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SPECTRA OF BP-LINEAR RELATIONS, v_n-SERIES, AND BP COHOMOLOGY OF EILENBERG-MAC LANE SPACES

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Abstract. On Brown-Peterson cohomology groups of a space, we introduce a natural inherent topology, BP topology, which is always complete Hausdorff for any space. We then construct a spectra map which calculates infinite BP-linear sums convergent with respect to the BP topology, and a spectrum which describes infinite sum BP-linear relations in BP cohomology. The mod p cohomology of this spectrum is a cyclic module over the Steenrod algebra with relations generated by products of exactly two Milnor primitives. We show a close relationship between BP-linear relations in BP cohomology and the action of the Milnor primitives on mod p cohomology. We prove main relations in the BP cohomology of Eilenberg–Mac Lane spaces. These are infinite sum BP-linear relations convergent with respect to the BP topology. Using BP fundamental classes, we define v_n-series which are v_n-analogues of the p-series. Finally, we show that the above main relations come from the v_n-series.

1. Introduction and summary of results

Generally speaking, the Brown-Peterson (BP) cohomology theory has more interesting structures and yet simpler descriptions than the BP homology theory.

In generalized cohomology theories we need to deal with infinite sums of elements, and to discuss convergences, we need a topology on these cohomology groups. For infinite dimensional CW complexes, the skeletal filtration topology is commonly used. However, this topology often fails to be complete Hausdorff, and there can exist elements of infinite filtration, so-called phantom elements, which vanish when restricted to any finite skeleton. For a generalized cohomology theory satisfying the Milnor's additivity axiom [M3], being Hausdorff and being complete Hausdorff are equivalent. When a topology on cohomology groups is not complete Hausdorff, convergence problems are tricky.

However, the good news is that for the BP cohomology theory the situation is very good. We show that there is a very natural and inherent topology on BP cohomology groups of any spectrum which is not necessarily a CW-spectrum. This topology is derived from the global structure of BP theory, namely the existence of

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the BP-tower. This is the following sequence of BP-module spectra and BP-module spectra maps:

$$\text{(1-1)} \qquad \mathrm{BP} \to \cdots \to \mathrm{BP}\langle n+1\rangle \to \mathrm{BP}\langle n\rangle \to \cdots \to \mathrm{BP}\langle 0\rangle = H\mathbb{Z}_{(p)} \to H\mathbb{Z}_p,$$

where $\pi_*(\mathrm{BP}\langle n\rangle) = \mathbb{Z}_{(p)}[v_1, v_2, \dots, v_n]$ with $|v_i| = 2(p^i-1)$ for $1 \le i \le n$, and $H\mathbb{Z}_{(p)}$ and $H\mathbb{Z}_p$ are Eilenberg–Mac Lane spectra for the ring $\mathbb{Z}_{(p)}$ of localized integers at p and the ring $\mathbb{Z}_p$ of mod p integers.

For any spectrum X and $k \in \mathbb{Z}$, we consider a decreasing filtration

$$\text{(1-2)} \quad \mathrm{BP}^k(X) \supset F^{-1} \supset F^0 \supset F^1 \supset \cdots \supset F^n \supset \cdots ,$$
$$\text{where } F^n = \mathrm{Ker}\,\{{\rho_{\langle n\rangle}}_* : \mathrm{BP}^k(X) \to \mathrm{BP}\langle n\rangle^k(X)\}.$$

The BP topology is defined to be the topology defined by this filtration. Although the BP topology on $\mathrm{BP}^*(X)$ can be defined for any spectrum X, it is inherently an unstable notion and it only exhibits nice properties when X is a space.

Proposition 1-1 [Proposition 2-1]. *For any $k \in \mathbb{Z}$, the* BP*-topology on* $\mathrm{BP}^k(X)$ *is always complete Hausdorff for any space X.*

Thus any Cauchy sequence in $\mathrm{BP}^k(X)$ always converges to a unique limit when X is a space. Although many results in this paper are valid for any spectrum X, we must assume that X is a space (not necessarily a CW complex) when we need convergences.

We are interested in infinite BP-linear sums in $\mathrm{BP}^*(X)$ of the following form:

$$\text{(1-3)} \qquad \sum_{n=0}^{\infty} v_n b_n = pb_0 + v_1 b_1 + \cdots + v_n b_n + \cdots ,$$

where $b_n \in \mathrm{BP}^{*+2p^n-1}(X)$ for $n \ge 0$.

Corollary 1-2 [Corollary 2-3]. *Any infinite sum of the form* (1-3) *always converges to a unique element in* $\mathrm{BP}^{k+1}(X)$ *with respect to the* BP *topology for any space X and for any collection of elements $b_n \in \mathrm{BP}^{k+2p^n-1}(X)$ for $n \ge 0$.*

We want to calculate the limit of the infinite sum (1-3). We consider the following composition κ of BP-module maps:

$$\text{(1-4)} \qquad \kappa : \prod_{i=0}^{\infty} \Sigma^{2(p^i-1)}\mathrm{BP} \xleftarrow{\ \cong\ } \bigvee_{i=0}^{\infty} \Sigma^{2(p^i-1)}\mathrm{BP} \xrightarrow{\ \vee v_i\ } \bigvee_{i=0}^{\infty} \mathrm{BP} \xrightarrow{\text{folding}} \mathrm{BP}.$$

Here, Σ^k denotes the k-fold formal suspension of spectra, and the first arrow is a homotopy equivalence [Lemma 2-4]. For any spectrum X, the induced map

$$\kappa_* : \prod_{i=0}^{\infty} \mathrm{BP}^{k+2p^i-1}(X) \longrightarrow \mathrm{BP}^{k+1}(X)$$

provides us with a well-defined element $\kappa_*(b_0, b_1, \dots, b_n, \dots)$ for any sequence of elements $\vec{b} = (b_0, b_1, \dots, b_n, \dots)$ of appropriate degrees. Note that $\kappa_*(\vec{b})$ is a well-defined element in BP cohomology of any spectrum X. However, when X is a space, we can identify this element as the limit of (1-3).

Theorem 1-3 [Theorem 2-7]. *For any elements $b_i \in \mathrm{BP}^{k+2p^i-1}(X)$ for $i \ge 0$, where X is a space, the limit of* (1-3) *is given by κ_*. Namely,*

$$\text{(1-5)} \qquad \kappa_*(b_0, b_1, \dots, b_i, \dots) = \sum_{i\ge 0} v_i b_i \qquad \textit{in} \quad \mathrm{BP}^{k+1}(X),$$

where the convergence on the right hand side is with respect to the BP*-topology.*

When X is an infinite dimensional CW complex, we can also consider the skeletal filtration topology on $\mathrm{BP}^*(X)$. Although these two topologies have very different origin, it turns out that the BP topology is finer than the skeletal filtration topology [Proposition 2-8]. So any convergent sequence with respect to the BP topology also converges with respect to the skeletal filtration topology, but not vice versa.

Let L be the cofibre spectrum of the spectra map κ. We have the following cofibre sequence:

$$\Sigma^{-1}L \xrightarrow{\prod q_i} \prod_{i=0}^{\infty} \Sigma^{2(p^i-1)}\mathrm{BP} \xrightarrow{\kappa} \mathrm{BP} \xrightarrow{\theta} L. \tag{1-6}$$

Since κ is a BP-module map, the spectrum L is a BP-module spectrum. This spectrum L turns out to have very interesting properties.

Theorem 1-4 [Theorem 3-1]. *Let X be a space and let $k \in \mathbb{Z}$.*

(I) *For any element $z \in L^k(X)$, let $b_i = q_{i*}(z) \in \mathrm{BP}^{k+2p^i-1}(X)$ for $i \geq 0$. Then*

$$pb_0 + v_1 b_1 + \cdots + v_n b_n + \cdots = 0 \quad in \quad \mathrm{BP}^{k+1}(X),$$

where the convergence is with respect to the BP*-topology.*

(II) *There exists a spectra map $\eta : L \to H\mathbb{Z}_p$ such that the following diagram strictly commutes (not up to an unknown nonzero constant in $\mathbb{Z}_p$) for any $i \geq 0$:*

$$\begin{array}{ccccc} L^k(X) & \xrightarrow{q_{i*}} & \mathrm{BP}^{k+2p^i-1}(X) & \xrightarrow{\theta_*} & L^{k+2p^i-1}(X) \\ \downarrow{\scriptstyle \eta_*} & & \downarrow{\scriptstyle \rho_*} & & \downarrow{\scriptstyle \eta_*} \\ H\mathbb{Z}_p^k(X) & \xrightarrow{Q_i} & H\mathbb{Z}_p^{k+2p^i-1}(X) & = & H\mathbb{Z}_p^{k+2p^i-1}(X), \end{array} \tag{1-7}$$

where Q_i is the i-th Milnor primitive in the Steenrod algebra, and $\rho : BP \to H\mathbb{Z}_p$ is the Thom map.

(III) *The mod p cohomology of the spectrum L is the following cyclic module over the Steenrod algebra $\mathcal{A}(p)$ generated by η:*

$$H\mathbb{Z}_p^*(L) \cong \Big[\mathcal{A}(p)\Big/\sum_{i,j\geq 0} \mathcal{A}(p)Q_iQ_j\Big] \cdot \eta. \tag{1-8}$$

Part (I) says that each element in $L^*(X)$ can be thought of as an infinite sum BP-linear relation in BP cohomology. Thus we call L the *spectrum of* BP*-linear relations*. Part (II) shows that in L-theory, there exist Milnor operations $\widehat{q}_i$, namely $\theta \circ q_i$ for $i \geq 0$. But any product among them is zero, since $q_j \circ \theta = 0$ for any j in the cofibre sequence (1-6). Part (III) is a reflection of this fact, and it shows that there are no other relations in the mod p cohomology of L.

We can also consider finite BP-linear sums in $\mathrm{BP}\langle n\rangle^*(X)$ of the form

$$pb_0 + v_1 b_1 + \cdots + v_n b_n. \tag{1-9}$$

The spectra map which calculates this summation is the following composition of BP-module maps:

$$\begin{aligned} \kappa_{\langle n\rangle} : \prod_{i=0}^{n} \Sigma^{2(p^i-1)}\mathrm{BP}\langle n\rangle &\xleftarrow{\cong} \bigvee_{i=0}^{n} \Sigma^{2(p^i-1)}\mathrm{BP}\langle n\rangle \\ &\xrightarrow{\vee v_i} \bigvee_{i=0}^{n} \mathrm{BP}\langle n\rangle \xrightarrow{\text{folding}} \mathrm{BP}\langle n\rangle. \end{aligned} \tag{1-10}$$

Let $L\langle n\rangle$ be the cofibre of $\kappa_{\langle n\rangle}$. Then $L\langle n\rangle$ is a BP-module spectrum with properties corresponding to a finite version of Theorem 1-4 [Theorem 4-1]. These BP-module spectra $L\langle n\rangle$ fit into the following tower [Proposition 4-5]:

$$L \to \cdots \to L\langle n+1\rangle \xrightarrow{\eta_{\langle n\rangle}^{\langle n+1\rangle}} L\langle n\rangle \to \cdots \to L\langle 1\rangle \to L\langle 0\rangle = H\mathbb{Z}_p. \tag{1-11}$$

This tower can be used to construct infinite sum BP-linear relations in $\mathrm{BP}^*(X)$ from finite sum BP-linear relations in $\mathrm{BP}\langle n\rangle^*(X)$ [Theorem 4-6].

The BP cohomology theory and mod p cohomology theory are closely related by the Thom map $\rho_*: \mathrm{BP}^*(X) \to H\mathbb{Z}_p^*(X)$. Through ρ_*, BP-linear relations in $\mathrm{BP}^*(X)$ translate into a certain property of the action of Milnor primitives on the mod p cohomology of X.

Proposition 1-5 [Proposition 5-1]. *Let X be a space and let k be a positive integer. Suppose we have*

$$pb_0 + v_1 b_1 + \cdots + v_n b_n + \cdots = 0 \quad in \quad \mathrm{BP}^{k+1}(X)$$

for some elements $b_n \in \mathrm{BP}^{k+2p^n-1}(X)$ for $n \geq 0$. Then there exists an element $x \in H\mathbb{Z}_p^k(X)$ such that in mod p cohomology we have

$$\rho_*(b_n) = Q_n(x) \qquad for\ all \quad n \geq 0. \tag{1-12}$$

Proposition 1-5 for finite sum BP-linear relations was first proved in [Y1] when X is a finite complex using a geometric method of manifolds with singularities. Our general result is proved in the stable category of spectra.

We remark that we can easily write down the corresponding BP homology version of the above proposition.

We consider a converse problem of constructing (infinite sum) BP-linear relations in BP cohomology from information on the action of the Milnor primitives on mod p cohomology.

Theorem 1-6 [Theorem 5-6]. *Let X be a space and let k, n be non-negative integers such that $k \leq 2(p^{n-1} + \cdots + p + 1)$. Then for any element*

$$x \in \mathrm{Im}\,\{\rho_*^{\langle n-1\rangle}: \mathrm{BP}\langle n-1\rangle^k(X) \longrightarrow H\mathbb{Z}_p^k(X)\}, \tag{1-13}$$

there exist elements $b_{n+j} \in \mathrm{BP}^{k+2p^{n+j}-1}(X)$ for $j \geq 0$ such that in $\mathrm{BP}^{k+1}(X)$,

$$\begin{gathered} v_n b_n + v_{n+1} b_{n+1} + \cdots + v_{n+j} b_{n+j} + \cdots = 0, \quad and \\ \rho_*(b_{n+j}) = Q_{n+j}(x) \quad for\ all \ \ j \geq 0. \end{gathered} \tag{1-14}$$

We apply our results to study the BP cohomology of Eilenberg–Mac Lane spaces. In this introductory summary, we describe our results for the integral Eilenberg–Mac Lane spaces localized at p, $K(\mathbb{Z}_{(p)}, n+2)$ with $n \geq 1$. Let

$$\mathcal{S}_n^+ = \{(s_1, s_2, \ldots, s_n) \in \mathbb{Z}^n \mid 0 < s_1 < \cdots < s_n\} \tag{1-15}$$

be the set of strictly increasing sequences of n positive integers. In [T1], we produced nontrivial elements $b_S \in \mathrm{BP}^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ for each $S \in \mathcal{S}_n^+$ with the property

$$\rho_*(b_S) = Q_S(\tau_{n+2}) = Q_{s_1} Q_{s_2} \cdots Q_{s_n}(\tau_{n+2}) \neq 0 \tag{1-16}$$

in $H\mathbb{Z}_p^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$. Here $\tau_{n+2} \in H\mathbb{Z}_p^{n+2}\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ is the mod p fundamental class. Our main result in [T1] was that for $X = K(\mathbb{Z}_{(p)}, n+2)$,

$$\mathrm{Im}\,\{\rho_*: \mathrm{BP}^*(X) \longrightarrow H\mathbb{Z}_p^*(X)\} = \mathbb{Z}_p[\,Q_S(\tau_{n+2}) \mid S \in \mathcal{S}_n^+]. \tag{1-17}$$

Here, the right hand side is a polynomial subalgebra of $H\mathbb{Z}_p^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$. In [RWY], they proved that (quoting results in [RW1]) these elements b_S actually generate the entire BP cohomology of $K(\mathbb{Z}_{(p)}, n+2)$. As the next step, we want to study infinite sum BP-linear relations in the BP cohomology of Eilenberg–Mac Lane spaces. Let $X = K(\mathbb{Z}_{(p)}, n+2)$. Repeatedly applying the connecting homomorphisms

$$\Delta_m : \mathrm{BP}\langle m-1\rangle^r(X) \longrightarrow \mathrm{BP}\langle m\rangle^{r+2p^m-1}(X)$$

in the Sullivan exact sequences to a $\mathbb{Z}_{(p)}$-lift $\widehat{\tau}_{n+2}$ of the mod p fundamental class τ_{n+2}, we can produce an element $z \in \mathrm{BP}\langle n-1\rangle^{2(p^{n-1}+\cdots+p+1)}(X)$ such that

$$\rho_*^{\langle n-1\rangle}(z) = Q_{n-1}\cdots Q_1(\tau_{n+2}) \neq 0 \quad \text{in } H\mathbb{Z}_p^*(X). \tag{1-18}$$

For a systematic study of the mod p cohomology of Eilenberg–Mac Lane spaces in terms of the Milnor basis, see [T2]. Applying Theorem 1-6, we then obtain an infinite sum BP-linear relation in the BP cohomology of $K(\mathbb{Z}_{(p)}, n+2)$.

Theorem 1-7 [Theorem 6-3]. *Let $n \geq 1$. There exist nontrivial elements*

$$b_{n+j} \in \mathrm{BP}^{2(p^{n+j}+p^{n-1}+\cdots+p+1)}\big(K(\mathbb{Z}_{(p)}, n+2)\big) \tag{1-19}$$

for $j \geq 0$ such that in $\mathrm{BP}^{2(p^{n-1}+\cdots+p+1)+2}\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ we have

$$\begin{aligned} &v_n b_n + v_{n+1}b_{n+1} + \cdots + v_{n+j}b_{n+j} + \cdots + = 0, \quad \textit{and} \\ &\rho_*(b_{n+j}) = Q_{n+j}Q_{n-1}\cdots Q_1(\tau_{n+2}) \neq 0 \quad \textit{for} \quad j \geq 0. \end{aligned} \tag{1-20}$$

Here $b_{n+j} = b_{(1,2,\ldots,n-1,n+j)}$ in our previous notation b_S for $S \in \mathcal{S}_n^+$.

We have corresponding statements for the mod p^j Eilenberg–Mac Lane spaces.

It is well known that the BP cohomology of $K(\mathbb{Z}_p, 1)$ is given by

$$\mathrm{BP}^*\big(K(\mathbb{Z}_p, 1)\big) \cong \mathrm{BP}^*[[\,x\,]]/\big([\,p\,](x)\big), \tag{1-21}$$

where $[\,p\,](x)$ is the p-series with respect to the formal group law in BP theory. In analogy, we would like to think of the BP-linear relation in (1-20) as a v_n-analogue of p-series. But then we must ask ourselves “what is a v_n-series?” Is there such a thing at all? If it exists, where does it live? We answer these questions by explicitly presenting a v_n-series with right properties.

Let $\{\underline{\mathrm{BP}}\langle n\rangle_*\}$ be the Ω-spectrum of $\mathrm{BP}\langle n\rangle$. By Wilson's the splitting theorem [W], $\underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)+t}$ is a factor space of $\underline{\mathrm{BP}}_{2(p^n+\cdots+p+1)+t}$ when $t \leq 0$. Let

(1-22)

$$\iota^{\langle n\rangle}_{2(p^n+\cdots+p+1)+t} : \underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)+t} \longrightarrow \underline{\mathrm{BP}}_{2(p^n+\cdots+p+1)+t}, \qquad t \leq 0,$$

be the inclusion map afforded by the Splitting Theorem. The map $\iota^{\langle n\rangle}$ above can be thought of as a BP cohomology class which we call a BP *fundamental class* of the space $\underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)+t}$ for $t \leq 0$. When $t > 0$, BP fundamental classes do not exist.

Definition 1-8 [Definition 6-4]. Let $\iota^{\langle n\rangle}_{2(p^{n-1}+\cdots+p+1)+2}$ be the BP fundamental class for the space $\underline{\mathrm{BP}}\langle n\rangle_{2(p^{n-1}+\cdots+p+1)+2}$. The pull-back of this class by the v_n-multiplication map

$$v_n : \underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)} \longrightarrow \underline{\mathrm{BP}}\langle n\rangle_{2(p^{n-1}+\cdots+p+1)+2} \tag{1-23}$$

is defined to be the v_n-*series* denoted by $[v_n]$. Namely,

$$[v_n] = v_n^*\big(\iota^{\langle n\rangle}_{2(p^{n-1}+\cdots+p+1)+2}\big) \in \mathrm{BP}^{2(p^{n-1}+\cdots+p+1)+2}\big(\underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)}\big). \tag{1-24}$$

Observe that when $n=0$, we have $\underline{\mathrm{BP}}\langle 0\rangle_2 = K(\mathbb{Z}_{(p)},2) = \mathbb{C}\mathrm{P}^\infty_{(p)}$ and $[v_0] \in \mathrm{BP}^2\big(\mathbb{C}\mathrm{P}^\infty_{(p)}\big)$ is the following map:

$$[\,p\,] = [v_0] : \mathbb{C}\mathrm{P}^\infty_{(p)} \xrightarrow{\ p\ } \mathbb{C}\mathrm{P}^\infty_{(p)} = \underline{\mathrm{BP}}\langle 0\rangle_2 \xrightarrow{\ \iota_2^{\langle 0\rangle}\ } \underline{\mathrm{BP}}_2. \tag{1-25}$$

Thus, the element $[\,p\,] \in \mathrm{BP}^2(\mathbb{C}\mathrm{P}^\infty_{(p)})$ defined by (1-24) coincides with the ordinary p-series $[\,p\,](x)$ when we choose $\iota_2^{\langle 0\rangle} = x \in \mathrm{BP}^2(\mathbb{C}\mathrm{P}^\infty_{(p)})$ to be the usual BP-orientation. To study the v_n-series (1-24) in detail, we introduce the following maps for $j \ge 1$:

$$\begin{aligned} -\theta^{\langle n\rangle}_{n+j} : \underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)} &\xrightarrow{\iota^{\langle n\rangle}} \underline{\mathrm{BP}}_{2(p^n+\cdots+p+1)} \xrightarrow{v_n} \underline{\mathrm{BP}}_{2(p^{n-1}+\cdots+p+1)+2} \\ &\xrightarrow{\mathrm{proj}\langle n+j\rangle} \underline{\mathrm{BP}}\langle n+j\rangle_{2(p^{n+j}+p^{n-1}+\cdots+p+1)} \xrightarrow{\iota^{\langle n+j\rangle}} \underline{\mathrm{BP}}_{2(p^{n+j}+p^{n-1}+\cdots+p+1)}. \end{aligned} \tag{1-26}$$

Here the map $\mathrm{proj}\langle n+j\rangle$ is the projection map afforded by Wilson's Splitting Theorem which, in this case, says

$$\begin{aligned} \underline{\mathrm{BP}}_{2(p^{n-1}+\cdots+p+1)+2} &\cong \underline{\mathrm{BP}}\langle n\rangle_{2(p^{n-1}+\cdots+p+1)+2} \\ &\quad \times \prod_{j\ge 1} \underline{\mathrm{BP}}\langle n+j\rangle_{2(p^{n+j}+p^{n-1}+\cdots+p+1)}. \end{aligned} \tag{1-27}$$

Note that $-\theta^{\langle n\rangle}_{n+j} \in \mathrm{BP}^{2(p^{n+j}+p^{n-1}+\cdots+p+1)}\big(\underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)}\big)$ for $j\ge 1$. Let $\theta^{\langle n\rangle}_n$ be the BP-fundamental class $\iota^{\langle n\rangle}_{2(p^n+\cdots+p+1)}$ of $\underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)}$.

Theorem 1-9 [Proposition 6-5, Proposition 6-6]. (I) *The v_n-series*

$$[v_n] \in \mathrm{BP}^{2(p^{n-1}+\cdots+p+1)+2}\big(\underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)}\big)$$

is of the following form:

$$[v_n] = v_n\theta^{\langle n\rangle}_n + v_{n+1}\theta^{\langle n\rangle}_{n+1} + \cdots + v_{n+j}\theta^{\langle n\rangle}_{n+j} + \cdots\,, \tag{1-28}$$

where the convergence is with respect to the BP *topology.*

(II) *The pull-back of the v_n-series $[v_n]$ by the map*

$$\Delta_n\circ\cdots\circ\Delta_1 : K(\mathbb{Z}_{(p)},n+2) \xrightarrow{\Delta_1} \underline{\mathrm{BP}}\langle 1\rangle_{2p+n+1} \xrightarrow{\Delta_2} \cdots \xrightarrow{\Delta_n} \underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)}$$

to the BP *cohomology of Eilenberg–Mac Lane space $K(\mathbb{Z}_{(p)},n+2)$ is equal to zero, and in $\mathrm{BP}^{2(p^{n-1}+\cdots+p+1)+2}\big(K(\mathbb{Z}_{(p)},n+2)\big)$ the v_n-series induces the following infinite sum* BP*-linear relation*:

$$v_nb_n + v_{n+1}b_{n+1} + \cdots + v_{n+j}b_{n+j} + \cdots = 0, \tag{1-29}$$

where $b_{n+j} = (\Delta_n\circ\cdots\circ\Delta_1)^*(\theta^{\langle n\rangle}_{n+j}) \in \mathrm{BP}^{2(p^{n+j}+p^{n-1}+\cdots+p+1)}\big(K(\mathbb{Z}_{(p)},n+2)\big)$ for $j\ge 0$ has the property

$$\rho_*(b_{n+j}) = Q_{n+j}Q_{n-1}\cdots Q_1(\tau_{n+2}) \ne 0. \tag{1-30}$$

Thus, our BP-linear relation (1-20), discovered by our general theory, actually comes from the v_n-series, which in turn comes from the BP fundamental class $\iota^{\langle n\rangle}$ of the space $\underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)}$. Thus the relation (1-29) can be appropriately called the *main relation* in the BP cohomology of Eilenberg–Mac Lane space $K(\mathbb{Z}_{(p)},n+2)$.

In [RWY], they give a description of $\mathrm{BP}^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ as a certain quotient. More precisely, they prove that

$$\widehat{\mathrm{BP}}{}^*_p\big(K(\mathbb{Z}_{(p)}, n+2)\big) \cong \widehat{\mathrm{BP}}{}^*_p\big(\underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)}\big) \;/\; (v_n^*), \tag{1-31}$$

where (v_n^*) is the ideal generated by the image of the following map induced by the v_n-multiplication map (1-23):

$$v_n^* : \widehat{\mathrm{BP}}{}^*_p\big(\underline{\mathrm{BP}}\langle n\rangle_{2(p^{n-1}+\cdots+p+1)+2}\big) \to \widehat{\mathrm{BP}}{}^*_p\big(\underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)}\big). \tag{1-32}$$

Here, $\widehat{\mathrm{BP}}_p$ is the p-adic completion of the BP-spectrum. Our definition of the v_n-series (1-24) was motivated by their result. Unfortunately, [RWY] is not very explicit about the ideal (v_n^*) which gives all the relations in the BP cohomology of Eilenberg–Mac Lane spaces. Our results on main relations and the v_n-series go some distance towards clarifying the ideal (v_n^*) of relations in the BP cohomology.

The organization of this paper is as follows. In §2, we introduce the BP topology on BP cohomology groups of any spectrum X, and we show that this topology is complete Hausdorff when X is a space. In §3 and §4, we introduce spectra L and $L\langle n\rangle$ of BP-linear relations and we show that these theories have Milnor primitives, and we calculate their mod p cohomology as modules over the Steenrod algebra. In §5, we demonstrate a close connection between BP-linear relations in BP cohomology and actions of Milnor primitives in mod p cohomology. Finally, in §6 we prove our main BP-linear relations in the BP cohomology of Eilenberg–Mac Lane spaces and show that these relations come from v_n-series.

2. BP topology and a spectra map which calculates infinite sums

For a generalized cohomology theory $h^*(X)$ of an infinite dimensional CW complex X, the skeletal filtration topology is commonly used. If the generalized cohomology theory $h^*(\,\cdot\,)$ further satisfies the additivity axiom of Milnor [M3], then we have the Milnor's exact sequence:

$$0 \to \varprojlim_m{}^1 h^{*-1}(X^{(m)}) \to h^*(X) \to \varprojlim_m h^*(X^{(m)}) \to 0,$$

where $\{h^*(X^{(m)})\}_{m\in\mathbb{Z}}$ is the inverse system formed by the skeletal filtration on X. In this exact sequence, $\varprojlim{}^1_m$-term describes the set of elements of infinite filtration. If elements of infinite filtration, also called phantom maps, do not exist in $h^*(X)$, then the topology is complete Hausdorff and Cauchy sequences converge to unique limits. Since in generalized cohomology theories, we must routinely consider infinite sums of elements, a non-Hausdorff topology on cohomology groups causes a serious problem in studying relations among generators of cohomology groups.

Fortunately, on BP cohomology groups there exists a very natural and inherent topology which is always complete Hausdorff for *any* space X, which may not even be a CW complex. We call this topology BP-topology which we now define. Recall that there exists a tower of BP-module spectra $\mathrm{BP}\langle n\rangle$ for $n \geq 0$ and BP-module maps [JW1]:

$$\mathrm{BP} \xrightarrow{\rho_{\langle n+1\rangle}} \mathrm{BP}\langle n+1\rangle \xrightarrow{\rho^{\langle n+1\rangle}_{\langle n\rangle}} \mathrm{BP}\langle n\rangle \xrightarrow{\rho^{\langle n\rangle}} \mathrm{BP}\langle -1\rangle = H\mathbb{Z}_p. \tag{2-1}$$

Using the BP-module map $\rho_{\langle n\rangle} : \mathrm{BP} \to \mathrm{BP}\langle n\rangle$, we let

$$F^n\big(\mathrm{BP}^k(X)\big) = \mathrm{Ker}\left\{{\rho_{\langle n\rangle}}_* : \mathrm{BP}^k(X) \longrightarrow \mathrm{BP}\langle n\rangle^k(X)\right\} \tag{2-2}$$

for any spectrum X and for any $n \geq -1$, $k \in \mathbb{Z}$. This defines a decreasing filtration on the BP cohomology of X:

$$\mathrm{BP}^k(X) \supset F^{-1} \supset F^0 \supset F^1 \supset \cdots \supset F^n \supset \cdots \supset \textstyle\bigcap_n F^n.$$

The topology on $\mathrm{BP}^k(X)$ for $k \in \mathbb{Z}$ defined by this filtration is the BP topology. That is, the base for the neighborhood system of an element $x \in \mathrm{BP}^k(X)$ is $\{x + F^n\}_n$. Thus a sequence of elements $\{x_i\}$ in $\mathrm{BP}^k(X)$ converges to an element x if for any integer n there exists an integer N such that $\rho_{\langle n\rangle_*}(x - x_m) = 0$ for all $m \geq N$. The BP topology is inherently an unstable notion.

Proposition 2-1. *Let X be any topological space (which may not even be a CW complex). Then the* BP*-topology on* $\mathrm{BP}^*(X)$ *is complete Hausdorff. Namely, we have the following*:

$$\bigcap_{n \geq -1} F^n\big(\mathrm{BP}^*(X)\big) = \{0\}, \tag{2-3}$$

$$\widehat{\rho}_* = \varprojlim_n \rho_{\langle n\rangle_*} \colon \mathrm{BP}^k(X) \xrightarrow{\cong} \varprojlim_n \mathrm{BP}\langle n\rangle^k(X). \tag{2-4}$$

For the proof of this proposition, we need Wilson's Splitting Theorem.

Theorem 2-2 ([W])**.** (i) *Let $k \leq 2(p^n + \cdots + p + 1)$. Then we have the following homotopy equivalence among spaces of the Ω-spectra of* BP *and* $\mathrm{BP}\langle n\rangle$*'s*:

$$\underline{\mathrm{BP}}_k \cong \underline{\mathrm{BP}}\langle n\rangle_k \times \prod_{j \geq n+1} \underline{\mathrm{BP}}\langle j\rangle_{k+2(p^j-1)}. \tag{2-5}$$

If $k < 2(p^n + \cdots + p + 1)$, then this equivalence is as H-spaces.

(ii) *Let $m \geq n$ and $k \leq 2(p^n + \cdots + p + 1)$. Then we have*

$$\underline{\mathrm{BP}}\langle m\rangle_k \cong \underline{\mathrm{BP}}\langle n\rangle_k \times \prod_{j=n+1}^{m} \underline{\mathrm{BP}}\langle j\rangle_{k+2(p^j-1)}. \tag{2-6}$$

If $k < 2(p^n + \cdots + p + 1)$, then this equivalence is as H-spaces.

Proof of Proposition 2-1. Wilson's Splitting Theorem shows that for a fixed k, $\mathrm{BP}\langle n\rangle^k(X)$ is a direct summand of $\mathrm{BP}^k(X)$ for n satisfying $k \leq 2(p^n + \cdots + p + 1)$, and for such an n, the induced map $\rho_{\langle n\rangle_*} \colon \mathrm{BP}^*(X) \to \mathrm{BP}\langle n\rangle^*(X)$ is surjective. We fix one such n_0. Then for any $m \geq n_0$, we have

$$\mathrm{BP}^k(X) \cong \mathrm{BP}\langle n_0\rangle^k(X) \times \prod_{j \geq n_0+1} \mathrm{BP}\langle j\rangle^{k+2(p^j-1)}(X), \tag{$*$}$$

$$\mathrm{BP}\langle m\rangle^k(X) \cong \mathrm{BP}\langle n_0\rangle^k(X) \times \prod_{j=n_0+1}^{m} \mathrm{BP}\langle j\rangle^{k+2(p^j-1)}(X). \tag{$**$}$$

To show injectivity of $\widehat{\rho}_*$, suppose $\widehat{\rho}_*(x) = 0$ for an element $x \in \mathrm{BP}^k(X)$. By definition, this means that $\rho_{\langle m\rangle_*}(x) = 0$ in $\mathrm{BP}\langle m\rangle^k(X)$ for any m. In the decomposition $(*)$, let the element in the right hand side corresponding to $x \in \mathrm{BP}^k(X)$ be $(x_{n_0}, x_{n_0+1}, \ldots, x_j, \ldots)$, where $x_j \in \mathrm{BP}\langle j\rangle^{k+2(p^j-1)}(X)$. Since the decompositions $(*)$ and $(**)$ are compatible, $\rho_{\langle m\rangle_*}(x) = 0$ implies that $x_{n_0} = \cdots = x_m = 0$. Since $m \geq n_0$ is arbitrary, we have $x_j = 0$ for all $j \geq n_0$. Thus, from $(*)$, this implies that $x = 0$. This proves that $\widehat{\rho}_*$ is injective. Since $\bigcap_n F^n\big(\mathrm{BP}^k(X)\big) = \bigcap_n \operatorname{Ker} \rho_{\langle n\rangle_*} = \operatorname{Ker} \widehat{\rho}_*$, (i) also follows.

Next we prove surjectivity of $\widehat{\rho}_*$. Let $\{y_n\}_n \in \varprojlim_n \mathrm{BP}\langle n\rangle^k(X)$ be any element in the inverse limit. This means that elements y_n are compatible in the sense that for any $m \geq n$, we have $\rho_{\langle n\rangle *}^{\langle m\rangle}(y_m) = y_n$. Thus, we only have to consider elements from the n_0-th term on. Let $m \geq n_0$. In $(**)$, let $(x_{n_0}, \dots, x_m)$ be the element in the right hand side corresponding to $y_m \in \mathrm{BP}\langle m\rangle^k(X)$. If we use different m' such that $m' \geq m$, $y_{m'}$ defines the same element $x_j \in \mathrm{BP}\langle j\rangle^{k+2(p^j-1)}(X)$ for $n_0 \leq j \leq m$, because $y_{m'}$ and y_m are compatible. Since m is arbitrary, we obtain an infinite sequence of elements

$$(x_{n_0}, x_{n_0+1}, \dots, x_j, \dots) \in \mathrm{BP}\langle n_0\rangle^k(X) \times \prod_{j \geq n_0+1} \mathrm{BP}\langle j\rangle^{k+2(p^j-1)}(X).$$

Let $x \in \mathrm{BP}^k(X)$ be the element in the left hand side of $(*)$ corresponding to the above sequence. We then have $\rho_{\langle m\rangle_*}(x) = y_m$ for any $m \geq n_0$, since both elements correspond to $(x_{n_0}, \dots, x_m)$ in the decomposition $(**)$. Hence $\widehat{\rho}_*(x) = \{y_m\}_m$ and $\widehat{\rho}_*$ is surjective. This completes the proof that $\widehat{\rho}_*$ is an isomorphism. $\square$

Corollary 2-3. *Let X be any topological space. Let $b_i \in \mathrm{BP}^{k+2(p^i-1)}(X)$ be any element for $i \geq 0$. Then $\sum_{i=0}^{\infty} v_i b_i$ always converges to a unique element in* $\mathrm{BP}^k(X)$ *with respect to the* BP*-topology.*

Proof. Let $x_n = \sum_{i=0}^{n} v_i b_i$ be a finite sum, and let $y_n = \rho_{\langle n\rangle_*}(x_n) \in \mathrm{BP}\langle n\rangle^k(X)$ for any $n \geq 0$. We claim that elements y_n define an element $\{y_n\}$ in the inverse limit $\varprojlim_n \mathrm{BP}\langle n\rangle^k(X)$. To see this, observe that

$$\begin{aligned} \rho_{\langle n\rangle_*}^{\langle n+1\rangle}(y_{n+1}) &= \rho_{\langle n\rangle_*}^{\langle n+1\rangle} \circ \rho_{\langle n+1\rangle_*}(x_{n+1}) = \rho_{\langle n\rangle_*}(x_n + v_{n+1}b_{n+1}) \\ &= \rho_{\langle n\rangle_*}(x_n) + \rho_{\langle n\rangle_*}(v_{n+1}) \cdot \rho_{\langle n\rangle_*}(b_{n+1}) = \rho_{\langle n\rangle_*}(x_n) = y_n, \end{aligned}$$

where the fourth equality holds because the BP-module map $\rho_{\langle n\rangle_*}$ has the property $\rho_{\langle n\rangle_*}(v_{n+1}) = 0$. Thus, the sequence $\{y_n\}_n$ defines an element in the inverse limit and, by Proposition 2-1, it defines a unique element $x \in \mathrm{BP}^k(X)$ such that $\rho_{\langle n\rangle_*}(x) = y_n$ for all n.

We now show that the sequence of finite sums $x_0, x_1, \dots, x_n, \dots$ converges to x in $\mathrm{BP}^k(X)$ with respect to the BP topology. We observe that for any $m \geq n$,

$$\rho_{\langle n\rangle_*}(x - x_m) = \rho_{\langle n\rangle_*}(x) - \rho_{\langle n\rangle_*}(x_m) = y_n - y_n = 0 \in \mathrm{BP}\langle n\rangle^k(X),$$

since $\rho_{\langle n\rangle_*}(v_k) = 0$ for $k > n$. This means that for any given n, we have $x - x_m \in F^n\big(\mathrm{BP}^k(X)\big)$ for all $m \geq n$. Hence the sequence $\{x_n\}$ converges to x in the BP topology. $\square$

Next, we construct a spectra map which automatically calculates infinite sums of elements of the form $\sum_{i\geq 0} v_i b_i$ in $\mathrm{BP}^*(X)$, where the convergence is with respect to the BP topology.

For this, we recall a few facts about a family of spectra. (For details, see Part III, §3 of Adams [Ad].) Let $\{X_\alpha\}_{\alpha\in A}$ be a family of CW spectra indexed by $\alpha \in A$. Then by E. H. Brown's Representability Theorem, we can consider the product spectrum $\prod_\alpha X_\alpha$ defined by the property

$$[Y, \prod_\alpha X_\alpha] = \prod_\alpha [Y, X_\alpha]$$

for any CW spectrum Y. The coproduct $\bigvee_\alpha X_\alpha$ is defined by the property

$$[\bigvee_\alpha X_\alpha, Y] = \prod_\alpha [X_\alpha, Y].$$

The coproduct can be taken to be the one point union spectrum. From the two properties above, we have a canonical map

$$\bigvee_\alpha X_\alpha \longrightarrow \prod_\alpha X_\alpha,$$

whose component $X_\alpha \to X_\beta$ is the identity map if $\alpha = \beta$, and 0 if $\alpha \neq \beta$. The following lemma is well known.

Lemma 2-4 [Ad, p. 157]**.** *Suppose for each n, we have $\pi_n(X_\alpha) = 0$ for all but finitely many α. Then the following canonical map is an equivalence*:

$$\bigvee_\alpha X_\alpha \longrightarrow \prod_\alpha X_\alpha.$$

We go back to constructing a spectra map calculating infinite sums. We consider the following composition κ of BP-module maps:

$$\kappa : \prod_{i\geq 0} \Sigma^{2(p^i-1)}\mathrm{BP} \xleftarrow{\cong} \bigvee_{i\geq 0} \Sigma^{2(p^i-1)}\mathrm{BP} \xrightarrow{\bigvee v_i} \bigvee_{i\geq 0} \mathrm{BP} \xrightarrow{\text{folding}} \mathrm{BP}. \tag{2-7}$$

The equivalence of the first map is due to Lemma 2-4. Components of the second map is induced by the v_i-multiplication for $i \geq 0$:

$$v_i : \Sigma^{2(p^i-1)}\mathrm{BP} \cong S^{2(p^i-1)} \wedge \mathrm{BP} \xrightarrow{v_i\wedge 1} \mathrm{BP}\wedge\mathrm{BP} \xrightarrow{\mu} \mathrm{BP}, \tag{2-8}$$

where μ is the multiplication map in BP, and $S^{2(p^i-1)}$ is a suspension of the sphere spectrum. The folding map is the map whose restriction to each component of the coproduct is the identity map. Namely,

$$\begin{aligned} [\bigvee_{i\geq 0} \mathrm{BP}, \mathrm{BP}] &= \prod_{i\geq 0} [\mathrm{BP},\mathrm{BP}] \\ \text{folding} &\longleftrightarrow \prod_{i\geq 0} \mathrm{Id}_{\mathrm{BP}} \end{aligned} \tag{2-9}$$

We also consider the following related spectra maps:

$$\kappa_n : \prod_{i=0}^{n} \Sigma^{2(p^i-1)}\mathrm{BP} \xleftarrow{\cong} \bigvee_{i=0}^{n} \Sigma^{2(p^i-1)}\mathrm{BP} \xrightarrow{\bigvee v_i} \bigvee_{i=0}^{n} \mathrm{BP} \xrightarrow{\text{folding}} \mathrm{BP}, \tag{2-10}$$

$$\begin{aligned} \kappa_{\langle n\rangle} : \prod_{i=0}^{n} \Sigma^{2(p^i-1)}\mathrm{BP}\langle n\rangle &\xleftarrow{\cong} \bigvee_{i=0}^{n} \Sigma^{2(p^i-1)}\mathrm{BP}\langle n\rangle \\ &\xrightarrow{\bigvee v_i} \bigvee_{i=0}^{n} \mathrm{BP}\langle n\rangle \xrightarrow{\text{folding}} \mathrm{BP}\langle n\rangle. \end{aligned} \tag{2-11}$$

Of course, the corresponding induced maps on the BP cohomology of a spectrum X are finite BP-linear sums:

$$\kappa_{n_*}: \prod_{i=0}^{n} \mathrm{BP}^{k+2(p^i-1)}(X) \to \mathrm{BP}^k(X),$$

$$(b_0, b_1, \dots, b_n) \longmapsto \sum_{i=0}^{n} v_i b_i \tag{2-12}$$

$$\kappa_{\langle n\rangle_*}: \prod_{i=0}^{n} \mathrm{BP}\langle n\rangle^{k+2(p^i-1)}(X) \to \mathrm{BP}\langle n\rangle^k(X),$$

$$(b'_0, b'_1, \dots, b'_n) \longmapsto \sum_{i=0}^{n} v_i b'_i. \tag{2-13}$$

We are most interested in the cohomology map induced by κ:

$$\begin{aligned} \kappa_*: \prod_{i\ge 0} \mathrm{BP}^{k+2(p^i-1)}(X) &\to \mathrm{BP}^k(X) \\ (b_0, b_1, \dots, b_i, \dots) &\longmapsto \kappa_*(b_0, b_1, \dots, b_i, \dots). \end{aligned} \tag{2-14}$$

When X is a general spectrum, we do not know whether $\mathrm{BP}^*(X)$ is a complete Hausdorff topological space or not, and an infinite sum of the form $\sum_{i=0}^{\infty} v_i b_i$ may or may not make sense as an element of $\mathrm{BP}^*(X)$. However, even for a general spectrum X, the element $\kappa_*(b_0, b_1, \dots, b_i, \dots)$ is always well defined in the BP cohomology.

We want to identify the element $\kappa_*(b_0, \dots, b_i, \dots)$ with the infinite sum $\sum_{i\ge 0} v_i b_i$ which is convergent with respect to the BP topology by Corollary 2-3 when X is a space.

We first consider the behavior of κ_* when almost all b_i's are zero.

Lemma 2-5. *For any spectrum X, let $b_i \in \mathrm{BP}^{k+2(p^i-1)}(X)$ for $0 \le i \le n$ be any elements. Then κ_* reduces to a finite* BP*-linear sum map. Namely,*

$$\kappa_*(b_0, b_1, \dots, b_n, 0, \dots) = \sum_{i=0}^{n} v_i b_i. \tag{2-15}$$

Proof. We consider the following commutative disgram of spectra and spectra maps:

$$\begin{array}{ccccccc} \kappa: \prod_{i=0}^{\infty} \Sigma^{2(p^i-1)}\mathrm{BP} & \xleftarrow{\cong} & \bigvee_{i=0}^{\infty} \Sigma^{2(p^i-1)}\mathrm{BP} & \xrightarrow{\bigvee v_i} & \bigvee_{i=0}^{\infty} \mathrm{BP} & \xrightarrow{\text{folding}} & \mathrm{BP} \\ {\scriptstyle\text{inclusion}}\uparrow & & {\scriptstyle\text{inclusion}}\uparrow & & {\scriptstyle\text{inclusion}}\uparrow & & \| \\ \kappa_n: \prod_{i=0}^{n} \Sigma^{2(p^i-1)}\mathrm{BP} & \xleftarrow{\cong} & \bigvee_{i=0}^{n} \Sigma^{2(p^i-1)}\mathrm{BP} & \xrightarrow{\bigvee v_i} & \bigvee_{i=0}^{n} \mathrm{BP} & \xrightarrow{\text{folding}} & \mathrm{BP}. \end{array}$$

Induced cohomology maps give the following commutative diagram:

$$\begin{array}{ccccc} (b_0, b_1, \dots, b_n, 0, \dots) & \in & \prod_{i=0}^{\infty} \mathrm{BP}^{k+2(p^i-1)}(X) & \xrightarrow{\kappa_*} & \mathrm{BP}^k(X) \\ \uparrow & & \uparrow & & \| \\ (b_0, b_1, \dots, b_n) & \in & \prod_{i=0}^{n} \mathrm{BP}^{k+2(p^i-1)}(X) & \xrightarrow{\kappa_{n*}} & \mathrm{BP}^k(X). \end{array}$$

The bottom row is a finite sum map (2-12) and we have $\kappa_{n*}(b_0, \dots, b_n) = \sum_{i=0}^n v_i b_i$. Hence the commutativity of the above diagram proves (2-15). $\square$

Thus, when there are only finitely many non-trivial elements, the map κ_* is really the summation map with v_i-coefficients. We cannot just let n tend to ∞ because we must deal with the convergence with respect to the BP topology. Since the BP topology is defined using the spectra map $\rho_{\langle n\rangle} : \mathrm{BP} \to \mathrm{BP}\langle n\rangle$, we examine the behavior of κ_* with respect to ${\rho_{\langle n\rangle}}_*$.

Lemma 2-6. *Let X be any spectrum. For arbitrary elements $b_i \in \mathrm{BP}^{k+2(p^i-1)}(X)$ for $i \geq 0$, let $x = \kappa_*(b_0, b_1, \dots, b_i, \dots) \in \mathrm{BP}^k(X)$. Then letting $b_i^{\langle n\rangle} = {\rho_{\langle n\rangle}}_*(b_i)$ for $i \geq 0$, we have*

$$
{\rho_{\langle n\rangle}}_*(x) = \sum_{i=0}^{n} v_i \cdot b_i^{\langle n\rangle} \quad \text{in} \ \ \mathrm{BP}\langle n\rangle^k(X). \tag{2-16}
$$

Proof. We consider the following commutative diagram:

$$
\begin{array}{ccccccc}
\prod_{i=0}^{\infty} \Sigma^{2(p^i-1)}\mathrm{BP} & \xleftarrow{\cong} & \bigvee_{i=0}^{\infty} \Sigma^{2(p^i-1)}\mathrm{BP} & \xrightarrow{\bigvee v_i} & \bigvee_{i=0}^{\infty} \mathrm{BP} & \xrightarrow{\text{folding}} & \mathrm{BP} \\
\Big\downarrow{\scriptstyle \prod_{i=0}^{\infty} \Sigma^{2(p^i-1)}\rho_{\langle n\rangle}} & & \Big\downarrow{\scriptstyle \bigvee_{i=0}^{\infty} \Sigma^{2(p^i-1)}\rho_{\langle n\rangle}} & & \Big\downarrow{\scriptstyle \bigvee_{i=0}^{\infty} \rho_{\langle n\rangle}} & & \Big\downarrow{\scriptstyle \rho_{\langle n\rangle}} \\
\prod_{i=0}^{\infty} \Sigma^{2(p^i-1)}\mathrm{BP}\langle n\rangle & \xleftarrow{\cong} & \bigvee_{i=0}^{\infty} \Sigma^{2(p^i-1)}\mathrm{BP}\langle n\rangle & \xrightarrow{\bigvee v_i} & \bigvee_{i=0}^{\infty} \mathrm{BP}\langle n\rangle & \xrightarrow{\text{folding}} & \mathrm{BP}\langle n\rangle \\
{\scriptstyle \text{proj}}\Big\downarrow & & {\scriptstyle \text{proj}}\Big\downarrow & & {\scriptstyle \text{inclusion}}\Big\uparrow & & \Big\| \\
\prod_{i=0}^{n} \Sigma^{2(p^i-1)}\mathrm{BP}\langle n\rangle & \xleftarrow{\cong} & \bigvee_{i=0}^{n} \Sigma^{2(p^i-1)}\mathrm{BP}\langle n\rangle & \xrightarrow{\bigvee v_i} & \bigvee_{i=0}^{n} \mathrm{BP}\langle n\rangle & \xrightarrow{\text{folding}} & \mathrm{BP}\langle n\rangle.
\end{array}
$$

The commutativity of squares is obvious, except possibly the lower middle one. This one commutes because the multiplication map $v_i : \Sigma^{2(p^i-1)}\mathrm{BP}\langle n\rangle \to \mathrm{BP}\langle n\rangle$ is a zero map for $i > n$. The induced cohomology diagram of a spectrum X is

$$
\begin{array}{ccccc}
\vec{b} = (b_0, \dots, b_n, b_{n+1}, \dots) \in & \prod_{i=0}^{\infty} \mathrm{BP}^{k+2(p^i-1)}(X) & \xrightarrow{\kappa_*} & \mathrm{BP}^k(X) & \ni x = \kappa_*(\vec{b}) \\
\Big\downarrow & \Big\downarrow & & {\scriptstyle {\rho_{\langle n\rangle}}_*}\Big\downarrow & \\
(b_0^{\langle n\rangle}, \dots, b_n^{\langle n\rangle}, 0, \dots) \in & \prod_{i=0}^{n} \mathrm{BP}\langle n\rangle^{k+2(p^i-1)}(X) & \xrightarrow{{\kappa_{\langle n\rangle}}_*} & \mathrm{BP}\langle n\rangle^k(X) & \ni \sum_{i=0}^{n} v_i b_i^{\langle n\rangle}.
\end{array}
$$

The commutativity of this diagram proves the result. $\square$

Now we show that the spectra map κ does calculate infinite sums with respect to the BP topology.

Theorem 2-7. *For any space X, let $b_i \in \mathrm{BP}^{k+2(p^i-1)}(X)$ for $i \geq 0$ be any elements. Let $x_n = \sum_{i=0}^n v_i b_i$ be a finite sum. Then the sequence $\{x_n\}$ converges to the element $\kappa_*(b_0, \dots, b_i, \dots)$ in the* BP*-topology. That is, in $\mathrm{BP}^k(X)$ we have*

$$
\kappa_*(b_0, b_1, \dots, b_i, \dots) = \sum_{i=0}^{\infty} v_i b_i. \tag{2-18}
$$

Proof. Let $x = \kappa_*(b_0, \dots, b_n, \dots) \in \mathrm{BP}^k(X)$. By Proposition 2-1, we know that the sequence $\{x_n\}$ converges to a unique element. We must show that this element is x. For any n, by Lemma 2-6, we have

$$\rho_{\langle n\rangle_*}(x) = \rho_{\langle n\rangle_*}\big(\kappa_*(b_0, \dots, b_n, b_{n+1}, \dots)\big) = \sum_{i=0}^{n} v_i \cdot \rho_{\langle n\rangle_*}(b_i).$$

Let $\vec{b}_m = (b_0, \dots, b_n, \dots, b_m, 0, \dots)$ for $m \geq n$. From Lemma 2-5, we have $\kappa_*(\vec{b}_m) = \sum_{i=0}^{m} v_i b_i = x_m$. Applying Lemma 2-6 to $\vec{b}_m$, we have

$$\rho_{\langle n\rangle_*}(x_m) = \rho_{\langle n\rangle_*}\big(\kappa_*(b_0, \dots, b_n, \dots, b_m, 0, \dots)\big) = \sum_{i=0}^{n} v_i \cdot \rho_{\langle n\rangle_*}(b_i).$$

Hence $\rho_{\langle n\rangle_*}(x) = \rho_{\langle n\rangle_*}(x_m)$ for all $m \geq n$. In other words, $x - x_m \in F^n\big(\mathrm{BP}^k(X)\big)$ for all $m \geq n$. This means that the sequence $\{x_n\}$ converges to x. $\square$

When X is an infinite dimensional CW complex, we can consider two different topologies on $\mathrm{BP}^*(X)$: the BP topology and the skeletal filtration topology. We compare these two topologies.

Recall that the skeletal filtration on $\mathrm{BP}^k(X)$ is a decreasing filtration

$$\begin{gathered} \mathrm{BP}^k(X) = G^0 \supset G^1 \supset \cdots \supset G^n \supset \cdots \\ \text{where} \quad G^n = G^n\big(\mathrm{BP}^k(X)\big) = \operatorname{Ker}\{r_{n-1}^*\colon \mathrm{BP}^k(X) \to \mathrm{BP}^k(X^{(n-1)})\}. \end{gathered} \tag{2-19}$$

Here $r_n\colon X^{(n)} \to X$ is the inclusion map of the n-skeleton of X. The skeletal filtration topology may not be complete Hausdorff due to the existence of phantom maps.

On the other hand, the BP topology is always complete Hausdorff for any infinite dimensional CW complex by Proposition 2-1. Although the definitions of these two topologies are very different, we show that we can compare these two topologies and, in fact, the BP topology is finer than the skeletal filtration topology. For convenience, let F^*_{BP} and F^*_{skeleton} denote the BP-filtration (2-2) and the skeletal filtration (2-19), respectively.

Proposition 2-8. *Let X be an infinite dimensional CW complex. Then the* BP-*topology is finer than the skeletal filtration topology. More precisely, for a given k, let m_0 be any integer such that $k \leq 2(p^{m_0} + \cdots + p + 1)$. Then for any $m \geq m_0$,*

$$F^m_{\mathrm{BP}}\big(\mathrm{BP}^k(X)\big) \subset F^{k+2(p^{m+1}-1)}_{\text{skeleton}}\big(\mathrm{BP}^k(X)\big). \tag{2-20}$$

Thus, any sequence convergent in the BP*-topology is also convergent in the skeletal filtration topology.*

Proof. Let m, n be as above. We consider the following diagram:

$$\begin{array}{ccc} \mathrm{BP}^k(X) & \xrightarrow{\rho_{\langle m\rangle_*}} & \mathrm{BP}\langle m\rangle^k(X) \\ \big\downarrow{\scriptstyle r^*} & & \big\downarrow{\scriptstyle r^*} \\ \mathrm{BP}^k\big(X^{(k+2p^{m+1}-3)}\big) & \xrightarrow{\rho_{\langle m\rangle_*}} & \mathrm{BP}\langle m\rangle^k\big(X^{(k+2p^{m+1}-3)}\big), \end{array}$$

where $r : X^{(k+2p^{m+1}-3)} \to X$ is the inclusion map. For the upper horizontal map, $\operatorname{Ker} \rho_{\langle m\rangle_*} = F^m_{\mathrm{BP}}\big(\mathrm{BP}^k(X)\big)$. Since $k \leq 2(p^m + \cdots + p + 1)$, by Wilson's Splitting

Theorem we have

$$\mathrm{BP}^k(X) \cong \mathrm{BP}\langle m\rangle^k(X) \times \prod_{j\geq m+1} \mathrm{BP}\langle j\rangle^{k+2(p^j-1)}(X).$$

For the $\big(k+2(p^{m+1}-1)-1\big)$-skeleton of X, for $j \geq m+1$ we have

$$\mathrm{BP}\langle j\rangle^{k+2(p^j-1)}\big(X^{(k+2p^{m+1}-3)}\big) = [X^{(k+2p^{m+1}-3)}, \underline{\mathrm{BP}}\langle j\rangle_{k+2(p^j-1)}] = 0,$$

since $\underline{\mathrm{BP}}\langle j\rangle_{k+2(p^j-1)}$ is at least $(k+2p^{m+1}-3)$-connected for $j \geq m+1$. Thus

$$\mathrm{BP}^k\big(X^{(k+2p^{m+1}-3)}\big) \xrightarrow[\cong]{\rho_{\langle n\rangle_*}} \mathrm{BP}\langle m\rangle^k\big(X^{(k+2p^{m+1}-3)}\big).$$

Then the above commutative diagram implies that

$$F^m_{\mathrm{BP}}\big(\mathrm{BP}^k(X)\big) = \operatorname{Ker}\rho_{\langle m\rangle_*} \subset \operatorname{Ker} r^* = F^{k+2(p^{m+1}-1)}_{\text{skeleton}}\big(\mathrm{BP}^k(X)\big),$$

for any $m \geq m_0$. This completes the proof. $\square$

3. Spectrum L of infinite sum BP-linear relations

We consider a spectrum which is closely related to the infinite BP-linear sum map κ of (2-7). Namely, let L be the cofibre spectrum of the spectra map κ. The resulting cofibre sequence is

$$\Sigma^{-1}L \xrightarrow{\prod q_i} \prod_{i=0}^{\infty} \Sigma^{2(p^i-1)}\mathrm{BP} \xrightarrow{\kappa} \mathrm{BP} \xrightarrow{\theta} L, \tag{3-1}$$

where $q_i : L \to \Sigma^{2p^i-1}\mathrm{BP}$ is the map to the i-th factor. We study the mod p cohomology of L and properties of q_i. The cofibre sequence (3-1) induces the following cohomology exact sequence for any spectrum X:

$$\begin{aligned} \cdots \longrightarrow L^{*-1}(X) \xrightarrow{\prod q_{i*}} \prod_{i=0}^{\infty} \mathrm{BP}^{*+2(p^i-1)}(X) \xrightarrow{\kappa_*} \mathrm{BP}^*(X) \\ \xrightarrow{\theta_*} L^*(X) \longrightarrow \cdots . \end{aligned} \tag{3-2}$$

The mod p cohomology exact sequence of the cofibre sequence (3-1) is of the form

$$\begin{aligned} \cdots \longleftarrow \prod_{i=0}^{\infty} H\mathbb{Z}_p^{*-2(p^i-1)}(\mathrm{BP}) \xleftarrow{\kappa^*} H\mathbb{Z}_p^*(\mathrm{BP}) \\ \xleftarrow{\theta^*} H\mathbb{Z}_p^*(L) \xleftarrow{\sum q_i{}^*} \prod_{i=0}^{\infty} H\mathbb{Z}_p^{*-2p^i+1}(\mathrm{BP}) \longleftarrow \cdots . \end{aligned} \tag{3-3}$$

When $*=0$, the map $\theta^* : H\mathbb{Z}_p^0(L) \to H\mathbb{Z}_p^0(\mathrm{BP})$ is an isomorphism by dimensional reason and by the fact that $p^*=0$ on mod p cohomology. Let $\eta : L \to H\mathbb{Z}_p$ be an element in $H\mathbb{Z}_p^0(L)$ corresponding to the Thom map $\rho : \mathrm{BP} \to H\mathbb{Z}_p$ in $H\mathbb{Z}_p^0(\mathrm{BP})$ under the isomorphism θ^*. That is, $\rho = \theta^*(\eta) = \eta\circ\theta$.

Let Q_i be the i-th Milnor primitive in the mod p Steenrod algebra $\mathcal{A}(p)$ [M1, M2]. These elements are defined by

$$\begin{cases} Q_0 = \text{Bockstein operator}, \\ Q_{n+1} = \mathcal{P}^{p^n}Q_n - Q_n\mathcal{P}^{p^n}, \quad n \geq 0. \end{cases} \tag{3-4}$$

The operation Q_i raises the cohomology degree by $2p^i-1$.

The purpose of this section is to prove the following theorem.

Theorem 3-1. *The spectrum L is a* BP*-module spectrum with the following properties*:

(I) *Let X be a space. For any $z \in L^k(X)$, let $b_i = q_{i*}(z) \in \mathrm{BP}^{k+2p^i-1}(X)$ for $i \geq 0$. Then we have*

$$pb_0 + v_1 b_1 + \cdots + v_n b_n + \cdots = 0 \quad in \quad \mathrm{BP}^{k+1}(X).$$

Here the convergence is with respect to the BP *topology.*

(II) *For any $i \geq 0$, the following diagram commutes* (*up to homotopy*):

$$\begin{array}{ccccc} L & \xrightarrow{q_i} & \Sigma^{2p^i-1}\mathrm{BP} & \xrightarrow{\Sigma^{2p^i-1}\theta} & \Sigma^{2p^i-1}L \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \Sigma^{2p^i-1}\rho} & & \downarrow{\scriptstyle \Sigma^{2p^i-1}\eta} \\ H\mathbb{Z}_p & \xrightarrow{Q_i} & \Sigma^{2p^i-1}H\mathbb{Z}_p & = & \Sigma^{2p^i-1}H\mathbb{Z}_p. \end{array} \tag{3-5}$$

The Milnor operation $\widehat{q}_i$ in L-theory can be defined by $\widehat{q}_i = \Sigma^{2p^i-1}\theta \circ q_i$ for $i \geq 0$, and they satisfy $\widehat{q}_i \circ \widehat{q}_j = 0$ for any $i, j \geq 0$.

(III) *The mod p cohomology of the spectrum L is the following cyclic module over the mod p Steenrod algebra $\mathcal{A}(p)$ generated by η*:

$$H\mathbb{Z}_p^*(L) \cong \Big[\mathcal{A}(p)\Big/\sum_{i,j\geq 0} \mathcal{A}(p)\cdot Q_iQ_j\Big]\cdot \eta. \tag{3-6}$$

(IV) *The coefficient group of L-theory is such that $L^* = 0$ when $* > 0$ and $L^0 \cong \mathbb{Z}_p$ spanned by η. When $* < 0$, the group L^* is torsion free and we have the following exact sequence*:

$$0 \to L^* \xrightarrow{\prod q_{i*}} \prod_{i\geq 0} \mathrm{BP}^{*+2p^i-1} \xrightarrow{\kappa_*} \mathrm{BP}^{*+1} \to 0, \qquad * < 0,$$

where $\kappa_*(\alpha_0, \alpha_1, \ldots, \alpha_i, \ldots) = \sum_{i\geq 0} v_i\alpha_i$ is a finite sum map.

Proof of (I) *and* (IV). (I) For a given $z \in L^k(X)$, by exactness of (3-2), we have $\kappa_*(b_0, \ldots, b_n, \ldots) = 0$ in $\mathrm{BP}^{k+1}(X)$. By Theorem 2-7, this means $\sum_{i\geq 0} v_ib_i = 0$. This proves (I).

(IV) The homotopy exact sequence of the cofibre sequence (3-1) is

$$\cdots \to L^{*-1} \xrightarrow{\prod q_{i*}} \prod_{i\geq 0} \mathrm{BP}^{*+2(p^i-1)} \xrightarrow{\kappa_*} \mathrm{BP}^* \xrightarrow{\theta_*} L^* \to \cdots.$$

We observe that $\mathrm{Im}\,(\kappa_*)$ is the ideal $I_\infty = (p, v_1, \ldots, v_n, \ldots) \subset \mathrm{BP}^*$ and $\mathrm{BP}^*/I_\infty \cong \mathbb{Z}_p$ concentrated in degree 0. Thus, when $* < 0$, κ_* is surjective and we obtain the short exact sequence in (IV). Note that κ_* reduces to a finite sum since $\mathrm{BP}^* \neq 0$ only for $* \leq 0$. ☐

Part (I) shows that any element $z \in L^*(X)$ gives rise to an infinite sum BP-linear relation in $\mathrm{BP}^*(X)$. This is why we call the spectrum L *the spectrum of* BP*-linear relations.*

Part (II) is proved by a sequence of lemmas. Note that the commutativity of the right square in (3-5) follows from the definition of the map η. We prove the commutativity of the left square.

To examine the spectra map κ, we compare it with the v_i-multiplication map on BP. We examine the following cofibre sequence:

$$\Sigma^{-1}\mathrm{BP}(v_i) \xrightarrow{\beta_i} \Sigma^{2(p^i-1)}\mathrm{BP} \xrightarrow{v_i} \mathrm{BP} \xrightarrow{j_i} \mathrm{BP}(v_i), \qquad i \geq 0. \tag{3-7}$$

Here $\mathrm{BP}(v_i)$ is the cofibre of the v_i-multiplication map. Its homotopy group is $\pi_*\big(\mathrm{BP}(v_i)\big) \cong \mathrm{BP}_*/(v_i)$. The mod p cohomology of the cofibre sequence (3-7) gives

$$\text{(3-8)} \quad \cdots \longleftarrow H\mathbb{Z}_p^{*+1}\mathrm{BP}(v_i) \xleftarrow{\beta_i^*} H\mathbb{Z}_p^{*-2(p^i-1)}\mathrm{BP} \xleftarrow{v_i^*} H\mathbb{Z}_p^*\mathrm{BP} \xleftarrow{j_i^*} H\mathbb{Z}_p^*\mathrm{BP}(v_i) \xleftarrow{\beta_i^*} H\mathbb{Z}_p^{*-2p^i+1}\mathrm{BP} \longleftarrow \cdots .$$

Observe that when $*=0$, j_i^* is an isomorphism. This is clear when $i>0$ by dimensional reason. When $i=0$, we get the same conclusion since $p^*=0$ in mod p cohomology. Let $\rho_i : \mathrm{BP}(v_i) \to H\mathbb{Z}_p$ be the map corresponding to the Thom map $\rho : \mathrm{BP} \to H\mathbb{Z}_p$ through the isomorphism j_i^*. Thus, $\rho = j_i^*(\rho_i) = \rho_i \circ j_i$ for $i \geq 0$.

The mod p cohomology modules of the spectra BP and $\mathrm{BP}(v_i)$ are known. Let $\mathcal{P}(p) = \mathcal{A}(p)/(Q_0)$ be the algebra of Steenrod reduced powers, where (Q_0) is the two-sided ideal generated by Q_0.

Lemma 3-2 [BM]**.** *As modules over the Steenrod algebra, the mod p cohomologies of BP and $\mathrm{BP}(v_i)$ are the following cyclic modules generated by ρ and ρ_i*:

(3-9)

$$H\mathbb{Z}_p^*(\mathrm{BP}) = \Big[\mathcal{A}(p)\big/\mathcal{A}(p)(Q_0, Q_1, \dots, Q_n, \dots)\Big]\rho \cong \mathcal{P}(p)\rho,$$
$$H\mathbb{Z}_p^*\big(\mathrm{BP}(v_i)\big) = \Big[\mathcal{A}(p)\big/\mathcal{A}(p)(Q_0, \dots, \widehat{Q}_i, Q_{i+1}, \dots)\Big]\rho_i \cong \mathcal{P}(p)\rho_i \oplus \mathcal{P}(p)Q_i(\rho_i).$$

Here $\mathcal{A}(p)(Q_j$'s) is the left ideal generated by Q_j's, and $\widehat{Q}_i$ means that Q_i is omitted.

It is known that the left ideal $(Q_0, Q_1, \dots, Q_i, \dots)$ coincides with the two-sided ideal (Q_0). Observe that $H\mathbb{Z}_p^*(\mathrm{BP})$ is even dimensional and $H\mathbb{Z}_p^{2p^i-1}\big(\mathrm{BP}(v_i)\big) \cong \mathbb{Z}_p$ is spanned by $Q_i(\rho_i)$.

We examine the exact sequence (3-8) in the light of Lemma 3-2. We have seen that $p^*=0$ by a trivial reason in mod p cohomology. It turns out that for all $i \geq 0$, we have $v_i^* = 0$ in (3-8).

Lemma 3-3. *The induced map v_i^* in* (3-8) *on mod p cohomology is trivial and we have the following short exact sequence*:

$$\text{(3-10)} \qquad 0 \to H\mathbb{Z}_p^*(\mathrm{BP}) \xrightarrow{\beta_i^*} H\mathbb{Z}_p^{*+2p^i-1}\big(\mathrm{BP}(v_i)\big) \xrightarrow{j_i^*} H\mathbb{Z}_p^{*+2p^i-1}(\mathrm{BP}) \to 0.$$

Here, both β_i^* and j_i^* are $\mathcal{A}(p)$-module maps such that

$$\text{(3-11)} \qquad \beta_i^*(\rho) = \lambda_i \cdot Q_i(\rho_i), \quad \textit{and} \quad j_i^*(\rho_i) = \rho$$

for some nonzero constant $\lambda_i \in \mathbb{Z}_p$ which may depend on i. In (3-10)*, β_i^* maps $\mathcal{P}(p)\cdot\rho$ isomorphically onto a summand $\mathcal{P}(p)\cdot Q_i(\rho_i)$, and j_i^* maps $\mathcal{P}(p)\cdot\rho_i$ isomorphically onto $\mathcal{P}(p)\cdot\rho$.*

Proof. By Lemma 3-2, both $H\mathbb{Z}_p^*(\mathrm{BP})$ and $H\mathbb{Z}_p^*\big(\mathrm{BP}(v_i)\big)$ are cyclic $\mathcal{A}(p)$-modules and we know that j_i^* maps the module generator ρ_i of $H\mathbb{Z}_p^*\big(\mathrm{BP}(v_i)\big)$ to the module generator ρ of $H\mathbb{Z}_p^*(\mathrm{BP})$. Hence j_i^* in (3-8) is surjective and, by exactness, v_i^* is a zero map. Thus, we have the short exact sequence (3-10). When $*=0$, we have an isomorphism $\beta_i^* : H\mathbb{Z}_p^0(\mathrm{BP}) \xrightarrow{\cong} H\mathbb{Z}_p^{2p^i-1}\big(\mathrm{BP}(v_i)\big) \cong \mathbb{Z}_p Q_i(\rho_i)$, since the mod p cohomology of BP is even dimensional. This proves the first formula in (3-11). The second one is the definition of ρ_i. Since both β_i^* and j_i^* are $\mathcal{A}(p)$-linear, we have the last statement. $\square$

We will show that the constants λ_i in (3-11) are independent of i, and in fact they are all equal to 1.

We combine the cofibre sequences (3-7) for all i, and compare it with the cofibre sequence (3-1). Consider the following diagram:

$$\begin{tikzcd}
\bigvee_{i=0}^{\infty} \Sigma^{-1}\mathrm{BP}(v_i) \arrow[r, "\bigvee \beta_i"] \arrow[d, "\Sigma^{-1}\tau"] & \bigvee_{i\geq 0} \Sigma^{2(p^i-1)}\mathrm{BP} \arrow[r, "\bigvee v_i"] \arrow[d, "\cong", "\text{h.e.}"'] & \bigvee_{i\geq 0} \mathrm{BP} \arrow[r, "\bigvee j_i"] \arrow[d, "\text{folding}"] & \bigvee_{i\geq 0} \mathrm{BP}(v_i) \arrow[d, "\tau"] \\
\Sigma^{-1}L \arrow[r, "\prod q_i"] & \prod_{i\geq 0} \Sigma^{2(p^i-1)}\mathrm{BP} \arrow[r, "\kappa"] & \mathrm{BP} \arrow[r, "\theta"] & L.
\end{tikzcd} \tag{3-12}$$

The commutativity of the middle square comes from the definition of κ. The map τ is a spectra map between cofibres of $\bigvee v_i$ and κ induced from the commutative middle square. With this definition of τ, the above diagram commutes.

We know the behavior of the mod p cohomology of the top row by Lemma 3-3. This implies the following result for the mod p cohomology of the bottom row.

Lemma 3-4. *The spectra map κ induces a zero map in mod p cohomology, and we have the following short exact sequence*:

$$0 \to \prod_{i\geq 0} H\mathbb{Z}_p^{*-2p^i+1}(\mathrm{BP}) \xrightarrow{(\prod q_i)^* = \sum q_i^*} H\mathbb{Z}_p^*(L) \xrightarrow{\theta^*} H\mathbb{Z}_p^*(\mathrm{BP}) \to 0. \tag{3-13}$$

With the same nonzero constant $\lambda_i \in \mathbb{Z}_p$ as in (3-11), *we have*

$$q_i^*(\rho) = \lambda_i \cdot Q_i(\eta) \quad \textit{for} \quad i \geq 0. \tag{3-14}$$

Proof. Considering the mod p cohomology of the middle square of (3-12), we have the following commutative diagram:

$$\begin{tikzcd}
\prod_{i\geq 0} H\mathbb{Z}_p^{*-2(p^i-1)}(\mathrm{BP}) \arrow[d, equal] & \prod_{i\geq 0} H\mathbb{Z}_p^*(\mathrm{BP}) \arrow[l, "\prod v_i^*"'] \\
\prod_{i\geq 0} H\mathbb{Z}_p^{*-2(p^i-1)}(\mathrm{BP}) & H\mathbb{Z}_p^*(\mathrm{BP}). \arrow[l, "\kappa^*"'] \arrow[u, "\text{diagonal}"']
\end{tikzcd}$$

Note that the cohomology map induced from the folding map is the diagonal map. Since $v_i^* = 0$ by Lemma 3-3, it follows that $\kappa^* = 0$. Thus we obtain the short exact sequence (3-13). Note that the induced map $(\prod_i q_i)^*$ is actually a finite sum map $\sum_i q_i^*$ by dimensional reason. From (3-12), we obtain the following diagram in which both rows are short exact:

$$\begin{tikzcd}
0 & \prod_{i\geq 0} H\mathbb{Z}_p^*(\mathrm{BP}) \arrow[l] & \prod_{i\geq 0} H\mathbb{Z}_p^*\big(\mathrm{BP}(v_i)\big) \arrow[l, "\prod j_i^*"'] & \prod_{i\geq 0} H\mathbb{Z}_p^{*-2p^i+1}(\mathrm{BP}) \arrow[l, "\prod \beta_i^*"'] \arrow[d, equal] & 0 \arrow[l] \\
0 & H\mathbb{Z}_p^*(\mathrm{BP}) \arrow[l] \arrow[u, "\text{diagonal}"'] & H\mathbb{Z}_p^*(L) \arrow[l, "\theta^*"'] \arrow[u, "\tau^*"'] & \prod_{i\geq 0} H\mathbb{Z}_p^{*-2p^i+1}(\mathrm{BP}) \arrow[l, "\sum q_i^*"'] & 0. \arrow[l]
\end{tikzcd}$$

First, we let $* = 0$ in this diagram. Then the right end groups are both zero because $H\mathbb{Z}_p^*(\mathrm{BP})$ is even dimensional. Thus, both θ^* and $\prod j_i^*$ are isomorphisms in this degree. Since $\theta^*(\eta) = \rho$ and $j_i^*(\rho_i) = \rho$ for $i \geq 0$ by definition, it follows that

$$\tau^*(\eta) = (\rho_0, \rho_1, \ldots, \rho_i, \ldots). \tag{3-15}$$

Next, we let $* = 2p^\ell - 1$ for some $\ell \geq 0$. This time, left end groups are zero and both maps $\prod \beta_i^*$ and $\sum q_i^*$ are isomorphisms in this case, and consequently τ^* is also an isomorphism in this degree. Let $\rho \in H\mathbb{Z}_p^0(\mathrm{BP})$ be in the ℓ-th factor of the right end group. We examine the behavior of this element in this diagram.

$$\begin{aligned}\tau^*\big(q_\ell^*(\rho)\big) &= (\textstyle\prod \beta_i^*)(0, \dots, 0, \rho, 0, \dots) = (0, \dots, 0, \beta_\ell^*(\rho), 0, \dots) \\ &= (0, \dots, 0, \lambda_\ell Q_\ell(\rho_\ell), 0, \dots),\end{aligned}$$

where the last equality is due to (3-11). On the other hand, $Q_\ell(\eta) \in H\mathbb{Z}_p^{2p^\ell - 1}(L)$, and by naturality of cohomology operations, we have

$$\tau^*\big(Q_\ell(\eta)\big) = Q_\ell\big(\tau^*(\eta)\big) = Q_\ell(\rho_0, \rho_1, \dots, \rho_\ell, \dots).$$

Since $Q_\ell \rho_i = 0$ when $i \neq \ell$ by Lemma 3-2, by derivation property of Q_ℓ, the above is further equal to $(0, \dots, 0, Q_\ell \rho_\ell, 0, \dots)$. Comparing this with the previous calculation, we see that $\tau^*\big(q_\ell^*(\rho)\big) = \tau^*\big(\lambda_\ell Q_\ell(\eta)\big)$. Since τ^* is an isomorphism in the degree we are working, we finally have $q_\ell^*(\rho) = \lambda_\ell \cdot Q_\ell(\eta)$. Since $\ell \geq 0$ is arbitrary, we get (3-14). This completes the proof. $\square$

This proves Part (II) of Theorem 3-1 up to nonzero constant multiples $\lambda_\ell \in \mathbb{Z}_p$.

Our next task is to show that all the constants λ_ℓ are equal to 1. For this, we need a preparation. We consider the following cofibre sequence:

(3-16) $$\cdots \to \Sigma^{-1} H\mathbb{Z}_p \xrightarrow{\beta} H\mathbb{Z}_{(p)} \xrightarrow{p} H\mathbb{Z}_{(p)} \xrightarrow{j} H\mathbb{Z}_p \to \cdots.$$

The Bockstein operator Q_0 is then defined by

(3-17) $$Q_0 = \beta^*(j) = j \circ \beta \in H\mathbb{Z}_p^1(H\mathbb{Z}_p).$$

We first show that λ_0 in (3-11) and (3-14) is equal to 1.

Lemma 3-5. *Let* $q_0 : \Sigma^{-1} L \to \mathrm{BP}$ *and* $\beta_0 : \Sigma^{-1}\mathrm{BP}(v_i) \to \mathrm{BP}$ *be as in* (3-1) *and* (3-7)*. Then in mod p cohomology,*

(3-18) $$\beta_0^*(\rho) = Q_0(\rho_0), \qquad q_0^*(\rho) = Q_0(\eta).$$

Proof. By Lemma 3-4, the first identity implies the second identity. To see the first identity, we consider the following commutative diagram between the cofibre sequence (3-16) and (3-7) with $i = 0$:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \Sigma^{-1}\mathrm{BP}(v_0) & \xrightarrow{\beta_0} & \mathrm{BP} & \xrightarrow{p} & \mathrm{BP} & \xrightarrow{j_0} & \mathrm{BP}(v_0) & \longrightarrow & \cdots \\ & & \Big\downarrow{\scriptstyle \Sigma^{-1}\rho_0} & & \Big\downarrow{\scriptstyle \rho_{\langle 0\rangle}} & & \Big\downarrow{\scriptstyle \rho_{\langle 0\rangle}} & & \Big\downarrow{\scriptstyle \rho_0} & & \\ \cdots & \longrightarrow & \Sigma^{-1}H\mathbb{Z}_p & \xrightarrow{\beta} & H\mathbb{Z}_{(p)} & \xrightarrow{p} & H\mathbb{Z}_{(p)} & \xrightarrow{j} & H\mathbb{Z}_p & \longrightarrow & \cdots. \end{array}$$

Here, $v_0 = p$. In the associated commutative diagram of mod p cohomologies, we have $p^* = 0$ and we obtain the following diagram:

$$\begin{array}{ccccccc} \cdots \longleftarrow & H\mathbb{Z}_p^1\big(\mathrm{BP}(v_0)\big) & \xleftarrow{\beta_0^*} & H\mathbb{Z}_p^0(\mathrm{BP}) & \xleftarrow{p^*=0} & 0 \\ & \Big\uparrow{\scriptstyle \rho_0^*} & & \Big\uparrow{\scriptstyle \rho_{\langle 0\rangle}^*} & & \\ \cdots \longleftarrow & H\mathbb{Z}_p^1(H\mathbb{Z}_p) & \xleftarrow{\beta^*} & H\mathbb{Z}_p^0(H\mathbb{Z}_{(p)}) & \xleftarrow{p^*=0} & 0. \end{array}$$

Since $\beta^*(j) = Q_0$ by (3-17) and $\rho^*_{\langle 0\rangle}(j) = \rho$, we have

$$\beta_0^*(\rho) = \beta_0^*\big(\rho^*_{\langle 0\rangle}(j)\big) = \rho_0^*\beta^*(j) = \rho_0^*\big(Q_0(1)\big) = Q_0\big(\rho_0^*(1)\big) = Q_0(\rho_0).$$

Here we used the naturality of Q_0. This proves the first identity of (3-18), and the second one follows from this. $\square$

Next, we show that $\lambda_\ell = 1$ for all $\ell \geq 0$. Since these constants are "universal" constants, we only have to prove this for a particular example. As such an example, we use the infinite dimensional lens space L_p. Recall that the mod p cohomology of the lens space is given by

$$\begin{aligned} &H\mathbb{Z}_p^*(L_p) \cong \mathbb{Z}_p[\,x\,] \otimes \textstyle\bigwedge_{\mathbb{Z}_p}(\alpha), \quad |x| = 2,\ |\alpha| = 1, \\ &\text{where} \quad Q_i(\alpha) = x^{p^i}, \qquad i \geq 0. \end{aligned} \tag{3-19}$$

The BP cohomology of L_p was calculated in [L] and it is given by

$$\begin{aligned} &\mathrm{BP}^*(L_p) = \mathrm{BP}^*[[\,x\,]]\,/\,\big([\,p\,]_{\mathrm{BP}}(x)\big), \quad x \in \mathrm{BP}^2(L_p) \\ &\text{where} \quad [\,p\,]_{\mathrm{BP}}(x) = x +_{\mathrm{BP}} x +_{\mathrm{BP}} \cdots +_{\mathrm{BP}} x \end{aligned} \tag{3-20}$$

is the p-series for the BP-formal group law [Ar, H]. One can easily show that

$$[\,p\,]_{\mathrm{BP}}(x) = \exp^{\mathrm{BP}}(px) +_{\mathrm{BP}} \sum_{i\geq 0}{}^{\mathrm{BP}} v_i x^{p^i}. \tag{3-21}$$

Let $I_\infty = (p, v_1, \dots, v_n, \dots)$ be the maximal ideal of BP^*. From (3-21), it follows that there exist elements $y_i \in \mathrm{BP}^*[[\,x\,]]$ for $i \geq 0$ such that

$$\begin{aligned} [\,p\,]_{\mathrm{BP}}(x) &= py_0 + v_1 y_1 + \cdots + v_i y_i + \cdots, \\ y_i &\equiv x^{p^i} \mod I_\infty \quad \text{for} \ \ i \geq 0. \end{aligned} \tag{3-22}$$

Note that we may take $y_0 = x$ exactly. From the cofibre sequence (3-1), we have the following induced cohomology exact sequence for the lens space L_p:

$$\cdots \to L^1(L_p) \xrightarrow{\prod q_{i*}} \prod_{i\geq 0} \mathrm{BP}^{2p^i}(L_p) \xrightarrow{\ \kappa_*\ } \mathrm{BP}^2(L_p) \to \cdots.$$

Since $\kappa_*(y_0, y_1, \dots, y_i, \dots) = \sum_{i\geq 0} v_i y_i = 0$ in $\mathrm{BP}^*(L_p)$ by (3-20) and (3-22), from the exactness of the above sequence, there exists an element $z \in L^1(L_p)$ such that $y_i = q_{i*}(z)$ for all $i \geq 0$. Formula (3-14) implies commutativity of the following diagram for each $\ell \geq 0$:

$$\begin{array}{ccc} L^1(L_p) & \xrightarrow{\ q_{\ell *}\ } & \mathrm{BP}^{2p^\ell}(L_p) \\ \big\downarrow{\scriptstyle \eta_*} & & \big\downarrow{\scriptstyle \rho_*} \\ H\mathbb{Z}_p^1(L_p) & \xrightarrow{\lambda_\ell Q_\ell} & H\mathbb{Z}_p^{2p^\ell}(L_p). \end{array} \tag{3-23}$$

To show that $\lambda_\ell = 1$ for all $\ell \geq 0$, it is necessary that all elements are chosen in a coherent way. Thus, we fix $x = x^{\mathrm{BP}} \in \mathrm{BP}^2(L_p)$ and define $x^H \in H\mathbb{Z}_p^2(L_p)$ by $x^H = \rho_*(x^{\mathrm{BP}})$. Then we fix $\alpha \in H\mathbb{Z}_p^1(L_p)$ by $Q_0(\alpha) = x^H$.

Lemma 3-6. *The diagram* (3-23) *commutes with* $\lambda_\ell = 1 \in \mathbb{Z}_p$ *for all* $\ell \geq 0$. *Thus*

$$q_\ell^*(\rho) = Q_\ell(\eta) \quad \textit{in} \quad H\mathbb{Z}_p^*(L) \quad \textit{for all} \ \ \ell \geq 0. \tag{3-24}$$

Proof. First we examine (3-23) with $\ell = 0$. From Lemma 3-5 we know that the diagram (3-23) commutes with $\lambda_0 = 1$. Hence

$$x = x^H = \rho_*(x^{\mathrm{BP}}) = \rho_*\big(q_{0*}(z)\big) = Q_0\big(\eta_*(z)\big).$$

Since $H\mathbb{Z}_p^1(L_p) \cong \mathbb{Z}_p\alpha$ and $Q_0(\alpha) = x$, we must have $\eta_*(z) = \alpha$ *exactly.* But then the commutativity of the diagram (3-23) for $\ell \geq 1$ implies

$$x^{p^\ell} = \rho_*(y_\ell) = \rho_*\big(q_{\ell*}(z)\big) = \lambda_\ell Q_\ell\big(\eta_*(z)\big) = \lambda_\ell Q_\ell(\alpha) = \lambda_\ell x^{p^\ell}.$$

Thus, we must have $\lambda_\ell = 1$ for all $\ell \geq 1$. This completes the proof. $\square$

This completes the proof of Part (II) of Theorem 3-1.

Proof of Part (III) *of Theorem* 3-1. First we prove the relations $Q_iQ_j(\eta) = 0$ for any $i, j \geq 0$. Part (II) gives $q_j^*(\rho) = Q_j(\eta)$ in $H\mathbb{Z}_p^*(L)$. By naturality of cohomology operations,

$$Q_iQ_j(\eta) = Q_i\big(q_j^*(\rho)\big) = q_j^*\big(Q_i(\rho)\big) = 0$$

since $Q_i(\rho) = 0$ for any $i \geq 0$ by Lemma 3-2. In the proof of Lemma 3-4, we had the following exact sequence:

$$0 \to \prod_{i\geq 0} H\mathbb{Z}_p^{*-2p^i+1}(\mathrm{BP}) \xrightarrow{\sum q_i^*} H\mathbb{Z}_p^*(L) \xrightarrow{\theta^*} H\mathbb{Z}_p^*(\mathrm{BP}) \to 0.$$

Let R be a sequence of non-negative integers almost all zero, and let $\mathcal{P}^R$ be the corresponding Milnor's Steenrod reduced power operation [M1]. By naturality $q_i^*\big(\mathcal{P}^R(\rho)\big) = \mathcal{P}^R\big(q_i^*(\rho)\big) = \mathcal{P}^R Q_i(\eta)$ for any sequence R and for $i \geq 0$. Since $\theta^*(\eta) = \rho$, as $\mathbb{Z}_p$-vector spaces (not as $\mathcal{A}(p)$-modules) we have

$$H\mathbb{Z}_p^*(L) = \bigoplus_{i\geq 0}\Big[\bigoplus_R \mathbb{Z}_p\mathcal{P}^R Q_i(\eta)\Big] \oplus \Big[\bigoplus_R \mathbb{Z}_p\mathcal{P}^R(\eta)\Big],$$

where the first summand is the monomorphic image of $\sum q_i^*$ and the second summand maps isomorphically onto $H\mathbb{Z}_p^*(\mathrm{BP})$ by θ^*. This shows that $Q_iQ_j(\eta) = 0$ for $i, j \geq 0$ are the only relations in $H\mathbb{Z}_p^*(L)$. This completes the proof of Part (III) of Theorem 3-1. $\square$

4. Spectrum $L\langle n\rangle$ of finite sum BP-linear relations and its relation to L

In §2, we introduced the following spectra map:

(4-1)
$$\kappa_{\langle n\rangle} : \prod_{i=0}^n \Sigma^{2(p^i-1)}\mathrm{BP}\langle n\rangle \xleftarrow{\cong} \bigvee_{i=0}^n \Sigma^{2(p^i-1)}\mathrm{BP}\langle n\rangle \xrightarrow{\bigvee v_i} \bigvee_{i=0}^n \mathrm{BP}\langle n\rangle \xrightarrow{\text{folding}} \mathrm{BP}\langle n\rangle.$$

The induced map on the cohomology of a spectrum X is a finite BP-linear sum of (2-13). In §3, we constructed the spectrum L of BP-linear relations as the cofibre of κ. Here, we define a spectrum $L\langle n\rangle$ as the cofibre of the spectra map $\kappa_{\langle n\rangle}$. We have the following cofibre sequence:

(4-2)
$$\Sigma^{-1}L\langle n\rangle \xrightarrow{\prod q_i\langle n\rangle} \prod_{i=0}^n \Sigma^{2(p^i-1)}\mathrm{BP}\langle n\rangle \xrightarrow{\kappa_{\langle n\rangle}} \mathrm{BP}\langle n\rangle \xrightarrow{\theta_{\langle n\rangle}} L\langle n\rangle.$$

Note that when $n = 0$, we have $\mathrm{BP}\langle n\rangle = H\mathbb{Z}_{(p)}$, and the cofibre sequence (4-2) reduces to the cofibre sequence (3-16) for the p-multiplication map $H\mathbb{Z}_{(p)} \xrightarrow{p} H\mathbb{Z}_{(p)}$, and consequently we have $L\langle 0\rangle = H\mathbb{Z}_p$.

We examine the following portion of the mod p cohomology of the cofibre sequence (4-2):

$$\cdots \to \prod_{i=0}^{n} H\mathbb{Z}_p^{-2p^i+1}(\mathrm{BP}\langle n\rangle) \xrightarrow{\sum q_i\langle n\rangle_*} H\mathbb{Z}_p^0(L\langle n\rangle)$$
$$\xrightarrow{\theta^*_{\langle n\rangle}} H\mathbb{Z}_p^0(\mathrm{BP}\langle n\rangle) \xrightarrow{\kappa^*_{\langle n\rangle}} \prod_{i=0}^{n} H\mathbb{Z}_p^{-2(p^i-1)}(\mathrm{BP}\langle n\rangle) \to \cdots$$

The left end group is zero by dimensional reason. The right end group is trivial except the $i = 0$ case, and the map $\kappa^*_{\langle n\rangle}$ reduces to p^* which is zero in mod p cohomology. Hence $\theta^*_{\langle n\rangle}$ is an isomorphism. Let $\eta^{\langle n\rangle} : L\langle n\rangle \to H\mathbb{Z}_p$ be the map corresponding to the generator $\rho^{\langle n\rangle} : \mathrm{BP}\langle n\rangle \to H\mathbb{Z}_p$ of the group $H\mathbb{Z}_p^0(\mathrm{BP}\langle n\rangle) \cong \mathbb{Z}_p$ under $\theta^*_{\langle n\rangle}$. That is, $\rho^{\langle n\rangle} = \theta^*_{\langle n\rangle}(\eta^{\langle n\rangle}) = \eta^{\langle n\rangle} \circ \theta_{\langle n\rangle}$.

We prove results for $L\langle n\rangle$ which correspond to Theorem 3-1 for L.

Theorem 4-1. (I) *Let X be a spectrum, and let $z \in L\langle n\rangle^k(X)$ be any element. Let $b_i = q_i\langle n\rangle_*(z) \in \mathrm{BP}\langle n\rangle^{k+2p^i-1}(X)$ for $1 \le i \le n$. Then we have*

$$pb_0 + v_1 b_1 + \cdots + v_n b_n = 0 \quad \text{in} \quad \mathrm{BP}\langle n\rangle^{k+1}(X).$$

(II) *There exists a map $\eta_{\langle n\rangle} : L \to L\langle n\rangle$ for $n \ge 0$ such that $\eta = \eta^{\langle n\rangle} \circ \eta_{\langle n\rangle}$, and the following diagram commutes for each $0 \le i \le n$:*

$$\begin{array}{ccccc} L & \xrightarrow{q_i} & \Sigma^{2p^i-1}\mathrm{BP} & \xrightarrow{\Sigma^{2p^i-1}\theta} & \Sigma^{2p^i-1}L \\ \downarrow{\scriptstyle \eta_{\langle n\rangle}} & & \downarrow{\scriptstyle \Sigma^{2p^i-1}\rho_{\langle n\rangle}} & & \downarrow{\scriptstyle \Sigma^{2p^i-1}\eta_{\langle n\rangle}} \\ L\langle n\rangle & \xrightarrow{q_i\langle n\rangle} & \Sigma^{2p^i-1}\mathrm{BP}\langle n\rangle & \xrightarrow{\Sigma^{2p^i-1}\theta_{\langle n\rangle}} & \Sigma^{2p^i-1}L\langle n\rangle \\ \downarrow{\scriptstyle \eta^{\langle n\rangle}} & & \downarrow{\scriptstyle \Sigma^{2p^i-1}\rho^{\langle n\rangle}} & & \downarrow{\scriptstyle \Sigma^{2p^i-1}\eta^{\langle n\rangle}} \\ H\mathbb{Z}_p & \xrightarrow{Q_i} & \Sigma^{2p^i-1}H\mathbb{Z}_p & = & \Sigma^{2p^i-1}H\mathbb{Z}_p. \end{array} \tag{4-3}$$

Milnor operations $\widehat{q}_i\langle n\rangle$ in $L\langle n\rangle$-theory can be defined by $\widehat{q}_i\langle n\rangle = \Sigma^{2p^i-1}\theta_{\langle n\rangle} \circ q_i\langle n\rangle$ for $0 \le i \le n$, and they satisfy $\widehat{q}_i\langle n\rangle \circ \widehat{q}_j\langle n\rangle = 0$ for $0 \le i, j \le n$.

(III) *The mod p cohomology of the spectrum $L\langle n\rangle$ is the following cyclic module over the mod p Steenrod algebra generated by $\eta^{\langle n\rangle}$:*

$$H\mathbb{Z}_p^*(L\langle n\rangle) \cong \Big[\mathcal{A}(p)\Big/\sum_{0\le i,j\le n}\mathcal{A}(p)Q_iQ_j\Big] \cdot \eta^{\langle n\rangle}. \tag{4-4}$$

(IV) *The coefficient group of $L\langle n\rangle$ cohomology theory is described as follows:*

$$L\langle n\rangle^* = 0 \quad \text{if } * > 0, \quad L\langle n\rangle^0 \cong \mathbb{Z}_p, \quad L\langle n\rangle^{-1} = L\langle n\rangle^{2k} = 0 \quad \text{for } k < 0.$$

In negative odd degrees less than -1, we have the following exact sequence:

$$0 \to L\langle n\rangle^{2k-1} \to \prod_{i=0}^{n} \mathrm{BP}\langle n\rangle^{2k+2p^i-2} \xrightarrow{\kappa_{\langle n\rangle *}} \mathrm{BP}\langle n\rangle^{2k} \to 0, \quad k < 0,$$

where $\kappa_{\langle n\rangle *}(b_0, b_1, \ldots, b_n) = \sum_{i=0}^{n} v_i b_i$.

In (I) of Theorem 4-1, we do not have to assume that X is a space as in Theorem 3-1 because we are dealing with finite sums and there are no problems of convergence. Because of the property (I), we call the spectrum $L\langle n\rangle$ *the spectrum of finite* BP*-linear relation in* $\mathrm{BP}\langle n\rangle$ *theory.*

For the proof, we need auxiliary spectra $\mathrm{BP}\langle n\rangle(v_i)$ for $0 \le i \le n$ defined as the cofibre of the v_i-multiplication map on $\mathrm{BP}\langle n\rangle$. We have the following cofibre sequence for $0 \le i \le n$:

$$\Sigma^{-1}\mathrm{BP}\langle n\rangle(v_i) \xrightarrow{\beta_i\langle n\rangle} \Sigma^{2p^i-1}\mathrm{BP}\langle n\rangle \xrightarrow{v_i} \mathrm{BP}\langle n\rangle \xrightarrow{j_i} \mathrm{BP}\langle n\rangle(v_i). \tag{4-5}$$

Examining the induced map j_i^* in mod p cohomology, we can easily see that j_i^* is an isomorphism in degree 0:

$$H\mathbb{Z}_p^0(\mathrm{BP}\langle n\rangle) \xleftarrow[\cong]{j_i^*} H\mathbb{Z}_p^0\big(\mathrm{BP}\langle n\rangle(v_i)\big).$$

In fact, when $i > 0$, this follows by dimensional reason, and when $i = 0$, we use $p^* = 0$ in mod p cohomology. Let $\rho_i^{\langle n\rangle}: \mathrm{BP}\langle n\rangle(v_i) \to H\mathbb{Z}_p$ be the map corresponding to $\rho^{\langle n\rangle}$ under j_i^*. Namely,

$$\rho^{\langle n\rangle} = j_i^*(\rho_i^{\langle n\rangle}) = \rho_i^{\langle n\rangle} \circ j_i. \tag{4-6}$$

Since $j_n = \rho^{\langle n\rangle}_{\langle n-1\rangle} : \mathrm{BP}\langle n\rangle \to \mathrm{BP}\langle n-1\rangle$, we actually have $\rho_n^{\langle n\rangle} = \rho^{\langle n-1\rangle}$.

The mod p cohomologies of the BP-module spectra $\mathrm{BP}\langle n\rangle$ and $\mathrm{BP}\langle n\rangle(v_i)$ as modules over the mod p Steenrod algebra are known.

Lemma 4-2 [BM]**.** *As modules over the mod p Steenrod algebra, the mod p cohomology modules of* $\mathrm{BP}\langle n\rangle$ *and* $\mathrm{BP}\langle n\rangle(v_i)$ *are the following cyclic modules generated by* $\rho^{\langle n\rangle}$ *and* $\rho_i^{\langle n\rangle}$:

$$\begin{aligned} H\mathbb{Z}_p^*(\mathrm{BP}\langle n\rangle) &\cong \Big[\mathcal{A}(p)\big/\mathcal{A}(p)(Q_0, Q_1, \dots, Q_n)\Big]\cdot \rho^{\langle n\rangle}, \\ H\mathbb{Z}_p^*\big(\mathrm{BP}\langle n\rangle(v_i)\big) &\cong \Big[\mathcal{A}(p)\big/\mathcal{A}(p)(Q_0, \dots, \widehat{Q}_i, \dots, Q_n)\Big]\cdot \rho_i^{\langle n\rangle}. \end{aligned} \tag{4-7}$$

Since $j_i^*(\rho_i^{\langle n\rangle}) = \rho^{\langle n\rangle}$ by (4-6), the induced map j_i^* on the mod p cohomology of the cofibre sequence (4-5) is surjective. Hence $v_i^* = 0$ by exactness for $0 \le i \le n$.

Next, we examine the following diagram where both rows are cofibre sequences:

(4-8)

$$\begin{tikzcd} \bigvee_{i=0}^n \Sigma^{-1}\mathrm{BP}\langle n\rangle(v_i) \arrow[r, "\bigvee \beta_i\langle n\rangle"] \arrow[d, "\Sigma^{-1}\tau"] & \bigvee_{i=0}^n \Sigma^{2(p^i-1)}\mathrm{BP}\langle n\rangle \arrow[r, "\bigvee v_i"] \arrow[d, "\cong"', "\text{h.e.}"] & \bigvee_{i=0}^n \mathrm{BP}\langle n\rangle \arrow[r, "\bigvee j_i"] \arrow[d, "\text{folding}"] & \bigvee_{i=0}^n \mathrm{BP}\langle n\rangle(v_i) \arrow[d, "\tau"] \\ \Sigma^{-1}L\langle n\rangle \arrow[r, "\prod q_i\langle n\rangle"] & \prod_{i=0}^n \Sigma^{2(p^i-1)}\mathrm{BP}\langle n\rangle \arrow[r, "\kappa_{\langle n\rangle}"] & \mathrm{BP}\langle n\rangle \arrow[r, "\theta_{\langle n\rangle}"] & L\langle n\rangle, \end{tikzcd}$$

where the middle square is commutative by the definition of $\kappa_{\langle n\rangle}$, and τ is the map induced on cofibres making the whole diagram commutative. A portion of the induced diagram of mod p cohomologies is

$$\begin{tikzcd}
\prod_{i=0}^{n} H\mathbb{Z}_p^{*-2(p^i-1)}\mathrm{BP}\langle n\rangle \arrow[d, equal] & \prod_{i=0}^{n} H\mathbb{Z}_p^{*}\mathrm{BP}\langle n\rangle \arrow[l, "\prod v_i^*"'] \\
\prod_{i=0}^{n} H\mathbb{Z}_p^{*-2(p^i-1)}\mathrm{BP}\langle n\rangle & H\mathbb{Z}_p^{*}\mathrm{BP}\langle n\rangle. \arrow[l, "\kappa_{\langle n\rangle}^*"'] \arrow[u, "\text{diagonal}"']
\end{tikzcd}$$

As noted in §3, the map induced from the folding map is the diagonal map. Since $v_i^* = 0$ for $0 \le i \le n$ as a consequence of Lemma 4-2, we must have $\kappa_{\langle n\rangle}^* = 0$. Thus the mod p cohomology exact sequence induced from the bottom row cofibre sequence of (4-8) splits into short exact sequences. We record these observations in the next lemma.

Lemma 4-3. (i) *In the mod p cohomology exact sequence of the cofibre sequence* (4-5)*, we have $v_i^* = 0$ for $0 \le i \le n$ and we have a short exact sequence*:

$$0 \to H\mathbb{Z}_p^{*-2p^i+1}(\mathrm{BP}\langle n\rangle) \xrightarrow{\beta_i\langle n\rangle^*} H\mathbb{Z}_p^*\big(\mathrm{BP}\langle n\rangle(v_i)\big) \xrightarrow{j_i^*} H\mathbb{Z}_p^*\big(\mathrm{BP}\langle n\rangle\big) \to 0.$$

(ii) *The map $\kappa_{\langle n\rangle}$ in the cofibre sequence* (4-2) *induces a zero map in mod p cohomology exact sequence, and we have the following short exact sequence*:

(4-9)
$$0 \to \prod_{i=0}^{n} H\mathbb{Z}_p^{*-2p^i+1}(\mathrm{BP}\langle n\rangle) \xrightarrow{\sum q_i\langle n\rangle^*} H\mathbb{Z}_p^*(L\langle n\rangle) \xrightarrow{\theta_{\langle n\rangle}^*} H\mathbb{Z}_p^*(\mathrm{BP}\langle n\rangle) \to 0.$$

In the diagram (4-8), we examine components of the map τ. For $0 \le i \le n$, let

$$\tau_i : \mathrm{BP}\langle n\rangle(v_i) \xrightarrow{\text{inclusion}} \bigvee_{i=0}^{n} \mathrm{BP}\langle n\rangle(v_i) \xrightarrow{\tau} L\langle n\rangle. \tag{4-10}$$

From the ℓ-th component of the right end square of the diagram (4-8), we have the following diagram of mod p cohomology groups:

$$\begin{tikzcd}
H\mathbb{Z}_p^0(\mathrm{BP}\langle n\rangle) \arrow[d, equal] & H\mathbb{Z}_p^0\big(\mathrm{BP}\langle n\rangle(v_\ell)\big) \arrow[l, "j_\ell^*"', "\cong"] \\
H\mathbb{Z}_p^0(\mathrm{BP}\langle n\rangle) & H\mathbb{Z}_p^0(L\langle n\rangle). \arrow[l, "\theta_{\langle n\rangle}^*"', "\cong"] \arrow[u, "\tau_\ell^*"']
\end{tikzcd}$$

Here the top horizontal map is an isomorphism by Lemma 4-2. We have already observed that $\theta_{\langle n\rangle}^*$ is an isomorphism in degree 0 prior to Theorem 4-1. For these maps we know that $j_\ell^*(\rho_\ell^{\langle n\rangle}) = \rho^{\langle n\rangle}$ by (4-6) and $\theta_{\langle n\rangle}^*(\eta^{\langle n\rangle}) = \rho^{\langle n\rangle}$ from the definition of $\eta^{\langle n\rangle}$. Hence we have

$$\rho_\ell^{\langle n\rangle} = \tau_\ell^*(\eta^{\langle n\rangle}) = \eta^{\langle n\rangle} \circ \tau_\ell. \tag{4-11}$$

The case $\ell = n$ is relevant to a Sullivan exact sequence, and (4-11) will be used later in Proposition 4-4.

Proof of Theorem 4-1. (I) The mod p cohomology exact sequence of the cofibre sequence (4-2) is of the form

$$\cdots \to L\langle n\rangle^k(X) \xrightarrow{\prod q_i\langle n\rangle_*} \prod_{i=0}^{n} \mathrm{BP}\langle n\rangle^{k+2p^i-1}(X) \xrightarrow{\kappa_{\langle n\rangle_*}} \mathrm{BP}\langle n\rangle^{k+1}(X) \to \cdots .$$

For an element $z \in L\langle n\rangle^k(X)$, we have $\prod_{i=0}^n q_i\langle n\rangle_*(z) = (b_0, b_1, \dots, b_n)$. Since ${\kappa_{\langle n\rangle}}_*(b_0, \dots, b_n) = \sum_i v_i b_i$ by (2-13), by exactness of the above sequence, we have $\sum_i v_i b_i = 0$ in $\mathrm{BP}\langle n\rangle^{k+1}(X)$. This proves (I).

(IV) We examine the homotopy exact sequence of the cofibre sequence (4-2). We observe that $\mathrm{Im}\,({\kappa_{\langle n\rangle}}_*) = (p, v_1, \dots, v_n) = I_{n+1}$, and consequently ${\kappa_{\langle n\rangle}}_*$ is surjective in degrees $* < 0$ and $\mathrm{BP}\langle n\rangle^*/I_{n+1} = \mathbb{Z}_p$ in degree 0. Thus, the long exact sequence splits into short exact sequences when $* < 0$. The result follows by noting that $\mathrm{BP}\langle n\rangle^*$ is even dimensional.

(II) The spectra map $\eta_{\langle n\rangle}\colon L \to L\langle n\rangle$ is defined as the induced map between the cofibres of κ and $\kappa_{\langle n\rangle}$. Thus, the following diagram commutes:

$$\text{(4-12)}\qquad \begin{array}{ccccccc} \Sigma^{-1}L & \xrightarrow{\prod q_i} & \prod_{i=0}^{\infty}\Sigma^{2(p^i-1)}\mathrm{BP} & \xrightarrow{\kappa} & \mathrm{BP} & \xrightarrow{\theta} & L \\ \downarrow{\scriptstyle \Sigma^{-1}\eta_{\langle n\rangle}} & & \downarrow & & \downarrow{\scriptstyle \rho_{\langle n\rangle}} & & \downarrow{\scriptstyle \eta_{\langle n\rangle}} \\ \Sigma^{-1}L\langle n\rangle & \xrightarrow{\prod q_i\langle n\rangle} & \prod_{i=0}^{n}\Sigma^{2p^i-1}\mathrm{BP}\langle n\rangle & \xrightarrow{\kappa_{\langle n\rangle}} & \mathrm{BP}\langle n\rangle & \xrightarrow{\theta_{\langle n\rangle}} & L\langle n\rangle, \end{array}$$

where the second vertical map is the obvious map: a projection onto the first n factors, followed by the map $\prod \Sigma^{2p^i-1}\rho_{\langle n\rangle}$. From this construction of the map $\eta_{\langle n\rangle}$, the commutativity of small squares in (4-3) is obvious, except the lower left square.

We consider mod p cohomology groups of degree zero for the right square of the above diagram. We have

$$\begin{array}{ccc} H\mathbb{Z}_p^0(\mathrm{BP}) & \xleftarrow[\cong]{\theta^*} & H\mathbb{Z}_p^0(L) \\ \uparrow{\scriptstyle \rho^*_{\langle n\rangle}} & & \uparrow{\scriptstyle \eta^*_{\langle n\rangle}} \\ H\mathbb{Z}_p^0(\mathrm{BP}\langle n\rangle) & \xleftarrow[\cong]{\theta^*_{\langle n\rangle}} & H\mathbb{Z}_p^0(L\langle n\rangle). \end{array}$$

Since $\eta^{\langle n\rangle} \in H\mathbb{Z}_p^0(L\langle n\rangle)$ is defined by $\theta^*_{\langle n\rangle}(\eta^{\langle n\rangle}) = \rho^{\langle n\rangle}$, and since $\rho^*_{\langle n\rangle}(\rho^{\langle n\rangle}) = \rho$, the commutativity of the above square implies that $\theta^*\big(\eta^*_{\langle n\rangle}(\eta^{\langle n\rangle})\big) = \rho^*_{\langle n\rangle} \circ \theta^*_{\langle n\rangle}(\eta^{\langle n\rangle}) = \rho = \theta^*(\eta)$ using the definition of η for the last equality. Since θ^* is an isomorphism in degree 0, we have $\eta = \eta^{\langle n\rangle} \circ \eta_{\langle n\rangle}$. This proves the first statement.

Now using Theorem 3-1 and the commutativity of small squares in (4-3), except the lower left square, we have

$$(Q_i \circ \eta^{\langle n\rangle}) \circ \eta_{\langle n\rangle} = Q_i \circ \eta = \rho \circ q_i = \rho^{\langle n\rangle} \circ \rho_{\langle n\rangle} \circ q_i = \rho^{\langle n\rangle} \circ q_i\langle n\rangle \circ \eta_{\langle n\rangle}.$$

Here, both elements $Q_i \circ \eta^{\langle n\rangle}$ and $\rho^{\langle n\rangle} \circ q_i\langle n\rangle$ are in $H\mathbb{Z}_p^{2p^i-1}(L\langle n\rangle)$ for $0 \le i \le n$. We claim that the map $\eta^*_{\langle n\rangle}$ in this degree is an isomorphism.

To see this, we consider the mod p cohomology diagram induced from (4-12) in relevant dimension:

$$\begin{array}{ccccccc}
H\mathbb{Z}_p^{2p^i-1}(\mathrm{BP}) & \xleftarrow{\theta^*} & H\mathbb{Z}_p^{2p^i-1}(L) & \xleftarrow{(\prod q_i)^*} & \prod_{j=0}^{\infty} H\mathbb{Z}_p^{2p^i-2p^j}(\mathrm{BP}) & \xleftarrow{\kappa^*=0} & \\
\uparrow{\scriptstyle \rho^*_{\langle n\rangle}} & & \uparrow{\scriptstyle \eta^*_{\langle n\rangle}} & & \uparrow & & \\
H\mathbb{Z}_p^{2p^i-1}(\mathrm{BP}\langle n\rangle) & \xleftarrow{\theta^*_{\langle n\rangle}} & H\mathbb{Z}_p^{2p^i-1}(L\langle n\rangle) & \xleftarrow{(\prod q_i\langle n\rangle)^*} & \prod_{j=0}^{n} H\mathbb{Z}_p^{2p^i-2p^j}(\mathrm{BP}\langle n\rangle) & \xleftarrow{\kappa^*_{\langle n\rangle}=0} & .
\end{array}$$

By dimensional reason, the infinite product at the upper right corner reduces to a finite product $\prod_{j=0}^{i} H\mathbb{Z}_p^{2p^i-2p^j}(\mathrm{BP})$, where $0 \le i \le n$. The same reduction occurs for the product at the lower right corner. By Lemma 3-2 and Lemma 4-2, the map

$$\rho^*_{\langle n\rangle} : H\mathbb{Z}_p^*(\mathrm{BP}\langle n\rangle) \longrightarrow H\mathbb{Z}_p^*(\mathrm{BP})$$

is an isomorphism in degree $* < 2p^{n+1} - 1$. Hence the right end vertical map is an isomorphism. Both of the left end odd degree cohomology groups are zero by Lemma 3-2 and Lemma 4-2. We know that $\kappa^* = 0$ and $\kappa^*_{\langle n\rangle} = 0$ by Lemma 3-4 and Lemma 4-3, respectively. Thus, by exactness, both $(\prod q_i)^*$ and $(\prod q_i\langle n\rangle)^*$ are isomorphisms in this degree. Hence the middle vertical map $\eta^*_{\langle n\rangle}$ is also an isomorphism. Thus we can cancel $\eta^*_{\langle n\rangle}$ from our previous calculation and we have $Q_i \circ \eta^{\langle n\rangle} = \rho^{\langle n\rangle} \circ q_i\langle n\rangle$ for $0 \le i \le n$. This proves the commutativity of the lower left square of (4-3).

For the remaining statement, we simply note that the composition $\widehat{q}_i\langle n\rangle \circ \widehat{q}_j\langle n\rangle = \theta_{\langle n\rangle} \circ (q_i\langle n\rangle \circ \theta_{\langle n\rangle}) \circ q_j\langle n\rangle = 0$, because the middle composition is zero by exactness of the cofibre sequence. This completes the proof of (II).

(III) First we prove the relation $Q_iQ_j(\eta^{\langle n\rangle}) = 0$ for $0 \le i, j \le n$. From Part (II), we have $q_i\langle n\rangle^*(\rho^{\langle n\rangle}) = Q_i(\eta^{\langle n\rangle})$ for $0 \le i \le n$. By naturality of cohomology operations, we have

$$Q_iQ_j(\eta^{\langle n\rangle}) = Q_i \cdot q_j\langle n\rangle^*(\rho^{\langle n\rangle}) = q_j\langle n\rangle^*\big(Q_i(\rho^{\langle n\rangle})\big) = 0,$$

where the last equality is due to Lemma 4-2. From the exact sequence (4-9), as a $\mathbb{Z}_p$-vector space (not as a $\mathcal{A}(p)$-module), $H\mathbb{Z}_p^*(L\langle n\rangle)$ is isomorphic to

$$\bigoplus_{i=0}^{n} \Big[\mathcal{A}(p)\big/\mathcal{A}(p)(Q_0, \ldots, Q_n)\Big] Q_i(\eta^{\langle n\rangle}) \oplus \Big[\mathcal{A}(p)\big/\mathcal{A}(p)(Q_0, \ldots, Q_n)\Big](\eta^{\langle n\rangle}).$$

Thus $H\mathbb{Z}_p^*(L\langle n\rangle)$ is a cyclic $\mathcal{A}(p)$-module generated by $\eta^{\langle n\rangle}$ whose only relations are $Q_iQ_j(\eta^{\langle n\rangle}) = 0$ for $0 \le i, j \le n$. This completes the proof of Part (III) and hence of Theorem 4-1. □

We give an immediate application of Part (II) of Theorem 4-1 to Sullivan exact sequences. Although this is only a slight improvement of a well known fact, it seems that this result has not been explicitly stated before.

The n-th Sullivan exact sequence is a (co)homology exact sequence associated to the following cofibre sequence:

$$\Sigma^{-1}\mathrm{BP}\langle n-1\rangle \xrightarrow{\Delta_n} \Sigma^{2(p^n-1)}\mathrm{BP}\langle n\rangle \xrightarrow{v_n} \mathrm{BP}\langle n\rangle \xrightarrow{\rho^{\langle n\rangle}_{\langle n-1\rangle}} \mathrm{BP}\langle n-1\rangle.$$

We are interested in the induced map Δ_{n*} in cohomology.

Proposition 4-4. *The connecting homomorphisms in Sullivan exact sequences correspond exactly to Milnor primitives. That is, the following diagram commutes exactly, not up to a nonzero constant in $\mathbb{Z}_p$:*

$$\begin{array}{ccc} \mathrm{BP}\langle n-1\rangle^k(X) & \xrightarrow{\Delta_{n*}} & \mathrm{BP}\langle n\rangle^{k+2p^n-1}(X) \\ \Big\downarrow{\scriptstyle \rho_*^{\langle n-1\rangle}} & & \Big\downarrow{\scriptstyle \rho_*^{\langle n\rangle}} \\ H\mathbb{Z}_p^k(X) & \xrightarrow{Q_n} & H\mathbb{Z}_p^{k+2p^n-1}(X). \end{array} \tag{4-13}$$

Proof. We consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{BP}\langle n-1\rangle & \xrightarrow{\Delta_n=\beta_n\langle n\rangle} & \Sigma^{2p^n-1}\mathrm{BP}\langle n\rangle \\ \Big\downarrow{\scriptstyle \tau_n} & & \Big\| \\ L\langle n\rangle & \xrightarrow{q_n\langle n\rangle} & \Sigma^{2p^n-1}\mathrm{BP}\langle n\rangle \\ \Big\downarrow{\scriptstyle \eta^{\langle n\rangle}} & & \Big\downarrow{\scriptstyle \rho^{\langle n\rangle}} \\ H\mathbb{Z}_p & \xrightarrow{Q_n} & \Sigma^{2p^n-1}H\mathbb{Z}_p. \end{array}$$

The upper square commutes because it is a component of the left square of the commutative diagram (4-8), and the commutativity of the lower square is due to Part(II) of Theorem 4-1. By (4-11) and the discussion right before Lemma 4-2, we have $\eta^{\langle n\rangle}\circ\tau_n=\rho_n^{\langle n\rangle}=\rho^{\langle n-1\rangle}$. Then the commutativity of the above diagram completes the proof. □

Next, we show that the family of spectra $\{L\langle n\rangle\}_n$ forms a tower similar to the BP-tower (1-1) and (2-1).

Proposition 4-5. *There exist spectra maps $\eta_{\langle n\rangle}^{\langle n+1\rangle}: L\langle n+1\rangle\to L\langle n\rangle$ for $n\geq 0$ with the following properties.*

(I) *The following diagram commutes for $0\leq i\leq n$:*

$$\begin{array}{ccc} L\langle n+1\rangle & \xrightarrow{q_i\langle n+1\rangle} & \Sigma^{2p^i-1}\mathrm{BP}\langle n+1\rangle \\ \Big\downarrow{\scriptstyle \eta_{\langle n\rangle}^{\langle n+1\rangle}} & & \Big\downarrow{\scriptstyle \rho_{\langle n\rangle}^{\langle n+1\rangle}} \\ L\langle n\rangle & \xrightarrow{q_i\langle n\rangle} & \Sigma^{2p^i-1}\mathrm{BP}\langle n\rangle. \end{array} \tag{4-14}$$

(II) *We have the following tower of spectra and compatible spectra maps:*

$$L \xrightarrow{\eta_{\langle n+1\rangle}} L\langle n+1\rangle \xrightarrow{\eta_{\langle n\rangle}^{\langle n+1\rangle}} L\langle n\rangle \xrightarrow{\eta^{\langle n\rangle}} L\langle 0\rangle = H\mathbb{Z}_p, \tag{4-15}$$

where $\eta_{\langle n\rangle}^{\langle n+1\rangle}\circ\eta_{\langle n+1\rangle}=\eta_{\langle n\rangle}$ and $\eta^{\langle n\rangle}\circ\eta_{\langle n\rangle}^{\langle n+1\rangle}=\eta^{\langle n+1\rangle}$ for $n\geq 0$.

Proof. The spectra map $\eta_{\langle n\rangle}^{\langle n+1\rangle}$ is defined by the induced map between the cofibres of maps $\kappa_{\langle n+1\rangle}$ and $\kappa_{\langle n\rangle}$. Namely, it is the cofibre extension of the bottom middle square of the following diagram:

$$
\begin{tikzcd}
 & \prod_{i=0}^{\infty} \Sigma^{2(p^i-1)}\mathrm{BP} \arrow[r, "\kappa"] \arrow[d] & \mathrm{BP} \arrow[r, "\theta"] \arrow[d, "\rho_{\langle n+1\rangle}"] & L \arrow[d, "\eta_{\langle n+1\rangle}"] \\
\Sigma^{-1}L\langle n+1\rangle \arrow[r, "\prod q_i\langle n+1\rangle"] \arrow[d, "\eta^{\langle n+1\rangle}_{\langle n\rangle}"] & \prod_{i=0}^{n+1} \Sigma^{2(p^i-1)}\mathrm{BP}\langle n+1\rangle \arrow[r, "\kappa_{\langle n+1\rangle}"] \arrow[d] & \mathrm{BP}\langle n+1\rangle \arrow[r, "\theta_{\langle n+1\rangle}"] \arrow[d, "\rho^{\langle n+1\rangle}_{\langle n\rangle}"] & L\langle n+1\rangle \arrow[d, "\eta^{\langle n+1\rangle}_{\langle n\rangle}"] \\
\Sigma^{-1}L\langle n+1\rangle \arrow[r, "\prod q_i\langle n\rangle"] & \prod_{i=0}^{n} \Sigma^{2(p^i-1)}\mathrm{BP}\langle n\rangle \arrow[r, "\kappa_{\langle n\rangle}"] & \mathrm{BP}\langle n\rangle \arrow[r, "\theta_{\langle n\rangle}"] & L\langle n\rangle,
\end{tikzcd}
$$

where the unnamed vertical maps are compositions of projection maps onto the first $n+1$ and n factors followed by $\prod_{i=0}^{n+1}\rho_{\langle n+1\rangle}$ and $\prod_{i=0}^{n}\rho^{\langle n+1\rangle}_{\langle n\rangle}$, respectively. From the commutativity of the lower left square, (I) follows.

For (II), we observe that $\eta_{\langle n+1\rangle}$ is the cofibre extension of the upper middle square, and $\eta^{\langle n+1\rangle}_{\langle n\rangle}$ is the cofibre extension of the lower middle square. Thus, their composition $\eta^{\langle n+1\rangle}_{\langle n\rangle}\circ\eta_{\langle n+1\rangle}$ is the cofibre extension of the combined middle squares, which is $\eta_{\langle n\rangle}$ by definition.

The other composition formula in (4-15) can be proved in a similar way. $\square$

Next, we consider an unstable property of the L-tower. Recall that the spectra L and $\langle n\rangle$ are spectra of BP linear relations.

Theorem 4-6. *Let X be a space. Suppose $k \le 2(p^{n-1}+\cdots+p+1)+1$. Then the homomorphism*

$$\rho_{\langle n\rangle *}\colon L^k(X) \longrightarrow L\langle n\rangle^k(X) \tag{4-16}$$

is an epimorphism. Consequently, whenever there is a finite BP*-linear relation*

$$pb_0^{\langle n\rangle}+v_1b_1^{\langle n\rangle}+\cdots+v_nb_n^{\langle n\rangle}=0 \quad \textit{in} \quad \mathrm{BP}\langle n\rangle^{k+1}(X), \tag{4-17}$$

there exists a corresponding infinite sum BP*-linear relation*

$$pb_0+v_1b_1+\cdots+v_nb_n+\cdots=0 \quad \textit{in} \quad \mathrm{BP}^{k+1}(X) \tag{4-18}$$

lifting the previous one in the sense that $\rho_{\langle n\rangle *}(b_\ell)=b_\ell^{\langle n\rangle}$ for $0\le \ell\le n$.

Proof. Let $x\in L\langle n\rangle^k(X)$ be an arbitrary element. For $0\le i\le n$, let $q_i\langle n\rangle_*(x)=b_i^{\langle n\rangle}\in \mathrm{BP}\langle n\rangle^{k+2p^i-1}(X)$. Then by Theorem 4-1 Part (I), we have a finite BP-linear relation $\sum_{i=0}^{n} v_ib_i^{\langle n\rangle}=0$ in $\mathrm{BP}\langle n\rangle^{k+1}(X)$. Since $|b_i^{\langle n\rangle}|\le 2(p^n+\cdots+p+1)$ for $0\le i\le n$, the elements $b_i^{\langle n\rangle}$ lift to elements b_i in the BP theory of X by Wilson's Splitting Theorem [Theorem 2-2]. Then we have

$$\rho_{\langle n\rangle *}(pb_0+v_1b_1+\cdots+v_nb_n)=\sum_{i=0}^{n} v_ib_i^{\langle n\rangle}=0\in \mathrm{BP}\langle n\rangle^{k+1}(X).$$

By induction, we construct elements $b_{n+1},\ \dots,\ b_m,\ \dots$ in BP theory such that for any $m\ge n$ we have

$$\rho_{\langle m\rangle *}(pb_0+v_1b_1+\cdots+v_mb_m)=0 \quad \text{in} \quad \mathrm{BP}\langle m\rangle^{k+1}(X). \tag{$*$}$$

As an inductive step, suppose we have chosen $b_{n+1}, \dots, b_m$ with the above property for some $m \geq n$. Let $\rho_{\langle m+1\rangle_*}(b_i) = b_i^{\langle m+1\rangle}$ for $0 \leq i \leq m$. Then by inductive hypothesis, we have $\rho^{\langle m+1\rangle}_{\langle m\rangle_*}\big(\sum_{i=0}^m v_i b_i^{\langle m+1\rangle})\big) = 0$, and by exactness of the Sullivan sequence, there exists an element $b_{m+1}^{\langle m+1\rangle} \in \mathrm{BP}\langle m+1\rangle^{k+2p^{m+1}-1}(X)$ such that

$$\sum_{i=0}^m v_i b_i^{\langle m+1\rangle} + v_{m+1} b_{m+1}^{\langle m+1\rangle} = 0 \quad \text{in} \quad \mathrm{BP}\langle m+1\rangle^{k+1}(X).$$

Since $|b_{m+1}^{\langle m+1\rangle}| \leq 2(p^{m+1} + p^m + \cdots + p + 1)$, this element lifts to an element b_{m+1} in the BP cohomology. With this choice of b_{m+1}, we have $\rho_{\langle m+1\rangle_*}\big(\sum_{i=0}^{m+1} v_i b_i\big) = 0$ in $\mathrm{BP}\langle m+1\rangle^*(X)$. This completes the inductive step and we have constructed elements $b_i \in \mathrm{BP}^*(X)$ for all $i \geq 0$ with the required property $(*)$ for all $m \geq n$. This property in turn means that the infinite sum $\sum_{i=0}^\infty v_i b_i$ converges to 0 in the BP topology. Hence there exists an element $z \in L^k(X)$ such that $q_{i*}(z) = b_i$ for all $i \geq 0$. We examine the difference $\eta_{\langle n\rangle_*}(z) - x$ in $L\langle n\rangle^k(X)$.

By the commutativity of the upper left square of the diagram (4-3), we have

$$q_i\langle n\rangle_*\big(\eta_{\langle n\rangle_*}(z) - x\big) = \rho_{\langle n\rangle_*} \circ q_{i*}(z) - q_i\langle n\rangle_*(x) = \rho_{\langle n\rangle_*}(b_i) - b_i^{\langle n\rangle} = 0,$$

for $0 \leq i \leq n$. Then from the exact sequence

$$\cdots \to \mathrm{BP}\langle n\rangle^k(X) \xrightarrow{\theta_{\langle n\rangle_*}} L\langle n\rangle^k(X) \xrightarrow{\prod q_i\langle n\rangle_*} \prod_{i=0}^n \mathrm{BP}\langle n\rangle^{k+2p^i-1}(X) \xrightarrow{\kappa_{\langle n\rangle_*}} \cdots,$$

we see that there exists an element $y \in \mathrm{BP}\langle n\rangle^k(X)$ such that $\theta_{\langle n\rangle_*}(y) = \eta_{\langle n\rangle_*}(z) - x$. Since $k \leq 2(p^n + \cdots + p + 1)$, the element y lifts to $\tilde{y} \in \mathrm{BP}^k(X)$ by the Splitting Theorem. Then in the commutative diagram

$$\begin{array}{ccc}
\mathrm{BP}^k(X) & \xrightarrow{\theta_*} & L^k(X) \\
\Big\downarrow{\scriptstyle \rho_{\langle n\rangle_*}} & & \Big\downarrow{\scriptstyle \eta_{\langle n\rangle_*}} \\
\mathrm{BP}\langle n\rangle^k(X) & \xrightarrow{\theta_{\langle n\rangle_*}} & L\langle n\rangle^k(X),
\end{array}$$

we have $\eta_{\langle n\rangle_*}\big(z - \theta_*(\tilde{y})\big) = \eta_{\langle n\rangle_*}(z) - \theta_{\langle n\rangle_*}(y) = x$. Hence the homomorphism $\eta_{\langle n\rangle_*} \colon L^k(X) \to L\langle n\rangle^k(X)$ is epic for $k \leq 2(p^{n-1} + \cdots + p + 1) + 1$. This completes the proof. $\square$

Remark 4-7. The above argument actually proves a slightly stronger statement. If we have $b_0^{\langle n\rangle} = \cdots = b_k^{\langle n\rangle} = 0$ for some $k \leq n$ in (4-17), then in (4-18) we can choose b_i's in such a way that $b_0 = \cdots = b_k = 0$ as well. This remark will be useful later in Theorem 5-6.

5. BP-linear relations and Milnor primitives

Existence of a BP-linear relation of the form $pb_0 + v_1 b_1 + \cdots + v_n b_n + \cdots = 0$ in the BP cohomology of a space X translates via Thom map ρ_* to a certain property of the action of the Milnor primitives on the mod p cohomology of the space X.

The next proposition was first proved in [Y1] for finite BP-linear relations when X has the homotopy type of a smooth manifold by a geometric method of manifolds with singularities [Ba, Mo]. Our method, using the stable homotopy theory, gives a simpler proof for a fully general result.

Proposition 5-1. *Let X be a space and let k be a positive integer. Suppose elements $b_i \in \mathrm{BP}^{k+2p^i-1}(X)$ for $i \geq 0$ satisfy the following relation*:

$$pb_0 + v_1 b_1 + \cdots + v_n b_n + \cdots = 0 \quad \textit{in} \quad \mathrm{BP}^{k+1}(X).$$

Then, there exists an element $x \in H\mathbb{Z}_p^k(X)$ such that in $H\mathbb{Z}_p^*(X)$ we have

$$\rho_*(b_i) = Q_i(x) \qquad \textit{for all} \quad i \geq 0, \tag{5-1}$$

where $\rho_* \colon \mathrm{BP}^*(X) \to H\mathbb{Z}_p^*(X)$ is the Thom map.

Proof. By our hypothesis, the element $\vec{b} = (b_0, b_1, \ldots, b_n, \ldots) \in \prod_{i=0}^{\infty} \mathrm{BP}^{k+2(p^i-1)}(X)$ has the property $\kappa_*(\vec{b}) = 0$ in the exact sequence

$$\cdots \to L^k(X) \xrightarrow{\prod q_{i*}} \prod_{i=0}^{\infty} \mathrm{BP}^{k+2p^i-1}(X) \xrightarrow{\kappa_*} \mathrm{BP}^{k+1}(X) \to \cdots.$$

Hence there exists an element $z \in L^k(X)$ such that $q_{i*}(z) = b_i$ for all $i \geq 0$. By Part (II) of Theorem 3-1, we have the following commutative diagram:

$$\begin{array}{ccc} L^k(X) & \xrightarrow{q_{i*}} & \mathrm{BP}^{k+2p^i-1}(X) \\ \downarrow{\scriptstyle \eta_*} & & \downarrow{\scriptstyle \rho_*} \\ H\mathbb{Z}_p^k(X) & \xrightarrow{Q_i} & H\mathbb{Z}_p^{k+2p^i-1}(X). \end{array}$$

Now by letting $x = \eta_*(z)$, we have $\rho_*(b_i) = Q_i\big(\eta_*(z)\big) = Q_i(x)$ for all $i \geq 0$. This completes the proof. $\square$

In the statement of Proposition 5-1, we need to assume that X is a space so that the infinite sum $\sum_i v_i b_i$ makes sense in the BP topology. However, when we consider finite BP-linear relations, X can be any spectrum.

Remark 5-2. A similar proposition can be easily stated for BP *homology* theory of any spectrum X. In BP homology, only finite BP-linear relations can exist by degree reason, and hence X does not have to be a space.

Now we consider the converse problem of constructing BP-linear relations in the BP cohomology of a space X starting with an element x in the mod p cohomology of X. Obviously not all elements x are related to BP-linear relations in a way described in Proposition 5-1. One sufficient condition on x is that $x \in \mathrm{Im}\,(\eta_*)$. Actually, we can use a slightly weaker but useful condition. We consider a set

$$J = \{y \in H\mathbb{Z}_p^*(X) \mid Q_i(y) = 0 \text{ for all } i \geq 0\} \tag{5-2}$$

of all elements $y \in H\mathbb{Z}_p^*(X)$ on which all Milnor primitives act trivially. We also define a related set

$$J_n = \{y \in H\mathbb{Z}_p^*(X) \mid Q_i(y) = 0 \text{ for } 0 \leq i \leq n\}, \tag{5-2$'$}$$

for any $n \geq 0$. By the derivation property of the Milnor primitives, the subsets J and J_n for any n are actually subalgebras of $H\mathbb{Z}_p^*(X)$. Now we define

$$\widehat{\mathrm{Im}}\,(\eta_*) = \mathrm{Im}\,(\eta_*) + J \subset H\mathbb{Z}_p^*(X). \tag{5-3}$$

Any element in this set behaves in the same way as an element in $\mathrm{Im}\,(\eta_*)$ as far as the action of the Milnor primitives is concerned.

When $x \in \operatorname{Im}(\eta_*)$, by Part (III) of Theorem 3-1 we have $Q_iQ_j(x) = 0$ for any $i, j \geq 0$. Thus, this is a necessary condition for a mod p cohomology element x to belong to $\widehat{\operatorname{Im}}(\eta_*)$.

The following converse result is more or less straightforward.

Proposition 5-3. *Let X be a space, and suppose an element $x \in H\mathbb{Z}_p^*(X)$ is such that $x \in \widehat{\operatorname{Im}}(\eta_*)$. Then there exist elements $b_i \in \mathrm{BP}^{k+2p^i-1}(X)$ for $i \geq 0$ such that*

$$\begin{aligned} &pb_0 + v_1b_1 + \cdots + v_nb_n + \cdots = 0 \quad \text{in} \quad \mathrm{BP}^{k+1}(X), \\ &\text{and} \quad \rho_*(b_i) = Q_i(x) \quad \text{for all} \quad i \geq 0. \end{aligned} \tag{5-4}$$

Proof. Since $x \in \widehat{\operatorname{Im}}(\eta_*)$, there exists an element $z \in L^k(X)$ such that $Q_i(x) = Q_i\big(\eta_*(z)\big)$ for all $i \geq 0$. Let $b_i = q_{i*}(z) \in \mathrm{BP}^{k+2p^i-1}(X)$ for $i \geq 0$. Then from the cohomology exact sequence

$$\cdots \longrightarrow L^k(X) \xrightarrow{\prod q_{i*}} \prod_{i=0}^{\infty} \mathrm{BP}^{k+2p^i-1}(X) \xrightarrow{\kappa_*} \mathrm{BP}^{k+1}(X) \longrightarrow \cdots,$$

we have $\sum_{i\geq 0} v_ib_i = \kappa_*(b_0, b_1, \ldots, b_n, \ldots) = 0$ using Theorem 2-7. Since $\rho \circ q_i = Q_i \circ \eta$ for all $i \geq 0$ by Theorem 3-1, we have $\rho_*(b_i) = Q_i\big(\eta_*(z)\big) = Q_i(x)$ for $i \geq 0$. This completes the proof. $\square$

Recall that we have the following tower of $L\langle n\rangle$ spectra [Proposition 4-5]:

$$\eta : L \xrightarrow{\eta_{\langle n+1\rangle}} L\langle n+1\rangle \xrightarrow{\eta^{\langle n+1\rangle}_{\langle n\rangle}} L\langle n\rangle \xrightarrow{\eta^{\langle n\rangle}} L\langle 0\rangle = H\mathbb{Z}_p. \tag{5-5}$$

Corresponding to this tower, we have the following nested subsets of $H\mathbb{Z}_p^*(X)$:

$$\begin{aligned} &\widehat{\operatorname{Im}}(\eta_*) \subset \cdots \subset \widehat{\operatorname{Im}}(\eta_*^{\langle n+1\rangle}) \subset \widehat{\operatorname{Im}}(\eta_*^{\langle n\rangle}) \subset \cdots \subset H\mathbb{Z}_p^*(X), \\ &\text{where} \quad \widehat{\operatorname{Im}}(\eta_*^{\langle n\rangle}) = \operatorname{Im}(\eta_*^{\langle n\rangle}) + J_n \quad \text{for} \quad n \geq 0. \end{aligned} \tag{5-6}$$

Here, J_n is defined in (5-2′). Let $J_n^k = J_n \cap H\mathbb{Z}_p^k(X)$.

Lemma 5-4. *Consider the action of the n-th Milnor primitive*:

$$Q_n : H\mathbb{Z}_p^k(X) \longrightarrow H\mathbb{Z}_p^{k+2p^n-1}(X).$$

Suppose $k < 2p^n$. Then for any $j \geq 0$, we have $Q_{n+j}\big(\operatorname{Ker} Q_n\big) = 0$. In particular, $Q_{n+j}(J_n^k) = 0$ for all $j \geq 0$.

Proof. By induction on j. When $j = 0$, our conclusion is obvious. Assume that $Q_{n+j}\big(\operatorname{Ker} Q_n\big) = 0$ for some $j \geq 0$. Since $k < 2p^n \leq 2p^{n+j}$, we have $\mathcal{P}^{p^{n+j}}(x) = 0$ for any $x \in H\mathbb{Z}_p^k(X)$. Then $Q_{n+j+1}(x) = [\mathcal{P}^{p^{n+j}}, Q_{n+j}](x) = \mathcal{P}^{p^{n+j}}Q_{n+j}(x) = 0$ by inductive hypothesis. This completes the inductive step and the proof is complete. $\square$

For elements in $\widehat{\operatorname{Im}}(\eta_*)$, we have Proposition 5-3. For elements in $\widehat{\operatorname{Im}}(\eta_*^{\langle n\rangle})$, we have the following result.

Proposition 5-5. *Let X be a space. Suppose $x \in H\mathbb{Z}_p^k(X)$ is such that $x \in \widehat{\operatorname{Im}}(\eta_*^{\langle n\rangle})$ for some n satisfying $k \leq 2(p^{n-1} + \cdots + p + 1) + 1$. Then there exist* *elements* $b_i \in \mathrm{BP}^{k+2p^i-1}(X)$ *for* $i \geq 0$ *such that in* $\mathrm{BP}^{k+1}(X)$,

$$\begin{aligned} pb_0 + v_1 b_1 + \cdots + v_n b_n + \cdots = 0 \quad \textit{and} \\ \rho_*(b_i) = Q_i(x) \quad \textit{for all} \quad i \geq 0. \end{aligned} \tag{5-7}$$

Proof. Since $x \in \widehat{\mathrm{Im}}\,(\eta_*^{\langle n\rangle})$, there exists an element $\bar{z} \in L\langle n\rangle^k(X)$ such that

$$Q_i(x) = Q_i\big(\eta_*^{\langle n\rangle}(\bar{z})\big) \quad \text{for} \;\; 0 \leq i \leq n. \tag{$*$}$$

Since $k \leq 2(p^{n-1} + \cdots + p + 1) + 1$, the element $\bar{z}$ lifts to an element $z \in L^k(X)$ by Theorem 4-6. If we let $b_i = q_{i*}(z) \in \mathrm{BP}^{k+2p^i-1}(X)$ for $i \geq 0$, we have

$$pb_0 + v_1 b_1 + \cdots + v_n b_n + \cdots = 0 \quad \text{in} \quad \mathrm{BP}^{k+1}(X),$$

and $\rho_*(b_i) = Q_i\big(\eta_*(z)\big) = Q_i\big(\eta_*^{\langle n\rangle}(\bar{z})\big)$ for all $i \geq 0$. By $(*)$, we have $\rho_*(b_i) = Q_i(x)$ for $0 \leq i \leq n$. Since $Q_i\big(\eta_*^{\langle n\rangle}(\bar{z}) - x\big) = 0$ for $0 \leq i \leq n$, we have $\eta_*(z) - x \in J_n$. Thus for $j \geq n+1$, we have

$$\rho_*(b_j) = Q_j\big(\eta_*(z)\big) \equiv Q_j(x) \mod Q_j(J_n^k).$$

Since $k < 2p^n$, we have $Q_j(J_n^k) = 0$ by Lemma 5-4, and we get the second identity in (5-7). This completes the proof. □

For Proposition 5-5 to be useful, we must find a way to produce an element $x \in H\mathbb{Z}_p^k(X)$ in $\widehat{\mathrm{Im}}\,(\eta_*^{\langle n\rangle})$ such that $k \leq 2(p^{n-1} + \cdots + p + 1) + 1$. Here is one such method.

Theorem 5-6. *Let* X *be a space, and suppose* $k \leq 2(p^{n-1} + \cdots + p + 1) + 1$. *Then for any element*

$$x \in \mathrm{Im}\,\{\rho_*^{\langle n-1\rangle} \colon \mathrm{BP}\langle n-1\rangle^k(X) \longrightarrow H\mathbb{Z}_p^k(X)\}, \tag{5-8}$$

there exist elements $b_{n+j} \in \mathrm{BP}^{k+2p^{n+j}-1}(X)$ *for* $j \geq 0$ *such that in* $\mathrm{BP}^{k+1}(X)$,

$$\begin{aligned} v_n b_n + v_{n+1} b_{n+1} + \cdots + v_{n+j} b_{n+j} + \cdots = 0 \quad \textit{and} \\ \rho_*(b_{n+j}) = Q_{n+j}(x) \quad \textit{for all} \;\; j \geq 0. \end{aligned} \tag{5-9}$$

Proof. Consider the following diagram whose top row is a portion of the Sullivan exact sequence:

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{BP}\langle n-1\rangle^k(X) \arrow[r, "\Delta_{n*}"] \arrow[d, "\rho_*^{\langle n-1\rangle}"] & \mathrm{BP}\langle n\rangle^{k+2p^n-1}(X) \arrow[r, "v_n"] \arrow[d, "\rho_*^{\langle n\rangle}"] & \mathrm{BP}\langle n\rangle^{k+1}(X) \arrow[r] & \cdots \\ & H\mathbb{Z}_p^k(X) \arrow[r, "Q_n"] & H\mathbb{Z}_p^{k+2p^n-1}(X). \end{tikzcd}$$

The commutativity of the square is due to Proposition 4-4. By our hypothesis, there exists an element $\widehat{x} \in \mathrm{BP}\langle n-1\rangle^k(X)$ such that $\rho_*^{\langle n-1\rangle}(\widehat{x}) = x$. Let $b_n^{\langle n\rangle} = \Delta_{n*}(\widehat{x}) \in \mathrm{BP}\langle n\rangle^{k+2p^n-1}(X)$. From the above diagram and Lemma 4-2, we have

$$\rho_*^{\langle n\rangle}(b_n^{\langle n\rangle}) = Q_n(x), \qquad Q_i(x) = 0 \quad \text{for} \quad 0 \leq i \leq n-1.$$

By exactness of the Sullivan sequence, we have $v_n b_n^{\langle n\rangle} = 0$ in $\mathrm{BP}\langle n\rangle^{k+1}(X)$. This is a finite BP-linear relation. From the exact sequence

$$\cdots \to L\langle n\rangle^k(X) \xrightarrow{\prod q_i\langle n\rangle_*} \prod_{i=0}^{n} \mathrm{BP}\langle n\rangle^{k+2p^i-1}(X) \xrightarrow{\kappa\langle n\rangle_*} \mathrm{BP}\langle n\rangle^{k+1}(X) \to \cdots,$$

we see that there exists an element $z_n \in L\langle n\rangle^k(X)$ such that

$$q_i\langle n\rangle_*(z_n) = 0 \quad \text{for} \quad 0 \le i \le n-1, \quad \text{and} \quad q_n\langle n\rangle_*(z_n) = b_n^{\langle n\rangle}.$$

Since $|b_n^{\langle n\rangle}| \le 2(p^n + p^{n-1} + \cdots + p + 1)$, it lifts an element $b_n \in \mathrm{BP}^{k+2p^n-1}(X)$. We let $b_0 = \cdots = b_{n-1} = 0$. Proceeding as in the proof of Theorem 4-6, we see that there exists a lift $z \in L^k(X)$ of z_n such that $q_{i*}(z) = 0$ for $0 \le i \le n-1$, and such that the elements $b_i = q_{i*}(z)$ for $i \ge n$ satisfy

$$\begin{gathered} v_n b_n + v_{n+1} b_{n+1} + \cdots + v_m b_m + \cdots = 0 \quad \text{in} \quad \mathrm{BP}^{k+1}(X), \quad \text{and} \\ \rho_*(b_i) = Q_i\big(\eta_*(z)\big) \quad \text{for} \quad i \ge n. \end{gathered}$$

The element $\eta_*(z)$ has the property $Q_i\big(\eta_*(z)\big) = \rho_*\big(q_{i*}(z)\big) = 0$ for $0 \le i \le n-1$, and $Q_n\big(\eta_*(z)\big) = \rho_*(b_n) = \rho_*^{\langle n\rangle}(b_n^{\langle n\rangle}) = Q_n(x)$. Thus, $Q_i\big(\eta_*(z) - x\big) = 0$ for $0 \le i \le n$, and $\eta_*(z) - x \in J_n^k$. Consequently, for $j \ge 1$ we have

$$\rho_*(b_{n+j}) = Q_{n+j}\big(\eta_*(z)\big) \equiv Q_{n+j}(x) \mod Q_{n+j}(J_n^k).$$

But $Q_{n+j}(J_n^k) = 0$ by Lemma 5-4, since $k < 2p^n$. This completes the proof. $\square$

In the next section, we apply Theorem 5-6 to obtain BP-linear relations in the BP cohomology of Eilenberg–Mac Lane spaces.

6. Main BP-linear relations in BP cohomology of Eilenberg–Mac Lane spaces: v_n-series

To apply Theorem 5-6 to obtain BP-linear relations in the BP cohomology of a space X, we need to produce elements in $\mathrm{BP}\langle n-1\rangle^*(X)$ and identify their images in the mod p cohomology of X. One way to do this is to use connecting homomorphisms in the Sullivan exact sequences [Proposition 4-4]. We have the following homotopy commutative diagram:

$$\text{(6-1)} \qquad \begin{CD} H\mathbb{Z}_{(p)}^{*+1}(X) @>{\Delta_1}>> \mathrm{BP}\langle 1\rangle^{*+2p}(X) @>{\Delta_2}>> \cdots @>{\Delta_{n-1}}>> \mathrm{BP}\langle n-1\rangle^{*+q-n}(X) \\ @VV{\rho_*^{\langle 0\rangle}}V @VV{\rho_*^{\langle 1\rangle}}V @. @VV{\rho_*^{\langle n-1\rangle}}V \\ H\mathbb{Z}_p^{*+1}(X) @>{Q_1}>> H\mathbb{Z}_p^{*+2p}(X) @>{Q_2}>> \cdots @>{Q_{n-1}}>> H\mathbb{Z}_p^{*+q-n}(X). \end{CD}$$

Here $q = 2(p^{n-1} + \cdots + p + 1)$. Since $\mathrm{Im}\,\{Q_0 : H\mathbb{Z}_p^*(X) \to H\mathbb{Z}_p^{*+1}(X)\} \subset \mathrm{Im}\,(\rho_*^{\langle 0\rangle})$, the commutativity of the above diagram shows that

$$\text{(6-2)} \qquad Q_{n-1} \cdots Q_1 Q_0\big(H\mathbb{Z}_p^*(X)\big) \subset Q_{n-1} \cdots Q_1\big(\mathrm{Im}\,(\rho_*^{\langle 0\rangle})\big) \subset \mathrm{Im}\,(\rho_*^{\langle n-1\rangle}).$$

To apply Theorem 5-6, we want elements in the image $\rho_*^{\langle n-1\rangle} : \mathrm{BP}\langle n-1\rangle^k(X) \to H\mathbb{Z}_p^k(X)$ satisfying the dimensional condition $k \le 2(p^{n-1} + \cdots + p + 1) + 1$. For $x \in H\mathbb{Z}_p^*(X)$ and $y \in \mathrm{Im}\,\rho_*^{\langle 0\rangle}$, the condition that elements $Q_{n-1} \cdots Q_1 Q_0(x)$ and $Q_{n-1} \cdots Q_1(y)$ have dimension less than or equal to $2(p^{n-1} + \cdots + p + 1) + 1$ implies that their degrees must satisfy $|x| \le n+1$ and $|y| \le n+2$. On the other hand, nontriviality of the action of products of Milnor primitives on x, y imposes another condition on the dimension of x, y. We recall the following special case of a result in [T2].

Theorem 6-1 [T2]**.** (I) *Let $\iota_r \in H\mathbb{Z}_p^r\big(K(\mathbb{Z}/p^\ell, r)\big)$ be the mod p fundamental class of the Eilenberg–Mac Lane space $K(\mathbb{Z}/p^\ell, r)$ in degree r. Then*

$$\begin{gathered} Q_{n-1}\cdots Q_1\delta_\ell(\iota_r) \neq 0 \iff r \geq n, \\ Q_{n+j}Q_{n-1}\cdots Q_1\delta_\ell(\iota_r) \neq 0 \quad \textit{for some} \quad j \geq 0 \iff r \geq n+1. \end{gathered} \tag{6-3}$$

(II) *Let $\tau_r \in H\mathbb{Z}_p^r\big(K(\mathbb{Z}_{(p)}, r)\big)$ be the mod p fundamental class of the integral Eilenberg–Mac Lane space $K(\mathbb{Z}_{(p)}, r)$ in degree r. Then*

$$\begin{gathered} Q_{n-1}\cdots Q_2Q_1(\tau_r) \neq 0 \iff r \geq n+1, \\ Q_{n+j}Q_{n-1}\cdots Q_1(\tau_r) \neq 0 \quad \textit{for some} \quad j \geq 0 \iff r \geq n+2. \end{gathered} \tag{6-4}$$

In (6-3), if $r \geq n+1$, then $Q_{n+j}Q_{n-1}\cdots Q_1\delta_\ell(\iota_r) \neq 0$ for all $j \geq 0$ [T2]. Similarly, for (6-4). Theorem 6-1 quickly implies the following sharper result.

Corollary 6-2. (I) *Let $x \in H\mathbb{Z}_p^*(X)$ be such that*

$$Q_{n-1}\cdots Q_1Q_0(x) \neq 0, \quad \textit{and} \quad Q_{n+j}Q_{n-1}\cdots Q_1Q_0(x) \neq 0 \quad \textit{for some} \quad j \geq 0.$$

If $|Q_{n-1}\cdots Q_1Q_0(x)| \leq 2(p^{n-1}+\cdots+p+1)+1$, then we have $|x| = n+1$.

(II) *Let $y \in H\mathbb{Z}_p^*(X)$ be a mod p reduction of an integral element such that*

$$Q_{n-1}\cdots Q_1(y) \neq 0, \quad \textit{and} \quad Q_{n+j}Q_{n-1}\cdots Q_1(y) \neq 0 \quad \textit{for some} \quad j \geq 0.$$

If $|Q_{n-1}\cdots Q_1(y)| \leq 2(p^{n-1}+\cdots+p+1)+1$, then $|y| = n+2$.

We see that the dimension $2(p^{n-1}+\cdots+p+1)+1$ is very special.

Now we prove the existence of certain infinite sum BP-linear relations in the BP cohomology of Eilenberg–Mac Lane spaces. To accommodate the condition in Theorem 5-6, we are forced to use spaces $K(\mathbb{Z}_{(p)}, n+2)$ and $K(\mathbb{Z}/p^\ell, n+1)$ as a consequence of Corollary 6-2.

Theorem 6-3 (Main Relations). (I) *Let $\tau_{n+2} \in H\mathbb{Z}_p^{n+2}\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ be the mod p fundamental class for $n \geq 1$. There exist nontrivial elements $b_{n+j} \in \mathrm{BP}^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ of degree $2(p^{n+j}+p^{n-1}+\cdots+p+1)$ for $j \geq 0$ such that*

$$\begin{gathered} v_nb_n + v_{n+1}b_{n+1} + \cdots + v_{n+j}b_{n+j} + \cdots = 0, \quad \textit{and} \\ \rho_*(b_{n+j}) = Q_{n+j}Q_{n-1}\cdots Q_1(\tau_{n+2}) \neq 0 \quad \textit{for} \quad j \geq 0. \end{gathered} \tag{6-5}$$

The above BP*-linear relation takes place in the* BP *cohomology group of degree $2(p^{n-1}+\cdots+p+1)+2$. Here, the convergence is with respect to the* BP *topology and $\rho_*\colon \mathrm{BP}^*(X) \to H\mathbb{Z}_p^*(X)$ is the Thom map.*

(II) *Let $\iota_{n+1} \in H\mathbb{Z}_p^{n+1}\big(K(\mathbb{Z}/p^\ell, n+1)\big)$ be the mod p fundamental class. There exist nontrivial elements $b_{n+j} \in \mathrm{BP}^*\big(K(\mathbb{Z}/p^\ell, n+1)\big)$ of degree $2(p^{n+j}+p^{n-1}+\cdots+p+1)$ for $j \geq 0$ such that*

$$\begin{gathered} v_nb_n + v_{n+1}b_{n+1} + \cdots + v_{n+j}b_{n+j} + \cdots = 0, \quad \textit{and} \\ \rho_*(b_{n+j}) = Q_{n+j}Q_{n-1}\cdots Q_1\delta_\ell(\iota_{n+1}) \neq 0 \quad \textit{for} \quad j \geq 0. \end{gathered} \tag{6-6}$$

The above BP*-linear relation takes place in the* BP *cohomology group of degree $2(p^{n-1}+\cdots+p+1)+2$. Here, the convergence is with respect to the* BP *topology, and δ_ℓ is the Bockstein map.*

Proof. Let $X = K(\mathbb{Z}_{(p)}, n+2)$. Let $\widehat{\tau}_{n+2} \in H\mathbb{Z}_{(p)}^{n+2}(X)$ be an integral class whose mod p reduction is τ_{n+2}. Then

$$z = \Delta_{n-1}\circ\cdots\circ\Delta_1(\widehat{\tau}_{n+2}) \in \mathrm{BP}\langle n-1\rangle^{2(p^{n-1}+\cdots+p+1)+1}(X).$$

This element is nontrivial since $\rho_*^{\langle n-1\rangle}(z) = Q_{n-1}\cdots Q_1\tau_{n+2} \neq 0$ in mod p cohomology due to the commutativity of the diagram (6-1) and Theorem 6-1. Applying Theorem 5-6 to z, we obtain elements b_{n+j} in the BP cohomology for $j \geq 0$ satisfying an infinite BP-linear relation as in (6-5) such that

$$\rho_*(b_{n+j}) = Q_{n+j}Q_{n-1}\cdots Q_1(\tau_{n+2}),$$

for all $j \geq 0$. These elements are nontrivial in $H\mathbb{Z}_p^*(X)$ by (6-4). Hence we have $b_{n+j} \neq 0$ in the BP cohomology.

The proof of (II) is obtained by pulling back results in (I) by the Bockstein map

$$\delta_\ell : K(\mathbb{Z}/p^\ell, n+1) \longrightarrow K(\mathbb{Z}_{(p)}, n+2).$$

This completes the proof. □

We call the infinite sum BP-linear relations (6-5) and (6-6) *main relations* in BP cohomology of Eilenberg–Mac Lane spaces. The reason for this name is explained later in the context of v_n-series. See Proposition 6-6 below. We can be very precise about the elements b_{n+j} in the BP cohomology. See (6-17) below.

In Theorem 6-3, the $n=0$ case is not treated. This case is well understood. In fact,

$$\text{(6-7)}\quad \begin{aligned} \mathrm{BP}^*\big(K(\mathbb{Z}_{(p)},2)\big) &= \mathrm{BP}^*[[\,x\,]], \quad \text{where } x \in \mathrm{BP}^2\big(K(\mathbb{Z}_{(p)},2)\big),\\ \mathrm{BP}^*\big(K(\mathbb{Z}/p^\ell,1)\big) &= \mathrm{BP}^*[[\,y\,]]/([\,p^\ell\,](y)), \quad \text{where } y \in \mathrm{BP}^2\big(K(\mathbb{Z}/p^\ell,1)\big).\end{aligned}$$

Here, $y = \delta_\ell^*(x)$. There is no nontrivial relation in the BP cohomology of $K(\mathbb{Z}_{(p)},2)$. For the case $K(\mathbb{Z}/p,1)$, (3-22) shows that (6-6) is still valid.

The infinite sum BP-linear relation in the BP cohomology of $K(\mathbb{Z}/p,1)$ is given by the p-series $[\,p\,](x)$. For higher dimensional Eilenberg–Mac Lane spaces, we want to interpret the BP-linear relations (6-5) and (6-6) as v_n-analogues of the p-series. Indeed, such an analogue with the right properties exists. We call it the v_n-series, which we now define.

We recall that the space $\underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)+t}$ of the Ω-spectrum of $\mathrm{BP}\langle n\rangle$ is a factor space of $\underline{\mathrm{BP}}_{2(p^n+\cdots+p+1)+t}$ if and only if $t \leq 0$ by Wilson's Splitting Theorem [Theorem 2-2]. We fix an inclusion map

$$\text{(6-8)}\quad \iota^{\langle n\rangle}_{2(p^n+\cdots+p+1)+t} : \underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)+t} \longrightarrow \underline{\mathrm{BP}}_{2(p^n+\cdots+p+1)+t}, \quad t \leq 0,$$

such that $\rho_{\langle n\rangle}\circ\iota^{\langle n\rangle} =$ identity afforded by the Splitting Theorem for each factor. The map $\iota^{\langle n\rangle}$ in (6-8) defines a BP cohomology class

$$\text{(6-8}'\text{)}\quad \iota^{\langle n\rangle}_{2(p^n+\cdots+p+1)+t} \in \mathrm{BP}^{2(p^n+\cdots+p+1)+t}\big(\underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)+t}\big), \quad t \leq 0.$$

We call this class *the* BP *fundamental class* for the space $\underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)+t}$ for $t \leq 0$. We omit the dimension when it is clear from the context. When $t > 0$, $\underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)+t}$ is not a factor of $\underline{\mathrm{BP}}_{2(p^n+\cdots+p+1)+t}$ and a BP fundamental class does not exist.

Now consider the following composition of maps which we call $[\,v_n\,]$:

$$\text{(6-9)}\quad \begin{aligned}[\,v_n\,] : \underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)} &\xrightarrow{v_n} \underline{\mathrm{BP}}\langle n\rangle_{2(p^{n-1}+\cdots+p+1)+2}\\ &\xrightarrow{\iota^{\langle n\rangle}} \underline{\mathrm{BP}}_{2(p^{n-1}+\cdots+p+1)+2}.\end{aligned}$$

Here the first map is the v_n-multiplication map, and the second map is the BP fundamental class. The cohomology class represented by this map is the v_n-series.

Definition 6-4 (v_n-series)**.** The v_n-series is a cohomology class defined by

$$\text{(6-9}'\text{)}\quad [\,v_n\,] = v_n^*\big(\iota^{\langle n\rangle}_{2(p^{n-1}+\cdots+p+1)+2}\big) \in \mathrm{BP}^{2(p^{n-1}+\cdots+p+1)+2}\big(\underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)}\big).$$

The above definition was motivated by [RWY], in which the importance of the v_n-multiplication map $v_n : \underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)} \to \underline{\mathrm{BP}}\langle n\rangle_{2(p^{n-1}+\cdots+p+1)+2}$ is emphasized in the description of the BP cohomology of Eilenberg–Mac Lane spaces. The above definition combines the v_n-multiplication map and the BP fundamental class. The resulting object is certain to be of basic importance.

We check that the v_n-series reduces to the usual p-series when $n = 0$. In this case, $[\,v_0\,]$ is defined as the following map:

$$\text{(6-10)}\qquad [\,v_0\,] : \underline{\mathrm{BP}}\langle 0\rangle_2 \xrightarrow{\ p\ } \underline{\mathrm{BP}}\langle 0\rangle_2 \xrightarrow{\ \iota_2^{\langle 0\rangle}\ } \underline{\mathrm{BP}}_2.$$

Note that $\underline{\mathrm{BP}}\langle 0\rangle_2 = \mathbb{C}\mathrm{P}^\infty_{(p)}$. The BP fundamental class $\iota_2^{\langle 0\rangle} = x \in \mathrm{BP}^2(\mathbb{C}\mathrm{P}^\infty_{(p)})$ is a complex orientation and $\mathrm{BP}^*(\mathbb{C}\mathrm{P}^\infty_{(p)}) = \mathrm{BP}^*[[\,x\,]]$. We may choose the map $\iota_2^{\langle 0\rangle}$ to be the usual orientation in BP theory coming from the MU-orientation $\mathbb{C}\mathrm{P}^\infty \xrightarrow{\cong} MU(1) \to \Sigma^2 MU$. Then, our v_0-series

$$\text{(6-11)}\qquad [\,v_0\,] = \iota_2^{\langle 0\rangle} \circ p = p^*(\iota_2^{\langle 0\rangle}) = [\,p\,](x) \in \mathrm{BP}^2(\mathbb{C}\mathrm{P}^\infty_{(p)})$$

is the usual p-series.

Next, we express the v_n-series as an infinite BP-linear sum in the BP cohomology of $\underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)}$. By Wilson's Splitting Theorem we have

$$\begin{aligned}\underline{\mathrm{BP}}_{2(p^{n-1}+\cdots+p+1)+2} &\cong \underline{\mathrm{BP}}\langle n\rangle_{2(p^{n-1}+\cdots+p+1)+2}\\ &\quad\times \prod_{j\ge 1} \underline{\mathrm{BP}}\langle n+j\rangle_{2(p^{n+j}+p^{n-1}+\cdots+p+1)}.\end{aligned}$$

Let $\mathrm{proj}\langle n+j\rangle$ be the projection map onto the factor $\underline{\mathrm{BP}}\langle n+j\rangle_*$ in the above splitting. We define a family of cohomology classes

$$\text{(6-12)}\qquad \theta^{\langle n\rangle}_{n+j} \in \mathrm{BP}^{2(p^{n+j}+p^{n-1}+\cdots+p+1)}\big(\underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)}\big), \qquad j \ge 0,$$

as follows. When $j = 0$,

$$\text{(6-12}'\text{)}\qquad \theta^{\langle n\rangle}_n : \underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)} \xrightarrow{\ \iota^{\langle n\rangle}\ } \underline{\mathrm{BP}}_{2(p^n+\cdots+p+1)}$$

is the BP fundamental class. Other classes $\theta^{\langle n\rangle}_{n+j}$ with $j \ge 1$ are defined as the following compositions:

$$\begin{aligned}\text{(6-12}''\text{)}\quad -\theta^{\langle n\rangle}_{n+j} &: \underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)} \xrightarrow{\ \iota^{\langle n\rangle}\ } \underline{\mathrm{BP}}_{2(p^n+\cdots+p+1)}\\ &\xrightarrow{\ v_n\ } \underline{\mathrm{BP}}_{2(p^{n-1}+\cdots+p+1)+2} \xrightarrow{\ \mathrm{proj}\langle n+j\rangle\ } \underline{\mathrm{BP}}\langle n+j\rangle_{2(p^{n+j}+p^{n-1}+\cdots+p+1)}\\ &\xrightarrow{\ \iota^{\langle n+j\rangle}\ } \underline{\mathrm{BP}}_{2(p^{n+j}+p^{n-1}+\cdots+p+1)}.\end{aligned}$$

Nontriviality of these cohomology elements will be proved in Proposition 6-6. With these notations, we have the following result.

Proposition 6-5. *The v_n-series in* (6-9) *can be written as an infinite* BP*-linear sum of the form*

$$\text{(6-13)}\qquad \begin{aligned}[\,v_n\,] &= v_n\theta^{\langle n\rangle}_n + v_{n+1}\theta^{\langle n\rangle}_{n+1} + \cdots + v_{n+j}\theta^{\langle n\rangle}_{n+j} + \cdots,\\ &\in \mathrm{BP}^{2(p^{n-1}+\cdots+p+1)+2}\big(\underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)}\big),\end{aligned}$$

where $\theta^{\langle n\rangle}_{n+j}$ are the elements defined above. Here, the convergence is with respect to the BP *topology.*

Proof. We recall that Wilson's Splitting Theorem [Theorem 2-2] of spaces of the Ω-spectrum of BP comes from the following tower of fibrations corresponding to Sullivan exact sequences:

$$
(*)\quad
\begin{array}{ccc}
\underline{\mathrm{BP}}_{2(p^{n-1}+\cdots+p+1)+2} & & \\
\downarrow & & \\
\vdots & & \\
\downarrow & & \\
\underline{\mathrm{BP}}\langle n+j\rangle_{2(p^{n-1}+\cdots+p+1)+2} & \xleftarrow{v_{n+j}} & \underline{\mathrm{BP}}\langle n+j\rangle_{2(p^{n+j}+p^{n-1}+\cdots+p+1)} \\
\downarrow{\scriptstyle \rho^{\langle n+j\rangle}_{\langle n+j-1\rangle}} & & \\
\underline{\mathrm{BP}}\langle n+j-1\rangle_{2(p^{n-1}+\cdots+p+1)+2} & \xleftarrow{v_{n+j-1}} & \underline{\mathrm{BP}}\langle n+j-1\rangle_{2(p^{n+j-1}+p^{n-1}+\cdots+p+1)} \\
\downarrow & & \\
\vdots & & \\
\downarrow & & \\
\underline{\mathrm{BP}}\langle n+1\rangle_{2(p^{n-1}+\cdots+p+1)+2} & \xleftarrow{v_{n+1}} & \underline{\mathrm{BP}}\langle n+1\rangle_{2(p^{n+1}+p^{n-1}+\cdots+p+1)} \\
\downarrow{\scriptstyle \rho^{\langle n+1\rangle}_{\langle n\rangle}} & & \\
\underline{\mathrm{BP}}\langle n\rangle_{2(p^{n-1}+\cdots+p+1)+2}. & &
\end{array}
$$

Although $\rho_{\langle m\rangle}:\underline{\mathrm{BP}}_*\to\underline{\mathrm{BP}}\langle m\rangle_*$ are BP-module maps for any $m\geq -1$, the BP fundamental classes $\iota^{\langle m\rangle}:\underline{\mathrm{BP}}\langle m\rangle_*\to\underline{\mathrm{BP}}_*$, when they exist, may *not*. We examine the failure of the homotopy commutativity of the following diagram:

$$
\text{(6-15)}\quad
\begin{array}{ccc}
\underline{\mathrm{BP}}_{2(p^{n}+\cdots+p+1)} & \xrightarrow{v_n} & \underline{\mathrm{BP}}_{2(p^{n-1}+\cdots+p+1)+2} \\
\uparrow{\scriptstyle \iota^{\langle n\rangle}} & & \uparrow{\scriptstyle \iota^{\langle n\rangle}} \\
\underline{\mathrm{BP}}\langle n\rangle_{2(p^{n}+\cdots+p+1)} & \xrightarrow{v_n} & \underline{\mathrm{BP}}\langle n\rangle_{2(p^{n-1}+\cdots+p+1)+2}.
\end{array}
$$

The difference of two elements $v_n\circ\iota^{\langle n\rangle}$ and $\iota^{\langle n\rangle}\circ v_n$ in $\mathrm{BP}^*\big(\underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)}\big)$ vanishes in $\mathrm{BP}\langle n\rangle^*(\,\cdot\,)$:

$$\rho_{\langle n\rangle *}(v_n\circ\iota^{\langle n\rangle}-\iota^{\langle n\rangle}\circ v_n)=v_n\cdot\big(\rho_{\langle n\rangle}\circ\iota^{\langle n\rangle}\big)-\big(\rho_{\langle n\rangle}\circ\iota^{\langle n\rangle}\big)\circ v_n=v_n-v_n=0.$$

Here, the first equality is because $\rho_{\langle n\rangle}$ is a BP-module map. From the Sullivan exact sequence (or from the fibration at the bottom of the diagram $(*)$), we have

$$\rho_{\langle n+1\rangle *}(v_n\circ\iota^{\langle n\rangle}-\iota^{\langle n\rangle}\circ v_n)=v_{n+1}\cdot\big(-\theta^{\langle n,n+1\rangle}_{n+1}\big),$$

for some element $-\theta^{\langle n,n+1\rangle}_{n+1}\in\mathrm{BP}\langle n+1\rangle^{2(p^{n+1}+p^{n-1}+\cdots+p+1)+2}\big(\underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)}\big)$. Wilson's Splitting Theorem says that all the fibrations in $(*)$ are trivial and hence they are Cartesian products. The above identity then means that for the map

$$\rho_{\langle n+1\rangle}\circ v_n\circ\iota^{\langle n\rangle}:\underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)}\longrightarrow\underline{\mathrm{BP}}\langle n+1\rangle_{2(p^{n-1}+\cdots+p+1)+2},$$

the $\underline{\mathrm{BP}}\langle n\rangle_{2(p^{n-1}+\cdots+p+1)+2}$-component is given by $\rho_{\langle n\rangle}\circ\iota^{\langle n\rangle}\circ v_n = v_n$, and the other $\underline{\mathrm{BP}}\langle n+1\rangle_{2(p^{n+1}+p^{n-1}+\cdots+p+1)+2}$-component is given by $-\theta^{\langle n,n+1\rangle}_{n+1}$. Lifting this element to $\underline{\mathrm{BP}}_{2(p^{n+1}+p^{n-1}+\cdots+p+1)+2}$ using the BP-fundamental class $\iota^{\langle n+1\rangle}$, we get the element $-\theta^{\langle n\rangle}_{n+1} = -\iota^{\langle n+1\rangle}\circ\theta^{\langle n,n+1\rangle}_{n+1}$. By construction, we have

$$(**)\qquad {\rho_{\langle n+1\rangle}}_*\big(v_n\circ\iota^{\langle n\rangle} - \iota^{\langle n\rangle}\circ v_n + v_{n+1}\theta^{\langle n\rangle}_{n+1}\big) = 0.$$

We repeat the same argument. Just to be more explicit, we construct the next element $\theta^{\langle n\rangle}_{n+2}$. By exactness of the Sullivan exact sequence, $(**)$ implies that

$${\rho_{\langle n+2\rangle}}_*\big(v_n\circ\iota^{\langle n\rangle} - \iota^{\langle n\rangle}\circ v_n + v_{n+1}\theta^{\langle n\rangle}_{n+1}\big) = v_{n+2}\big(-\theta^{\langle n,n+2\rangle}_{n+2}\big),$$

for some $-\theta^{\langle n,n+2\rangle}_{n+2} \in \mathrm{BP}\langle n+2\rangle^{2(p^{n+2}+p^{n-1}+\cdots+p+1)+2}\big(\underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)}\big)$. Because the fibrations in $(*)$ are trivial, this element can be realized as the map

$$\begin{aligned}-\theta^{\langle n,n+2\rangle}_{n+2} : \underline{\mathrm{BP}}_{2(p^n+\cdots+p+1)} &\xrightarrow{v_n\circ\iota^{\langle n\rangle}} \underline{\mathrm{BP}}_{2(p^{n-1}+\cdots+p+1)+2}\\ &\xrightarrow{\mathrm{proj}\langle n+2\rangle} \underline{\mathrm{BP}}\langle n+2\rangle_{2(p^{n+2}+p^{n-1}+\cdots+p+1)+2}.\end{aligned}$$

By lifting this element to $\underline{\mathrm{BP}}_{2(p^{n+2}+p^{n-1}+\cdots+p+1)+2}$ using the BP-fundamental class $\iota^{\langle n+2\rangle}$, we obtain $-\theta^{\langle n\rangle}_{n+2}$. By construction, we have

$${\rho_{\langle n+2\rangle}}_*\big(v_n\circ\iota^{\langle n\rangle} - \iota^{\langle n\rangle}\circ v_n + v_{n+1}\theta^{\langle n\rangle}_{n+1} + v_{n+2}\theta^{\langle n\rangle}_{n+2}\big) = 0.$$

Now by induction, we obtain BP cohomology elements $\theta^{\langle n\rangle}_{n+j}$ for $j\ge 0$ such that

$$\begin{aligned}&{\rho_{\langle n+j\rangle}}_*\big(v_n\circ\iota^{\langle n\rangle} - \iota^{\langle n\rangle}\circ v_n + v_{n+1}\theta^{\langle n\rangle}_{n+1} + \cdots + v_{n+j}\theta^{\langle n\rangle}_{n+j}\big)\\ &\qquad\qquad = {\rho_{\langle n+j\rangle}}_*\big(v_n\circ\iota^{\langle n\rangle} - \iota^{\langle n\rangle}\circ v_n + \sum_{j=1}^{\infty} v_{n+j}\theta^{\langle n\rangle}_{n+j}\big) = 0,\end{aligned}$$

for all $j\ge 0$. This means that with respect to the BP topology, we have

$$[\, v_n\,] = \iota^{\langle n\rangle}\circ v_n = v_n\circ\iota^{\langle n\rangle}_{2(p^n+\cdots+p+1)} + \sum_{j=1}^{\infty} v_{n+j}\theta^{\langle n\rangle}_{n+j},$$

in $\mathrm{BP}^{2(p^{n-1}+\cdots+p+1)+2}\big(\underline{\mathrm{BP}}_{2(p^n+\cdots+p+1)}\big)$. This completes the proof. □

Next, we show that the v_n-series (6-13) gives rise to the main relations (6-5) and (6-6) in the BP cohomology of Eilenberg–Mac Lane spaces $K(\mathbb{Z}_{(p)}, n+2)$ and $K(\mathbb{Z}/p^\ell, n+1)$. First we note that the following composition is a zero map by exactness of the Sullivan sequence:

$$(6\text{-}16)\qquad \begin{aligned}\underline{\mathrm{BP}}\langle n-1\rangle_{2(p^{n-1}+\cdots+p+1)+1} &\xrightarrow{\Delta_n} \underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)}\\ &\xrightarrow{v_n} \underline{\mathrm{BP}}\langle n\rangle_{2(p^{n-1}+\cdots+p+1)+2}.\end{aligned}$$

Hence $\Delta_n^*\big([\,v_n\,]\big) = \iota^{\langle n\rangle}\circ\big(v_n\circ\Delta_n\big) = 0 \in \mathrm{BP}^*\big(\underline{\mathrm{BP}}\langle n-1\rangle_{2(p^{n-1}+\cdots+p+1)+1}\big)$. Now we pull back the v_n-series to the BP cohomology of Eilenberg–Mac Lane spaces. We consider the composition of the following maps:

$$\begin{aligned}(6\text{-}17)\qquad & b_{n+j} : K(\mathbb{Z}_{(p)}, n+2)\\ & \xrightarrow{\Delta_1} \underline{\mathrm{BP}}\langle 1\rangle_{2p+n+1} \xrightarrow{\Delta_2} \cdots \xrightarrow{\Delta_{n-1}} \underline{\mathrm{BP}}\langle n-1\rangle_{2(p^{n-1}+\cdots+p+1)+1}\\ & \xrightarrow{\Delta_n} \underline{\mathrm{BP}}\langle n\rangle_{2(p^n+\cdots+p+1)} \xrightarrow{\theta^{\langle n\rangle}_{n+j}} \underline{\mathrm{BP}}_{2(p^{n+j}+p^{n-1}+\cdots+p+1)}.\end{aligned}$$

In terms of BP cohomology, elements b_{n+j} are pull-backs of $\theta^{\langle n\rangle}_{n+j}$ by $\Delta_n\circ\cdots\circ\Delta_1$:

$$(6\text{-}17')\qquad b_{n+j} = \Delta_1^*\circ\cdots\circ\Delta_n^*\big(\theta^{\langle n\rangle}_{n+j}\big) \in \mathrm{BP}^{2(p^{n+j}+p^{n-1}+\cdots+p+1)}\big(K(\mathbb{Z}_{(p)}, n+2)\big),$$

for $j\ge 0$. Pulling back the v_n-series (6-13) by the BP-module map $(\Delta_n\circ\cdots\circ\Delta_1)^*$, and using $\Delta_n^*([\,v_n\,]) = 0$, we have

$$(6\text{-}18)\qquad v_nb_n + v_{n+1}b_{n+1} + \cdots + v_{n+j}b_{n+j} + \cdots = 0,$$

in $\mathrm{BP}^{2(p^{n-1}+\cdots+p+1)+2}\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ with respect to the BP topology. We claim that (6-18) *is* the main relation (6-5) by showing that elements b_{n+j} have the required properties.

Proposition 6-6. *With respect to the Thom map*

$$\rho_* : \mathrm{BP}^*\big(K(\mathbb{Z}_{(p)}, n+2)\big) \longrightarrow H\mathbb{Z}_p^*\big(K(\mathbb{Z}_{(p)}, n+2)\big),$$

the elements b_{n+j} in (6-17) *for $j\ge 0$ have the property*

$$(6\text{-}19)\qquad \rho_*(b_{n+j}) = Q_{n+j}Q_{n-1}\cdots Q_1(\tau_{n+2}) \ne 0.$$

Hence elements $\theta^{\langle n\rangle}_{n+j}$ of (6-12) *and b_{n+j} are nontrivial for all $j\ge 0$.*

Proof. Proposition 5-1 says that given a BP-linear relation (6-18), there exists an element $x\in H\mathbb{Z}_p^{2(p^{n-1}+\cdots+p+1)+1}\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ such that $\rho_*(b_{n+j}) = Q_{n+j}(x)$ for $j\ge 0$. By (6-17), we have $b_n = \iota^{\langle n\rangle}\circ\Delta_n\circ\cdots\circ\Delta_1$. Since $\rho\circ\iota^{\langle n\rangle} = \rho^{\langle n\rangle}$ and Δ_j corresponds to Q_j by Proposition 4-4, we have

$$\rho_*(b_n) = \rho^{\langle n\rangle}\circ\Delta_n\circ\cdots\circ\Delta_1 = Q_nQ_{n-1}\cdots Q_1(\tau_{n+2}).$$

Thus, we have $Q_n\big(x - [Q_{n-1}\cdots Q_1(\tau_{n+2})]\big) = 0$. Since the dimension of this difference element is $2(p^{n-1}+\cdots+p+1)+1 < 2p^n$, Lemma 5-4 applies and the effect of Q_{n+j} on x and $Q_{n-1}\cdots Q_1(\tau_{n+2})$ are the same for all $j\ge 0$. Hence we have

$$\rho_*(b_{n+j}) = Q_{n+j}(x) = Q_{n+j}Q_{n-1}\cdots Q_1(\tau_{n+2}),\quad j\ge 0.$$

This completes the proof. □

By pulling back results in Proposition 6-6 by the Bockstein map δ_ℓ, we get a corresponding statement for $K(\mathbb{Z}/p^\ell, n+1)$. The above results show that the main relations we found in Theorem 6-4, by a general theory of BP-linear relations, actually come from the v_n-series, just like the only relation in $\mathrm{BP}^*\big(K(\mathbb{Z}/p,1)\big)$ comes from the p-series.

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