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THE FARRELL-JONES ISOMORPHISM CONJECTURE FOR FINITE COVOLUME HYPERBOLIC ACTIONS AND THE ALGEBRAIC K-THEORY OF BIANCHI GROUPS

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Abstract. We prove the Farrell-Jones Isomorphism Conjecture for groups acting properly discontinuously via isometries on (real) hyperbolic n-space $\mathbb{H}^n$ with finite volume orbit space. We then apply this result to show that, for any Bianchi group Γ, $Wh(\Gamma)$, $\tilde{K}_0(\mathbb{Z}\Gamma)$, and $K_i(\mathbb{Z}\Gamma)$ vanish for $i \leq -1$.

1. Introduction

Let Γ be a discrete group, let $\mathbb{Z}\Gamma$ denote its integral group ring, and let $K_i(\mathbb{Z}\Gamma)$ be the algebraic K-theory groups of the ring $\mathbb{Z}\Gamma$, where $i \in \mathbb{Z}$. It has been conjectured that these K-groups may be computed from the corresponding K-groups of certain subgroups of Γ. More precisely, the Farrell-Jones Isomorphism Conjecture [10] states that the algebraic K-theory of $\mathbb{Z}\Gamma$ may be computed from the algebraic K-theory of the virtually cyclic subgroups of Γ via an appropriate "assembly map" (see Section 2 for a precise statement and definitions), where a group G is called virtually cyclic if it either is finite or fits into an extension $1 \to \mathbb{Z} \to G \to F \to 1$, with F a finite group. In [10] Farrell and Jones prove the Isomorphism Conjecture in lower algebraic K-theory for subgroups of discrete cocompact groups contained in virtually connected Lie groups, and for certain discrete cocompact groups acting properly discontinuously by isometries on a simply connected symmetric Riemannian manifold M with everywhere nonpositive curvature.

Let Γ be a discrete group acting properly discontinuously on hyperbolic space $\mathbb{H}^n$ via isometries whose orbit space has finite volume. In particular, there is a representation of Γ in $Isom(\mathbb{H}^m)$ with finite kernel K and image $\hat{\Gamma}$ (that $\hat{\Gamma}$ is virtually torsion-free follows from [22] (6.11 and 13.21) and the fact that $Isom(\mathbb{H}^n)$ embeds in $GL(m, \mathbb{R})$ for some m). We prove in Section 3 of this paper that the Farrell-Jones Isomorphism Conjecture holds for such a group Γ. We state this precisely:

Theorem A. *The Isomorphism Conjecture is true for the functors $\mathcal{P}_*$ and $\mathcal{P}_*^{Diff}$ on the space X provided that there exists a properly discontinuous finite covolume group action by isometries of $\Gamma = \pi_1(X)$ on a hyperbolic space $\mathbb{H}^n$.*

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The technique of proof consists of modifying the ends of the finite volume orbit manifold to construct a new manifold which is compact. This modification of our original manifold changes its sectional curvature, but we are able to control it in such a way that it stays nonpositive. The resulting extension of the Isomorphism Conjecture yields a wide variety of examples on which one may reduce the computation of the lower K-groups to a family of proper subgroups. We mention a particular collection of groups to which this theorem applies. Let $\mathcal{O}_d$ denote the ring of integers in an imaginary quadratic number field $\mathbb{Q}(\sqrt{d})$, with $d \in \mathbb{Z}, d < 0$ and square free. The Bianchi group Γ_d is defined as $\Gamma_d = PSL_2(\mathcal{O}_d) \subset PSL_2(\mathbb{C})$; these groups comprise a family of infinite discrete subgroups of $PSL_2(\mathbb{C})$. In this paper we classify all virtually cyclic subgroups contained in Bianchi groups, then show that the lower algebraic K-theory of all the virtually cyclic subgroups vanishes, obtaining the following application of Theorem A:

Theorem B. *Let Γ_d be a Bianchi group. Then $K_i(\mathbb{Z}\Gamma_d) = 0$ for $i \leq -1$, $\tilde{K}_0(\mathbb{Z}\Gamma_d) = 0$, and $Wh(\Gamma_d) = 0$.*

This paper is organized as follows. In the second section we state the Isomorphism Conjecture of Farrell-Jones and their theorem for the case of cocompact actions. Next we recall the concept of a warped product of two Riemannian manifolds, a technical device we need to extend the result. In the third section we prove the Farrell-Jones Isomorphism Conjecture for certain actions with finite covolume. This implies that the lower algebraic K-theory of the Bianchi groups comes from their virtually cyclic subgroups. In the fourth section we classify these virtually cyclic subgroups and show that their lower algebraic K-groups vanish.

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2. Preliminaries

This section has two parts. In the first part we recall the definitions, notation, and terminology needed to state the Farrell-Jones Isomorphism Theorem. In the second part we recall the concept of a *warped product* of two Riemannian manifolds. The main references are [10], [21] and [8] for the first section; [3], [4], and [18] for the second.

2.1. **Background to the Farrell-Jones Isomorphism Conjecture.**

The Farrell-Jones Isomorphism Conjecture concerns four functors from Top to Ω-$Spectra$. Let $\mathcal{P}_*$ ($\mathcal{P}_*^{Diff}$) denote the functor that maps a space X to the Ω-spectrum of stable topological (smooth) pseudo-isotopies on X. Denote by $\mathcal{K}_*$ the functor that maps X to the algebraic K-theoretic (non-connective) Ω-spectrum for the integral group ring $\mathbb{Z}\pi_1(X)$ (see [12], [20]), and by $L_*^{-\infty}$ the $L^{-\infty}$-surgery functor. We use $\mathcal{S}_*$ to denote any of these four functors.

Definition 1. A group G is *virtually cyclic* if it has a cyclic subgroup of finite index, i.e., if it is either finite or contains an infinite cyclic subgroup of finite index.

The philosophy of the Isomorphism Conjecture is that the spectrum $\mathcal{S}_*(X)$ should be computable in a simple way from the spectra $\{\mathcal{S}_*(X_H)\}$, where H denotes a virtually cyclic subgroup of $\pi_1 X$ and X_H is the covering space of X corresponding to the subgroup H.

We establish some notation. Let $G = \pi_1(X)$ and $\mathcal{E}(X) = \tilde{X} \times_G E_F G = \tilde{X} \times E_F G/\sim$. Here $\tilde{X}$ is the universal cover of X, $E_F G$ is the universal F-classifying space for a family F of subgroups of G, and $(x\gamma, y) \sim (x, \gamma y)$. Let $\rho : \mathcal{E}(X) \to \mathcal{B}(X)$ be projection onto the second factor, and let $f : \mathcal{E}(X) \to X$ be the composition of projection onto the first factor followed by the covering map $\tilde{X} \to X$. The map ρ is a stratified fibration with fibers homeomorphic to the spaces $\{X_H, H \in F\}$.

A simplicially stratified fibration $g : Y \to X$ gives rise to an Ω-spectrum $\mathcal{H}_*(X; \mathcal{S}_*(g))$ and an *assembly map* $A_* : \mathcal{H}_*(X; \mathcal{S}_*(g)) \to \mathcal{S}_*(Y)$, which is the classical assembly map $A_* : \mathcal{H}_*(X; \mathcal{S}_*(pt)) \to \mathcal{S}_*(X)$ in the special case of $g = id : X \to X$ (see [10], [21]; also [8] for a different formulation).

Let X be a connected CW-complex and $\rho : \mathcal{E}(X) \to \mathcal{B}(X)$ the simplicially stratified fibration constructed from the family of virtually cyclic subgroups of $G = \pi_1(X)$. Conjecture (1.6) from [10] states

Farrell-Jones Isomorphism Conjecture. *The composite*

$$\mathcal{H}_*(\mathcal{B}(X), \mathcal{S}_*(\rho)) \stackrel{\mathcal{S}_*(f)\circ A_*}{\longrightarrow} \mathcal{S}_*(X)$$

is an equivalence of Ω-spectra, where A_* is the assembly map $A_* : \mathcal{H}_*(\mathcal{B}(X); \mathcal{S}_*(\rho)) \to \mathcal{S}_*(\mathcal{E}(X))$, and $\mathcal{S}_*(f)$ is the image of the map $f : \mathcal{E}(X) \to X$ under $\mathcal{S}_*$.

There is also the following stronger, "fibered" version of the conjecture:

Fibered Isomorphism Conjecture. *Let X be a connected CW-complex, $\xi = Y \to X$ a Serre fibration, and $\mathcal{E}(\xi)$ the total space of the pullback of ξ along the map $f : \mathcal{E}(X) \to X$. Also let $\rho(\xi) : \mathcal{E}(\xi) \to \mathcal{B}(\xi)$ be the composition $\mathcal{E}(\xi) \to \mathcal{E}(X) \to \mathcal{B}(X)$, and $f(\xi) : \mathcal{E}(\xi) \to Y$ the map that covers $f : \mathcal{E}(X) \to X$. Then the composite*

$$\mathcal{H}_*(\mathcal{B}(\xi), \mathcal{S}_*(\rho(\xi))) \stackrel{\mathcal{S}_*(f(\xi))\circ A_*}{\longrightarrow} \mathcal{S}_*(Y)$$

is an equivalence of Ω-spectra.

Later in [10] (2.1, A.8, 2.2.1 and 2.3) Farrell and Jones prove the following fundamental results:

Farrell-Jones Isomorphism Theorem. *The Fibered Conjecture is true for the functors $\mathcal{P}_*$ and $\mathcal{P}_*^{Diff}$ on the space X provided that there exists a simply connected symmetric Riemannian manifold M with nonpositive sectional curvature everywhere such that M admits a properly discontinuous action of a group Γ via isometries of M, with $\pi_1(X) \subset \Gamma$ and compact orbit space M/Γ.*

Theorem 2. *Let X be a connected CW-complex such that $\pi_1(X)$ is a subgroup of a cocompact discrete subgroup of a virtually connected Lie group. Then the Fibered Conjecture is true for the functors $\mathcal{P}_*$ and $\mathcal{P}_*^{Diff}$ on the space X.*

Proposition 3. *If the Fibered Isomorphism Conjecture holds for the functor $\mathcal{S}_*$ on a connected aspherical CW-complex C, then it holds for $\mathcal{S}_*$ on any connected CW-complex X with $\pi_1(X) \cong \pi_1(C)$.*

In light of the above, we say the Isomorphism Conjecture is *true for* Γ if it is true for some classifying space $B\Gamma$.

The relation between $\mathcal{P}_*$ and lower algebraic K-theory is given by the work of Anderson and Hsiang [1]. They show

$$\pi_j(\mathcal{P}_*(X)) = \begin{cases} K_{j+2}(\mathbb{Z}\pi_1(X)) & \text{if } j \leq -3, \\ \tilde{K}_0(\mathbb{Z}\pi_1(X)) & \text{if } j = -2, \\ Wh(\pi_1(X)) & \text{if } j = -1. \end{cases}$$

We make use of this relationship in Theorem 7.

2.2. **Warped Products.** The main reference for this section is [18]. Setting up some notation, let B be an n-dimensional Riemannian manifold, let p be a point in B, and let u, v and w be tangent vectors in T_pB. Denote by $\langle u, v\rangle_B$ the inner product at T_pB, and by $R_{uv}w$ the curvature operator on T_pM induced by the curvature tensor R of B.

Definition 4. Let B and F be Riemannian manifolds, and let $\varphi : B \to \mathbb{R}$ be a positive $\mathcal{C}^\infty$ function. The *warped product*

$$B \underset{\varphi}{\times} F$$

is the manifold $M = B \times F$ with the following metric: for each $x \in TM$ at $(p, q) \in B \times F$

$$\langle x, x\rangle_M \;=\; \langle d\pi(x), d\pi(x)\rangle_B + \;\varphi^2(p)\langle d\sigma(x), d\sigma(x)\rangle_F,$$

where $\pi : M \to B$ and $\sigma : M \to F$ are the corresponding projections, and $\langle\;\rangle_B$ and $\langle\;\rangle_F$ are the metrics on B and F respectively.

It is possible to express the curvature tensor of a warped product in terms of the curvature tensors of each factor and the warping function. In a warped product

$$M = B \underset{\varphi}{\times} F$$

we denote by R the curvature tensor on M. Let X, Y, and Z be lifts of vector fields on B to M, and let U, V, and W be lifts of vector fields on F to M. Moreover, denote by ${}^FR$ the lift of the curvature tensor of F to M. We use the following identities (see [18], chapter 7):

$$R_{VX}Y = (H^\varphi(X, Y)/\varphi)V, \quad \text{where } H^\varphi \text{ is the Hessian of } \varphi; \tag{1}$$

$$R_{VW}(U) = \;{}^FR_{V,W}U - \left(\frac{\langle\nabla\varphi, \nabla\varphi\rangle}{\varphi^2}\right)(\langle V, U\rangle W - \langle W, U\rangle V), \tag{2}$$

where ∇ denotes the gradient of φ.

Our next proposition is the key technical device in modifying an end (cusp) of a hyperbolic manifold.

Proposition 5. *Let $\varphi : (c, \infty) \to \mathbb{R}$ be a positive $\mathcal{C}^\infty$ function which is concave up (i.e., $\varphi''(x) \geq 0$ for all $x \in (c, \infty)$). Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the standard metric. Then the warped product*

$$(c, \infty) \underset{\varphi}{\times} \mathbb{R}^n$$

has nonpositive sectional curvature everywhere.

Proof. Recall that the sectional curvature, κ, in a manifold M is defined as

$$\kappa(v,w) = \langle R_{vw}v, w\rangle / \; Q(v,w),$$

where v and w are linearly independent vectors in T_pM, R is the curvature operator on T_pM, and $Q(v,w) = \langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle^2$. The sectional curvature κ does not depend on the pair of vectors v, w as long as they span the same plane in T_pM. In order to compute the sectional curvature of our warped product, we first consider the following two special cases in which both v and w are orthogonal unit vectors.

Case 1. Both vectors, v and w, are tangent to the $\mathbb{R}^n$ factor. By identity (2), we have

$$\kappa(v,w) = \langle {}^{\mathbb{R}^n}R_{vw}v - \left(\frac{\varphi'}{\varphi}\right)^2 w, w\rangle = -\left(\frac{\varphi'}{\varphi}\right)^2 \leq 0.$$

Case 2. The vector w is tangent to the $\mathbb{R}$ factor and v is tangent to the $\mathbb{R}^n$ factor. Since $\langle R_{vw}v, w\rangle = -\langle R_{vw}w, v\rangle$, identity (1) implies

$$\kappa(v,w) = -\langle \left(\frac{H^{\varphi}(w,w)}{\varphi}\right) v, v\rangle = -\frac{\varphi''}{\varphi} \leq 0.$$

We now consider the remaining case.

Case 3. $v = v_0 + v_1$ and $w = w_0 + w_1$, where the pair $v_0,\ w_0$ are as in Case 1 and the vectors $v_1,\ w_1$ are both tangent to the $\mathbb{R}$ factor. Since $Q(v,w) > 0$, it suffices to show that

$$\langle R_{vw}v, w\rangle \;\leq 0.$$

To accomplish this, expand the 4-tensor $\langle R_{vw}v, w\rangle$ into the 16-terms

$$\langle R_{v_{\alpha(1)}w_{\alpha(2)}}v_{\alpha(3)}, w_{\alpha(4)}\rangle.$$

(Here α varies over all functions from $\{1,2,3,4\}$ to $\{0,1\}$.) Observe that each of these terms is non-positive. This observation for the three terms

$$\langle R_{v_0w_0}v_0, w_0\rangle, \qquad \langle R_{v_0w_1}v_0, w_1\rangle, \qquad \langle R_{v_1w_0}v_1, w_0\rangle$$

is a consequence of cases 1 and 2. The remaining 13 terms are 0. This is easily seen by using identities (1) and (2) together with the basic symmetries of the curvature tensor (cf. [18], pg. 75, Prop. 36, identities (1), (2), and (4)). □

3. The Farrell-Jones Conjecture for finite covolume hyperbolic actions

In this section we use the warped product construction to prove the conjecture for discrete groups acting on hyperbolic space with finite volume quotient. Our strategy is to manufacture a group $\mathcal{G}$ with cocompact action which essentially satisfies the hypotheses of the Farrell-Jones Isomorphism Theorem and which contains an isomorphic copy of our original group.

Proposition 6. *Let Γ be a discrete, finite covolume subgroup of Isom$(\mathbb{H}^{n+1})$ such that Γ fits into an extension*

$$1 \to \Pi \to \Gamma \to G \to 1,$$

where Π is a torsion free subgroup and G is a finite group. Then there exist a simply connected, complete, nonpositively curved Riemannian manifold Z and a subgroup $\mathcal{G} \subseteq Isom(Z)$ with the properties that $\Gamma \subseteq \mathcal{G}$ and $\mathcal{G}$ acts properly discontinuously *on Z with compact orbit space via its natural isometric action. Such a $\mathcal{G}$ fits into an extension as above, $1 \to \Pi' \to \mathcal{G} \to G \to 1$, with Π' torsion free.*

Proof. The method of this proof is to start with a finite volume manifold with cusps. We flatten the cusps, truncate them, then glue the resulting manifold with boundary to a copy of itself. We perform these operations so that the resulting doubled manifold is compact and has nonpositive sectional curvature.

We start with $M = \mathbb{H}^{n+1}/\Pi$. This is a complete finite volume Riemannian manifold with constant curvature -1. Let μ be the Margulis constant for n-dimensional manifolds of nonpositive curvature (see [3] or [4]). This splits M into two parts: M_1, the *thin* part, and M_2, the *thick* part. By picking a suitable ϵ smaller than μ we can assume that M_1 consists entirely of a finite number of disjoint unbounded components, and that each component is isotopic to $U_i = (c, \infty) \underset{e^{-t}}{\times} N_i^n$, for some flat compact n-dimensional manifold N_i^n (see [4], section D.3). Thus

$$M_1 = \bigsqcup_{i=1}^{s} U_i.$$

Now rescale the Riemannian metrics on N_i^n so that each has volume equal to 1. Then

$$U_i = (d_i, \infty) \underset{e^{-t}}{\times} N_i^n$$

for certain numbers $d_1, d_2, \ldots, d_s$. Choose real numbers c_1 and c_2 such that $c_1 > c_2 > \max\{d_i\}$. Let $c_3 \in (c_2, c_1)$. Using an elementary real variable argument, we construct $\varphi : \mathbb{R} \to \mathbb{R}$, a positive C^∞ function which equals e^{-t} for $t \leq c_2$, equals e^{-c_3} for $t \geq c_1$, and is concave up everywhere.

For each U_i we "cut off" the end

$$(c_2, \infty) \underset{e^{-t}}{\times} N_i^n$$

and replace it with the warped product

$$(c_2, \infty) \underset{\varphi(t)}{\times} N_i^n.$$

This yields an open $(n+1)$-manifold $\tilde{Y}^{n+1}$, homeomorphic to M but with different curvature. Locally $\tilde{Y}^{n+1}$ is either $\mathbb{H}^{n+1}$ or the warped product

$$\mathbb{R} \underset{\varphi}{\times} \mathbb{R}^n,$$

where φ satisfies the hypotheses of Proposition 5. As sectional curvature is a local concept, it follows that $\tilde{Y}$ has nonpositive sectional curvature everywhere. This new manifold $\tilde{Y}$ also has ends of the form

$$(c_1, \infty) \underset{d}{\times} N_i,$$

where d is a constant function.

Intuitively it is clear how to construct the double of $\tilde{Y}$. We take two disjoint copies of $\tilde{Y}$ and cut off each cusp at $c_1 + 1$ (in the first coordinate) to form a collar which is the product $[0, 1] \times N_i$. We then glue corresponding truncated cusps of $\tilde{Y}$ together in twos along their boundaries. The resulting Riemannian manifold Y is compact, as there are only finitely many cusps, and it is clearly complete with nonpositive sectional curvature everywhere.

Let Z be the universal cover of Y. Then Z is a simply connected complete Riemannian manifold. Consider $\pi_1(Y)$, regarded as the group of deck transformations. Notice that $\pi_1(Y) \subseteq Isom(Z)$, it acts cocompactly and properly discontinuously on Z, and that $\pi_1(M) = \Pi$ and injects into $\pi_1(Y)$. Thus Z seems to be a good candidate for the manifold in this proposition.

We have a couple of details left to prove. The elements of G, the quotient of Γ, act on M. We need to show that these are isometries of $\tilde{Y}$, then show that the doubling contruction respects the action of these elements, so that they induce isometries of Y.

From Section 10 in [3], given a component U_i of M_1 and an element $\gamma \in \Gamma$, either $\gamma U_i = U_i$ or $\gamma U_i \cap U_i = \emptyset$. Take an element $g \in G$, and also denote by g the isometry it determines on M. Then g acts on the ends U_i, perhaps by permuting them. Assume first that $g \cdot U_i = U_i$, and consider the geodesic

$$\alpha(t) = (t, x)$$

for a fixed point $x \in N_i$, and $t_0 \leq t < \infty$ where $t_0 > d_i$. Then $g \cdot \alpha$ is also a geodesic, as g is an isometry. Define $(t_1, x_1) \in (d_i, \infty) \times N_i^n$ by

$$(t_1, x_1) = g(\alpha(t_0)).$$

Comparing $g \cdot \alpha$ to the geodesic β defined by

$$\beta(t) = (t + (t_1 - t_0), x_1),$$

we see they start at the same point. We lift these geodesics to $\mathbb{H}^{n+1}$, the universal cover of M, and observe that they determine two geodesics that go through some point x_0, and that determine the same point at infinity. Thus these geodesics coincide, i.e., we have $\beta(t) = g \cdot \alpha(t)$.

Hence g preserves the foliation of U_i by the set of lines $\{(d_i, \infty) \times x \mid x \in N_i^n\}$. Consequently, g also preserves the perpendicular foliation $\{t \times N_i^n \mid t \in (d_i, \infty)\}$, since g acts isometrically. But each leaf $t \times N_i^n$ has a different volume, so g must map $t \times N_i^n$ to $t \times N_i^n$. Thus the action of g on

$$U_i = (d_i, \infty) \underset{e^{-t}}{\times} N_i$$

is only on the N_i factor, and is also an isometry of

$$(d_i, \infty) \underset{\varphi}{\times} N_i.$$

It is also possible that $g \cdot U_i = U_j, i \neq j$, and a similar argument shows that g acts only on the right factor

$$g : (d_i, \infty) \underset{e^{-t}}{\times} N_i \to (d_j, \infty) \underset{e^{-t}}{\times} N_j$$

(note this implies that $d_i = d_j$). Then g is also an isometry

$$g : (d_i, \infty) \underset{\varphi}{\times} N_i \to (d_j, \infty) \underset{\varphi}{\times} N_j,$$

so the action of G on M is also an action by isometries on $\tilde{Y}$. Also, since the action restricted to $\bigsqcup_i U_i$ is only on the fiber and not on the leaf (the $\mathbb{R}$ factor), all the glueings required in the construction of the double can be completed smoothly. Therefore $G \subseteq Isom(Y)$.

We define the group $\mathcal{G} \subseteq Isom(Z)$ to consist of all possible lifts of $g : Y \to Y$ to $\bar{g} : Z \to Z$, where $g \in G \subseteq Isom(Y)$. Set Π' in this proposition equal to $\pi_1(Y)$.

A covering space argument, making use of van Kampen's theorem, produces a subgroup of $\mathcal{G}$ that is isomorphic to Γ. ☐

We now complete the proof of the Farrell-Jones Conjecture for discrete finite covolume groups.

Theorem A. *The Isomorphism Conjecture is true for the functors $\mathcal{P}_*$ and $\mathcal{P}_*^{Diff}$ on the space X, provided that there exists a properly discontinuous finite covolume group action by isometries of $\Gamma = \pi_1(X)$ on a hyperbolic space $\mathbb{H}^n$.*

Proof. Such a Γ satisfies the hypotheses of Proposition 6. Hence Γ embeds in $\mathcal{G}$, where the group $\mathcal{G}$ acts cocompactly and properly discontinuously on Z via isometries and Z is a complete non-positively curved simply connected Riemannian manifold. If Z were a symmetric space, then Theorem A would be an immediate consequence of these facts and the Farrell-Jones Isomorphism Theorem as stated in Section 2. Unfortunately, Z is not a symmetric space. But Theorem A still follows directly from the variant of the Farrell-Jones Isomorphism Theorem where the word "symmetric" is replaced by "complete." It is also required that the image of the representation of Γ in $Isom(Z)$ be virtually torsion free. The proof of this variant for Z constructed above then follows almost word for word that given for Proposition 2.3 in [10] (substituting Z for M and $\mathcal{G}$ for $\pi_1(X)$). As a consequence, the Fibered Isomorphism Conjecture 1.7 of [10] holds for any space with fundamental group $\mathcal{G}$. The proof of Theorem A follows from A.8 of [10] since $\Gamma \subset \mathcal{G}$, and from Proposition 3. ☐

4. Algebraic K-theory of Bianchi groups

As an application of our extension of the Farrell-Jones Isomorphism Theorem, consider the Bianchi groups, mentioned in the introduction. The Bianchi groups satisfy the hypotheses of Theorem A; we use this to calculate their lower algebraic K-groups.

Theorem 7. *Let Γ be a group for which the Isomorphism Conjecture holds for the functor $\mathcal{P}_*$, and suppose, for every virtually cyclic subgroup G of Γ, that $Wh(G)$, $\tilde{K}_0(\mathbb{Z}G)$, and $K_i(\mathbb{Z}G)$ for $i \leq -1$ all vanish. Then $Wh(\Gamma)$, $\tilde{K}_0(\mathbb{Z}\Gamma)$, and $K_i(\mathbb{Z}\Gamma)$ for $i \leq -1$ all vanish.*

Proof. This essentially follows from [10], Section 1.6.5. If the Isomorphism Conjecture for $\mathcal{P}_*$ holds for Γ, then, using the correspondence of [1],

$$\pi_n(\mathcal{H}_*(\mathcal{B}(X), \mathcal{P}_*(\rho))) \cong \pi_n(\mathcal{P}_*(X)) \cong Wh_{n+2}(\pi_1(X)) = Wh_{n+2}(\Gamma)$$

for $n \leq -1$, where $Wh_0(\Gamma)$ means $\tilde{K}_0(\mathbb{Z}\Gamma)$ and $Wh_n(\Gamma) = K_n(\mathbb{Z}\Gamma)$ for n negative. There is a spectral sequence converging to $\pi_{i+j}(\mathcal{H}_*(\mathcal{B}(X), \mathcal{P}_*(\rho)))$ whose $E^2_{i,j}$-term involves homology with coefficients in $\pi_j(\mathcal{P}_*(\rho))$ (see [21]). If $Wh_n(G)$ vanishes for $n \leq 1$ for all virtually cyclic subgroups G of Γ, then, using [1] again, $\pi_j(\mathcal{P}_*(\rho)) = 0$ for $j \leq -1$. Thus $E^2_{i,j} = 0$ for $j \leq -1$, and so $\pi_n(\mathcal{H}_*(\mathcal{B}(X), \mathcal{P}_*(\rho))) = 0$ for $n \leq -1$. Consequently $Wh_n(\Gamma) = 0$ for $n \leq 1$. ☐

We now prove

Theorem B. *Let Γ_d be a Bianchi group. Then $K_i(\mathbb{Z}\Gamma) = 0$ for $i \leq -1$, $\tilde{K}_0(\mathbb{Z}\Gamma) = 0$, and $Wh(\Gamma) = 0$.*

The proof requires two steps. We must classify all of the virtually cyclic subgroups of the Bianchi groups. We then need to show that their lower algebraic K-theory vanishes. Once we have done this, the result follows at once from Theorem A and Theorem 7. We accomplish these remaining steps in the next two sections.

4.1. **Virtually cyclic subgroups of Bianchi groups.** In this section we classify up to isomorphism all possible virtually cyclic groups G which occur as subgroups of Bianchi groups. In particular we list all such subgroups in the five Euclidean Bianchi groups, Γ_d, for $d = -1, -2, -3, -7, -11$.

Recall that a virtually cyclic group either is finite or fits into an extension $1 \to \mathbb{Z} \to G \to F \to 1$. The finite subgroups of Bianchi groups are well-known: 1, $\mathbb{Z}/2$, $\mathbb{Z}/3$, $\mathbf{D}_2$ (the elementary abelian group of order 4), $\mathbf{S}_3$ (the symmetric group on three letters), and $\mathbf{A}_4$ (the alternating group on four letters) [13].

To classify the infinite virtually cyclic subgroups we use the following structure theorem due to P. Scott and T. Wall:

Theorem 8 ([24])**.** *The following conditions on a finitely generated group G are equivalent:*

TF1 *G contains an infinite cyclic subgroup of finite index.*
TF2 *G contains a finite normal subgroup with quotient isomorphic to $\mathbb{Z}$ or D_∞.*
TF3 *G is isomorphic to an HNN extension of the form $F \underset{F}{*}$ with F a finite group when the quotient is $\mathbb{Z}$, and to an amalgamated product*

$$A \underset{F}{*} B$$

where F is finite and $|A : F| = |B : F| = 2$ when the quotient is $\mathbf{D}_\infty$.

Condition TF1 gives the usual definition of a virtually infinite cyclic group. In our case, condition TF2 yields a significant reduction in the problem of classification, as only the five non-trivial finite groups listed above appear as subgroups of the Bianchi groups.

Consider the virtually cyclic groups that fit into short exact sequences with finite kernel and $\mathbf{D}_\infty$ as a quotient. By TF3 above, there are only three possibilities: the amalgamated products $\mathbb{Z}/2 \underset{1}{*} \mathbb{Z}/2 \cong \mathbf{D}_\infty$, $\mathbf{D}_2 \underset{\mathbb{Z}/2}{*} \mathbf{D}_2 \cong \mathbf{D}_\infty \times \mathbb{Z}/2$, and $\mathbf{S}_3 \underset{\mathbb{Z}/3}{*} \mathbf{S}_3$. These subgroups do in fact appear in various Bianchi groups.

Virtually cyclic groups that arise from extensions of the form $1 \to F \to G \to \mathbb{Z} \to 1$ are a little harder to classify. We first show that the kernel of this short exact sequence cannot equal either $\mathbf{D}_2$ or $\mathbf{A}_4$. We will need a couple of lemmas.

The following is well-known (see [14]):

Proposition 9. *Two non-trivial elements f and g of $PSL_2(\mathbb{C})$ commute if and only if either*

(i) *they have exactly the same fixed point set, or*
(ii) *each is elliptic of order two, and each interchanges the fixed points of the other.*

Although elements of $PSL_2(\mathbb{C})$ are matrices, they may also be thought of as fractional linear transformations, a subgroup of isometries of the complex plane.

Specifically,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longleftrightarrow z \longmapsto \frac{az+b}{cz+d}.$$

The advantage of this point of view is that it allows us to use geometry in describing algebraic properties of $PSL_2(\mathbb{C})$. For example, recall that fractional linear transformations are triply transitive, that is, they can send any three complex values to any other three complex values. In fact, this uniquely determines the transformation.

Lemma 10. *If G is a subgroup of $PSL_2(\mathbb{C})$ which fits into an extension $1 \to F \to G \to \mathbb{Z} \to 1$ where F is a finite group, then F is abelian.*

Proof. First we observe that any such G has a subgroup isomorphic to $\mathbb{Z} \times F$. Say $\mathbb{Z} = \langle t \rangle$. Then t acts on F by conjugation and has finite order, say n. Notice that t^n acts trivially on (commutes with) F, so $\langle t^n, F \rangle \cong \mathbb{Z} \times F$. Let g and h be non-trivial elements of F. Since g and t^n commute and t^n has infinite order, by Proposition 9, g and t^n must have the same fixed point set. By an identical argument, h and t^n have the same fixed point set. Therefore g and h have the same fixed point set, so they must commute, and so F is abelian. $\square$

To limit the possibilities even further,

Lemma 11. *The group $PSL_2(\mathbb{C})$ does not contain a subgroup isomorphic to $G = \mathbb{Z} \times \mathbf{D}_2$.*

Proof. The subgroup G is generated by an element t of infinite order and two elements a, b of order two, and they all commute. We claim that a and b do not share fixed points. A non-identity element of order two in $PSL_2(\mathbb{C})$ is of the form $\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$ with determinant one. Considered as a fractional linear transformation, the fixed points of this element are $\frac{\alpha \pm i}{\gamma}$. It is now easy to confirm that only one element of order two has these fixed points. Thus case (i) in Proposition 9 does not apply, so a and b are as in case (ii), with each interchanging the other's fixed points.

Now for t to commute with both a and b it must, by (ii), share their fixed points. However there are four of them, forcing t to be the identity element. $\square$

Corollary 12. *The finite groups $\mathbf{D}_2$, $\mathbf{S}_3$ and $\mathbf{A}_4$ do not appear as kernels of $1 \to F \to G \to \mathbb{Z} \to 1$.*

The only possibilities left are $F = \mathbb{Z}/2$ or $\mathbb{Z}/3$. We note that the classifying space of $\mathbb{Z}$ is $\mathbb{S}^1$. Therefore, $H^2(\mathbb{Z}; A) = 0$ for any coefficients A. This implies that each action of $\mathbb{Z}$ on F will yield a split extension which is unique up to equivalence in the sense of Brown [6].

Case 1: $F = \mathbb{Z}/2$. The quotient $\mathbb{Z}$ acts trivially on $\mathbb{Z}/2$, and $H^2(\mathbb{Z}; \mathbb{Z}/2) = 0$. The only possibility is $G \cong \mathbb{Z} \times \mathbb{Z}/2$.

Case 2: $F = \mathbb{Z}/3$. The finite group $\mathbb{Z}/3$ has two possible $\mathbb{Z}$-module structures, the trivial action and the action that sends a generator of $\mathbb{Z}/3$ to its square. Thus either $G \cong \mathbb{Z} \times \mathbb{Z}/3$ (trivial action), or $G \cong \mathbb{Z}/3 \rtimes \mathbb{Z}$ (twisting action). We summarize:

Theorem 13. *Any infinite virtually cyclic subgroup of a Bianchi group must be isomorphic to one of the following seven groups: $\mathbb{Z}, \mathbb{Z} \times \mathbb{Z}/2, \mathbb{Z} \times \mathbb{Z}/3, \mathbf{D}_\infty, \mathbf{D}_\infty \times \mathbb{Z}/2, \mathbb{Z}/3 \rtimes \mathbb{Z}$, and $\mathbf{S}_3 \underset{\mathbb{Z}/3}{*} \mathbf{S}_3$.*

For the five Euclidean Bianchi groups, we provide a specific list below.

Theorem 14. *Up to isomorphism, the following is a complete list of the infinite virtually cyclic subgroups* G *of* Γ_d *for* $d = -1, -2, -3, -7, -11$:

d	G
-1	$\mathbb{Z}, \mathbb{Z}\times\mathbb{Z}/3, D_\infty, G_2$
-2	$\mathbb{Z}, \mathbb{Z}\times\mathbb{Z}/2, \mathbb{Z}\times\mathbb{Z}/3, D_\infty, G_1$
-3	$\mathbb{Z}, \mathbb{Z}\times\mathbb{Z}/2, D_\infty$
-7	$\mathbb{Z}, \mathbb{Z}\times\mathbb{Z}/2, \mathbb{Z}\times\mathbb{Z}/3, D_\infty$
-11	$\mathbb{Z}, \mathbb{Z}\times\mathbb{Z}/2, \mathbb{Z}\times\mathbb{Z}/3, D_\infty$

where $G_1 = D_\infty \times \mathbb{Z}/2$ *and* $G_2 = \mathbf{S}_3 \underset{\mathbb{Z}/3}{*} \mathbf{S}_3$.

Proof. To verify the results listed in the table, we construct infinite virtually cyclic subgroups by examining the normalizers and centralizers of conjugacy classes of torsion elements. We give an overview of the case Γ_{-1}; full information on normalizers and stabilizers for the five Bianchi groups can be found in [5].

For Γ_{-1}, the existence of subgroups $\mathbb{Z}$ and $\mathbb{Z}\times\mathbb{Z}/3$ follows directly from the computations in [5]. The subgroups $G \cong \mathbb{Z}\times\mathbb{Z}/2$ and $\mathbf{D}_\infty \times \mathbb{Z}/2$ do not occur, as every element of order two in Γ_{-1} has finite centralizer. Let $A1 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $A2 = \left(\begin{smallmatrix} -i & -1+i \\ 0 & i \end{smallmatrix}\right)$ be elements in Γ_{-1}. Then it is immediate to verify that $G = \langle A1, A2\rangle \cong \mathbf{D}_\infty$. On the other hand, let $C = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1, \end{smallmatrix}\right), A = \left(\begin{smallmatrix} -1-2i & 1-2i \\ 2 & 1+2i \end{smallmatrix}\right)$, and $D = \left(\begin{smallmatrix} -i & -i \\ 0 & i \end{smallmatrix}\right)$ be elements in Γ_{-1}. Then $\langle A, C\rangle \cong \mathbf{S}_3$, $\langle C, D\rangle \cong \mathbf{S}_3$, and $\langle A, C, D\rangle \cong \mathbf{S}_3 \underset{\mathbb{Z}/3}{*} \mathbf{S}_3$. The explicit calculations in [5] show that these are the only possible virtually cyclic subgroups. □

4.2. **Algebraic K-theory of the virtually cyclic subgroups of Γ_d.** In this section we provide the final step in proving Theorem B: the K-theory vanishing result for the virtually cyclic subgroups classified in the last section. The calculations require a variety of techniques, so we either supply a summary of the calculation or a reference. In what follows, $\mathbb{F}_2$ and $\mathbb{F}_3$ are the fields with 2 and 3 elements respectively.

Proposition 15. *Let* G *be any virtually cyclic subgroup of a Bianchi group. Then* $\tilde{K}_0(\mathbb{Z}G) = 0$, $Wh(G) = 0$, *and* $K_i(\mathbb{Z}G) = 0$ *for* $i \le -1$.

Proof. From our classification theorem, G is either one of the six finite groups: 1, $\mathbb{Z}/2$, $\mathbb{Z}/3$, $\mathbf{D}_2 \cong \mathbb{Z}/2\oplus\mathbb{Z}/2$, $\mathbf{S}_3 \cong D_3$, $\mathbf{A}_4$; or one of the seven infinite virtually cyclic groups: $\mathbb{Z}$, $\mathbb{Z}\times\mathbb{Z}/2$, $\mathbb{Z}\times\mathbb{Z}/3$, $\mathbb{Z}/3\rtimes\mathbb{Z}$, $D_\infty \cong \mathbb{Z}/2*\mathbb{Z}/2$, $D_\infty\times\mathbb{Z}/2$, $\mathbf{S}_3 \underset{Z/3}{*} \mathbf{S}_3$.

The finite groups 1, $\mathbb{Z}/2$, $\mathbb{Z}/3$, $\mathbf{D}_2$, $\mathbf{S}_3$, $\mathbf{A}_4$.

The Whitehead groups $Wh(G)$ are trivial for H cyclic of order 1, 2, and 3 by [25]. For the other finite groups, a result of Bass and Murthy [2] states that $Wh(G) = \mathbb{Z}^y \oplus SK_1(\mathbb{Z}G)$, where y is the number of irreducible real representations of G minus the number of irreducible rational representations of G. For $G = \mathbf{D}_2$, $\mathbf{S}_3$, and $\mathbf{A}_4$, y is zero, and a result of Oliver [17] proves that $SK_1(\mathbb{Z}G)$ vanishes for each of these three groups as well.

For the other K-groups, Carter shows in [7] that the groups $K_i(\mathbb{Z}G)$ vanish for $i < -1$. The formula given in ([2], p. 695) shows that $K_{-1}(\mathbb{Z}G)$ is zero for the finite abelian subgroups listed. Carter's formula [7] applied in [19] shows that $K_{-1}(\mathbb{Z}\mathbf{S}_3)$

$= 0$, and Dress induction as used in Theorem 11.2 of [17] shows that $K_{-1}(\mathbb{Z}\mathbf{A}_4) = 0$. Finally, Reiner shows in [23] that $\tilde{K}_0(\mathbb{Z}G) = 0$ for all the finite subgroups in question.

The infinite virtually cyclic groups $\mathbb{Z}$, $\mathbb{Z}\times\mathbb{Z}/2$, $\mathbb{Z}\times\mathbb{Z}/3$, $\mathbb{Z}/3\rtimes\mathbb{Z}$, D_∞, $D_\infty\times\mathbb{Z}/2$, $\mathbf{S}_3 \underset{\mathbb{Z}/3}{*} \mathbf{S}_3$.

When G is infinite virtually cyclic, Farrell and Jones show in [11] that $K_i(\mathbb{Z}G)$ is zero for $i < -1$ and that $K_{-1}(\mathbb{Z}G)$ is generated by the images of $K_{-1}(\mathbb{Z}F)$ where F ranges over all finite subgroups $F \subset G$. As discussed above, $K_{-1}(\mathbb{Z}F)$ is trivial for all such possible F, so all that remains is to show that $\tilde{K}_0(\mathbb{Z}G) = 0$ and $Wh(G) = 0$ for all the infinite virtually cyclic subgroups in the classification. This result is well-known for the cases $G = \mathbb{Z}$ and $D_\infty = \mathbb{Z}/2 * \mathbb{Z}/2$.

The cases $G = \mathbb{Z}\times\mathbb{Z}/2$ **or** $D_\infty\times\mathbb{Z}/2$. Let G equal either $\mathbb{Z}$ or D_∞. Then one can build a Rim square

$$\begin{array}{ccc} \mathbb{Z}[G\times\mathbb{Z}/2] & \longrightarrow & \mathbb{Z}G \\ \big\downarrow & & \big\downarrow \\ \mathbb{Z}G & \longrightarrow & \mathbb{F}_2[G] \end{array}$$

with all maps surjective. This results in a Mayer-Vietoris sequence of K-groups. The terms $K_i(\mathbb{F}_2[G])$ can be easily computed with the Bass-Heller-Swan Fundamental Theorem when $G = \mathbb{Z}$ and with the Gersten-Stallings-Waldhausen theorem for free products when $G = D_\infty$. In either case, $Wh_i(G)$ is zero for $G \cong \mathbb{Z}\times\mathbb{Z}/2$, $D_\infty\times\mathbb{Z}/2$ for $i \le 1$.

The case $G = \mathbb{Z}\times\mathbb{Z}/3$. We write $\mathbb{Z}[\mathbb{Z}\times\mathbb{Z}/3]$ as $\mathbb{Z}[\mathbb{Z}/3][\mathbb{Z}]$, and apply the Fundamental Theorem of K-theory (following the notation of Bass [2]):

$$K_i(\mathbb{Z}[\mathbb{Z}\times\mathbb{Z}/3]) \cong K_i(\mathbb{Z}[\mathbb{Z}/3]) \oplus K_{i-1}(\mathbb{Z}[\mathbb{Z}/3]) \oplus 2NK_i(\mathbb{Z}[\mathbb{Z}/3]).$$

To compute $NK_i(\mathbb{Z}[\mathbb{Z}/3])$ for $i = 0$ or 1, let ξ_3 be a primitive third root of unity and use the Rim square (see, for example, [15]):

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{Z}/3] & \longrightarrow & \mathbb{Z}[\xi_3] \\ \big\downarrow & & \big\downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{F}_3. \end{array}$$

This square induces a long exact sequence on Nil-groups (see [2], p. 677). As all the other terms in the square are regular rings, their Nil terms are zero; hence the $NK_i(\mathbb{Z}[\mathbb{Z}/3])$ are also zero. Thus

$$\begin{aligned} K_1(\mathbb{Z}[\mathbb{Z}\times\mathbb{Z}/3]) &\cong K_1(\mathbb{Z}[\mathbb{Z}/3]) \oplus K_0(\mathbb{Z}[\mathbb{Z}/3]) \cong \mathbb{Z}/3\oplus\mathbb{Z}, \\ K_0(\mathbb{Z}[\mathbb{Z}\times\mathbb{Z}/3]) &\cong K_0(\mathbb{Z}[\mathbb{Z}/3]) \oplus K_{-1}(\mathbb{Z}[\mathbb{Z}/3]) \cong \mathbb{Z}, \end{aligned}$$

so both $Wh(\mathbb{Z}\times\mathbb{Z}/3)$ and $\tilde{K}_0(\mathbb{Z}[\mathbb{Z}\times\mathbb{Z}/3])$ vanish.

The case $G = \mathbb{Z}/3\rtimes\mathbb{Z}$. We use a twisted version of the previous square. Let $R = \mathbb{Z}[\xi_3]$, and let α be the automorphism of R defined by $\alpha(\xi_3) = {\xi_3}^{-1}$. We have

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{Z}/3 \rtimes \mathbb{Z}] & \longrightarrow & R_\alpha[\mathbb{Z}] \\ \downarrow & & \downarrow \\ \mathbb{Z}[\mathbb{Z}] & \longrightarrow & \mathbb{F}_3[\mathbb{Z}], \end{array}$$

which again yields a Mayer-Vietoris sequence of K-groups.

The projection map $pr_* : K_i(\mathbb{Z}) \to K_i(\mathbb{F}_3)$ is an isomorphism for $i = 0, 1$ and is a surjection for $i = 2$. The same is true for $pr_* : K_i(\mathbb{Z}[\mathbb{Z}]) \to K_i(\mathbb{F}_3[\mathbb{Z}])$, due to the naturality of the Fundamental Theorem and the fact that $\mathbb{Z}$ and $\mathbb{F}_3$ are regular rings.

Thus the Mayer-Vietoris sequence shows that $K_i(\mathbb{Z}[\mathbb{Z}/3 \rtimes \mathbb{Z}]) \to K_i(R_\alpha[\mathbb{Z}])$ is an isomorphism for $i = 0, 1$. As $R = \mathbb{Z}[\xi_3]$ is a Dedekind domain and a regular ring, $K_i(R_\alpha[\mathbb{Z}])$ can be computed in a straightforward manner. Farrell and Hsiang prove in [9] that $K_1(R_\alpha[\mathbb{Z}]) \cong X \oplus \tilde{C}(R, \alpha) \oplus \tilde{C}(R, \alpha^{-1})$, and that

$$0 \to K_1(R)/I(\alpha_*) \to X \to (K_0(R))^{\alpha_*} \to 0 \quad \text{is exact,}$$

where $I(\alpha_*) = \{\gamma - \alpha_*(\gamma) \mid \gamma \in K_1(R)\}$. Furthermore, it is stated in [9] that the exotic terms $\tilde{C}(R, \alpha)$ and $\tilde{C}(R, \alpha^{-1})$ vanish when R is a regular ring. In our situation, then,

$$0 \to K_1(R)/I(\alpha_*) \to K_1(R_\alpha[\mathbb{Z}]) \to (K_0(R))^{\alpha_*} \to 0 \quad \text{is exact.}$$

The map α_* acts trivially on $K_0(R) \cong \mathbb{Z}$, $K_1(R) \cong \mathbb{Z}/6$, and the order of $I(\alpha_*)$ is 3. Hence $K_1(\mathbb{Z}[\mathbb{Z}/3 \rtimes \mathbb{Z}]) = \mathbb{Z} \oplus \mathbb{Z}/2$; one easily deduces that $Wh(\mathbb{Z}/3 \rtimes \mathbb{Z}) = 0$.

The vanishing of $\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/3 \rtimes \mathbb{Z}])$ is easier to show. From [9], if R is a regular ring, then the inclusion map $\tilde{K}_0(R) \to \tilde{K}_0(R_\alpha[\mathbb{Z}])$ is a surjection. Since $\tilde{K}_0(R)$ vanishes for $R = \mathbb{Z}[\xi_3]$, we have the desired result.

The case $G = \mathbf{S}_3 \underset{\mathbb{Z}/3}{*} \mathbf{S}_3$. We thank S. Prassidis, who kindly informed us of the following results and made them accessible. In [16] he and Munkholm show that if $G = G_1 \underset{\mathbb{Z}/p}{*} G_2$, then there is a Mayer-Vietoris exact sequence

$$Wh(\mathbb{Z}/p) \to Wh(G_1) \oplus Wh(G_2) \to Wh(G) \to \tilde{K}_0(\mathbb{Z}[\mathbb{Z}/p]) \to \cdots .$$

Applying this to $G = \mathbf{S}_3 \underset{\mathbb{Z}/3}{*} \mathbf{S}_3$, it immediately follows that $Wh(G)$ and $\tilde{K}_0(\mathbb{Z}G)$ are trivial. $\square$

Remark. We note that the vanishing result for the case $G = \mathbf{S}_3 \underset{\mathbb{Z}/3}{*} \mathbf{S}_3$ can also be deduced independently of [16] from our previous vanishing results (i.e., those demonstrated above for the other virtually cyclic subgroups of Bianchi groups) and [11], Section 2.

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