Since $A \subseteq E$, we can assume that $\forall s\ \hat{a}_s \leq \hat{e}_s$.

Construction of $G, (\delta_n)_{n\in\mathbb{N}}, \mathcal{P}$ *and* $(p_{\beta,s})_{\beta\in 3^{<\omega}, s\in\mathbb{N}}$.

Stage $s_0 = 0$. Initialize all the strategies β by declaring $p_{\beta,0}$ to be undefined. Let δ_0 be the empty string.

Stage $s = s_n = h(s_{n-1}) + 1$, $n > 0$.

Go through the substages i, for $i < n$, thereby defining $\delta_n \restriction (i+1)$. After that, initialize all the strategies β such that $\delta_n <_L \beta$ by declaring p_{β,s_n} to be undefined.

Substage i. Suppose that $\alpha = \delta_n \restriction i$ has been defined. Let $l = |\{m < n : \alpha \subseteq \delta_m\}|$. If l is not an active stage in the Definition 6.2 of X_i, let $\delta_n(i) = 2$ and terminate substage i. Now suppose l is active. Let $\widetilde{p}_{\alpha,s}$ be the largest possible new candidate for $p_{\alpha,s}$, namely

$$\tilde{p}_{\alpha,s} = \hat{x}_{i,s} \wedge \bigwedge_{\gamma 0\subseteq\alpha \,\vee\, \gamma 1\subseteq\alpha} [\sup(X_{|\gamma|,s}) \vee \sup(\overline{X}_{|\gamma|,s})] \tag{14}$$

$$- g(\alpha,s) \tag{15}$$

$$- b(\alpha,s) \tag{16}$$

$$- \hat{e}_t, \tag{17}$$

where $t < s$ is greatest such that α was initialized at stage t, $g(\alpha,s) = \sup\{p_{\beta,s} : \beta 1 \subseteq \alpha\}$, and $b(\alpha,s)$ is the supremum of all the elements which have been enumerated into G at stages $< s$ by a strategy $\beta >_L \alpha 1$.

1. If $p_{\alpha,s}$ is undefined, let $p_{\alpha,s} = \tilde{p}_{\alpha,s}$. Let $\delta_n(i) = 1$ and end substage i.
2. If $p_{\alpha,s}$ is defined but $p_{\alpha,s} \in H_s$, declare $p_{\alpha,s}$ to be undefined. Let $\delta_n(i) = 0$ and end substage i.
3. Otherwise enumerate $p_{\alpha,s} - \hat{g}_s \wedge F_s^*$ into $G_{h(s)}$ and let $\delta_n(i) = 1$.

Verification. Before verifying (i) and (ii) we need some preliminary facts. Let $f \in 3^\omega$ be the true path. *In the following let* $\alpha = f \upharpoonright i$. Then $f(i) = 2 \Leftrightarrow X_i \vee \overline{X}_i \neq A$, and if $f(i) < 2$ then $f(i) = 1 \Leftrightarrow \lim_s p_{\alpha,s}$ exists. Suppose that $i \geq 0$ and that $f(i) \neq 2$. Let

$$k = \text{the least number } > 0 \text{ such that } \forall n \geq k[\alpha \leq \delta_n]. \tag{18}$$

Since α is not initialized at a stage $\geq s_n$,

$$\lim_s p_{\alpha,s} \text{ exists} \Leftrightarrow \text{ for some stage } s = s_n \geq s_k,\ p_{\alpha,s} \notin H.$$

First we verify that if $X_i \not\sqsubset E$, then the strategy α eventually makes a choice $p_\alpha \notin H$. Subsequently we show that actually $p_\alpha \notin G$.

Lemma 6.6. *If $X_i \vee \overline{X}_i = A$ and $X_i \not\sqsubset E$, then, for some $n \geq k$, p_{α,s_n} is newly defined at stage s_n and $p_{\alpha,s_n} \notin H$. In particular, $p_{\alpha,s}$ reaches its limit p_α at s_n and $f(i) = 1$.*

Proof. Since $X_i \vee \overline{X}_i = A$, there are infinitely many active stages for X_i in Definition 6.2, and hence infinitely many stages s where $p_{\alpha,s}$ is defined. It is sufficient to show that $p_{\alpha,s}$ is redefined only finitely often. Suppose not. Let t be greatest such that α was initialized at stage t (in fact $t = s_{k-1}$). Since A is locally principal in E, $A \cap \hat{e}_t = a$ for some $a \in A$. Let R be the ideal generated by $a, g(\alpha, s_k)$ and all the elements which are enumerated into G by strategies β, $\alpha 1 <_L \beta$, and let R_{s_n} be the approximation by the end of stage s_n. We claim that $R \in \mathcal{R}_E(A)$. First, $R \subseteq A$ because $\hat{e}_t \cap A$ is principal and $g(\alpha, s_k) \in A$. Moreover $R \sqsubset E$ by Fact 6.1: given m, to obtain b_m compute an $n > m, k$ such that $\alpha 0 \subseteq \delta_n$ or $\alpha 1 \subseteq \delta_n$. Since strategies β such that $\alpha 1 <_L \beta$ are initialized at stage s_n, they can at stages $\geq s_n$ only enumerate elements x such that $x \wedge \hat{e}_{s_n} = 0$. So $R \sqsubset E$ by Fact 6.1, where $b_m = e_m \wedge \sup(R_{s_n})$.

Since $R \in \mathcal{R}_E(A)$, $X_i \subseteq (X_i \cap \mathrm{Cpl}_A(R)) \vee R$ and $\mathcal{R}_E(A)$ is an IDEAL of $\mathcal{B}(A)$, $X_i \not\sqsubset E$ implies that $X_i \cap \mathrm{Cpl}_A(R) \not\subseteq H$. So we can choose $u \notin H$, $u \in X_i \cap \mathrm{Cpl}_A(R)$ and $s^* \geq s_k$ such that $u \in X_{i,s^*}$ and also $u \in X_{e,s^*} \vee \overline{X}_{e,s^*}$ for all $e < i$ such that $f(e) \neq 2$.

Suppose that $p_{\alpha,s}$ is redefined at a stage $s = s_n \geq s^*$. Let $g = g(\alpha, s), b = b(\alpha, s)$, and recall that $a = A \cap \hat{e}_t$. Since $u \in \mathrm{Cpl}_A(R) \subseteq A$ and $b \vee g \vee a \in R$, it follows that $u - b - g - \hat{e}_t = u - b - g - a \notin H$. Moreover, $u \leq \hat{x}_{e,s}$ and

$$u \leq \bigwedge_{\gamma 0 \subseteq \alpha \,\vee\, \gamma 1 \subseteq \alpha} [\sup(X_{|\gamma|,s}) \vee \sup(\overline{X}_{|\gamma|,s})].$$

Since at Stage s of the construction we define $p_{\alpha,s} \geq u - b - g - \hat{e}_t$, we can conclude that $p_{\alpha,s} \notin H$. $\square$

We now prove that actually $p_\alpha \notin G$. To do so, we analyze how it can happen that some $d_\sigma \notin H$ such that $n = |\sigma|$ is an $\alpha 0$- or $\alpha 1$-stage is enumerated into G. This is the threat for the R_i strategy α trying to define at s_n a successful value for $p_{\alpha,s} \geq d_\sigma$. We make two observations.

1. If $|\sigma| = s_n \geq s_k$, then $d_\sigma \notin G$ if d_σ is not already in G by stage s_n, up to H. Here "up to H" means that $d_\sigma - \hat{g}_{s_n} \in H$. (This is in general *not* a decidable statement.)
2. If d_σ is in G at stage s_n up to H, and also $d_\sigma \leq \tilde{p}_{\alpha,s_n}$, then there is a shorter string $\nu = \sigma \restriction s'$ which has the similar property $d_\nu \leq \tilde{p}_{\alpha,s'}$, where s' is an $\alpha 0$- or $\alpha 1$-stage, $s_k \leq s' < s_n$.

The argument why the R_i-strategy α will define a value $p_\alpha \notin G$ provided that $X_i \not\subseteq E$ is as follows. By the preceding lemma, α will eventually define a value $p_\alpha \notin H$. So some d_σ, $|\sigma| = s_n$, is not in H, and $d_\sigma \leq \tilde{p}_{\alpha,s_n} = p_{\alpha,s_n} =: p$. If $d_\sigma \in G$, then, by the first observation, $d_\sigma - \hat{g}_s \in H$. So a *minimal* $\nu \subseteq \sigma$ obtained using the second observation must have the property that $d_\nu - \hat{g}_{s'} \notin H$ (where $s' = |\nu|$). Then, by the first observation once again, $d_\nu \notin G$. Now $d_\nu - p \leq \tilde{p}_{\alpha,s'} - p \in G$, because by the choice of k this difference consists solely of elements enumerated into G by strategies to the right of $\alpha 1$ at stages between s' and s. Therefore $p \notin G$.

We now state and verify the two observations in detail.

Claim 6.7. *Suppose $n \geq k$ (k was defined in* (18)*) and $|\sigma| = s_n$. If $d_\sigma - \hat{g}_{s_n} \notin H$, then in fact $d_\sigma \notin G$.*

Proof. We prove by induction on $m \geq n$ that $d_\sigma - \hat{g}_{s_m} \notin H$. For $m = n$ this is our assumption. Now suppose inductively that $d_\sigma - \hat{g}_{s_m} \notin H$ for an $m \geq n$. Let $s = s_m$. We can pick $\nu \supseteq \sigma$, $|\nu| = s$, such that $d_\nu \notin H$ and $d_\nu \wedge \hat{g}_s = 0$. Then $d_\nu - F_s^* \notin H$. Because $d_\nu \wedge \sup(G_s) = 0$ and all elements enumerated into G at stage $h(s)$ are $\leq F_s^*$, we can conclude that $d_\nu - \hat{g}_{s_{m+1}} \notin H$. $\square$

Claim 6.8. *Suppose $s = s_n \geq s_k$ is an $\alpha 0$- or $\alpha 1$-stage. Suppose $|\sigma| = s$, $d_\sigma \notin H$ and $d_\sigma \leq \tilde{p}_{\alpha,s}$. Then $d_\sigma - \hat{g}_s \in H \Rightarrow \exists$ $\alpha 0$-or $\alpha 1$-stage $s', s_k \leq s' < s$, such that $d_\nu \leq \tilde{p}_{\alpha,s'}$, where $\nu = \sigma \restriction s'$.*

Proof. If $d_\sigma - \hat{g}_s \in H$, an element $\geq d_\sigma$ is enumerated at a stage $s^* < s_n$ into G by a strategy β: Else all elements enumerated into G at stages $< s_n$ are disjoint from d_σ, so $d_\sigma - \hat{g}_s \in H$ implies that $d_\sigma \in H$. Thus $d_\sigma \leq p' := p_{\beta,s'}$, where $s' < s^*$ is the stage where $p_{\beta,s'}$ was defined. Let $\nu = \sigma \restriction s'$.

First we prove that $s' \geq s_k$. By definition, $k \neq 0$ (see (18)). If $s' < s_k$ then $s' \leq t = s_{k-1}$, the stage where α is initialized for the last time. So $p_{\beta,s'} \leq \hat{e}_t$ by our assumption that $\hat{a}_{s'} \leq \hat{e}_{s'}$, since $p_{\beta,s'} \leq \hat{x}_{|\beta|,s'} \leq \hat{a}_{s'} \leq \hat{e}_{s'} \leq \hat{e}_t$. But $d_\sigma \wedge \hat{e}_t = 0$ because of clause (17). Thus $s' \geq s_k$.

If $\alpha = \beta$, then s' is an $\alpha 1$-stage and $d_\sigma \leq \tilde{p}_{\alpha,s'}$, so $d_\nu = d_\sigma \restriction s' \leq \tilde{p}_{\alpha,s'}$. Otherwise, we show that $\alpha 0 \subseteq \beta$ or $\alpha 1 \subseteq \beta$ (so that s' is an $\alpha 0$- or an $\alpha 1$-stage).

- If $\alpha 1 <_L \beta$, then because $\tilde{p}_{\alpha,s} \wedge b(\alpha, s) = 0$, d_σ is disjoint from any element enumerated by β into G at a stage $< s$.
- If $\beta 0 \subset \alpha$ then, since α is on the true path, $p' \in H$, so $d_\sigma \not\leq p'$ because $d_\sigma \notin H$.
- If $\beta 1 \subset \alpha$ then p_β has reached its limit already at stage s_{k-1} (otherwise α is initialized at a stage $\geq s_k$), so p_β cannot be newly defined at $s' \geq s_k$. One can argue in a similar way if $\beta 2 \subseteq \alpha$.
- $\beta <_L \alpha$ is not possible because $s' \geq s_k$.

We conclude that $\alpha 0 \subseteq \beta$ or $\alpha 1 \subseteq \beta$. Then $p' \leq \sup(X_{i,s'}) \vee \sup(\overline{X}_{i,s'})$ (recall that $i = |\alpha|$). So if $\nu = \sigma \upharpoonright s'$, then $d_\nu \notin H$ implies that $d_\nu \in X_{i,s'}$ or $d_\nu \in \overline{X}_{i,s'}$ by (7). But then, since $d_\sigma \in X_{i,s'}$ and $d_\sigma \notin H$, we have $d_\nu \in X_{i,s'}$. Moreover, $d_\nu \leq p'$. Since $\alpha \subseteq \beta$, $b(\alpha, s') \leq b(\beta, s')$, it follows that $g(\alpha, s') \leq g(\beta, s')$, and the largest stage $< s'$ where β is initialized is $\geq s_{k-1}$. So $d_\nu \leq p' \leq \tilde{p}_{\alpha,s'}$. □

We now complete the argument sketched above.

Lemma 6.9. *Suppose at a stage $s = s_n \geq s_k$, $p = p_{\alpha,s}$ is newly defined. If $p \notin H$, then $p \notin G$.*

Proof. First note that if $s_k \leq s' \leq s$ and s' is an $\alpha 0$- or $\alpha 1$-stage, then

$$\widetilde{p}_{\alpha,s'} - p \in G. \tag{19}$$

For $g(\alpha, s) = g(\alpha, s')$ by the definition of k, and $b(\alpha, s) - b(\alpha, s')$ consists of the elements put into G by strategies $>_L \alpha 1$ at stages t, $s' \leq t < s$.

Suppose that $p \notin H$ but $p \in G$. Choose a string σ of length s_n such that $d_\sigma \leq p(= \tilde{p}_{\alpha,s})$ and $d_\sigma \notin H$. If $d_\sigma \notin G$ then we are done. Otherwise $d_\sigma - \hat{g}_s \in H$ by Claim 6.7. By Claim 6.8, choose the minimal $\alpha 0$- or $\alpha 1$-stage s', $s_k \leq s' < s$, such that $\nu = \sigma \upharpoonright s' \leq \tilde{p}_{\alpha,s'}$. Then $d_\nu - \hat{g}_{s'} \notin H$, else we could find a yet smaller stage s'. So $d_\nu \notin G$ by Claim 6.7. But $d_\nu - p \in G$, since $d_\nu - p \leq \tilde{p}_{\alpha,s'} - p \in G_s$ by (19). Because G is an ideal, we can conclude that $d_\nu \wedge p \notin G$. □

By the preceding two lemmas, if $X_i \not\subset E$ then $p = \lim_t p_{\alpha,t} \notin G$, where $\alpha = f \upharpoonright i$. Thus the G-enumeration of α at a stage s where the limit has been reached will ensure that $p \cap G \not\sqsubset 1$. Indeed, suppose $H \vee t = p \cap G$. Then $p - t \notin H$ (else $p \in G$), so $F_s^* \wedge p - t \notin H$ and $F_s^* \wedge p - t$ will be enumerated into G, a contradiction.

We are now ready to verify (i) and (ii).

(i). We give a procedure to obtain the trace $p_X = p_i$ from $X = X_i \sqsubset A$, using $\emptyset''$ as an oracle. First determine $\alpha = f \upharpoonright i$. If $f(i) \neq 1$, let $p_i = 0$. Else determine $n(i)$ minimal such that, where $s = s_{n(i)}$, $\forall t \geq s\ p_{\alpha,t} = p_{\alpha,s}$, and define

$$p_i = p_{\alpha,s} - \hat{g}_s \tag{20}$$

(note that s is an $\alpha 1$-stage). Then $p_i \in X_i$. Since $p_{\alpha,s} \cap G \not\sqsubset 1$, also $p_i \cap G \not\sqsubset 1$. Because $p_i \in E$, this implies that actually $p_i \cap G \not\sqsubset E$. Clearly $i \mapsto p_i$ is recursive in $\emptyset''$.

(ii). Let

$$\mathcal{P} = \{p_{\alpha,s} - \hat{g}_s :\ p_{\alpha,s} \text{ is newly defined at stage } s\},$$

and for each $p = p_{\alpha,s} - \hat{g}_s \in \mathcal{P}$, let G_p be the ideal generated by H and the elements the strategy α enumerates into G at stages $\geq s$ but before (if ever) p_α is declared undefined. Clearly (ii.a) is satisfied, and $\forall i\ p_i \in \mathcal{P} \cup \{0\}$. For (ii.c), suppose $p = p_{\alpha,s} - \hat{g}_s, q = p_{\beta,t} - \hat{g}_t$, where $p_{\alpha,s}, p_{\beta,t}$ are newly defined at the stages

$s = s_n, t = s_m$, respectively. Then $\alpha 1 \subseteq \delta_n$ and $\beta 1 \subseteq \delta_m$. If $n = m$, then $\alpha \neq \beta$. Say $\alpha 1 \subseteq \beta$. Then $p_{\alpha,s} \leq g(\beta, s)$ and therefore $p \wedge q = 0$.

Now suppose that $n < m$ (the case that $m > n$ is handled analogously). We can suppose that $p_{\alpha,s}$ is not declared undefined at a stage t', $s < t' \leq t$, else by the definition of G_p we would have $G_p = \sup\ G_{p,t'} \subseteq \hat{g}_t$, and so $q \cap G_p = H$. We distinguish four cases.

- $\alpha \subseteq \beta$. Then $\alpha \neq \beta$, else we would not have to redefine p_β at stage t. Moreover $\alpha 0 \not\subseteq \beta$, else p_α is declared undefined at stage t. If $\alpha 1 \subseteq \beta$, then $p_{\alpha,s} = p_{\alpha,t} \leq g(\beta, t)$ and therefore $p \wedge q = 0$. If $\alpha 2 \subseteq \beta$ then β is initialized at stage s; thus $p \leq \hat{e}_s \leq \hat{e}_t$ and $q \wedge \hat{e}_t = 0$.
- $\alpha <_L \beta$. Then at stage s, β is initialized. Now argue as before.
- $\beta <_L \alpha$. Then at stage t, α is initialized, contrary to our assumption.
- $\beta \subset \alpha$. If $\beta 2 \subseteq \alpha$, then α is initialized at stage t. If $\beta 1 \subseteq \alpha$, then, since we are assuming that p_β is newly defined at t, there is a stage $s_{m'}$, $n < m' < m$, such that $\beta 0 \subseteq \delta_{m'}$. Then α is initialized at $s_{m'}$. Finally, suppose that $\beta 0 \subseteq \alpha$. Since $q \wedge \hat{g}_t = 0$ (recall that $t = s_m$), a G-enumeration of α which can contribute elements $\notin H$ to $G_p \cap G_q$ must be at a stage $h(s_{m'})$, $m' \geq m$, when p_α has not yet been declared undefined. Therefore α and hence β have not yet been initialized. Since $\beta 0 \subseteq \delta_{m'}$, the last value p_β had before stage $s_{m'}$ is in H. Either this value equals q, or the earlier value q had to be abandoned because it was found to be in H. In any case $q \in H$.

This shows that $G_p \cap G_q = H$. By (ii.a) we immediately obtain that $G \cap p = G_p$. This completes the proof of the Trace Lemma. $\square$

Lemma 6.10. *Suppose $N \geq 3$, A is locally principal in E, and G has been obtained by the Trace Lemma. Then there is a C, which is locally principal in G, and a 3-acceptable$_G$ IDEAL $\boldsymbol{M} \triangleleft \mathcal{B}(C)$ with the following property. For each $N{+}2$-acceptable$_E$ IDEAL $\boldsymbol{I} \triangleleft \mathcal{B}(A)$, there is an N-acceptable$_G$ IDEAL $\boldsymbol{J} \triangleleft \mathcal{B}(C)$ such that, for each $X \sqsubset A$,*

$$X \in \boldsymbol{I} \Leftrightarrow \exists R \in \boldsymbol{M} \forall S \in \boldsymbol{M}[S \cap R = H \Rightarrow X \cap S \in \boldsymbol{J}]. \tag{21}$$

Proof. For each $p \in \mathcal{P}$, apply the Subideal Lemma 4.6 to the ideal G_p given by (ii) of the Trace Lemma. We obtain a u.c.e. sequence $(C_p)_{p \in \mathcal{P}}$ of ideals. Let $C \subseteq G$ be the (c.e.) ideal generated by all the ideals C_p.

First we show that C is locally principal in G. If $g \in G$, then by (ii.a) of the Trace Lemma, for some $q_1, \dots, q_m \in \mathcal{P}$ and $g_j \in G_{q_j}$ we have $g = \sup_{1 \leq j \leq m} g_j$. Since $\mathcal{I}(\mathcal{B})$ is distributive, $C \cap g = \sup_{1 \leq j \leq m}(C \cap g_j)$. By (ii.c) of the Trace Lemma, $C \cap q_j = C_{q_j}$. So $C \cap g_j = C_{q_j} \cap g_j \sqsubset 1$ for each j. Thus $C \cap g \sqsubset 1$.

Next we define a $\emptyset''$-sequence (u_n) of elements of A. Let $\mathcal{B}_{\leq e}$ be a finite set of indices for the subalgebra of $\mathcal{B}(A)$ generated by $\{X_0, \dots, X_{e+1}\}$ ($\mathcal{B}_{\leq e}$ can be obtained from i using a $\emptyset''$-oracle). Let $u_0 = 0$ and

$$u_{n+1} = \bigvee \{p_{Z - \hat{u}_n} : Z \in \mathcal{B}_{\leq n}\} \tag{22}$$

(recall that we write p_X instead of p_i if $X = X_i$), where $\hat{u}_n = u_0 \vee \ldots \vee u_n$. Clearly $u_i \cap u_j = 0$ for $i \neq j$, and (u_n) is a $\emptyset''$ sequence by (ii) of the Trace Lemma. Let $S_n = u_n \cap C$ and let

$$\boldsymbol{M} \triangleleft \mathcal{B}(C) \ = \ \text{the IDEAL generated by } \mathcal{R}_G(C) \cup \{S_n : n \in \mathbb{N}\}. \tag{23}$$

Then $\boldsymbol{M}$ is 3-acceptable$_G$.

We make use of a relativizable lemma from [10, Appendix, Lemma 4].

Table 1.

Object	Defined in	Function	Compl.
$\boldsymbol{I} \lhd \mathcal{B}(A)$	given		$N+2$-acc.$_E$
$G \subseteq A$	Trace Lemma to A, E	G, C create a frame-	
$C \subseteq G$	Lemma 4.6 to all G_p	work for reduction	
$\boldsymbol{J} \lhd \mathcal{B}(C)$	(24)	Reduce $\boldsymbol{I}$ to $\boldsymbol{J}$	N-acc.$_G$
$\boldsymbol{M} \lhd \mathcal{B}(C)$	(23)	Codes true path f	3-acc.$_G$

Lemma 6.11 ([10]). *If P is a Σ^0_3–set, then there is a uniformly c.e. sequence (Z_i), $Z_i \subseteq \{0, \dots, i\}$, such that $e \in P \Rightarrow$ a.e. i $[e \in Z_i]$ and $\exists^\infty i$ $[Z_i \subseteq P]$.* □

Relativized to $\emptyset^{(N-1)}$, the lemma states that, if P is Σ^0_{N+2}, then there is a $\emptyset^{(N)}$-sequence of Σ^0_N sets (Z_i) with the properties above. Applying this to $P = \mathrm{Ind}(\boldsymbol{I}) := \{e : X_i \in \boldsymbol{I}\}$, we obtain

- $X_i \in \boldsymbol{I} \Rightarrow$ a.e. i $[e \in Z_i]$,
- $\exists^\infty i$ $[Z_i \subseteq \mathrm{Ind}(\boldsymbol{I})]$.

Let $\boldsymbol{J} \lhd \mathcal{B}(C)$ be the ideal generated by $\mathcal{R}_G(C)$ and

$$\{X_e \cap S_n : e \in Z_n\}. \tag{24}$$

Clearly $\boldsymbol{J}$ is N-acceptable$_G$. In Table 1 we summarize our definitions of ideals and ideals .

We now verify (21).

"$\Rightarrow$" Suppose that $X_e \in \boldsymbol{I}$. Choose $\widetilde{n}$ such that $\forall n > \widetilde{n}$ $(e \in Z_n)$ and let $R = S_0 \vee \ldots \vee S_{\widetilde{n}}$. If $S \in \boldsymbol{M}$ and $S \cap R = H$, then, for some $n > \widetilde{n}$ and $W \in \mathcal{R}_G(C)$, $S \subseteq S_n \vee \ldots \vee S_j \vee W$. But $X_e \cap S_n \in \boldsymbol{J}$ for all $n > \widetilde{n}$ and $X_e \cap W \in \mathcal{R}_G(C) \subseteq \boldsymbol{J}$. Therefore $X \cap S \in \boldsymbol{J}$.

"$\Leftarrow$" Now suppose that $X_e \notin \boldsymbol{I}$. Given $R \in \boldsymbol{M}$, choose k such that $R \subseteq S_0 \vee \ldots \vee S_k \vee W$ for some $W \in \mathcal{R}_G(C)$. Choose $n > k$ such that $Z_n \subseteq \mathrm{Ind}(\boldsymbol{I})$ and also $n > e+1$. We show that the witness S_n is a counterexample to the right hand side in (21), i.e. $X_e \cap S_n \notin \boldsymbol{J}$.

Let $V = X_e \cap \mathrm{Cpl}_A(\bigvee_{j \in Z_n} X_j) - \hat{u}_{n-1}$. Then $V \not\subseteq E$: else, since $\hat{u}_{n-1} \in \mathcal{R}_E(A) \subseteq \boldsymbol{I}$ and $\bigvee_{j \in Z_n} X_j \in \boldsymbol{I}$, we could infer that $X_e \in \boldsymbol{I}$. Therefore, by (i) of the Trace Lemma, $G \cap p_V \not\subseteq E$. But by (ii.c) of the Trace Lemma, $p_V \cap G = G_{p_V}$ and hence $p_V \cap C = C_{p_V}$. Thus $G_{p_V} \not\subseteq 1$ and, by the Subideal Lemma 4.6, $C_{p_V} \not\subseteq G_{p_V}$. We can conclude that $p_V \cap C \not\subseteq G$.

Also $Z = X_e \cap \mathrm{Cpl}_A(\bigvee_{j \in Z_n} X_j) \in \mathcal{B}_{\leq n-1}$, so V occurs in the disjunction (22) where u_n (and hence S_n) is defined. Hence we see that $p_V \cap C \subseteq S_n \cap V$ and $S_n \cap (X_e - \bigvee_{j \in Z_n} X_j) \not\subseteq G$. But this implies that S_n is a counterexample as desired: if $X_e \cap S_n \in \boldsymbol{J}$, then by the fact that the ideals (S_k) have pairwise meet H we have $X \cap S_n \subseteq W \vee \bigvee_{j \in Z_n} X_j$ for some $W \in \mathcal{R}_G(C)$. This means that

$$S_n \cap (X_e - \bigvee_{j \in Z_n} X_j) \subseteq W \sqsubset G$$

whence $S_n \cap (X_e - \bigvee_{j \in Z_n} X_j) \sqsubset G$, a contradiction. □

We conclude the inductive step by determining a formula ψ_{N+2} which shows uniform definability of the class of $N+2$-acceptable$_E$ IDEALS . By Lemma 6.3, $\boldsymbol{M}$ is definable via the formula ψ_3 introduced in (12). By the inductive hypothesis, $\boldsymbol{J}$ is definable via a fixed formula ψ_N. Let $\overline{P}$ be the list of parameters (including G and C) needed to define $\boldsymbol{M}, \boldsymbol{J}$, and let $\widetilde{\psi}_{N+2}(X; \overline{P}, A, E)$ be the formula derived from the right hand side in (21), but with $\boldsymbol{M}, \boldsymbol{J}$ replaced by their definitions via ψ_3, ψ_N, and the constant symbol 0 in our language of lattices replacing H. Then $\boldsymbol{I}$ is definable via $\widetilde{\psi}_{N+2}$.

On the other hand, if a subset $\boldsymbol{I}$ of $\mathcal{B}(A)$ is defined via $\widetilde{\psi}_{N+2}(X; \overline{P}, A, E)$, where $\overline{P}$ is an arbitrary list of parameters of the appropriate length, then $\boldsymbol{I}$ is an IDEAL of $\mathcal{B}(A)$ and has a Σ^0_{N+2} index set, since by the inductive hypothesis any set $\boldsymbol{J}$ defined by ψ_N is N-acceptable$_G$. However, it may not be the case that $\mathcal{R}_E(A) \subseteq \boldsymbol{I}$. To enforce this, let

$$\psi_{N+2}(X; \overline{P}, A, E) \Leftrightarrow \exists U \in \mathcal{R}_E(A)\ \widetilde{\psi}_{N+2}(X \cap \mathrm{Cpl}_A(U); \overline{P}, A, E).$$

Then the class of $N+2$-acceptable$_E$ IDEALS of $\mathcal{B}(A)$ is uniformly definable via ψ_{N+2}. $\square$

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