

MULTIPLE SOLUTIONS FOR QUASI-LINEAR PDES INVOLVING THE CRITICAL SOBOLEV AND HARDY EXPONENTS

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ABSTRACT. We use variational methods to study the existence and multiplicity of solutions for the following quasi-linear partial differential equation:

$$\begin{cases} -\Delta_p u = \lambda |u|^{r-2} u + \mu \frac{|u|^{q-2}}{|x|^s} u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where λ and μ are two positive parameters and Ω is a smooth bounded domain in \mathbf{R}^n containing 0 in its interior. The variational approach requires that $1 < p < n$, $p \leq q \leq p^*(s) \equiv \frac{n-s}{n-p}p$ and $p \leq r \leq p^* \equiv p^*(0) = \frac{np}{n-p}$, which we assume throughout. However, the situations differ widely with q and r , and the interesting cases occur either at the critical Sobolev exponent ($r = p^*$) or in the Hardy-critical setting ($s = p = q$) or in the more general Hardy-Sobolev setting when $q = \frac{n-s}{n-p}p$. In these cases some compactness can be restored by establishing Palais-Smale type conditions around appropriately chosen *dual sets*. Many of the results are new even in the case $p = 2$, especially those corresponding to singularities (i.e., when $0 < s \leq p$).

1. INTRODUCTION

Consider the following quasi-linear partial differential equation:

$$(P_{\lambda,\mu}) \quad \begin{cases} -\Delta_p u = \lambda |u|^{r-2} u + \mu \frac{|u|^{q-2}}{|x|^s} u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where λ and μ are two positive parameters and Ω is a smooth bounded domain in \mathbf{R}^n containing 0 in its interior. We shall assume throughout that $0 \leq s \leq p < n$.

The starting point of the variational approach to these problems is the following *Sobolev-Hardy inequality*, which is essentially due to Caffarelli, Kohn and Nirenberg [8]. Assume that $1 < p < n$ and that $q \leq p^*(s) \equiv \frac{n-s}{n-p}p$; then there is a constant $C > 0$ such that

$$C \left(\int_{\Omega} \frac{|u|^q}{|x|^s} dx \right)^{\frac{p}{q}} \leq \int_{\Omega} |\nabla u|^p dx \quad \text{for all } u \in H_0^{1,p}(\Omega).$$

We use $\mu_{s,q}(\Omega)$ to denote the best *Sobolev-Hardy* constant, i.e. the largest constant C satisfying the above inequality for all $u \in H_0^{1,p}(\Omega)$; that is,

$$\mu_{s,q}(\Omega) = \inf_{u \in H_0^{1,p}(\Omega), u \neq 0} \frac{\int_{\Omega} |\nabla u|^p dx}{\left(\int_{\Omega} \frac{|u|^q}{|x|^s} dx \right)^{\frac{p}{q}}}.$$

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In the important case where $q = p^*(s)$, we shall simply denote $\mu_{s,p^*(s)}$ as μ_s . Note that μ_0 is nothing but the best constant in the *Sobolev inequality* while μ_p is the best constant in the *Hardy inequality*, i.e.,

$$\mu_p(\Omega) = \inf_{u \in H_0^{1,p}(\Omega), u \neq 0} \frac{\int_{\Omega} |\nabla u|^p dx}{\int_{\Omega} \frac{|u|^p}{|x|^p} dx}.$$

We shall always assume that $p \leq r \leq p^* \equiv p^*(0) = \frac{np}{n-p}$ for the non-singular term in such a way that the functional

$$E_{\lambda,\mu}(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{\lambda}{r} \int_{\Omega} |u|^r dx - \frac{\mu}{q} \int_{\Omega} \frac{|u|^q}{|x|^s} dx$$

is then well defined on the Sobolev space $H_0^{1,p}(\Omega)$. The (weak) solutions of the problem $(P_{\lambda,\mu})$ are then the critical points of the functional $E_{\lambda,\mu}$.

Another relevant parameter will be the first “eigenvalue” of the p -Laplacian $-\Delta_p$, defined as

$$\lambda_1(\Omega) \equiv \mu_{0,p}(\Omega) = \inf \left\{ \int_{\Omega} |\nabla w|^p dx : w \in H_0^{1,p}(\Omega), \int_{\Omega} |w|^p dx = 1 \right\}.$$

Here are the main results of this paper.

Theorem 1.1 (Hardy-Sobolev subcritical singular and non-singular terms). *Suppose $1 < p \leq q < p^*(s)$ and $r < p^*$. Assume one of the following conditions holds:*

- (1) (*High order singular term*) $p < q$, $p \leq r$, $\lambda > 0$ and $\mu > 0$.
- (2) (*Low order singular term*) $p = q$, $p < r$, $\lambda > 0$ and $\mu_{s,p} > \mu > 0$.

Then $(P_{\lambda,\mu})$ has infinitely many solutions. Moreover, $(P_{\lambda,\mu})$ has an everywhere positive solution with least energy and another one that is sign-changing.

Theorem 1.2 (Hardy-critical singular term). *Suppose $1 < p = q = p^*(s)$ (i.e., $s = p$).*

1. (*Subcritical non-singular term*) *If $r < p^*$, then $(P_{\lambda,\mu})$ has infinitely many solutions –at least one of them being positive– for any $\lambda > 0$ and $0 < \mu < \mu_p$.*
2. (*Critical non-singular term*) *If $r = p^*$ and Ω is star-shaped. Then $(P_{\lambda,\mu})$ has no non-trivial solution for any $\lambda > 0, \mu > 0$.*

Theorem 1.3 (Hardy-Sobolev critical singular term). *Suppose $1 \leq p < q = p^*(s)$ (i.e., $s < p$).*

1. (*High order non-singular term*) *Assume $p < r < p^*$ and $\lambda > 0, \mu > 0$.*
 - *If $n > \frac{p(p-1)r+p^2}{p+(p-1)(r-p)}$ (in particular if $n \geq p^2$), then $(P_{\lambda,\mu})$ has a positive solution.*
 - *If $n > \frac{p(p-1)r+p}{1+(p-1)(r-p)}$ (in particular if $n > p^3 - p^2 + p$), then $(P_{\lambda,\mu})$ has also a sign-changing solution.*
2. (*Low order non-singular term*) *Assume $p = r < p^*$ and $0 < \lambda < \lambda_1, \mu > 0$.*
 - *If $n \geq p^2$, then $(P_{\lambda,\mu})$ has a positive solution.*
 - *If $n > p^3 - p^2 + p$, then $(P_{\lambda,\mu})$ has also a sign-changing solution.*
3. (*Sobolev-critical non-singular term*) *Assume $r = p^*$ and Ω is star-shaped, then $(P_{\lambda,\mu})$ has no non-trivial solution for any $\lambda > 0$ and any $\mu > 0$.*

Theorem 1.4 (Sobolev-critical non-singular term and subcritical singular term). *Suppose $1 < p \leq q < p^*(s)$ and $r = p^*$.*

1. (High order singular term) Assume that $p < q$ and $\lambda > 0, \mu > 0$.
 - If $n > \frac{p(p-1)(q-s)+p^2}{p+(p-1)(q-p)}$ (in particular if $n \geq p^2 - (p-1)s$), then $(P_{\lambda,\mu})$ has a positive solution.
 - If $n > \frac{p(p-1)(q-s)+p}{1+(p-1)(q-p)}$ (in particular if $n > p(p-1)(p-s) + p$), then $(P_{\lambda,\mu})$ has also a sign-changing solution.
2. (Low order non-singular term) Assume $p = q$ and $\lambda > 0, \mu_{s,p} > \mu > 0$.
 - If $n \geq p^2 - (p-1)s$, then $(P_{\lambda,\mu})$ has a positive solution.
 - If $n > p((p-1)(p-s) + 1)$, then $(P_{\lambda,\mu})$ has also a sign-changing solution.

The following tables summarize our results.

TABLE 1. Sobolev-subcritical non-singular term

Singular term	Parameters	Non-singular term	Dimension	# of solutions
(HS-subcritical) $p < q < p^*(s)$ $p = q < p^*(s)$	$\lambda > 0; \mu > 0$ $\mu_{s,p} > \mu > 0; \lambda > 0$	$1 \leq p \leq r < p^*$ $1 \leq p < r < p^*$	$n > p$ $n > p$	Infinite One positive One positive
(H-critical) $p = q = p^*(s)$	$\lambda > 0; \mu_p > \mu > 0$	$1 \leq p < r < p^*$	$n > p$	Infinite (One positive)
(HS-critical) $p < q = p^*(s)$	$\lambda > 0; \mu > 0$ — $\lambda_1 > \lambda > 0; \mu > 0$ —	$1 \leq p < r < p^*$ $2 \leq p < r < p^*$ $1 \leq p = r < p^*$ $2 \leq p = r < p^*$	$n > \frac{p(p-1)r+p^2}{p+(p-1)(r-p)}$ $n > \frac{p(p-1)r+p}{1+(p-1)(r-p)}$ $n \geq p^2$ $n > p^3 - p^2 + p$	One positive Two One positive Two

TABLE 2. Sobolev-critical non-singular term

Singular term	Parameters	Non-singular term	Dimension	# of solutions
$1 \leq p = q < p^*(s)$ $2 \leq p = q < p^*(s)$	$\mu_{s,p} > \mu > 0$ and $\lambda > 0$	$r = p^*$ —	$n > p^2 - (p-1)s$ $n > p((p-1)(p-s) + 1)$	One positive Two
$1 \leq p < q < p^*(s)$ $2 \leq p < q < p^*(s)$	$\lambda > 0; \mu > 0$ —	$r = p^*$ —	$n > \frac{p(p-1)(q-s)+p^2}{p+(p-1)(q-p)}$ $n > \frac{p(p-1)(q-s)+p}{1+(p-1)(q-p)}$	One positive Two
$p \leq q = p^*(s)$	$\lambda > 0, \mu > 0$	$r = p^*$	$n > p$	None

2. A POHOZAEV-TYPE IDENTITY

In this section, we start by identifying the constraints on the problem of existence of solutions for $(P_{\lambda,\mu})$. Here is the main result.

Theorem 2.1. *If Ω is a star-shaped domain in \mathbf{R}^n , then problem $(P_{\lambda,\mu})$ has no solution in the doubly critical case: That is, for $r = p^*$ and $q = p^*(s) = \frac{n-s}{n-p}p$, the problem $(P_{\lambda,\mu})$ has no non-trivial solution.*

Assume Ω is a star-shaped domain. Then, if we let v denote the outwards normal to $\partial\Omega$, then $\langle x, v \rangle > 0$ on $\partial\Omega$. We assume we have the necessary regularity in the following operations; otherwise, we can use an approximation argument as in Guedda and Veron [20].

Multiplying the equation $(P_{\lambda,\mu})$ by $\langle x, \nabla u \rangle$ on both sides and integrate by parts, we get

$$\frac{p-1}{p} \int_{\partial\Omega} |\nabla u|^p \langle x, v \rangle dx + \frac{n-p}{p} \int_{\Omega} |\nabla u|^p dx = \mu \frac{n-s}{q} \int_{\Omega} \frac{|u|^q}{|x|^s} dx + \lambda \frac{n}{r} \int_{\Omega} |u|^r dx.$$

On the other hand, multiplying the equation by u and integrating, we get

$$\int_{\Omega} |\nabla u|^p = \mu \int_{\Omega} \frac{|u|^q}{|x|^s} dx + \lambda \int_{\Omega} |u|^r dx.$$

Putting the two identities together, we have

$$\frac{p-1}{p} \int_{\partial\Omega} |\nabla u|^p \langle x, \nu \rangle d\sigma = \mu \left(\frac{n-s}{q} - \frac{n-p}{p} \right) \int_{\Omega} \frac{|u|^q}{|x|^s} dx + \lambda \left(\frac{n}{r} - \frac{n-p}{p} \right) \int_{\Omega} |u|^r.$$

So if $r = \frac{np}{n-p} = p^*$ and $q = \frac{n-s}{n-p}p$, the problem has no non-trivial solution.

3. THE EXTREMAL FUNCTIONS IN THE HARDY-SOBOLEV INEQUALITIES

In this section, we summarize the needed results concerning the Hardy-Sobolev inequalities. We first recall the *Hardy* inequality.

Lemma 3.1 ([13]). *Assume that $1 < p < n$ and $u \in H^{1,p}(\mathbf{R}^n)$. Then:*

- (1) $\frac{u}{|x|} \in L^p(\mathbf{R}^n)$.
- (2) (*Hardy Inequality*) $\int_{\mathbf{R}^n} \frac{|u|^p}{|x|^p} dx \leq C_{n,p} \int_{\mathbf{R}^n} |\nabla u|^p dx$, where $C_{n,p} = \left(\frac{p}{n-p}\right)^p$.
- (3) *The constant $C_{n,p}$ is optimal.*

The following extension of the Hardy and Sobolev inequalities is essentially due to Caffarelli, Kohn and Nirenberg[8].

Lemma 3.2 (Sobolev-Hardy Inequality). *Assume that $1 < p < n$ and that $p \leq q \leq p^*(s) := \frac{n-s}{n-p}p$. Then:*

- (1) *There exists a constant $C > 0$ such that for any $u \in H_0^p(\Omega)$,*

$$\left(\int_{\Omega} \frac{|u|^q}{|x|^s} dx \right)^p \leq C \left(\int_{\Omega} |\nabla u|^p dx \right)^q.$$

- (2) *The map $u \rightarrow \frac{u}{|x|^{s/q}}$ from $H_0^p(\Omega)$ into $L^q(\Omega)$ is compact provided $q < p^*(s)$.*

Proof. (1) For $s = 0$ or $s = p$, this is just the Sobolev (resp., the Hardy) inequality. Since $p^*(s) \geq p$, we have $0 \leq s \leq p$. We can therefore only consider the case where $0 < s < p$. By the Hardy, Sobolev and Hölder inequalities, we have

$$\begin{aligned} \int_{\mathbf{R}^n} \frac{|u|^{p^*(s)}}{|x|^s} &= \int_{\mathbf{R}^n} \frac{|u|^s}{|x|^s} \cdot |u|^{p^*(s)-s} \\ &\leq \left(\int_{\mathbf{R}^n} \frac{|u|^p}{|x|^p} \right)^{\frac{s}{p}} \left(\int_{\mathbf{R}^n} |u|^{(p^*(s)-s)\frac{p}{p-s}} \right)^{\frac{p-s}{p}} \\ &= \left(\int_{\mathbf{R}^n} \frac{|u|^p}{|x|^p} \right)^{\frac{s}{p}} \left(\int_{\mathbf{R}^n} |u|^{p^*} \right)^{\frac{p-s}{p}} \\ &\leq C_1 \left(\int_{\mathbf{R}^n} |\nabla u|^p \right)^{\frac{s}{p}} \left(\int_{\mathbf{R}^n} |\nabla u|^p \right)^{\frac{p^*}{p} \cdot \frac{p-s}{p}} \\ &= C_1 \left(\int_{\mathbf{R}^n} |\nabla u|^p \right)^{\frac{n-s}{n-p}}. \end{aligned}$$

□

Remark 3.1. If Ω is the whole space, one can show that the conditions $p \leq q = p^*(s) := \frac{n-s}{n-p}p$ are also necessary for the above inequality to hold. Indeed, a standard scaling argument shows that q must be equal to $p^*(s)$. On the other hand,

if we insert into the inequality the following function (ρ and $\theta \in S^{n-1}$ being the polar coordinates),

$$u(x) = \begin{cases} 0 & \text{for } |x| \geq 1, \\ |x|^{\frac{p-n}{p}} \log \frac{1}{|x|} & \text{for } \varepsilon \leq |x| < 1, \\ \varepsilon^{\frac{p-n}{p}} \log \frac{1}{\varepsilon} & \text{for } |x| \leq \varepsilon, \end{cases}$$

and

$$\frac{du(x)}{d\rho} = \begin{cases} 0, & |x| \geq 1, \\ 0, & |x| \leq \varepsilon, \\ (1 - \frac{n}{p})\rho^{-\frac{n}{p}} \log \frac{1}{\rho} - \rho^{-\frac{n}{p}}, & \varepsilon \leq |x| < 1, \end{cases}$$

we get

$$\int_{\mathbf{R}^n} |\nabla u|^p \sim \int_{\varepsilon}^1 \rho^{-1} (1 + (\frac{n}{p} - 1) \log \frac{1}{\rho})^p d\rho.$$

By L'Hospital's rule, we have

$$\lim_{\varepsilon \rightarrow 0} \frac{\int_{\varepsilon}^1 \rho^{-1} (1 + (\frac{n}{p} - 1) \log \frac{1}{\rho})^p d\rho}{\log^{1+p} \frac{1}{\varepsilon}} = \frac{\frac{n}{p} - 1}{1 + p},$$

and also

$$\int_{\mathbf{R}^n} \frac{|u|^q}{|x|^s} \sim \int_{\varepsilon}^1 \rho^{-s} \log^q \frac{1}{\rho} \rho^{\frac{p-n}{p}q} \rho^{n-1} = \int_{\varepsilon}^1 \frac{1}{\rho} \log^q \frac{1}{\rho} \sim \log^{1+q} \frac{1}{\varepsilon}.$$

Thus from the inequality

$$\log^{1+\frac{1}{q}} \frac{1}{\varepsilon} \leq \log^{1+\frac{1}{p}} \frac{1}{\varepsilon},$$

we get that $q \geq p$.

The following is an extension of what is well known in the case $p = 2$ and $s = 0$.

Theorem 3.1. *Suppose $1 < p < n$, $0 \leq s < p$ and $q = p^*(s)$. Then the following hold:*

- (1) $\mu_s(\Omega)$ is independent of Ω (and will henceforth be denoted by μ_s).
- (2) μ_s is attained when $\Omega = \mathbf{R}^n$ by the functions

$$y_a(x) = (a \cdot (n - s) (\frac{n-p}{p-1})^{p-1})^{\frac{n-p}{p(p-s)}} (a + |x|^{\frac{p-s}{p-1}})^{\frac{p-n}{p-s}}$$

for some $a > 0$. Moreover the functions y_a are the only positive radial solutions of

$$-\operatorname{div}(|\nabla u|^{p-2} \nabla u) = \frac{u^{p^*(s)-1}}{|x|^s}$$

in \mathbf{R}^n . Hence,

$$\mu_s (\int_{\mathbf{R}^n} \frac{|y_a|^q}{|x|^s})^{\frac{p}{q}} = \|\nabla y_a\|_p^p = \int_{\mathbf{R}^n} \frac{|y_a|^q}{|x|^s} = \mu_s^{\frac{n-s}{p-s}}.$$

Proof. We prove (2). We show the best constants are attained at functions

$$u_s(x) = c(\lambda_0 + |x|^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}} \quad (0 \leq s < p), \text{ where } \lambda_0 > 0 \text{ is a constant.}$$

For any f , let f^* be its Schwarz symmetrization (or rearrangement) [21]. Then we have

$$\int_{\mathbf{R}^n} |\nabla f^*|^p \leq \int_{\mathbf{R}^n} |\nabla f|^p \quad \text{and} \quad \int_{\mathbf{R}^n} \frac{|f^*|^q}{|x|^t} \geq \int_{\mathbf{R}^n} \frac{|f|^q}{|x|^t},$$

assuming the above integrals are well defined (refer to Lieb [21], [22]). By these inequalities, we may restrict our discussion to radial symmetric functions. Thus we may consider the following variational problem:

$$\text{Maximize } I(g) = \int_0^\infty |g(r)|^q r^{n-s-1} dr, \quad \text{when } J(g) = \int_0^\infty |g'(r)|^p r^{n-1} dr = C.$$

where C is a given constant. The *Euler-Lagrange* equation is

$$(*) \quad (r^{n-1}|u'(r)|^{p-2}u'(r))_r + kr^{n-s-1}|u|^{q-1} = 0.$$

It can be easily verified that the functions

$$u_s(x) = (\lambda + |x|^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}} \quad (0 \leq s < p)$$

are solutions of $(*)$, where $\lambda > 0$. To continue, we need the following lemma of Bliss ([1], [2]).

Lemma 3.3. *Let $h(x) \geq 0$ be a measurable, real-valued function defined on \mathbf{R} such that the integral $J_0 = \int_0^\infty h^{p_0}(x)dx$ is finite and given. Set $g(x) = \int_0^x h(t)dt$. Then $I_0 = \int_0^\infty g^{q_0}(x)x^{\alpha-q_0}dx$ attains its maximum value at the functions $h(x) = (\lambda x^\alpha + 1)^{-\frac{\alpha+1}{\alpha}}$, with p_0 and q_0 two constants satisfying $q_0 > p_0 > 1$, $\alpha = \frac{q_0}{p_0} - 1$, and $\lambda > 0$, a real number.*

By this lemma and with the change of variables $x = r^{\frac{p-n}{p-1}}$ we can deduce that $I(\cdot)$ attains its maximum at the functions

$$u_s(x) = (\lambda + |x|^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}} \quad (0 \leq s < p).$$

Note that if

$$h(x) = (\lambda x^\alpha + 1)^{-\frac{\alpha+1}{\alpha}},$$

then

$$g(x) = \int_0^x h(t)dt = (\lambda + x^{-\alpha})^{-\frac{1}{\alpha}}.$$

And if $q = \frac{n-s}{n-p}p$, then $\alpha = \frac{q}{p} - 1 = \frac{p-s}{n-p}$. The theorem is thus proved. \square

Remark 3.2. As expected, the compactness of the embedding $u \rightarrow \frac{u}{x^{s/q}}$ in Lemma 3.2.(2) above does not hold when $q = p^*(s)$. Indeed, let

$$f_k(x) = \left(\frac{1}{k}\right)^{\frac{n-p}{p(p-s)}} \left(\frac{1}{k} + |x|^{\frac{p-s}{p-1}}\right)^{\frac{p-n}{p-s}},$$

and set $\|\nabla f_k(x)\|_p^p = A$ and $\int_\Omega \frac{|f_k(x)|^{p^*(s)}}{|x|^s} dx = C$. Let

$$h_k(x) = f_k(x) - \left(\frac{1}{k}\right)^{\frac{n-p}{p(p-s)}} \left(\frac{1}{k} + 1\right)^{\frac{p-n}{p-s}}$$

for $|x| \leq 1$, so that $h_k(x) \in H_0^{1,p}(B)$ and $\|h_k\|_{H_0^{1,p}(\Omega)} \rightarrow A^{\frac{1}{p}}$. Hence $\{h_k\}$ is bounded in $H_0^{1,p}(\Omega)$ and $\left\|\frac{h_k}{|x|^{s/p^*(s)}}\right\|_{L^{p^*(s)}} \rightarrow C^{1/p^*(s)}$. Now, $h_k(x) \rightarrow 0$ for $|x| \neq 0$ and 0 is the only possible cluster point of $\left\{\frac{h_k}{|x|^{s/p^*(s)}}\right\}$ in $L^{p^*(s)}(\Omega)$, which is impossible since $C \neq 0$.

4. THE COMPACTNESS LEMMAS

This section deals with the compactness properties of the functional

$$E_{\lambda,\mu}(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{\lambda}{r} \int_{\Omega} |u|^r dx - \frac{\mu}{q} \int_{\Omega} \frac{|u|^q}{|x|^s} dx.$$

We recall the following standard definition.

Definition 4.1. A C^1 -functional E on Banach space X satisfies the Palais-Smale condition at the level c (in short $(PS)_c$), if every sequence $(u_n)_n$ satisfying $\lim_n E(u_n) = c$ and $\lim_n \|E'(u_n)\| = 0$ has a convergent subsequence.

Define the following function:

$$L(\mu, s) = \begin{cases} \frac{p-s}{p(n-s)} \left(\frac{\mu_s^{n-s}}{\mu^{n-p}}\right)^{\frac{1}{p-s}} & \text{if } s < p, \\ +\infty & \text{if } s = p \text{ and } \mu < \mu_p, \\ 0 & \text{if } s = p \text{ and } \mu \geq \mu_p. \end{cases}$$

Theorem 4.1. Assume $0 \leq s \leq p < n$, $p \leq q \leq p^*(s)$ and $p \leq r \leq p^*$.

- (1) If $p \leq q < p^*(s)$ and $r < p^*$, then for any $\lambda > 0$ and any $\mu > 0$, the functional $E_{\lambda,\mu}$ satisfies $(PS)_c$ for all c .
- (2) If $p \leq q = p^*(s)$ and $r < p^*$, then for any $\lambda > 0$ and any $\mu > 0$, the functional $E_{\lambda,\mu}$ satisfies $(PS)_c$ for all $c < L(\mu, s)$.
- (3) If $p \leq q < p^*(s)$ and $r = p^*$, then for any $\lambda > 0$ and any $\mu > 0$, the functional $E_{\lambda,\mu}$ satisfies $(PS)_c$ for all $c < L(\lambda, 0) = \frac{1}{n} \left(\frac{\mu_0^n}{\lambda^{n-p}}\right)^{\frac{1}{p}}$.

Note that statement 2 above yields that $E_{\lambda,\mu}$ also satisfies $(PS)_c$ for all c when $p = q = q^*(s)$ (i.e., when $s = p$) as long as $\mu < \mu_p$. This solves a problem in [13] and [25], where only a certain singular Palais-Smale condition is established. On the other hand, when $\mu = 1$, $p = 2$ and $s = 0$, we recover the (by now well known) restricted compactness properties that appears in Yamabe-type problems, [4].

We first recall a few known results.

Lemma 4.1 ([25]). Let $x, y \in \mathbf{R}^n$, and let $\langle \cdot, \cdot \rangle_e$ be the standard scalar product in \mathbf{R}^n . Then

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y \rangle_e \geq \begin{cases} C_p |x - y|^p, & \text{if } p \geq 2, \\ C_p \frac{|x-y|^2}{(|x|+|y|)^{2-p}}, & \text{if } 1 < p < 2. \end{cases}$$

The following result of Brezis and Lieb ([4]) will be useful in the sequel.

Lemma 4.2. Suppose $f_n \rightarrow f$ a.e. and $\|f_n\|_p \leq C < \infty$ for all n and for some $0 < p < \infty$. Then

$$\lim_{n \rightarrow \infty} \{ \|f_n\|_p^p - \|f_n - f\|_p^p \} = \|f\|_p^p.$$

Lemma 4.3. Let $(u_n)_n$ be a bounded sequence in $H_0^{1,p}(\Omega)$ and let $(q_n)_n$ be a sequence such that $p < q_n \leq p^*(s)$, $q_n \rightarrow p^*(s)$ as $n \rightarrow \infty$. Then there exists a subsequence (without loss of generality still denoted by $(u_n)_n$) such that:

- (1) $u_n \rightarrow u$ weakly in $H_0^{1,p}(\Omega)$.
- (2) $u_n \rightarrow u$ in $L^r(\Omega)$ if $1 < r < p^* = \frac{np}{n-p}$.
- (3) $u_n \rightarrow u$ almost everywhere.
- (4) $\frac{u_n}{x} \rightarrow \frac{u}{x}$ weakly in $L^p(\Omega)$.

(5) For any $f \in H_0^{1,p}(\Omega)$,

$$\int_{\Omega} \frac{|u_n|^{q_n-2} u_n}{|x|^{s_n}} f \rightarrow \int_{\Omega} \frac{|u|^{p^*(s)-2} u}{|x|^s} f.$$

(6) If $p \geq 2$, then

$$\int_{\Omega} |u_n|^{q_n} \leq \int_{\Omega} |u|^{q_n} + \int_{\Omega} |u_n - u|^{q_n} + o(1).$$

(7)

$$\int_{\Omega} \frac{|u_n - u|^{p^*(s)}}{|x|^s} = \int_{\Omega} \frac{|u_n|^{p^*(s)}}{|x|^s} - \int_{\Omega} \frac{|u|^{p^*(s)}}{|x|^s} + o(1).$$

Proof. These are standard applications of the Hardy-Sobolev embedding theorem and the Brezis-Lieb result. We just give the proofs of (5) and (6). Without loss of generality, we may assume that $q_n < p^*(s)$. For (5), it is clear that

$$\frac{|u_n|^{q_n-2} u_n}{|x|^{s \frac{q_n-1}{q_n}}} \rightarrow \frac{|u|^{p^*(s)-2} u}{|x|^{s \frac{p^*(s)-1}{p^*(s)}}} \text{ a.e.,}$$

and that the integral

$$\begin{aligned} \int_{\Omega} \left| \frac{|u_n|^{q_n-1}}{|x|^{s(1-\frac{1}{p^*(s)})}} \right|^{\frac{p^*(s)}{p^*(s)-1}} &= \int_{\Omega} \frac{|u_n|^{p^*(s) \frac{q_n-1}{p^*(s)-1}}}{|x|^{s \frac{q_n-1}{p^*(s)-1}}} \cdot \frac{1}{|x|^{s(1-\frac{q_n-1}{p^*(s)-1})}} \\ &\leq \left(\int_{\Omega} \frac{|u_n|^{p^*(s)}}{|x|^s} \right)^{\frac{q_n-1}{p^*(s)-1}} \cdot \left(\int_{\Omega} \frac{1}{|x|^s} \right)^{\frac{p^*(s)-q_n}{p^*(s)-1}} \end{aligned}$$

is uniformly bounded in n . Since $f/|x|^{\frac{s}{p^*(s)}} \in L^{p^*(s)}(\Omega)$ for any $f \in H_0^{1,p}(\Omega)$, the conclusion follows.

In order to prove (6), we need the following easy lemma.

Calculus Lemma. For every $1 \leq q \leq 3$, there exists a constant C (depending on q) such that for $\alpha, \beta \in \mathbf{R}$ we have

$$|\alpha + \beta|^q - |\alpha|^q - |\beta|^q - q\alpha\beta(|\alpha|^{q-2} + |\beta|^{q-2})| \leq \begin{cases} C|\alpha||\beta|^{q-1} & \text{if } |\alpha| \geq |\beta|, \\ C|\alpha|^{q-1}|\beta| & \text{if } |\alpha| \leq |\beta|. \end{cases}$$

For $q \geq 3$, there exists a constant C (depending on q) such that for $\alpha, \beta \in \mathbf{R}$ we have

$$|\alpha + \beta|^q - |\alpha|^q - |\beta|^q - q\alpha\beta(|\alpha|^{q-2} + |\beta|^{q-2})| \leq C(|\alpha|^{q-2}\beta^2 + \alpha^2|\beta|^{q-2}).$$

From this inequality, we can actually deduce the following more convenient result for any $q \geq 1$:

$$|\alpha + \beta|^q - |\alpha|^q - |\beta|^q - q\alpha\beta(|\alpha|^{q-2} + |\beta|^{q-2})| \leq 2C(|\alpha|^{q-1}\beta + \alpha|\beta|^{q-1}).$$

Now, back to the proof of (6). Let $w_n = u_n - u$; then $w_n \rightarrow 0$ weakly in $H_0^{1,p}(\Omega)$. By the above calculus lemma,

$$\frac{|u_n|^{q_n}}{|x|^s} = \frac{|w_n + u|^{q_n}}{|x|^s} \leq \frac{|w_n|^{q_n}}{|x|^s} + \frac{|u|^{q_n}}{|x|^s} + C_1 \frac{|u||w_n|^{q_n-1}}{|x|^s} + C_2 \frac{|w_n||u|^{q_n-1}}{|x|^s}.$$

In view of (5), we only need to show that

$$\lim_n \int_{\Omega} \frac{|w_n||u|^{q_n-1}}{|x|^s} = \lim_n \int_{\Omega} \frac{|w_n||u|^{q_n-2}}{|x|^{s(1-\frac{1}{p^*(s)})}} \cdot \frac{|u|}{|x|^{\frac{s}{p^*(s)}}} = 0.$$

For that, we check that

$$\begin{aligned} \int_{\Omega} \left(\frac{|w_n||u|^{q_n-2}}{|x|^{s(1-\frac{1}{p^*(s)})}} \right)^{\frac{p^*(s)}{p^*(s)-1}} &= \int_{\Omega} \frac{|w_n|^{\frac{p^*(s)}{p^*(s)-1}}}{|x|^{s\frac{1}{p^*(s)-1}}} \cdot \frac{|u|^{\frac{q_n-2}{p^*(s)-1}p^*(s)}}{|x|^{s\frac{q_n-2}{p^*(s)-1}}} \cdot \frac{1}{|x|^{s\frac{p^*(s)-q_n}{p^*(s)-1}}} \\ &\leq \left(\int_{\Omega} \frac{|u_n|^{p^*(s)}}{|x|^s} \right)^{\frac{1}{p^*(s)-1}} \left(\int_{\Omega} \frac{|u|^{p^*(s)}}{|x|^s} \right)^{\frac{q_n-2}{p^*(s)-1}} \left(\int_{\Omega} \frac{1}{|x|^s} \right)^{\frac{p^*(s)-q_n}{p^*(s)-1}}. \end{aligned}$$

Hence it is uniformly bounded in n , and the claim follows. □

Lemma 4.4. *Let $E_n(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{\mu}{q_n} \int_{\Omega} \frac{|u|^{q_n}}{|x|^s} dx - \frac{\lambda}{r} \int_{\Omega} |u|^r dx$ ($\lambda > 0, \mu > 0$), where q_n satisfy the conditions in the previous lemma and $1 < p \leq r < p^*$. Assume the sequence $\{u_n\}$ satisfies $E_n(u_n) \rightarrow c, E'_n(u_n) \rightarrow 0$. Then, there exists a subsequence, still denoted by $\{u_n\}$, such that for some $u \in H_0^{1,p}(\Omega)$:*

- (1) $u_n \rightarrow u$ weakly in $u \in H_0^{1,p}(\Omega)$.
- (2) $\nabla u_n \rightarrow \nabla u$ a.e.
- (3) $\int_{\Omega} |\nabla u_n - \nabla u|^p = \int_{\Omega} |\nabla u_n|^p - \int_{\Omega} |\nabla u|^p + o(1)$.
- (4) $|\nabla u_n|^{p-2} \nabla u_n \rightarrow |\nabla u|^{p-2} \nabla u$ weakly in $[L^{\frac{p}{p-1}}(\Omega)]^n$.

Proof. Since

$$\lim_n E_n(u_n) = c \text{ and } \lim_n E'_n(u_n) = 0,$$

and

$$\langle E'_n(u_n), u_n \rangle = \int_{\Omega} |\nabla u_n|^p - \mu \int_{\Omega} \frac{|u_n|^{q_n}}{|x|^s} - \lambda \int_{\Omega} |u_n|^r,$$

we have

$$\begin{aligned} o(1)(1 + \|u_n\|) + p|c| &\geq pE_n(u_n) - \langle E'_n(u_n), u_n \rangle \\ &= \begin{cases} \mu(1 - \frac{p}{q_n}) \int_{\Omega} \frac{|u|^{q_n}}{|x|^s} dx + \lambda(1 - \frac{p}{r}) \int_{\Omega} |u_n|^r dx, & r > p, \\ (1 - \frac{p}{q_n}) \int_{\Omega} \frac{|u|^{q_n}}{|x|^s} dx, & r = p. \end{cases} \end{aligned}$$

Since Ω is bounded, we have

$$\int_{\Omega} |u_n|^p = \int_{\Omega} \frac{|u_n|^p}{|x|^{ps/q_n}} \cdot |x|^{ps/q_n} dx \leq M \left(\int_{\Omega} \frac{|u_n|^{q_n}}{|x|^s} \right)^{p/q_n},$$

and

$$\|\nabla u_n\|_p^p = pE_n(u_n) + \mu \frac{p}{q_n} \int_{\Omega} \frac{|u_n|^{q_n}}{|x|^s} dx + \lambda \frac{p}{r} \int_{\Omega} |u_n|^r dx.$$

We conclude that $\{u_n\}$ is a bounded sequence in $H_0^{1,p}(\Omega)$.

We therefore can assume that $\{u_n\}$ satisfies all of the conclusions in Lemma 4.3. Now we use a technique initiated by Boccardo and Murat and already used by Garcia and Peral in a related context.

Define the functions

$$\tau_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ ks/|s| & \text{if } |s| \geq k. \end{cases}$$

We may assume also that $\tau_k(u_n - u) \rightarrow 0$ weakly in $H_0^{1,p}(\Omega)$ for any fixed positive k , since $\tau_k(u_n - u) \rightarrow 0$ a.e. and it is bounded. Then from the assumption we get

$$\begin{aligned} o(1) &= \langle E'_n(u_n) - (E'_\lambda)^'(u), \tau_k(u_n - u) \rangle + o(1) \\ &= \int_{\Omega} \langle |\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u, \nabla \tau_k(u_n - u) \rangle_e \\ &\quad - \lambda \int_{\Omega} \left(\frac{|u_n|^{q_n-2}}{|x|^s} u_n - \frac{|u|^{p^*(s)-2}}{|x|^s} u \right) \tau_k(u_n - u). \end{aligned}$$

Since

$$\frac{|u_n|^{q_n-2}}{|x|^s} u_n \rightarrow \frac{|u|^{p^*(s)-2}}{|x|^s} u$$

in the weak star topology of $H^{-1,p'}(\Omega)$ (by Lemma 4.3), we have

$$\left| \int_{\Omega} \left(\frac{|u_n|^{q_n-2}}{|x|^s} u_n - \frac{|u|^{p^*(s)-2}}{|x|^s} u \right) \tau_k(u_n - u) \right| \leq Ck,$$

and

$$\limsup_{n \rightarrow \infty} \int_{\Omega} \langle |\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u, \nabla \tau_k(u_n - u) \rangle_e dx \leq Ck.$$

Let $e_n(x) = \langle |\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u, \nabla \tau_k(u_n - u) \rangle_e$; then $e_n(x) \geq 0$ by Lemma 3.1, and is uniformly bounded in $L^1(\Omega)$. Take $0 < \theta < 1$ and split Ω into

$$S_n^k = \{x \in \Omega \mid |u_n - u| \leq k\}, \quad G_n^k = \{x \in \Omega \mid |u_n - u| > k\}.$$

Then

$$\begin{aligned} \int_{\Omega} e_n^\theta dx &= \int_{S_n^k} e_n^\theta dx + \int_{G_n^k} e_n^\theta dx \\ &\leq \left(\int_{S_n^k} e_n dx \right)^\theta |S_n^k|^{1-\theta} + \left(\int_{G_n^k} e_n dx \right)^\theta |G_n^k|^{1-\theta}. \end{aligned}$$

Now, for fixed k , $|G_n^k| \rightarrow 0$ as $n \rightarrow \infty$, and from the uniform boundedness in L^1 we get

$$\limsup_n \int_{\Omega} e_n^\theta dx \leq (Ck)^\theta |\Omega|^{1-\theta}.$$

Letting $k \rightarrow 0$, we get that $e_n^\theta \rightarrow 0$ strongly in L^1 . By Lemma 4.1,

$$\nabla u_n \rightarrow \nabla u \quad \text{in } L^q$$

for $1 < q < p$. By passing to a subsequence, we have

$$\nabla u_n \rightarrow \nabla u \quad \text{a.e.}$$

Thus (1) holds. As for (2), just apply Lemma 4.2. The proof of this lemma is thus complete. \square

Proof of Theorem 4.1.(1): If $p \leq q < p^*(s)$ and $r < p^*$, it is standard to show that the compactness of the Hardy-Sobolev embedding and of the Sobolev embedding imply that for any $\lambda > 0$ and any $\mu > 0$, the functional $E_{\lambda,\mu}$ satisfies $(PS)_c$ for all c . \square

Proof of Theorem 4.1.(2): Recall that

$$L(\mu, s) = \begin{cases} \frac{p-s}{p(n-s)} \left(\frac{\mu_s^{n-s}}{\mu^{n-p}}\right)^{\frac{1}{p-s}} & \text{if } s < p, \\ +\infty & \text{if } s = p \text{ and } \mu < \mu_s, \\ 0 & \text{if } s = p \text{ and } \mu \geq \mu_s, \end{cases}$$

and assume that $p \leq q = p^*(s)$ and $r < p^*$. We need to show that

$$E_{\lambda,\mu}(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{\mu}{p^*(s)} \int_{\Omega} \frac{|u|^{p^*(s)}}{|x|^s} - \frac{\lambda}{r} \int_{\Omega} |u|^r dx$$

satisfies the Palais-Smale condition at any energy level less than $L(\mu, s)$.

For that, assume $\{u_n\}$ is a sequence in $H_0^{1,p}(\Omega)$ satisfying

$$E_{\lambda,\mu}(u_n) \rightarrow c < L(\mu, s) \text{ and } E'_{\lambda,\mu}(u_n) \rightarrow 0.$$

By Lemma 4.4, we may assume that $\{u_n\}$ satisfies the conclusions of both Lemma 4.2 and Lemma 4.3. For any $v \in C_0^\infty(\Omega)$,

$$\langle E'_{\lambda,\mu}(u_n), v \rangle = \int_{\Omega} (\langle |\nabla u_n|^{p-2} \nabla u_n, \nabla v \rangle - \lambda |u_n|^{r-2} u_n v - \mu \frac{|u_n|^{p^*(s)-2} u_n}{|x|^s} v) dx,$$

which converges as $n \rightarrow \infty$ to

$$0 = \int_{\Omega} (\langle |\nabla u|^{p-2} \nabla u, \nabla v \rangle - \lambda |u|^{r-2} uv - \mu \frac{|u|^{p^*(s)-2} u}{|x|^s} v) dx = \langle E'_{\lambda,\mu}(u), v \rangle.$$

Hence $u \in H_0^{1,p}(\Omega)$ is a weak solution of (P_λ, μ) . Choosing $v = u$, we have

$$0 = \langle E'_{\lambda,\mu}(u), u \rangle = \int_{\Omega} (|\nabla u|^p - \lambda |u|^r - \mu \frac{|u|^{p^*(s)}}{|x|^s}) dx,$$

and thus

$$E_{\lambda,\mu}(u) = \lambda \left(\frac{1}{p} - \frac{1}{r}\right) \int_{\Omega} |u|^r + \mu \left(\frac{1}{p} - \frac{1}{p^*(s)}\right) \int_{\Omega} \frac{|u|^{p^*(s)}}{|x|^s} dx \geq 0.$$

By Lemmas 4.3 and 4.4, we have

$$E_{\lambda,\mu}(u_n) = E_{\lambda,\mu}(u) + E_{0,\mu}(u_n - u) + o(1)$$

and

$$\begin{aligned} o(1) = \langle E'_{\lambda,\mu}(u_n), u_n - u \rangle &= \langle E'_{\lambda,\mu}(u_n) - E'_{\lambda,\mu}(u), u_n - u \rangle \\ &= \int_{\Omega} (|\nabla u_n - \nabla u|^p - \mu \frac{|u_n - u|^{p^*(s)}}{|x|^s}) + o(1). \end{aligned}$$

If $s = p = p^*(s)$ and $\mu < \mu_p$, then

$$o(1) = \int_{\Omega} (|\nabla u_n - \nabla u|^p - \mu \frac{|u_n - u|^p}{|x|^p}) + o(1) \leq (1 - \frac{\mu}{\mu_p}) \int_{\Omega} (|\nabla u_n - \nabla u|^p + o(1));$$

that is, $u_n \rightarrow u$ strongly.

If $s < p$ (i.e., $p < p^*(s)$), we have, for large n ,

$$\begin{aligned} E_{0,\mu}(u_n - u) &= E_{\lambda,\mu}(u_n) - E_{\lambda,\mu}(u) + o(1) \\ &\leq E_{\lambda,\mu}(u_n) + o(1) \leq c < L(\mu, s). \end{aligned}$$

Thus, for such n ,

$$\left(\frac{1}{p} - \frac{1}{p^*(s)}\right) \|\nabla u_n - \nabla u\|^p \leq c < \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} \left(\frac{1}{\mu}\right)^{\frac{n-p}{p-s}}.$$

By the Sobolev-Hardy inequality, we finally get

$$\begin{aligned}
 o(1) &= \int_{\Omega} (|\nabla u_n - \nabla u|^p - \mu \frac{|u_n - u|^{p^*(s)}}{|x|^s}) dx \\
 &\geq \int_{\Omega} |\nabla u_n - \nabla u|^p - \mu \mu_s^{-\frac{p^*(s)}{p}} \left(\int_{\Omega} |\nabla u_n - \nabla u|^p \right)^{\frac{p^*(s)}{p}} \\
 &= \left(\int_{\Omega} |\nabla u_n - \nabla u|^p \right) [1 - \mu \mu_s^{-\frac{p^*(s)}{p}} \left(\int_{\Omega} |\nabla u_n - \nabla u|^p \right)^{\frac{p^*(s)-p}{p}}] \\
 &\geq C \int_{\Omega} |\nabla u_n - \nabla u|^p dx.
 \end{aligned}$$

So again $u_n \rightarrow u$ in $H_0^{1,p}(\Omega)$ strongly. \square

Proof of Theorem 4.1.(3): Suppose now that $p \leq q < p^*(s)$ and $r = p^*$; then we have compactness in the singular term but we will be dealing with a non-singular term involving the critical Sobolev exponent. We have again

$$\begin{aligned}
 E_0(u_n - u) &= E_{\lambda,\mu}(u_n) - E_{\lambda,\mu}(u) + o(1) \\
 &\leq E_{\lambda,\mu}(u_n) + o(1) \leq c < L(\lambda, 0) = \frac{1}{n} \mu_0^{\frac{n}{p}} \left(\frac{1}{\lambda} \right)^{\frac{n-p}{p}}
 \end{aligned}$$

Thus, for such n ,

$$\left(\frac{1}{p} - \frac{1}{p^*} \right) \|\nabla u_n - \nabla u\|^p \leq c < \frac{1}{n} \mu_0^{\frac{n}{p}} \left(\frac{1}{\lambda} \right)^{\frac{n-p}{p}},$$

so that this time we get, from the Sobolev inequality,

$$\begin{aligned}
 o(1) &= \langle E'_{\lambda,\mu}(u_n), u_n - u \rangle = \langle E'_{\lambda,\mu}(u_n) - E'_{\lambda,\mu}(u), u_n - u \rangle \\
 &= \int_{\Omega} (|\nabla u_n - \nabla u|^p - \lambda \int_{\Omega} |u_n - u|^{p^*}) + o(1) \\
 &= \left(\int_{\Omega} |\nabla u_n - \nabla u|^p \right) [1 - \lambda \mu_0^{-\frac{p^*}{p}} \left(\int_{\Omega} |\nabla u_n - \nabla u|^p \right)^{\frac{p^*}{p}-1}] + o(1) \\
 &\geq C \int_{\Omega} |\nabla u_n - \nabla u|^p dx.
 \end{aligned}$$

So again $u_n \rightarrow u$ in $H_0^{1,p}(\Omega)$ strongly. \square

5. MIN-MAX PRINCIPLES AND DUAL SETS ASSOCIATED TO $E_{\lambda,\mu}$

For Banach spaces X and Y , we use $C(X, Y)$ to denote the space of all continuous maps from X to Y .

Definition 5.1. Let X be a Banach space and B be a closed subset of X . We say that a class \mathcal{F} of compact subsets of X is a *homotopy-stable family with boundary B* provided that

- (1) every set in \mathcal{F} contains B , and
- (2) for any set A in \mathcal{F} and any $\eta \in C([0, 1] \times X; X)$ satisfying $\eta(t, x) = x$ for all (t, x) in $(\{0\} \times X) \cup ([0, 1] \times B)$ we have that $\eta(\{1\} \times A) \in \mathcal{F}$.

We say that the class \mathcal{F} is Z_2 -homotopy stable if all sets in \mathcal{F} are symmetric and if we only require stability under odd homotopies η (i.e., $\eta(t, -x) = -\eta(t, x)$).

We say that a closed set M is dual to the family \mathcal{F} if

$$M \cap B = \emptyset \text{ and } M \cap A \neq \emptyset \text{ for all } A \in \mathcal{F}.$$

We shall need the following weakened version of the Palais-Smale condition.

Definition 5.2. A C^1 -functional E on Banach space X satisfies the Palais-Smale condition at level c and around the set M (in short, $(PS)_{M,c}$), if every sequence $(u_n)_n$ satisfying $\lim_n E(u_n) = c$, $\lim_n \|E'(u_n)\| = 0$ and $\lim_n \text{dist}(u_n, M) = 0$ has a convergent subsequence.

The following theorem of Ghoussoub [17] will be frequently used in the sequel.

Theorem 5.1. Let E be a C^1 -functional on X and consider a homotopy stable family \mathcal{F} of compact subsets of X with a closed boundary B . Let M be a dual set to \mathcal{F} such that

$$\inf_{x \in M} E(x) = c := c(E, \mathcal{F}) = \inf_{A \in \mathcal{F}} \max_{x \in A} E(x).$$

If E satisfies $(PS)_{M,c}$, then $M \cap K_c \neq \emptyset$, where K_c is the set of all critical points of E at level c .

If \mathcal{F} is only Z_2 -homotopy stable, then the result still holds true as long as the functional E is even and the dual set M is symmetric.

Note that the above theorem includes the classical min-max principle which holds under the assumption that $\sup_{x \in B} E(x) < c$. It is enough to notice that in that case $M = \{x \in X; E(x) \geq c\}$ is a dual set.

Consider again the functional

$$E_{\lambda,\mu}(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{\lambda}{r} \int_{\Omega} |u|^r dx - \frac{\mu}{q} \int_{\Omega} \frac{|u|^q}{|x|^s} dx.$$

Recall that we assume that $1 < p < n$, $0 \leq s \leq p$, $0 \leq q \leq p^*(s) \equiv \frac{n-s}{n-p}p$ and that $p \leq r \leq p^* \equiv \frac{np}{n-p}$, so that E is a C^1 -functional on the Sobolev space $H_0^{1,p}(\Omega)$.

A first dual set: Define the Mountain Pass class to be

$$\mathcal{F}_1 = \{\gamma \in C([0, 1]; H_0^{1,p}(\Omega)); \gamma(0) = 0, \gamma(1) \neq 0 \text{ and } E(\gamma(1)) \leq 0\},$$

which is clearly homotopy-stable with boundary $B = \{E \leq 0\}$. Let

$$M_1 = \{u \in H_0^{1,p}(\Omega); u \neq 0, \langle E'(u), u \rangle = 0\}.$$

The following Nehari-type duality property is by now standard.

Theorem 5.2. Assume $p \leq q \leq p^*(s)$ and one of the following cases:

- (1) $p = r$, $p < q$ and $0 < \lambda < \lambda_1$, $\mu > 0$.
- (2) $p < r \leq p^*$, $p = q$ and $0 < \mu < \mu_{s,p}$, $\lambda > 0$.
- (3) $p < r \leq p^*$, $p < q$ and $\mu > 0$, $\lambda > 0$.

The set M_1 is then closed, is dual to \mathcal{F}_1 and satisfies

$$\inf_{M_1} E_{\lambda,\mu} = c_1 := c(E_{\lambda,\mu}, \mathcal{F}_1)$$

Proof. By definition,

$$\langle E'_{\lambda,\mu}(u), u \rangle = \int_{\Omega} (|\nabla u|^p - \lambda|u|^r - \mu \frac{|u|^q}{|x|^s}) dx.$$

Note that $B \cap M_1 = \emptyset$, since for every $u \in M_1$ we have

$$E_{\lambda,\mu}(u) = \lambda\left(\frac{1}{p} - \frac{1}{r}\right) \int_{\Omega} |u|^r dx + \mu\left(\frac{1}{p} - \frac{1}{q}\right) \int_{\Omega} \frac{|u|^q}{|x|^s} dx > 0,$$

under the assumption that either r or q is different from p . We also show that under this assumption, we have the estimate $c_1 \leq \inf_{u \in M_1} E_{\lambda,\mu}(u)$.

Let $u \neq 0$ in M_1 , and consider the straight path $\gamma(t) = tu$. We have

$$E_{\lambda,\mu}(tu) = \frac{t^p}{p} \int_{\Omega} |\nabla u|^p - \frac{\lambda t^r}{r} \int_{\Omega} |u|^r - \mu \frac{t^q}{q} \int_{\Omega} \frac{|u|^q}{|x|^s}.$$

Since $\lim_{t \rightarrow \infty} E_{\lambda,\mu}(tu) = -\infty$, we have that $c_1 \leq \sup_{0 \leq t < \infty} E_{\lambda,\mu}(tu) = E_{\lambda,\mu}(t_0 u)$. From

$$\frac{dE_{\lambda,\mu}(tu)}{dt} = t^{p-1} \int_{\Omega} |\nabla u|^p - \lambda t^{r-1} \int_{\Omega} |u|^r - \mu t^{q-1} \int_{\Omega} \frac{|u|^q}{|x|^s}$$

and $\frac{dE_{\lambda,\mu}(tu)}{dt}(t_0) = 0$, we get

$$\int_{\Omega} |\nabla u|^p = t_0^{r-p} \cdot \lambda \int_{\Omega} |u|^r + \mu t_0^{q-p} \int_{\Omega} \frac{|u|^q}{|x|^s}.$$

Since $u \in M_1$, we should have

$$\int_{\Omega} |\nabla u|^p = \lambda \int_{\Omega} |u|^r + \mu \int_{\Omega} \frac{|u|^q}{|x|^s}.$$

Thus, t_0 must be equal to 1 as long as either r or q is distinct from p . This clearly shows that under any of the 3 conditions above, we have

$$c_1 \leq \inf_{u \in M_1} E_{\lambda,\mu}(u).$$

For the rest, we have to distinguish the 3 cases.

Case (1). $2 \leq p = r$, $p < q$ and $0 < \lambda < \lambda_1$.

To prove that M_1 is closed, use the Sobolev-Hardy inequality and the definition of λ_1 to find a constant $c > 0$ such that

$$\begin{aligned} \langle E'_{\lambda,\mu}(u), u \rangle &= \left(1 - \frac{\lambda}{\lambda_1}\right) \|u\|_{H_0^{1,p}(\Omega)}^p - c \|u\|_{H_0^{1,p}(\Omega)}^q \\ &= \|u\|_{H_0^{1,p}(\Omega)}^p \left(1 - \frac{\lambda}{\lambda_1} - c \|u\|_{H_0^{1,p}(\Omega)}^{q-p}\right). \end{aligned}$$

Choose some $\beta > 0$ such that if $\|u\| < \beta$, then $1 - \frac{\lambda}{\lambda_1} - c \|u\|_{H_0^{1,p}(\Omega)}^{q-p} > 0$. This means that we can find some constant $\beta > 0$ such that for any $u \in M_1$, we have $\|u\| \geq \beta$. So M_1 is closed.

To prove the intersection property, fix $\gamma \in \mathcal{F}_1$ joining 0 to v , where $v \neq 0$ and $E_{\lambda,\mu}(v) \leq 0$. Note that since $\lambda < \lambda_1$, we have $\langle E'_{\lambda,\mu}(\gamma(t)), \gamma(t) \rangle > 0$ for t close to 0 (same proof as for the closedness of M_1). On the other hand, since $v \neq 0$, we have

$$\langle E'_{\lambda,\mu}(v), v \rangle < p E_{\lambda,\mu}(v) \leq 0.$$

It follows from the intermediate value theorem that there exists t_0 such that $\gamma(t_0) \in M_1$. This proves the duality, and consequently $c_1 \geq \inf\{E_{\lambda,\mu}(u) : u \in M_1\}$.

Case (2). $1 < p < r \leq p^*$, $p = q \leq p^*(s)$ and $0 < \mu < \mu_{s,q}$.

To prove that M_1 is closed, use the Sobolev-Hardy inequality with its best constant $\mu_{s,p}$ and the Sobolev inequality to get

$$\begin{aligned} \langle E'_{\lambda,\mu}(u), u \rangle &= \int_{\Omega} |\nabla u|^p - \lambda \int_{\Omega} |u|^r - \mu \int_{\Omega} \frac{|u|^q}{|x|^s} \\ &\geq \int_{\Omega} |\nabla u|^p - c \left(\int_{\Omega} |\nabla u|^p \right)^{\frac{r}{p}} - \frac{\mu}{\mu_{s,q}} \int_{\Omega} |\nabla u|^p \\ &= \left(1 - \frac{\mu}{\mu_{s,q}} \right) \|u\|_{H_0^{1,p}(\Omega)}^p - c \|u\|_{H_0^{1,p}(\Omega)}^r \\ &= \|u\|_{H_0^{1,p}(\Omega)}^p \left(1 - \frac{\mu}{\mu_{s,q}} - c \|u\|_{H_0^{1,p}(\Omega)}^{r-p} \right). \end{aligned}$$

Choose some $\beta > 0$ such that if $\|u\| < \beta$, then $1 - \frac{\mu}{\mu_{s,q}} - c \|u\|_{H_0^{1,p}(\Omega)}^{r-p} > 0$. This means that we can find some constant $\beta > 0$ such that for any $u \in M_1$, we have $\|u\| \geq \beta$. So M_1 is closed.

To prove the intersection property, fix $\gamma \in \mathcal{F}_1$ joining 0 to v , where $v \neq 0$ and $E_{\lambda,\mu}(v) \leq 0$. Note that since $\mu < \mu_{s,q}$, we have $\langle E'_{\lambda,\mu}(\gamma(t)), \gamma(t) \rangle > 0$ for t close to 0 (same proof as for the closedness of M_1). On the other hand, since $v \neq 0$, we have $\langle E'_{\lambda,\mu}(v), v \rangle < pE_{\lambda,\mu}(v) \leq 0$. It follows from the intermediate value theorem that there exists t_0 such that $\gamma(t_0) \in M_1$. This proves the duality, and consequently $c_1 \geq \inf\{E_{\lambda,\mu}(u) : u \in M_1\}$.

Case (3). $2 \leq p < r \leq p^*$ and $\lambda > 0$.

To prove that M_1 is closed, again use the Sobolev-Hardy inequality and the Sobolev embedding to find constants $c' > 0$, $c'' > 0$ such that

$$\begin{aligned} \langle E'_{\lambda,\mu}(u), u \rangle &\geq \|u\|^p - c' \lambda \|u\|^r - c'' \|u\|^q \\ &= \|u\|^p (1 - c' \lambda \|u\|^{r-p} - c'' \|u\|^{q-p}). \end{aligned}$$

Since both r and q are distinct from p , we may choose $\gamma > 0$ such that for any $u \in H_0^{p-1}(\Omega)$ with $\|u\| < \gamma$, we have $1 - c_1 \lambda \|u\|^{r-p} - c_2 \|u\|^{q-p} > 0$. This means that $\|u\| \geq \gamma$ for any $u \in M_1$; hence M_1 is closed.

For the intersection property, consider any $\gamma \in \mathcal{F}_1$ joining 0 and v . Since $p < r$, again the proof above of the closedness of M_1 yields that $\langle E_{\lambda,\mu}(\gamma(t)), \gamma(t) \rangle > 0$ for t close to 0. Also since $v \neq 0$, we have

$$\langle E'_{\lambda,\mu}(v), v \rangle < pE_{\lambda,\mu}(v) \neq 0.$$

Then again, by the intermediate value theorem, we conclude that there exists t_0 such that $\gamma(t_0) \in M_1$. This proves the duality and the inequality

$$c_1 \geq \inf\{E_{\lambda,\mu}(u), u \in M_1\}.$$

□

Another dual set: Denote by S_ρ the sphere $S_\rho = \{u \in H_0^{1,p}(\Omega); \|u\|_{H_0^{1,p}(\Omega)} = \rho\}$ and by \mathcal{H} the set

$$\mathcal{H} = \{h : H_0^{1,p}(\Omega) \rightarrow H_0^{1,p}(\Omega) \text{ an odd homeomorphism}\}.$$

Let γ_{Z_2} denote the Krasnoselskii genus, defined for every closed symmetric subset D of $H_0^{1,p}(\Omega)$ as

$$\gamma_{Z_2}(D) = \inf\{n; \text{there exists an odd and continuous map } h : D \rightarrow \mathbf{R}^n \setminus \{0\}\},$$

and consider the class

$$\mathcal{F}_2 = \{A; A \text{ closed symmetric with } \gamma_{Z_2}(h(A) \cap S_\rho) \geq 2, \forall h \in \mathcal{H}\}.$$

It is easy to verify that \mathcal{F}_2 is a Z_2 -homotopy stable class. Let

$$c_2 = \inf_{A \in \mathcal{F}_2} \sup_A E_{\lambda, \mu}.$$

We shall now consider an appropriate dual set to \mathcal{F}_2 . First, we recall a few facts about the following weighted eigenvalue problem ($1 < p < \infty$):

$$(*) \quad \begin{cases} -\Delta_p u = \lambda b(x)|u|^{p-2}u, \\ u \in H_0^{1,p}(\Omega), \quad u \neq 0. \end{cases}$$

We will say that $\lambda \in \mathbf{R}$ is the eigenvalue and $u \in H_0^{1,p}(\Omega)$, $u \neq 0$, is the corresponding eigenfunction of the above problem if the equality

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx = \lambda \int_{\Omega} b(x)|u|^{p-2} u \varphi \, dx$$

holds for any $\varphi \in H_0^{1,p}(\Omega)$. The following lemma is well known.

Lemma 5.1 ([25] [11]). *Assume $b(x) \geq 0$, $b(x) \in L^t(\Omega)$, and $|\{x \in \Omega : b(x) > 0\}| \neq 0$, where $t \geq 1$ if $p > n$, $t > 1$ if $p = n$ and $t > \frac{n}{p} > 1$ otherwise. Let $\lambda_0 = \inf\{\int_{\Omega} |\nabla v|^p; \int_{\Omega} b(x)|v|^p = 1\}$. Then:*

- (1) $\lambda_0 > 0$ is the first eigenvalue of the problem (*).
- (2) λ_0 is simple, and there exists precisely one pair of normalized eigenfunctions corresponding to λ_0 which do not change sign in Ω . Here, v being normalized means that $\int_{\Omega} b(x)|v|^p = 1$.

We use the lemma to prove the following fact:

Lemma 5.2. *For $2 \leq p \leq r < p^*$, $p \leq q < p^*(s)$, $\lambda > 0, \mu > 0$ and any $u \in H_0^{1,p}(\Omega)$, $u \neq 0$, there exists a unique $v = v(u) \in H_0^{1,p}(\Omega)$ such that*

- (a) $\int_{\Omega} (\lambda|u|^{r-p} + \mu \frac{|u|^{q-p}}{|x|^s}) v^p = 1, v \geq 0;$
- (b) $\|\nabla v\|_p^p = \inf\{\|\nabla \omega\|_p^p : \int_{\Omega} (\lambda|u|^{r-p} + \mu \frac{|u|^{q-p}}{|x|^s}) |\omega|^p = 1\}.$

Furthermore, the map $u \rightarrow v(u)$ is continuous from $L^r(\Omega) \rightarrow H_0^{1,p}(\Omega)$.

Remark 5.1. It is quite unfortunate that the above lemma is not applicable –unless $p = 2$ – whenever $r = p^*$ or when $q = p^*(s)$. This will create additional complications in the search for a second solution of the critical problems.

Proof. Since $\frac{np^*}{sp^*+(q-p)n} > \frac{n}{p}$, choose $\frac{n}{p} < t < \frac{np^*}{sp^*+(q-p)n}$; then $st \frac{p^*}{p^*-(q-p)t} < n$. Because

$$\int_{\Omega} \frac{|u|^{(q-p)t}}{|x|^{st}} \leq \left(\int_{\Omega} |u|^{p^*} \right)^{\frac{t(q-p)}{p^*}} \left(\int_{\Omega} \frac{1}{|x|^{st \cdot \frac{p^*}{p^*-(q-p)t}}} \right)^{\frac{p^*-(q-p)t}{p^*}},$$

we get that $|u|^{q-p}/|x|^s \in L^t(\Omega)$.

The functional $\psi(u) = \|u\|_p^p = \int_{\Omega} |\nabla u|^p \, dx$ is clearly weakly lower semicontinuous and coercive. Moreover, the constraint set

$$C = \{\omega \in H : \int_{\Omega} (\lambda|u|^{r-p} + \mu \frac{|u|^{q-p}}{|x|^s}) |\omega|^p \, dx = 1\}$$

is weakly closed in $H_0^{1,p}(\Omega)$ and $\psi(\cdot)$ is bounded below on C . Therefore, by the direct methods of the calculus of variations (Struwe [26], p. 4), the infimum in (b) is achieved and this infimum is the first eigenvalue of (*) and thus is simple. Any function where such an infimum is achieved is the eigenfunction corresponding to the first eigenvalue of (*). By Lemma 5.1, it cannot change sign in Ω . This gives the uniqueness of $v(u)$ and therefore its continuity for non-zero u . \square

Note that $(\nu_1(u), v(u))$ corresponds to the first eigenpair of the (weighted) eigenvalue problem

$$(**) \quad \begin{cases} -\Delta_p v = \nu(\lambda|u|^{r-p} + \mu \frac{|u|^{q-p}}{|x|^s})|v|^{p-2}v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Now let

$$M_2 = M_1 \cap \{u \in H_0^{1,p}(\Omega); \int_{\Omega} (\lambda|u|^{r-p} + \mu \frac{|u|^{q-p}}{|x|^s})v(u)^{p-1}u = 0\}.$$

The following duality result was first noticed by G. Tarantello [28] in the case when $s = 0$ and $p = 2$.

Theorem 5.3. *Assume $p \leq q < p^*(s)$ and $r < p^*$. Then M_2 is a closed set that is dual to \mathcal{F}_2 , and*

$$\inf_{M_2} E_{\lambda,\mu} = c_2 := c(E_{\lambda,\mu}, \mathcal{F}_2)$$

as long as we are in one of the following cases:

- (1) $p = r, p < q$ and $0 < \lambda < \lambda_1, 0 < \mu$.
- (2) $p < r, p = q$ and $0 < \mu < \mu_{s,p}, 0 < \lambda$.
- (3) $p < r, p < q$ and $0 < \mu, 0 < \lambda$.

Proof. In the 3 cases, we get from Theorem 5.2 (and its proof) that M_1 is closed and that for any $u \neq 0$, there exists a unique $t(u) > 0$ such that $t(u)u \in M_1$. Clearly, $t(u) = t(|u|) = t(-u)$ and

$$E_{\lambda,\mu}(t(u)u) = \max_{t \geq 0} E_{\lambda,\mu}(tu).$$

The uniqueness of $t(u)$ and its properties tell us that the map $u \rightarrow t(u)$ is continuous on $H_0^{1,p}(\Omega)$ and that the map $u \rightarrow t(u)u$ defines an odd homeomorphism between S_ρ and M_1 which gives that $\gamma_{Z_2}(A \cap M_1) \geq 2$ for all $A \in \mathcal{F}_2$.

On the other hand, the map $h : A \cap M_1 \rightarrow \mathbf{R}$ given by

$$h(u) = \int_{\Omega} (\lambda|u|^{r-p} + \mu \frac{|u|^{q-p}}{|x|^s})v(u)^{p-1}u dx$$

defines an odd and continuous map. Since $\gamma_{Z_2}(h(A \cap M_1)) \geq 2$, we get that $0 \in h(A \cap M_1)$ which means that $A \cap M_2 \neq \emptyset$ and M_2 is dual to \mathcal{F}_2 . In particular, $c_2 \geq \inf_{u \in M_2} E_{\lambda,\mu}(u)$.

To prove the reverse inequality, take $u \in M_2$ and let $v(u)$ be such that

$$\int_{\Omega} (\lambda|u|^{r-p} + \mu \frac{|u|^{q-p}}{|x|^s})v(u)^{p-1}u dx = 0.$$

Let $\omega(u)$ be a minimizer for the problem:

$$\mu_2 = \inf \left\{ \psi(\omega); \omega \in H_0^{1,p}, \int_{\Omega} (\lambda|u|^{r-p} + \mu \frac{|u|^{q-p}}{|x|^s}) v(u)^{p-1} \omega = 0, \right. \\ \left. \int_{\Omega} (\lambda|u|^{r-p} + \mu \frac{|u|^{q-p}}{|x|^s}) |\omega|^p = 1 \right\}.$$

Since $u \in M_1$, we obtain

$$\mu_2 \leq \frac{\|\nabla u\|_p^p}{\int_{\Omega} (\lambda|u|^r + \mu \frac{|u|^q}{|x|^s})} = 1.$$

Define $A = \text{span}\{v(u), \omega(u)\} \in \mathcal{F}_2$. Then, clearly,

$$1 \geq \mu_2 \geq \frac{\|\nabla \omega\|_p^p}{\int_{\Omega} (\lambda|u|^{r-p} + \mu \frac{|u|^{q-p}}{|x|^s}) |\omega|^p}, \quad \forall \omega \in A, \omega \neq 0.$$

For $\omega_0 \in A$ satisfying $E_{\lambda,\mu}(\omega_0) = \sup_A E_{\lambda,\mu} \geq c_2$, we have $\omega_0 \neq 0$ and $\omega_0 \in M_1$. From the above inequality, we derive

$$\int_{\Omega} (\lambda|u|^{r-p} + \mu \frac{|u|^{q-p}}{|x|^s}) |\omega_0|^p \geq \|\nabla \omega_0\|_p^p.$$

This implies

$$\frac{1}{p} \int_{\Omega} (\lambda|u|^{r-p} + \mu \frac{|u|^{q-p}}{|x|^s}) (|\omega_0|^p - |u|^p) \geq \frac{\|\nabla \omega_0\|_p^p}{p} - \frac{\|\nabla u\|_p^p}{p}.$$

Applying the inequality (valid for $t \geq p$ and $x, y \in \mathbf{R}$)

$$\frac{1}{t} (|x|^t - |y|^t) \geq \frac{1}{p} (|x|^p - |y|^p) |y|^{t-p}$$

with $t = r$ (resp. $t = q$), we conclude that

$$\frac{\lambda}{r} \int_{\Omega} |\omega_0|^r + \mu \frac{1}{q} \int_{\Omega} \frac{|\omega_0|^q}{|x|^s} - \frac{\lambda}{r} \int_{\Omega} |u|^r - \mu \frac{1}{q} \int_{\Omega} \frac{|u|^q}{|x|^s} \geq \frac{\|\nabla \omega_0\|_p^p}{p} - \frac{\|\nabla u\|_p^p}{p},$$

that is,

$$E_{\lambda,\mu}(u) \geq E_{\lambda,\mu}(\omega_0) \geq c_2.$$

This finishes the proof of the theorem. \square

6. THE SOLUTIONS IN THE CASE OF AN HS-SUBCRITICAL SINGULAR TERM

In this section, we consider the problem

$$(P_{\lambda,\mu}) \quad \begin{cases} -\Delta_p u = \lambda|u|^{r-2}u + \frac{|u|^{q-2}}{|x|^s}u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $0 \leq s \leq p < n$, in the presence of a subcritical singular term ($1 < p \leq q < p^*(s)$) and a subcritical non-singular term ($1 < p \leq r < p^*$).

Theorem 6.1 (Hardy-Sobolev subcritical singular term). *Suppose $1 < p \leq q < p^*(s)$ and $r < p^*$. Also assume one of the following conditions:*

- (1) $p < q$, $p \leq r$ and $\lambda > 0$, $\mu > 0$.
- (2) $p = q$, $p < r$ and $\lambda > 0$, $\mu_{s,p} > \mu > 0$.

Then the equation $(P_{\lambda,\mu})$ has infinitely many solutions. Moreover, it has an everywhere positive solution u_1 with minimal energy and a sign-changing solution u_2 that satisfies

$$\int_{\Omega} (\lambda|u_2|^{r-p} + \mu \frac{|u_2|^{q-p}}{|x|^s}) v(u_2)^{p-1} u_2 = 0,$$

where $v(u_2)$ is the first eigenvector of the (weighted) eigenvalue problem

$$\begin{cases} -\Delta_p v = \nu(\lambda|u_2|^{r-p} + \mu \frac{|u_2|^{q-p}}{|x|^s}) |v|^{p-2} v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof. Now that under these conditions the functional $E_{\lambda,\mu}$ satisfies $(PS)_c$ for any c . It is now enough to apply Theorem 5.1 to \mathcal{F}_1 and its dual set M_1 (resp., to \mathcal{F}_2 and its dual set M_2) to get a solution u_1 (resp. u_2) which minimizes the energy functional on M_1 (resp. M_2).

To obtain other solutions, we need the following result of Rabinowitz ([17]).

Lemma 6.1. *Let E be an even C^1 -functional satisfying the Palais-Smale condition on a Banach space $X = Y \oplus Z$ with $\dim(Y) < \infty$. Assume $E(0) = 0$, as well as the following conditions:*

- (1) *There is $\rho > 0$ such that $\inf_{S_\rho(Z)} E \geq 0$.*
- (2) *There exists an increasing sequence $\{Y_n\}_n$ of finite dimensional subspaces of X , all containing Y , such that $\lim_n \dim(Y_n) = \infty$ and for each n , $\sup_{S_{R_n}(Y_n)} E \leq 0$ for some $R_n > \rho$.*

Then E has an unbounded sequence of critical values.

We now show that the functional

$$E(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p - \frac{\mu}{q} \int_{\Omega} \frac{|u|^q}{|x|^s} - \frac{\lambda}{r} \int_{\Omega} |u|^r$$

satisfies the hypothesis of the lemma.

Without loss of generality, we assume that $\Omega = (0, 1)^n$. Let Y_k be the k -dimensional subspace of $X = H_0^{1,p}(\Omega)$, generated by the first k functions of the basis

$$\{(\sin k_1 \pi x_1, \dots, \sin k_n \pi x_n), k_i \in \mathbf{N}, i = 1, \dots, n\}.$$

Let Z_k denote the complement of Y_k in X , that is, the set generated by the base vectors not in Y_k . For any $u \in Y_{k-1}^c$, the topological complement of Y_{k-1} ,

$$\|u\|_p \leq C \|\nabla u\|_p / k^{\frac{1}{n}} \quad (\text{Peral [25]}).$$

Claim 1. For k sufficiently large, there exists $\rho > 0$ such that $E(u) \geq 1$ for all $u \in Z_{k-1}$ with $\|u\|_{H_0^{1,p}} = \rho$

Proof of Claim 1. We first consider the case where $p = q, p < r$ and $\mu < \mu_{s,p}$:

$$E(u) \geq (1 - \frac{\mu}{\mu_{s,p}}) \frac{1}{p} \int_{\Omega} |\nabla u|^p - C \int_{\Omega} |u|^r.$$

By the Gagliardo-Nirenberg inequality,

$$\left(\int_{\Omega} |u|^r\right)^{\frac{1}{r}} \leq C_1 \left(\int_{\Omega} |\nabla u|^p\right)^{\frac{\alpha}{p}} \left(\int_{\Omega} |u|^p\right)^{\frac{1-\alpha}{p}}$$

with $a = \frac{n}{p}(1 - \frac{2}{r})$. Hence, for $u \in \partial B_\rho \cap Y_{k-1}^c$,

$$\begin{aligned} E(u) &\geq C_2 \int_\Omega |\nabla u|^p - C_1 \left(\int_\Omega |\nabla u|^p \right)^{\frac{ra}{p}} \left(\int_\Omega |u|^p \right)^{\frac{r(1-a)}{p}} \\ &= \rho^p (C_2 - C_1 \rho^{ra} \left(\frac{C_1 \rho}{k^{\frac{1}{n}}} \right)^{r(1-a)} \rho^{-p}) \\ &= \rho^p (C_2 - C_3 \rho^{r-p} \frac{1}{k^{r(1-a)/n}}). \end{aligned}$$

Choosing $\rho = \left(\frac{C_2}{2C_3} k^{\frac{r(1-a)}{n}} \right)^{\frac{1}{r-p}}$, we get that $E(u) \geq \frac{1}{2} C_2 \rho^p = C_4 k^{\frac{pr(1-a)}{n(r-p)}} \geq 1$, for k large enough. This completes the proof of Claim 1 in the first case.

We turn to the case where $p < q$: Since $q < p^*(s)$, choose $\varepsilon > 0$ such that $q < \frac{n-s-\varepsilon}{n-p} p$; then

$$\int_\Omega \frac{|u|^q}{|x|^s} \leq C_0 \left(\int_\Omega |u|^{q \frac{n}{n-s-\varepsilon}} \right)^{\frac{n-s-\varepsilon}{n}}.$$

If $q \frac{n}{n-s-\varepsilon} \leq r$, then

$$\int_\Omega |u|^{q \frac{n}{n-s-\varepsilon}} \leq C_1 \left(\int_\Omega |u|^r \right)^{\frac{q \frac{n}{n-s-\varepsilon}}{r}}.$$

Thus

$$\int_\Omega \frac{|u|^q}{|x|^s} \leq C_2 \left(\int_\Omega |u|^r \right)^{\frac{q}{r}}.$$

If $q \frac{n}{n-s-\varepsilon} \geq r$, then

$$\int_\Omega |u|^r \leq C_3 \left(\int_\Omega |u|^{q \frac{n}{n-s-\varepsilon}} \right)^{\frac{n-s-\varepsilon}{n} \cdot \frac{r}{q}}.$$

Because of these relationships, we could combine the last two terms of the functional E together. In this sense, we may assume that

$$E(u) \geq \frac{1}{p} \int_\Omega |\nabla u|^p - C \int_\Omega |u|^r,$$

and the rest is as in case (1). \square

Let $Y = Y_k$ with the k chosen in Claim 1. We now show the following.

Claim 2. In both cases, there exist for each finite dimensional subspace $Y_k \subset H_0^{1,p}(\Omega)$, positive constants C_1, C_2 (depending on Y_k) such that

$$\sup_{u \in \partial B_R(Y_k)} E(u) \leq C_1 R^p - C_2 R^r.$$

Indeed, for any $u \in H_0^{1,p}(\Omega)$ and any $R > 0$, we have

$$E(Ru) \leq \frac{R^p}{p} \|u\|_{H_0^{1,p}(\Omega)}^p - \frac{R^r}{r} \|u\|_r^r.$$

Since Y_k is a finite dimensional space, it is closed and the two norms $\|\cdot\|_r$ and $\|\cdot\|_{H_0^{1,p}(\Omega)}$ on Y_k are equivalent. This implies the Claim.

Now we can apply Lemma 6.1 to conclude that $E_{\lambda,\mu}$ has an unbounded sequence of critical values. Theorem 6.1 is proved. \square

7. THE SOLUTIONS IN THE CASE OF
A HARDY-CRITICAL SINGULAR TERM

Theorem 7.1 (Hardy-critical singular term). *Suppose $1 < p = q = p^*(s)$ (i.e., $s = p$).*

- (1) *If $p < r < p^*$ (high order non-singular term), then $(P_{\lambda,\mu})$ has infinitely many solutions –at least one of them being positive– for any $\lambda > 0$ and any $0 < \mu < \mu_p$.*
- (2) *If $r = p^*$ (critical non-singular term) and Ω is star-shaped, then $(P_{\lambda,\mu})$ has no non-trivial solution for any $\lambda > 0, \mu > 0$.*

Proof. If $r < p^*$, then by Theorem 4.1.2 the functional $E_{\lambda,\mu}$ satisfies $(PS)_c$ for any c as long as $\mu < \mu_p$. Since $p < r$, the proof is the same as in Theorem 6.1.(2), while the second case of the theorem is covered in section 2. □

Remark 7.1. The case when $p = q = r$ is really an eigenvalue problem. There are solutions for $(P_{\lambda,\mu})$ as long as λ is an eigenvalue of the problem $-\Delta_p u - \frac{\mu|u|^{p-2}u}{|x|^p} = \lambda|u|^{p-2}u$ in $H_0^{1,p}(\Omega)$.

If $0 \leq \mu < \mu_p$, one can show that there is an infinite number of eigenvalues for the above problem. Indeed, these correspond to the critical levels of the restriction of the functional

$$\tilde{E}(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{\mu}{p} \int_{\Omega} \frac{|u|^p}{|x|^p} dx$$

to the submanifold $\{u; \int_{\Omega} |u|^p dx = 1\}$. But a slight variation of Theorem 4.1.(2) shows that in this case \tilde{E} has $(PS)_c$ for any c , and therefore a standard application of Ljusternik-Schnirelmann theory applied to the genus $\gamma_{\mathbb{Z}_2}$ will yield the result.

8. A POSITIVE SOLUTION IN THE CASE OF
A HARDY-SOBOLEV CRITICAL SINGULAR TERM

In this section, we consider the first solution for the problem $(P_{\lambda,\mu})$ with the critical *Sobolev-Hardy* exponent.

Theorem 8.1 (Hardy-Sobolev critical singular term). *Suppose $1 < p < q = p^*(s)$ (i.e., $s < p$) in the equation:*

$$(P_{\lambda,\mu}) \quad \begin{cases} -\Delta_p u = \lambda|u|^{r-2}u + \mu \frac{|u|^{p^*(s)-2}u}{|x|^s} & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

- *If $r < p^*$, then $(P_{\lambda,\mu})$ has a solution that is strictly positive everywhere on Ω , under any one of the following conditions:*
 - (1) $p = r < p^*$ and $n \geq p^2$, $0 < \lambda < \lambda_1$ and $\mu > 0$.
 - (2) $p < r < p^*$, λ is large enough and $\mu > 0$.
 - (3) $p < r < p^*$ and $n > \frac{p(p-1)r+p^2}{p+(p-1)(r-p)}$, $\lambda > 0, \mu > 0$.
- *If $r = p^*$ and Ω is star-shaped, then $(P_{\lambda,\mu})$ has no non-trivial solution for any $\lambda > 0, \mu > 0$.*

Proof. Note that the last case ($r = p^*$) was covered in section 2. Now if $r < p^*$, then Theorem 5.2 asserts that any one of the 3 conditions yields that the set

$$M_1 = \{u \in H_0^{1,p}(\Omega); u \neq 0, \langle E'_{\lambda,\mu}(u), u \rangle = 0\},$$

is closed, that it is dual to the Mountain Pass class

$$\mathcal{F}_1 = \{\gamma \in C([0, 1]; H_0^{1,p}(\Omega)); \gamma(0) = 0, \gamma(1) \neq 0 \text{ and } E_{\lambda,\mu}(\gamma(1)) \leq 0\},$$

and that

$$\inf_{M_1} E_{\lambda,\mu} = c_1 := c(E_{\lambda,\mu}, \mathcal{F}_1).$$

On the other hand, Theorem 4.1.(2) yields that $E_{\lambda,\mu}$ satisfies the $(PS)_c$ condition for any

$$c < \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}}.$$

Therefore, we should be able to apply Theorem 5.1 and obtain our desired assertion, if only we can prove the following case. \square

Lemma 8.1. *In any one of the above three cases, we have*

$$c_1 < \frac{p-s}{p(n-s)} \left(\frac{\mu_s^{n-s}}{\mu^{n-p}} \right)^{\frac{1}{p-s}}.$$

Proof. We may assume without loss of generality that $\mu = 1$. We first consider the following case:

Case (1). $p < r$ and λ is large.

In order to estimate the energy level c_1 , we consider the functions

$$g(t) = E_{\lambda,\mu}(tv_\varepsilon) = \frac{t^p}{p} \int_{\Omega} |\nabla v_\varepsilon|^p - \frac{t^{p^*(s)}}{p^*(s)} - \frac{\lambda t^r}{r} \int_{\Omega} |v_\varepsilon|^r$$

and

$$\bar{g}(t) = \frac{t^p}{p} \int_{\Omega} |\nabla v_\varepsilon|^p - \frac{t^{p^*(s)}}{p^*(s)},$$

where v_ε is the extremal function defined in the appendix. Note that $\lim_{t \rightarrow \infty} g(t) = -\infty$ and $g(t) > 0$ when t is close to 0, so that $\sup_{t \geq 0} g(t)$ is attained for some $t_\varepsilon > 0$. From

$$0 = g'(t_\varepsilon) = t_\varepsilon^{p-1} \left(\int_{\Omega} |\nabla v_\varepsilon|^p - t_\varepsilon^{p^*(s)-p} - \lambda t_\varepsilon^{r-p} \int_{\Omega} |v_\varepsilon|^r \right)$$

we have

$$\int_{\Omega} |\nabla v_\varepsilon|^p = t_\varepsilon^{p^*(s)-p} + \lambda t_\varepsilon^{r-p} \int_{\Omega} |v_\varepsilon|^r > t_\varepsilon^{p^*(s)-p},$$

and therefore

$$t_\varepsilon \leq \left(\int_{\Omega} |\nabla v_\varepsilon|^p \right)^{\frac{1}{p^*(s)-p}}.$$

Thus

$$\int_{\Omega} |\nabla v_\varepsilon|^p \leq t_\varepsilon^{p^*(s)-p} + \lambda \left(\int_{\Omega} |\nabla v_\varepsilon|^p \right)^{\frac{r-p}{p^*(s)-p}} \left(\int_{\Omega} |v_\varepsilon|^r \right).$$

Choose ε small enough so that by (1) and (6) of Lemma 11.1 we have $t_\varepsilon^{p^*(s)-p} \geq \frac{\mu_s}{2}$. That is, we get a lower bound for t_ε , which is independent of ε .

Now we estimate $g(t_\varepsilon)$. The function $\bar{g}(t)$ attains its maximum at

$$t = \left(\int_{\Omega} |\nabla v_\varepsilon|^p \right)^{\frac{1}{p^*(s)-p}}$$

and is increasing in the interval

$$\left[0, \left(\int_{\Omega} |\nabla v_\varepsilon|^p \right)^{\frac{1}{p^*(s)-p}} \right].$$

By Lemma 11.1, we have

$$\begin{aligned} g(t_\varepsilon) &= \bar{g}(t_\varepsilon) - \frac{\lambda}{r} t_\varepsilon^r \int_\Omega |v_\varepsilon|^r \\ &\leq \bar{g}\left(\left(\int_\Omega |\nabla v_\varepsilon|^p\right)^{\frac{1}{p^*(s)-p}}\right) - \frac{\lambda}{r} t_\varepsilon^r \int_\Omega |v_\varepsilon|^r \\ &= \left(\frac{1}{p} - \frac{1}{p^*(s)}\right) \left(\int_\Omega |\nabla v_\varepsilon|^p\right)^{\frac{p^*(s)}{p^*(s)-p}} - \frac{\lambda}{r} t_\varepsilon^r \int_\Omega |v_\varepsilon|^r \\ &\leq \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} + O(\varepsilon^{\frac{n-p}{p-s}}) - \frac{\lambda}{r} \left(\frac{\mu_s}{2}\right)^{\frac{r}{p^*(s)-p}} \int_\Omega |v_\varepsilon|^r. \end{aligned}$$

So for λ large enough, we have

$$g(t_\varepsilon) < \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}}.$$

Case (2). $p < r < p^*$ and $n > \frac{p(p-1)r+p^2}{p+(p-1)(r-p)}$, $\lambda > 0$.

Note first that the above condition is equivalent to $\max\{p, p^* - \frac{p}{p-1}\} < r < p^*$ and $\lambda > 0$.

For any $\lambda > 0$, the above estimate on $g(t_\varepsilon)$ and Lemma 6.1 yield

$$g(t_\varepsilon) \leq \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} + O(\varepsilon^{\frac{n-p}{p-s}}) - O(\varepsilon^{\frac{p-1}{p-s}(n-\frac{r(n-p)}{p})}),$$

so that if r is chosen in such a way that

$$\frac{n-p}{p-s} > \frac{p-1}{p-s} \left(n - \frac{r(n-p)}{p}\right),$$

i.e. $r > p^* - \frac{p}{p-1}$, then

$$g(t_\varepsilon) < \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}}.$$

Case (3). $p = r$, $0 < \lambda < \lambda_1$ and $n \geq p^2$.

We still use the function $g(t)$. Since $\lambda < \lambda_1$, we have $g(t) > 0$ when t is close to 0, and $\lim_{t \rightarrow \infty} g(t) = -\infty$. So again $g(t)$ attains its maximum at some $t_\varepsilon > 0$. From

$$g'(t) = t^{p-1} \left(\int_\Omega |\nabla v_\varepsilon|^p - t^{p^*(s)-p} - \lambda \int_\Omega |v_\varepsilon|^p\right) = 0$$

we get

$$t_\varepsilon = \left(\int_\Omega |\nabla v_\varepsilon|^p - \lambda \int_\Omega |v_\varepsilon|^p\right)^{\frac{1}{p^*(s)-p}}.$$

Thus

$$\begin{aligned} g(t_\varepsilon) &= \left(\frac{1}{p} - \frac{1}{p^*(s)}\right) \left(\int_\Omega |\nabla v_\varepsilon|^p - \lambda \int_\Omega |v_\varepsilon|^p\right)^{\frac{p^*(s)}{p^*(s)-p}} \\ &= \begin{cases} \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} + O(\varepsilon^{\frac{n-p}{p-s}}) - O(\varepsilon^{\frac{p(p-1)}{p-s}}), & p > p^* \left(1 - \frac{1}{p}\right), \\ \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} + O(\varepsilon^{\frac{n-p}{p-s}}) - O(\varepsilon^{\frac{n-p}{p-s}} |\log \varepsilon|), & p = p^* \left(1 - \frac{1}{p}\right). \end{cases} \end{aligned}$$

In the case where $p > p^*(1 - \frac{1}{p})$, we require that $\frac{n-p}{p-s} > \frac{p(p-1)}{p-s}$, but both are equivalent to $p^2 < n$. In the case where $p = p^*(1 - \frac{1}{p})$ we have $p^2 = n$, and the proof the lemma is now complete. \square

Remark 8.1. (1) If $p^2 \leq n$, then $p^* - \frac{p}{p-1} \leq p^*(1 - \frac{1}{p}) \leq p$, so that $r > p^* - \frac{p}{p-1}$ whenever $p < r$. In this case, r can take any value between p and p^* .
 (2) If $p^2 > n$, then $p < p^*(1 - \frac{1}{p}) < p^* - \frac{p}{p-1}$, and then we require that $p^* - \frac{p}{p-1} < p < p^*$.

9. A SIGN CHANGING SOLUTION IN THE HARDY-SOBOLEV CRITICAL CASE

In this section, we extend the arguments of Tarantello [28] to establish the following.

Theorem 9.1 (Hardy-Sobolev critical singular term). *Suppose $2 \leq p < q = p^*(s)$ and $r < p^*$ in the equation*

$$(P_{\lambda,\mu}) \quad \begin{cases} -\Delta_p u = \lambda |u|^{r-2} u + \mu \frac{|u|^{p^*(s)-2}}{|x|^s} u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

Assume any one of the following conditions:

- (1) $p = r < p^*$, $n > p^3 - p^2 + p$, $\mu > 0$ and $0 < \lambda < \lambda_1$.
- (2) $p < r < p^*$, $\mu > 0$ and λ large enough.
- (3) $p < r < p^*$, $n > \frac{p(p-1)r+p}{1+(p-1)(r-p)}$ and $\mu > 0, \lambda > 0$.

Then $(P_{\lambda,\mu})$ has also a changing-sign solution u that satisfies

$$\int_{\Omega} (\lambda |u|^{r-p} + \mu \frac{|u|^{p^*(s)-p}}{|x|^s}) v(u)^{p-1} u = 0,$$

where $v(u)$ is the first eigenvector of the (weighted) eigenvalue problem

$$\begin{cases} -\Delta_p v = \nu (\lambda |u|^{r-p} + \mu \frac{|u|^{p^*(s)-p}}{|x|^s}) |v|^{p-2} v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof. We assume without loss of generality that $\mu = 1$. Theorem 5.2 asserts that for any $q < p^*(s)$, any one of the 3 conditions yields that the closed set

$$M_2^q = M_1^q \cap \{u \in H_0^{1,p}(\Omega); \int_{\Omega} (\lambda |u|^{r-p} + \frac{|u|^{q-p}}{|x|^s}) v(u)^{p-1} u = 0\}.$$

is dual to the class \mathcal{F}_2 , and that

$$\inf_{M_2^q} E_{\lambda,q} = c_{2,q} := c(E_{\lambda,q}, \mathcal{F}_2),$$

where for each $q < q^*(s)$, the sets M_1^q (resp. M_2^q) denote the dual sets associated to the functional

$$E_{\lambda,q} := \frac{1}{p} \int_{\Omega} |\nabla u|^p - \frac{1}{q} \int_{\Omega} \frac{|u|^q}{|x|^s} - \frac{\lambda}{r} \int_{\Omega} |u|^r.$$

E_{λ} will denote $E_{\lambda,p^*(s)}$ and M_1 (resp. M_2) will denote $M_1^{p^*(s)}$ (resp. $M_2^{p^*(s)}$). Note that by Theorem 5.2, M_1 is dual to \mathcal{F}_1 , but the same cannot be said about M_2 and \mathcal{F}_2 unless $p = 2$. Therefore, to establish the theorem above, we shall resort to a limiting argument as $q \rightarrow p^*(s)$.

Lemma 9.1. *Under any one of the 3 conditions in Theorem 9.1, we have:*

- (1) $c_{i,q} \rightarrow c_i$ ($i = 1, 2$) as $q \rightarrow p^*(s)$.
- (2) *There exist $\sigma > 0$ and $\delta_0 > 0$ such that for $0 < |q - p^*(s)| < \delta_0$, we have $c_{2,q} \leq c_{1,q} + \frac{1}{n}S^{\frac{n}{p}} - \sigma$.*

Proof. (1) First we prove that $(c_{1,q})_q$ and $(c_{2,q})_q$ are uniformly bounded in q . We shall only show it for $c_{2,q}$. For any $u \in M_2^q$,

$$\int_{\Omega} |\nabla u|^p - \int_{\Omega} \frac{|u|^q}{|x|^s} - \lambda \int_{\Omega} |u|^r = 0.$$

Thus

$$E_{\lambda,q}(u) = \lambda \left(\frac{1}{p} - \frac{1}{r}\right) \int_{\Omega} |u|^r + \left(\frac{1}{p} - \frac{1}{q}\right) \int_{\Omega} \frac{|u|^q}{|x|^s} dx \geq 0,$$

i.e., $c_{2,q} \geq \inf_{M_2^q} E_{\lambda,q} \geq 0$. Since now

$$\begin{aligned} E_{\lambda,q}(u) &= \frac{1}{p} \int_{\Omega} |\nabla u|^p - \frac{1}{q} \int_{\Omega} \frac{|u|^q}{|x|^s} - \frac{\lambda}{r} \int_{\Omega} |u|^r \\ &\leq \frac{1}{p} \int_{\Omega} |\nabla u|^p - \frac{\lambda}{r} \int_{\Omega} |u|^r \equiv E(u), \end{aligned}$$

and for any $u_0, v_0 \in H_0^{1,p}(\Omega)$,

$$\begin{aligned} &\lim_{\alpha \rightarrow \infty, \beta \rightarrow \infty} E(\alpha u_0 + \beta v_0) \\ &= \lim_{\alpha \rightarrow \infty, \beta \rightarrow \infty} \left(\frac{1}{p} \int_{\Omega} |\nabla(\alpha u_0 + \beta v_0)|^p - \frac{\lambda}{r} \int_{\Omega} |\alpha u_0 + \beta v_0|^r \right) = -\infty, \end{aligned}$$

$E(\alpha u_0 + \beta v_0)$ attains its maximum at some finite α_0 and β_0 . This means

$$0 \leq c_{2,q} \leq E(\alpha_0 u_0 + \beta_0 v_0),$$

which is independent of q .

This implies the existence of constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 \leq \int_{\Omega} \frac{|u_{1,q}|^q}{|x|^s} \leq C_2.$$

Similar estimates also hold for $\|\nabla u_{1,q}\|_p$ and $\|u_{1,q}\|_r$. Notice that for every $u \neq 0$, there exist unique $t_q(u) > 0$ and $t(u) > 0$ such that

$$t(u)u \in M_1 \quad \text{and} \quad t_q(u)u \in M_1^q.$$

Furthermore, $t_q(u) \rightarrow t(u)$ as $q \rightarrow p^*(s)$. Set $s_q = t(u_{1,q})$ so that $s_q u_{1,q} \in M_1$. We have

$$\begin{aligned} c_1 &\leq E_{\lambda}(s_q u_{1,q}) \\ &= \frac{1}{p} \int_{\Omega} |\nabla s_q u_{1,q}|^p - \frac{\lambda}{r} \int_{\Omega} |s_q u_{1,q}|^r - \frac{1}{p^*(s)} \int_{\Omega} \frac{|s_q u_{1,q}|^{p^*(s)}}{|x|^s} \\ &= \left(\frac{1}{p} - \frac{1}{p^*(s)}\right) \int_{\Omega} |\nabla s_q u_{1,q}|^p + \lambda \left(\frac{1}{p^*(s)} - \frac{1}{r}\right) \int_{\Omega} |s_q u_{1,q}|^r. \end{aligned}$$

Since $u_{1,q} \in M_1^q$, we have

$$\begin{aligned} E_{\lambda,q}(u_{1,q}) &= \frac{1}{p} \int_{\Omega} |\nabla u_{1,q}|^p - \frac{\lambda}{r} \int_{\Omega} |u_{1,q}|^r - \frac{1}{q} \int_{\Omega} \frac{|u_{1,q}|^q}{|x|^s} \\ &= \left(\frac{1}{p} - \frac{1}{q}\right) \int_{\Omega} |\nabla u_{1,q}|^p + \lambda \left(\frac{1}{q} - \frac{1}{r}\right) \int_{\Omega} |u_{1,q}|^r. \end{aligned}$$

Thus

$$\begin{aligned} c_1 &\leq E_{\lambda}(s_q u_{1,q}) \\ &= \left(\frac{1}{p} - \frac{1}{q}\right) \int_{\Omega} s_q^p |\nabla u_{1,q}|^p + \left(\frac{1}{q} - \frac{1}{p^*(s)}\right) \int_{\Omega} s_q^p |\nabla u_{1,q}|^p \\ &\quad + \lambda \left(\frac{1}{p^*(s)} - \frac{1}{r}\right) \int_{\Omega} |s_q u_{1,q}|^r \\ &= \left(\frac{1}{p} - \frac{1}{q}\right) \int_{\Omega} |\nabla u_{1,q}|^p + \left(\frac{1}{p} - \frac{1}{q}\right) (s_q^p - 1) \int_{\Omega} |\nabla u_{1,q}|^p \\ &\quad + \left(\frac{1}{q} - \frac{1}{p^*(s)}\right) \int_{\Omega} s_q^p |\nabla u_{1,q}|^p + \lambda \left(\frac{1}{p^*(s)} - \frac{1}{r}\right) \int_{\Omega} |s_q u_{1,q}|^r \\ &= E_{\lambda,q}(u_{1,q}) + \left(\frac{1}{p} - \frac{1}{q}\right) (s_q^p - 1) \int_{\Omega} |\nabla u_{1,q}|^p \\ &\quad + \left(\frac{1}{q} - \frac{1}{p^*(s)}\right) \int_{\Omega} s_q^p |\nabla u_{1,q}|^p \\ &\quad + \lambda \left(\frac{1}{q} - \frac{1}{r}\right) (s_q^r - 1) \int_{\Omega} |u_{1,q}|^r + \lambda \left(\frac{1}{p^*(s)} - \frac{1}{q}\right) \int_{\Omega} s_q^r |u_{1,q}|^r. \end{aligned}$$

Note that $s_q \rightarrow 1$ as $q \rightarrow p^*(s)$; therefore

$$c_1 \leq c_{1,q} + o(1).$$

To obtain the reverse inequality, set $t_q = t_q(u_1) > 0$. Thus, $t_q u_1 \in M_1^q$, $t_q \rightarrow 1$ as $q \rightarrow p^*(s)$, and

$$\begin{aligned} c_{1,q} \leq E_{\lambda,q}(t_q u_1) &= E_{\lambda}(t_q u_1) + \frac{1}{p^*(s)} \int_{\Omega} \frac{|u|^{p^*(s)}}{|x|^s} - \frac{1}{q} \int_{\Omega} \frac{|u|^q}{|x|^s} \\ &= E_{\lambda}(u_1) + o(1) = c_1 + o(1). \end{aligned}$$

This completes part (1) of the lemma.

We prove part (2) by estimating $\sup_{A_{\varepsilon}} E_{\lambda,q}$, where $A_{\varepsilon} = \text{span}\{u_1, v_{\varepsilon}\} \in \mathcal{F}_2$ and u_1 is the first solution of the critical problem. For that, we need the smoothness of u_1 ; but this cannot be guaranteed unless $p = 2$. However, an easy approximation argument, and the fact that $\sup_{t \geq 0} E_{\lambda}(tu) \rightarrow \sup_{t \geq 0} E_{\lambda}(tu_1)$ as $u \rightarrow u_1$ strongly, allow us to assume that u_1 has the required smoothness.

Therefore we may suppose that $u_1, \nabla u_1 \in L^{\infty}(\Omega)$. We shall consider the case where $r > p$ first.

Case (1). Assume $p < r < p^*$ and $\lambda > 0$.

For $\varepsilon > 0$ and q sufficiently close to $p^*(s)$, and by the calculus lemma,

$$\begin{aligned}
 & E_{\lambda,q}(\alpha u_1 + \beta v_\varepsilon) \\
 &= \frac{1}{p} \int_{\Omega} |\nabla(\alpha u_1 + \beta v_\varepsilon)|^p - \frac{\lambda}{r} \int_{\Omega} |\alpha u_1 + \beta v_\varepsilon|^r - \frac{1}{q} \int_{\Omega} \frac{|\alpha u_1 + \beta v_\varepsilon|^q}{|x|^s} \\
 &\leq \frac{1}{p} \int_{\Omega} |\nabla(\alpha u_1)|^p - \frac{\lambda}{r} \int_{\Omega} |\alpha u_1|^r - \frac{1}{q} \int_{\Omega} \frac{|\alpha u_1|^q}{|x|^s} \\
 &\quad + \frac{1}{p} \int_{\Omega} |\nabla(\beta v_\varepsilon)|^p - \frac{\lambda}{r} \int_{\Omega} |\beta v_\varepsilon|^r - \frac{1}{q} \int_{\Omega} \frac{|\beta v_\varepsilon|^q}{|x|^s} \\
 &\quad + A_1 \left[\int_{\Omega} |\nabla \alpha u_1| |\nabla \beta v_\varepsilon|^{p-1} + |\nabla \alpha u_1|^{p-1} |\nabla \beta v_\varepsilon| \right] \\
 &\quad + B_1 \left[\int_{\Omega} |\alpha u_1| |\beta v_\varepsilon|^{r-1} + |\alpha u_1|^{r-1} |\beta v_\varepsilon| \right] \\
 &\quad + C_1 \left[\int_{\Omega} |\alpha u_1| \frac{|\beta v_\varepsilon|^{q-1}}{|x|^s} + |\alpha u_1|^{q-1} \frac{|\beta v_\varepsilon|}{|x|^s} \right] \\
 &\leq \frac{1}{p} \int_{\Omega} |\nabla(\alpha u_1)|^p - \frac{\lambda}{r} \int_{\Omega} |\alpha u_1|^r - \frac{1}{q} \int_{\Omega} \frac{|\alpha u_1|^q}{|x|^s} \\
 &\quad + \frac{1}{p} \int_{\Omega} |\nabla(\beta v_\varepsilon)|^p - \frac{\lambda}{r} \int_{\Omega} |\beta v_\varepsilon|^r - \frac{1}{q} \int_{\Omega} \frac{|\beta v_\varepsilon|^q}{|x|^s} \\
 &\quad + A_2 (|\alpha|^p + |\beta|^p) \varepsilon^{\frac{n-p}{p(p-s)}} + B_2 (|\alpha|^r + |\beta|^r) \varepsilon^{\frac{p-1}{p-s} (n - \frac{(r-1)(n-p)}{p})} \\
 &\quad + C_2 (|\alpha|^q + |\beta|^q) \varepsilon^{\frac{n-p}{p(p-s)}}.
 \end{aligned}$$

Therefore, for ε sufficiently small,

$$\lim_{\alpha, \beta \rightarrow \infty} E_{\lambda,q}(\alpha u_1 + \beta v_\varepsilon) = -\infty.$$

So we may assume that α and β are in a bounded set.

As in the study of the first solution, let us consider the function

$$g(t) = E_\lambda(t v_\varepsilon) = \frac{t^p}{p} \int_{\Omega} |\nabla v_\varepsilon|^p - \frac{t^{p^*(s)}}{p^*(s)} - \frac{\lambda t^r}{r} \int_{\Omega} |v_\varepsilon|^r$$

again. As in the previous section, we have

$$g(t_\varepsilon) \leq \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} + C \varepsilon^{\frac{n-p}{p-s}} - \frac{\lambda}{r} \left(\frac{\mu_s}{2} \right)^{\frac{r}{p^*(s)-p}} \int_{\Omega} |v_\varepsilon|^r.$$

If now $r > p^* - \frac{1}{p-1} > p^* - 1$, we have

$$\begin{aligned}
E_{\lambda,q}(\alpha u_1 + \beta v_\varepsilon) &\leq E_{\lambda,q}(\alpha u_1) + E_\lambda(\beta v_\varepsilon) + \frac{|\beta|^{p^*(s)}}{p^*(s)} - \frac{|\beta|^q}{q} \int_\Omega \frac{|v_\varepsilon|^q}{|x|^s} \\
&\quad + A_3 \varepsilon^{\frac{n-p}{p(p-s)}} + B_3 \varepsilon^{\frac{p-1}{p-s} (n - \frac{(r-1)(n-p)}{p})} \\
&\leq c_{1,q} + \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} + C \varepsilon^{\frac{n-p}{p-s}} + \frac{|\beta|^{p^*(s)}}{p^*(s)} \\
&\quad - \frac{|\beta|^q}{q} \int_\Omega \frac{|v_\varepsilon|^q}{|x|^s} - \frac{\lambda}{r} \left(\frac{\mu_s}{2}\right)^{\frac{r}{p^*(s)-p}} \int_\Omega |v_\varepsilon|^r \\
&\quad + A_3 \varepsilon^{\frac{n-p}{p(p-s)}} + B_3 \varepsilon^{\frac{p-1}{p-s} (n - \frac{(r-1)(n-p)}{p})} \\
&\leq c_{1,q} + \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} + C \varepsilon^{\frac{n-p}{p-s}} + A_3 \varepsilon^{\frac{n-p}{p(p-s)}} \\
&\quad + B_3 \varepsilon^{\frac{p-1}{p-s} (n - \frac{(r-1)(n-p)}{p})} - C_4 \varepsilon^{\frac{p-1}{p-s} (n - \frac{r(n-p)}{p})} \\
&\quad + \frac{|\beta|^{p^*(s)}}{p^*(s)} - \frac{|\beta|^q}{q} \int_\Omega \frac{|v_\varepsilon|^q}{|x|^s}.
\end{aligned}$$

Choose ε small enough so that

$$C \varepsilon^{\frac{n-p}{p-s}} + A_3 \varepsilon^{\frac{n-p}{p(p-s)}} + B_3 \varepsilon^{\frac{p-1}{p-s} (n - \frac{(r-1)(n-p)}{p})} - C_4 \varepsilon^{\frac{p-1}{p-s} (n - \frac{r(n-p)}{p})} \leq -2\sigma$$

for some constant $\sigma > 0$. Now choose $\delta_0 > 0$ small enough so that

$$\frac{|\beta|^{p^*(s)}}{p^*(s)} - \frac{|\beta|^q}{q} \int_\Omega \frac{|v_\varepsilon|^q}{|x|^s} < \sigma \quad \text{for } 0 < |q - p^*(s)| < \delta_0.$$

Thus the case when $r > p$ is established.

Case (2). $r = p$, $p^3 - p^2 + p < n$ and $0 < \lambda < \lambda_1$.

The assumption $p^3 - p^2 + p < n$ implies that $p^2 < n$, $p > p^*(1 - \frac{1}{p})$ and $p-1 < p^*(1 - \frac{1}{p})$. We assume that α and β are in a bounded set, and we estimate $E_\lambda(\alpha u_1 + \beta v_\varepsilon)$. Again let

$$g(t) = E_\lambda(t v_\varepsilon) = \frac{t^p}{p} \int_\Omega |\nabla v_\varepsilon|^p - \frac{t^{p^*(s)}}{p^*(s)} - \frac{\lambda t^p}{p} \int_\Omega |v_\varepsilon|^p;$$

then the maximum $g(t_\varepsilon)$ of $g(t)$ satisfies

$$g(t_\varepsilon) = \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} + O(\varepsilon^{\frac{n-p}{p-s}}) - O(\varepsilon^{\frac{p(p-1)}{p-s}}).$$

Thus we have

$$\begin{aligned}
 & E_{\lambda,q}(\alpha u_1 + \beta v_\varepsilon) \\
 &= \frac{1}{p} \int_{\Omega} |\nabla(\alpha u_1 + \beta v_\varepsilon)|^p - \frac{\lambda}{p} \int_{\Omega} |\alpha u_1 + \beta v_\varepsilon|^p - \frac{1}{q} \int_{\Omega} \frac{|\alpha u_1 + \beta v_\varepsilon|^q}{|x|^s} \\
 &\leq \frac{1}{p} \int_{\Omega} |\nabla(\alpha u_1)|^p - \frac{\lambda}{p} \int_{\Omega} |\alpha u_1|^p - \frac{1}{q} \int_{\Omega} \frac{|\alpha u_1|^q}{|x|^s} \\
 &\quad + \frac{1}{p} \int_{\Omega} |\nabla(\beta v_\varepsilon)|^p - \frac{\lambda}{p} \int_{\Omega} |\beta v_\varepsilon|^p - \frac{1}{q} \int_{\Omega} \frac{|\beta v_\varepsilon|^q}{|x|^s} \\
 &\quad + A_1 \left[\int_{\Omega} |\nabla \alpha u_1| |\nabla \beta v_\varepsilon|^{p-1} + |\nabla \alpha u_1|^{p-1} |\nabla \beta v_\varepsilon| \right] \\
 &\quad + B_1 \left[\int_{\Omega} |\alpha u_1| |\beta v_\varepsilon|^{p-1} + |\alpha u_1|^{p-1} |\beta v_\varepsilon| \right] \\
 &\quad + C_1 \left[\int_{\Omega} |\alpha u_1| \frac{|\beta v_\varepsilon|^{q-1}}{|x|^s} + |\alpha u_1|^{q-1} \frac{|\beta v_\varepsilon|}{|x|^s} \right] \\
 &\leq \frac{1}{p} \int_{\Omega} |\nabla(\alpha u_1)|^p - \frac{\lambda}{p} \int_{\Omega} |\alpha u_1|^p - \frac{1}{q} \int_{\Omega} \frac{|\alpha u_1|^q}{|x|^s} \\
 &\quad + \frac{1}{p} \int_{\Omega} |\nabla(\beta v_\varepsilon)|^p - \frac{\lambda}{p} \int_{\Omega} |\beta v_\varepsilon|^p - \frac{1}{q} \int_{\Omega} \frac{|\beta v_\varepsilon|^q}{|x|^s} \\
 &\quad + A_2 (|\alpha|^p + |\beta|^p) \varepsilon^{\frac{n-p}{p(p-s)}} + B_2 (|\alpha|^p + |\beta|^p) \varepsilon^{\frac{(n-p)(p-1)}{p(p-s)}} \\
 &\quad + C_2 (|\alpha|^q + |\beta|^q) \varepsilon^{\frac{n-p}{p(p-s)}} \\
 &\leq E_{\lambda,q}(\alpha u_1) + E_{\lambda}(\beta v_\varepsilon) + \frac{|\beta|^{p^*(s)}}{p^*(s)} - \frac{|\beta|^q}{q} \int_{\Omega} \frac{|v_\varepsilon|^q}{|x|^s} \\
 &\quad + A_3 \varepsilon^{\frac{n-p}{p(p-s)}} + B_3 \varepsilon^{\frac{(n-p)(p-1)}{p(p-s)}} \\
 &\leq c_{1,q} + \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} + C \varepsilon^{\frac{n-p}{p-s}} + \frac{|\beta|^{p^*(s)}}{p^*(s)} - \frac{|\beta|^q}{q} \int_{\Omega} \frac{|v_\varepsilon|^q}{|x|^s} - O(\varepsilon^{\frac{p(p-1)}{p-s}}) \\
 &\quad + A_3 \varepsilon^{\frac{n-p}{p(n-s)}} + B_3 \varepsilon^{\frac{(n-p)(p-1)}{p(p-s)}} \\
 &\leq c_{1,q} + \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} + C \varepsilon^{\frac{n-p}{p-s}} + A_3 \varepsilon^{\frac{n-p}{p(p-s)}} + B_3 \varepsilon^{\frac{(n-p)(p-1)}{p(p-s)}} \\
 &\quad - C_4 \varepsilon^{\frac{p(p-1)}{p-s}} + \frac{|\beta|^{p^*(s)}}{p^*(s)} - \frac{|\beta|^q}{q} \int_{\Omega} \frac{|v_\varepsilon|^q}{|x|^s}.
 \end{aligned}$$

Since $p^3 - p^2 + p < n$, we may choose ε small enough so that

$$C \varepsilon^{\frac{n-p}{p-s}} + A_3 \varepsilon^{\frac{n-p}{p(p-s)}} + B_3 \varepsilon^{\frac{(n-p)(p-1)}{p(p-s)}} - C_4 \varepsilon^{\frac{p(p-1)}{p-s}} \leq -2\sigma$$

for some constant $\sigma > 0$. Now choose $\delta_0 > 0$ small enough so that

$$\frac{|\beta|^{p^*(s)}}{p^*(s)} - \frac{|\beta|^q}{q} \int_{\Omega} \frac{|v_\varepsilon|^q}{|x|^s} < \sigma \text{ for } 0 < |q - p^*(s)| < \delta_0.$$

The proof of the lemma is now complete. □

Proof of Theorem 9.1. In order to get the second solution, we shall consider the second solutions $u_{2,q}$ of the problems corresponding to $q < p^*(s)$ and we will find a

limit as $q \rightarrow p^*(s)$. The location of $u_{2,q}$ on the dual sets M_2^q will be crucial for the compactness.

Since $c_{2,q}$ is bounded uniformly in q , there is $K > 0$ such that

$$\|\nabla u_{2,q}\|_p \leq K \text{ whenever } 0 < |q - p^*(s)| < \delta_0.$$

For $x \in \Omega$, define $(u_{2,q})^+(x) = \max\{u_{2,q}(x), 0\}$ and $(u_{2,q})^-(x) = \max\{-u_{2,q}(x), 0\}$. Since $u_{2,q} \in M_2^q$, both $(u_{2,q})^+$ and $(u_{2,q})^-$ are non-zero and belong to $H_0^{1,p}(\Omega)$. In addition,

$$\|\nabla(u_{2,q})^\pm\| \leq K \text{ whenever } 0 < |q - p^*(s)| < \delta_0.$$

Thus, we can find q_n such that $q_n \rightarrow p^*(s)$ as $n \rightarrow +\infty$, $u^+, u^- \in H_0^{1,p}$ and

$$(u_{2,q_n})^\pm \rightharpoonup u^\pm \text{ weakly in } H_0^{1,p} \text{ as } n \rightarrow +\infty.$$

We claim that $u^+ \neq 0$ and $u^- \neq 0$. To shorten notation, set $u_n^\pm = (u_{2,q_n})^\pm$, $c_{1,n} = c_{1,q_n}$, $E_n = E_{\lambda,q_n}$ and $\Gamma_n = M_1^{q_n}$. Since u_n is the solution of the corresponding sub-critical problem, we have that $u_n^\pm \in \Gamma_n$. In particular,

$$E_n(u_n^\pm) \geq c_{1,n}.$$

From Lemma 9.1, we also know that

$$E_n(u_n^+) + E_n(u_n^-) = E_n(u_n) = c_{2,q_n} \leq c_{1,n} + \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} - \sigma$$

for n large. Necessarily,

$$E_n(u_n^\pm) \leq \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} - \sigma$$

for n large. From the fact that $u_n^\pm \in \Gamma_n$ and $c_1^n \rightarrow c_1 > 0$, we derive

$$K_1 \leq \int_\Omega \frac{|u_n^\pm|^{q_n}}{|x|^s} \leq K_2$$

with suitable positive constants K_1 and K_2 .

Arguing by contradiction, assume, for example, that $u^+ = 0$. From the above and the fact that $u_n^\pm \in \Gamma_n$, we obtain

$$\frac{1}{p} \|\nabla u_n^+\|_p^p - \frac{1}{q_n} \int_\Omega \frac{|u_n^+|^{q_n}}{|x|^s} \leq \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} - \sigma + o(1)$$

and

$$\|\nabla u_n^+\|_p^p - \int_\Omega \frac{|u_n^+|^{q_n}}{|x|^s} = o(1).$$

Consequently,

$$\begin{aligned} & \mu_s \left(\int_\Omega \frac{|u_n^+|^{p^*(s)}}{|x|^s} \right)^{\frac{p}{p^*(s)}} \\ & \leq \|\nabla u_n^+\|_p^p = \int_\Omega \frac{|u_n^+|^{q_n}}{|x|^s} + o(1) = \int_\Omega \frac{|u_n^+|^{q_n}}{|x|^{\frac{q_n}{p^*(s)}s}} \cdot \frac{1}{|x|^{s(1-\frac{q_n}{p^*(s)})}} \\ & \leq \left(\int_\Omega \frac{|u_n^+|^{q_n}}{|x|^s} \right)^{\frac{q_n}{p^*(s)}} \left(\int_\Omega \frac{1}{|x|^s} \right)^{\frac{p^*(s)-q_n}{p^*(s)}} + o(1). \end{aligned}$$

Since $\int_\Omega \frac{|u_n^+|^{q_n}}{|x|^s}$ is bounded away from zero, we conclude that

$$\left(\int_\Omega \frac{|u_n^+|^{q_n}}{|x|^s} \right)^{\frac{q_n-p}{p^*(s)}} \geq \left(\int_\Omega \frac{1}{|x|^s} \right)^{\frac{q_n-p^*(s)}{p^*(s)}} \mu_s + o(1).$$

That is,

$$\int_{\Omega} \frac{|u_n^+|^{q_n}}{|x|^s} \geq \mu_s^{\frac{n-s}{p-s}} + o(1).$$

Thus, we have

$$\begin{aligned} \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} + o(1) &\leq \frac{p-s}{p(n-s)} \int_{\Omega} \frac{|u_n^+|^{q_n}}{|x|^s} \\ &= \frac{1}{p} \|\nabla u_n^+\|_p^p - \frac{1}{q_n} \int_{\Omega} \frac{|u_n^+|^{q_n}}{|x|^s} + o(1) \leq \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} - \sigma + o(1). \end{aligned}$$

This is a contradiction, and we conclude that $u^+ \neq 0$. Similarly, $u^- \neq 0$.

Set $u = u^+ - u^-$; that is, u changes sign in Ω and

$$u_n := u_{q_n} \rightharpoonup u \text{ weakly in } H_0^{1,p}(\Omega).$$

So, $\langle E'_\lambda(u), w \rangle = 0$ for any $w \in H_0^{1,p}(\Omega)$, i.e. u is a weak solution of (P_λ) . Now, we prove that a subsequence of $\{u_n\}$ converges to u strongly in $H_0^{1,p}(\Omega)$ and conclude that u is a solution of $(P_{\lambda,p^*(s)})$ that is located on M_2 .

Since $\{E(u_n)\}$ is bounded and $E'_n(u_n) \rightarrow 0$, we may assume that the conclusions of Lemma 4.4 hold for the sequence $(u_n)_n$.

Note that $u \in M_1$; hence $E(u) \geq c_1$. Set $u_n = u + w_n$, with $w_n \rightharpoonup 0$ weakly in $H_0^{1,p}$. We have

$$\begin{aligned} c_{1,n} + \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} - \sigma &\geq E_n(u + w_n) \\ &\geq \frac{1}{p} \|\nabla u\|_p^p - \frac{1}{q_n} \int_{\Omega} \frac{|u|^{q_n}}{|x|^s} - \frac{\lambda}{r} \|u\|_r^r + \frac{1}{p} \|\nabla w_n\|_p^p - \frac{1}{q_n} \int_{\Omega} \frac{|w_n|^{q_n}}{|x|^s} + o(1) \\ &\geq c_1 + \frac{1}{p} \|\nabla w_n\|_p^p - \frac{1}{q_n} \int_{\Omega} \frac{|w_n|^{q_n}}{|x|^s} + o(1). \end{aligned}$$

Since $|c_{1,n} - c_1| = o(1)$, we derive

$$\frac{1}{p} \|\nabla w_n\|_p^p - \frac{1}{q_n} \int_{\Omega} \frac{|w_n|^{q_n}}{|x|^s} \leq \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} - \sigma + o(1).$$

Furthermore,

$$0 = \langle E'_n(u_n), u_n \rangle = \langle E'(u), u \rangle + \|\nabla w_n\|_p^p - \int_{\Omega} \frac{|w_n|^{q_n}}{|x|^s} + o(1);$$

i.e.

$$\|\nabla w_n\|_p^p - \int_{\Omega} \frac{|w_n|^{q_n}}{|x|^s} = o(1).$$

The last two relations show that the sequence $\|\nabla w_n\|_p$ cannot be bounded away from zero, and therefore a subsequence of $\{w_n\}$ converges strongly to zero. \square

10. SOBOLEV CRITICAL NON-SINGULAR TERM

In this section, we prove Theorem 1.4. We reformulate it as follows.

Theorem 10.1. *Suppose $1 < p \leq q < p^*(s)$ and $r = p^*$ in the equation*

$$(P_{\lambda,\mu}) \quad \begin{cases} -\Delta_p u = \lambda |u|^{p^*-2} u + \mu \frac{|u|^{q-2}}{|x|^s} u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

Then $(P_{\lambda,\mu})$ has a positive solution if any one of the following conditions holds:

- (1) $p < q$, $n > \frac{p(p-1)(q-s)+p^2}{p+(p-1)(q-p)}$ and $\lambda > 0, \mu > 0$.
 (2) $p = q$, $n \geq p^2 - (p-1)s$ and $\lambda > 0, \mu_{s,p} > \mu > 0$.

If one of the following conditions holds:

- (1') $p < q$, $n > \frac{p(p-1)(q-s)+p}{1+(p-1)(q-p)}$ and $\lambda > 0, \mu > 0$,
 (2') $p = q$, $n > p((p-1)(p-s) + 1)$ and $\lambda > 0, \mu_{s,p} > \mu > 0$,

then $(P_{\lambda,\mu})$ has also a sign-changing solution.

Remark 10.1. The existence of a positive solution under condition (2) above has already been noticed in [13] in the case where $p = q$.

By scaling, we can always assume that $\lambda = 1$. The corresponding functional is again

$$E_\mu(u) = \frac{1}{p} \int_\Omega |\nabla u|^p - \frac{1}{p^*} \int_\Omega |u|^{p^*} - \frac{\mu}{q} \int_\Omega \frac{|u|^q}{|x|^s}.$$

Recall that under any one of the above conditions, the set

$$M_1 = \{u \in H_0^{1,p}(\Omega); u \neq 0, \langle E'_\mu(u), u \rangle = 0\}$$

is dual to the class

$$\mathcal{F}_1 = \{\gamma \in C([0, 1]; H_0^{1,p}(\Omega)); \gamma(0) = 0, \gamma(1) \neq 0 \text{ and } E_\mu(\gamma(1)) \leq 0\}.$$

Moreover, the energy level $c_1 = \inf_{A \in \mathcal{F}_1} \sup_{u \in A} E_\mu(u)$ is equal to $\inf_{u \in M_1} E_\mu(u)$.

By Theorem 4.1.(3), E_μ satisfies $(PS)_c$ for any $c < \frac{1}{n} \mu_0^{\frac{n}{p}}$. So, the existence of the first positive solution follows immediately from the following estimates.

Lemma 10.1. *If $r = p^*$, then $c_1 < \frac{1}{n} \mu_0^{\frac{n}{p}}$ in any one of the following three cases:*

- (1) $q = p$, $0 < \mu < \mu_{s,p}$ and $n \geq p^2 - (p-1)s$.
 (2) $p < q < p^*(s)$ and μ large enough.
 (3) $p < q < p^*(s)$ and $n > \frac{p(p-1)(q-s)+p^2}{p+(p-1)(q-p)}$.

Proof. Take v_ε to be the function as in Lemma 11.2 of the appendix. Then, as in the proof of Lemma 9.1, we consider:

Case $q > p$. We have

$$\begin{aligned} \max_{0 \leq t < \infty} E_\mu(tv_\varepsilon) &\leq \frac{1}{n} \mu_0^{\frac{n}{p}} + O(\varepsilon^{\frac{n-p}{p}}) - \frac{\mu}{q} \left(\frac{\mu_0}{s}\right)^{\frac{q}{p^*-p}} \int_\Omega \frac{|v_\varepsilon|^q}{|x|^s} \\ &= \frac{1}{n} \mu_0^{\frac{n}{p}} + O(\varepsilon^{\frac{n-p}{p}}) - O(\varepsilon^{\frac{(n-p)(p-1)}{p^2} (p^*(s)-q)}). \end{aligned}$$

where we require that $q > \frac{n-s}{n-p}(p-1)$. The estimate then follows.

Case $q = p$. We have

$$\begin{aligned} \max_{0 \leq t < \infty} E_\mu(tv_\varepsilon) &= \frac{1}{n} \left(\int_\Omega |\nabla v_\varepsilon|^p - \frac{|v_\varepsilon|^p}{|x|^s} \right)^{\frac{n}{p}} \\ &= \begin{cases} \frac{1}{n} \mu_0^{\frac{n}{p}} + O(\varepsilon^{\frac{n-p}{p}}) - O(\varepsilon^{\frac{(n-p)(p-1)}{p^2} (p^*(s)-p)}), & p > \frac{n-s}{n-p}(p-1), \\ \frac{1}{n} \mu_0^{\frac{n}{p}} + O(\varepsilon^{\frac{n-p}{p}}) - O(\varepsilon^{\frac{n-p}{p}} |\log \varepsilon|), & p = \frac{n-s}{n-p}(p-1). \end{cases} \end{aligned}$$

Since

$$\frac{(n-p)(p-1)}{p^2} (p^*(s) - p) = \frac{(p-s)(p-1)}{p},$$

the conclusions now follow immediately.

For the sign changing solution, we shall proceed as in the case of the Hardy-Sobolev critical exponent. First we find appropriate sign changing solutions for the sub-critical problem, i.e. when $r < p^*$, and then we pass to the limit as $r \rightarrow p^*$.

Write again

$$E_{\mu,r}(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p - \frac{1}{r} \int_{\Omega} |u|^r - \frac{\mu}{q} \int_{\Omega} \frac{|u|^q}{|x|^s},$$

$$\mathcal{F}_1^r = \{\gamma \in C([0, 1]; H_0^{1,p}(\Omega)); \gamma(0) = 0, \gamma(1) \neq 0 \text{ and } E_{\mu,r}(\gamma(1)) \leq 0\},$$

$$M_1^r = \{u \in H_0^{1,p}(\Omega); u \neq 0, \langle (E_{\mu,r})'(u), u \rangle = 0\},$$

and

$$c_{1,r} = \inf_{A \in \mathcal{F}_1} \sup_{u \in A} E_{\mu,r}(u).$$

Then, as previously, we know that M_1^r is dual to \mathcal{F}_1^r and $c_{1,r} = \inf_{u \in M_1} E_{\mu}(u)$. Also define

$$c_{2,r} = \inf_{A \in \mathcal{F}_2} \sup_{u \in A} E_{\mu,r}(u),$$

where \mathcal{F}_2 is defined in section 5. We write c_2 (resp. E_{μ}) for c_{2,p^*} (resp. E_{μ,p^*}). \square

Lemma 10.2. *Under either one of the following conditions,*

- (1) $p = q, 0 < \mu < \mu_{s,p}$ and $n > p(p - 1)(p - s) + p,$
- (2) $p < q, \mu > 0$ and $n > \frac{p(p-1)(q-s)+p}{1+(p-1)(q-p)},$

we have

- (i) $c_{i,r} \rightarrow c_i$ ($i = 1, 2$) as $r \rightarrow p^*,$
- (ii) $c_{2,r} \leq c_{1,r} + \frac{1}{n} \mu_0^{\frac{n}{p}} - \sigma$ for some $\sigma > 0$ and r sufficiently close to $p^*.$

Proof. For the first conclusion, the proof is exactly the same as in the last section. For the second one, we can assume that the first solution u_1 is smooth and $\nabla u_1 \in L^\infty(\Omega).$

For $\varepsilon > 0$ and q sufficiently close to $p^*,$ apply the calculus lemma to obtain

$$\begin{aligned} E_{\mu,r}(\alpha u_1 + \beta v_\varepsilon) &\leq E_{\mu,r}(\alpha u_1) + E_{\mu,r}(\beta v_\varepsilon) \\ &\quad + A_1 \left[\int_{\Omega} |\nabla \alpha u_1| |\nabla \beta v_\varepsilon|^{p-1} + |\nabla \alpha u_1|^{p-1} |\nabla \beta v_\varepsilon| \right] \\ &\quad + B_1 \left[\int_{\Omega} |\alpha u_1| |\beta v_\varepsilon|^{r-1} + |\alpha u_1|^{r-1} |\beta v_\varepsilon| \right] \\ &\quad + C_1 \left[\int_{\Omega} |\alpha u_1| \frac{|\beta v_\varepsilon|^{q-1}}{|x|^s} + |\alpha u_1|^{q-1} \frac{|\beta v_\varepsilon|}{|x|^s} \right] \\ &\leq E_{\mu,r}(\alpha u_1) + E_{\mu,r}(\beta v_\varepsilon) + A_2 (|\alpha|^p + |\beta|^p) \varepsilon^{\frac{n-p}{p^2}} \\ &\quad + B_2 (|\alpha|^r + |\beta|^r) \varepsilon^{\frac{p-1}{p} (n - \frac{(r-1)(n-p)}{p})} \\ &\quad + C_2 (|\alpha|^q + |\beta|^q) \varepsilon^{\frac{n-p}{p^2}}. \end{aligned}$$

Again, we note that for ε sufficiently small,

$$\lim_{\alpha, \beta \rightarrow \infty} E_{\mu,r}(\alpha u_1 + \beta v_\varepsilon) = -\infty.$$

So we may assume that α and β are in a bounded set.

Case 1. Assume $p < q, 0 < \mu$ and $n > \frac{p(p-1)(q-s)+p}{1+(p-1)(q-p)}.$

Then, by the calculus lemma,

$$\begin{aligned}
& E_{\mu,r}(\alpha u_1 + \beta v_\varepsilon) \\
& \leq E_{\mu,r}(\alpha u_1) + E_\mu(\beta v_\varepsilon) + \frac{|\beta|^{p^*}}{p^*} - \frac{|\beta|^r}{r} \int_\Omega |v_\varepsilon|^r \\
& \quad + A_3 \varepsilon^{\frac{n-p}{p^2}} + B_3 \varepsilon^{\frac{p-1}{p}(n-\frac{(r-1)(n-p)}{p})} \\
& \leq c_{1,r} + \frac{1}{n} S^{\frac{n}{p}} + C \varepsilon^{\frac{n-p}{p}} + \frac{|\beta|^{p^*}}{p^*} - \frac{|\beta|^r}{r} \int_\Omega |v_\varepsilon|^r - \frac{\mu}{q} \left(\frac{S}{2}\right)^{\frac{q}{p^*-p}} \int_\Omega \frac{|v_\varepsilon|^q}{|x|^s} \\
& \quad + A_3 \varepsilon^{\frac{n-p}{p^2}} + B_3 \varepsilon^{\frac{p-1}{p}(n-\frac{(r-1)(n-p)}{p})} \\
& \leq c_{1,r} + \frac{1}{n} S^{\frac{n}{p}} + C \varepsilon^{\frac{n-p}{p}} + A_3 \varepsilon^{\frac{n-p}{p^2}} + B_3 \varepsilon^{\frac{p-1}{p}(n-\frac{(r-1)(n-p)}{p})} \\
& \quad - C_4 \varepsilon^{\frac{(n-p)(p-1)}{p^2}(p^*(s)-q)} + \frac{|\beta|^{p^*}}{p^*} - \frac{|\beta|^r}{r} \int_\Omega |v_\varepsilon|^r,
\end{aligned}$$

where we require $q > \frac{n-s}{n-p}(p-1)$. From

$$\frac{p-1}{p} \left(n - \frac{(r-1)(n-p)}{p}\right) > \frac{(n-p)(p-1)}{p^2} (p^*(s) - q)$$

we get $q > r - 1 - \frac{ns}{n-p}$. From

$$\frac{n-p}{p^2} > \frac{(n-p)(p-1)}{p^2} (p^*(s) - q),$$

we get that $q > p^*(s) - \frac{1}{p-1}$. Since

$$p^*(s) - \frac{1}{p-1} \geq p^*(s) - 1 = p^* - 1 - \frac{ps}{n-p} > r - 1 - \frac{ns}{n-p},$$

the hypothesis $q > p^*(s) - \frac{1}{p-1}$ (i.e. $n > \frac{p(p-1)(q-s)+p}{1+(p-1)(q-p)}$) is sufficient to allow us to choose ε small enough so that

$$C \varepsilon^{\frac{n-p}{p}} + A_3 \varepsilon^{\frac{n-p}{p^2}} + B_3 \varepsilon^{\frac{p-1}{p}(n-\frac{(r-1)(n-p)}{p})} - C_4 \varepsilon^{\frac{(n-p)(p-1)}{p^2}(p^*(s)-q)} \leq -2\sigma$$

for some constant $\sigma > 0$. Now choose $\delta_0 > 0$ small enough so that

$$\frac{|\beta|^{p^*}}{p^*} - \frac{|\beta|^r}{r} \int_\Omega |v_\varepsilon|^r < \sigma \quad \text{for } 0 < |r - p^*| < \delta_0.$$

Thus we have proved the case for $q > p$.

Case 2. $q = p$, $n > p(p-1)(p-s) + p$ and $0 < \mu < \mu_{s,p}$.

We assume that α and β are in a bounded set, and we estimate $E_\lambda(\alpha u_1 + \beta v_\varepsilon)$:

$$\begin{aligned} & E_{\mu,r}(\alpha u_1 + \beta v_\varepsilon) \\ & \leq E_{\mu,r}(\alpha u_1) + E_{\mu,r}(\beta v_\varepsilon) \\ & \quad + A_1 \left[\int_\Omega |\nabla \alpha u_1| |\nabla \beta v_\varepsilon|^{p-1} + |\nabla \alpha u_1|^{p-1} |\nabla \beta v_\varepsilon| \right] \\ & \quad + B_1 \left[\int_\Omega |\alpha u_1| \frac{|\beta v_\varepsilon|^{p-1}}{|x|^s} + |\alpha u_1|^{p-1} \frac{|\beta v_\varepsilon|}{|x|^s} \right] \\ & \quad + C_1 \left[\int_\Omega |\alpha u_1| |\beta v_\varepsilon|^{r-1} + |\alpha u_1|^{r-1} |\beta v_\varepsilon| \right] \\ & \leq E_{\mu,r}(\alpha u_1) + E_\mu(\beta v_\varepsilon) + \frac{|\beta|^{p^*}}{p^*} - \frac{|\beta|^r}{r} \int_\Omega |v_\varepsilon|^r \\ & \quad + A_3 \varepsilon^{\frac{n-p}{p^2}} + B_3 \varepsilon^{\frac{p-1}{p} (n - \frac{(r-1)(n-p)}{p})} \\ & \leq c_{1,r} + \frac{1}{n} S^{\frac{n}{p}} + C \varepsilon^{\frac{n-p}{p}} + \frac{|\beta|^{p^*}}{p^*} - \frac{|\beta|^r}{r} \int_\Omega |v_\varepsilon|^r - O(\varepsilon^{\frac{(p-s)(p-1)}{p}}) \\ & \quad + A_3 \varepsilon^{\frac{n-p}{p^2}} + B_3 \varepsilon^{\frac{p-1}{p} (n - \frac{(r-1)(n-p)}{p})} \\ & \leq c_{1,r} + \frac{1}{n} S^{\frac{n}{p}} + C \varepsilon^{\frac{n-p}{p}} + A_3 \varepsilon^{\frac{n-p}{p^2}} + B_3 \varepsilon^{\frac{p-1}{p} (n - \frac{(r-1)(n-p)}{p})} \\ & \quad - C_4 \varepsilon^{\frac{(p-s)(p-1)}{p}} + \frac{|\beta|^{p^*}}{p^*} - \frac{|\beta|^r}{r} \int_\Omega |v_\varepsilon|^r. \end{aligned}$$

Note that we have required that $p > \frac{n-s}{n-p}(p-1)$. By the assumption, we can choose ε small enough so that

$$C \varepsilon^{\frac{n-p}{p}} + A_3 \varepsilon^{\frac{n-p}{p^2}} + B_3 \varepsilon^{\frac{p-1}{p} (n - \frac{(r-1)(n-p)}{p})} - C_4 \varepsilon^{\frac{(p-s)(p-1)}{p}} \leq -2\sigma$$

for some constant $\sigma > 0$. Now choose $\delta_0 > 0$ small enough so that

$$\frac{|\beta|^{p^*}}{p^*} - \frac{|\beta|^r}{r} \int_\Omega |v_\varepsilon|^r < \sigma \text{ for } 0 < |q - p^*| < \delta_0.$$

The proof of this lemma is now complete. □

The rest of the proof of the theorem is now very similar to Theorem 9.1. The details are left for the interested reader. □

11. APPENDIX: ESTIMATES ON THE EXTREMAL SOBOLEV-HARDY FUNCTIONS

Assume, without loss of generality, that $0 \in \Omega$, and let

$$U_\varepsilon(x) = (\varepsilon + |x|^{\frac{p-s}{p-1}})^{\frac{p-n}{p-s}}.$$

$U_\varepsilon(x)$ is a function in $H^{1,p}(\mathbf{R}^n)$ where the best constant in the Sobolev-Hardy inequality is attained. They are, modulo translation and dilations, the unique positive ones where the best constant is achieved. (See section 2.)

Let $0 \leq \phi(x) \leq 1$ be a function in $C_0^\infty(\Omega)$ defined as

$$\phi(x) = \begin{cases} 1 & \text{if } |x| \leq R, \\ 0 & \text{if } |x| \geq 2R, \end{cases}$$

where $B_{2R}(0) \subset \Omega$. Set $u_\varepsilon(x) = \phi(x)U_\varepsilon(x)$. For $\varepsilon \rightarrow 0$, the behavior of u_ε has to be the same as that of U_ε but we need precise estimates of the error terms.

Lemma 11.1. *Assume $0 \leq s < p$, $p \geq 2$ and $q = \frac{n-s}{n-p}p$. By taking*

$$v_\varepsilon = \frac{u_\varepsilon}{\left(\int_\Omega \frac{|u_\varepsilon|^q}{|x|^s}\right)^{\frac{1}{q}}}$$

so that $\int_\Omega \frac{|v_\varepsilon|^q}{|x|^s} = 1$, we have the following estimates:

- (1) $\|\nabla v_\varepsilon\|_p^p = \mu_s + O(\varepsilon^{\frac{n-p}{p-s}})$,
- (2) $\int_\Omega |\nabla v_\varepsilon|^\alpha = O(\varepsilon^{\frac{(n-p)\alpha}{p(p-s)}})$, for $\alpha = 1, 2, p-2, p-1$.
- (3) if $r > p^*(1 - \frac{1}{p})$, then

$$C_1 \varepsilon^{\frac{(p-1)(n-\frac{r(n-p)}{p})}{p-s}} \leq \|v_\varepsilon\|_r^r \leq C_2 \varepsilon^{\frac{(p-1)(n-\frac{r(n-p)}{p})}{p-s}},$$

- (4) if $r = p^*(1 - \frac{1}{p})$, then

$$C_1 \varepsilon^{\frac{(n-p)r}{p(p-s)}} |\log \varepsilon| \leq \|v_\varepsilon\|_r^r \leq C_2 \varepsilon^{\frac{(n-p)r}{p(p-s)}} |\log \varepsilon|,$$

- (5) if $r < p^*(1 - \frac{1}{p})$, then

$$C_1 \varepsilon^{\frac{(n-p)r}{p(p-s)}} \leq \|v_\varepsilon\|_r^r \leq C_2 \varepsilon^{\frac{(n-p)r}{p(p-s)}},$$

- (6) if $p < r < p^*$, then $\|v_\varepsilon\|_r^r \rightarrow 0$ (as $\varepsilon \rightarrow 0$),

(7)

$$\int_\Omega \frac{|v_\varepsilon|^{q-1}}{|x|^s} = O(\varepsilon^{\frac{p-1}{p}(\frac{n-p}{p-s})}).$$

(8)

$$\int_\Omega \frac{|v_\varepsilon|}{|x|^s} = O(\varepsilon^{\frac{n-p}{p(p-s)}}).$$

where $C_1, C_2 > 0$ are constants.

Proof. Let

$$k(\varepsilon) = (\varepsilon \cdot (n-s) \left(\frac{n-p}{p-1}\right)^{p-1})^{\frac{n-p}{p(p-s)}}.$$

Then $y_\varepsilon(x) = k(\varepsilon)U_\varepsilon(x)$ is the extremal function in the Sobolev-Hardy inequality. Furthermore,

$$k(\varepsilon)^p \|\nabla U_\varepsilon(x)\|_p^p = \|\nabla y_\varepsilon(x)\|_p^p = \mu_s^{\frac{n-s}{p-s}}$$

The gradient of $u_\varepsilon(x)$ is given by

$$\begin{aligned} \nabla u_\varepsilon(x) &= (\varepsilon + |x|^{\frac{p-s}{p-1}})^{\frac{p-n}{p-s}} \nabla \phi(x) + \frac{p-n}{p-1} \cdot \frac{x\phi(x)}{(\varepsilon + |x|^{\frac{p-s}{p-1}})^{\frac{n-s}{p-s}} |x|^{\frac{p-2}{p-1}}} \\ &= \begin{cases} \frac{p-n}{p-1} \cdot \frac{x}{(\varepsilon + |x|^{\frac{p-s}{p-1}})^{\frac{p-s}{p-s}} |x|^{\frac{p-2}{p-1}}} & \text{if } |x| \leq R, \\ 0 & \text{if } |x| \geq 2R. \end{cases} \end{aligned}$$

Thus we have

$$\begin{aligned} \int_\Omega |\nabla u_\varepsilon|^p &= O(1) + \int_{|x| \leq R} |\nabla U_\varepsilon(x)|^p dx = O(1) + \int_{\mathbf{R}^n} |\nabla U_\varepsilon(x)|^p dx \\ &= O(1) + \|\nabla U_\varepsilon(x)\|_p^p, \end{aligned}$$

and

$$\int_{\Omega} \frac{|u_{\varepsilon}|^p}{|x|^s} = O(1) + \int_{\mathbf{R}^n} \frac{|U_{\varepsilon}|^q}{|x|^s} = O(1) + \int_{\mathbf{R}^n} \frac{|y_{\varepsilon}|^q}{|x|^s} \cdot k^{-q}(\varepsilon) = O(k^{-q}(\varepsilon)).$$

From this we further get

$$\begin{aligned} \|\nabla v_{\varepsilon}\|_p^p &= \frac{\|\nabla u_{\varepsilon}\|_p^p}{\left(\int_{\Omega} \frac{|u_{\varepsilon}|^q}{|x|^s}\right)^{\frac{p}{q}}} = \frac{O(1) + \|\nabla U_{\varepsilon}\|_p^p}{\left(\int_{\Omega} \frac{|u_{\varepsilon}|^q}{|x|^s}\right)^{\frac{p}{q}}} \\ &= \frac{O(1) + \mu_s^{\frac{n-s}{p-s}} k(\varepsilon)^{-p}}{\left(\int_{\Omega} \frac{|u_{\varepsilon}|^q}{|x|^s}\right)^{\frac{p}{q}}} = \frac{O(1) + \mu_s^{\frac{n-s}{p-s}} k(\varepsilon)^{-p}}{O(1) + k(\varepsilon)^{-p} \mu_s^{\frac{p(n-s)}{(p-s)q}}} \\ &= O(k(\varepsilon)^p) + \mu_s^{\frac{n-s}{p-s} - \frac{p(n-s)}{(p-s)q}} = \mu_s + O(\varepsilon^{\frac{n-p}{p-s}}). \end{aligned}$$

(1) is thus proved. For (2), let ω_n denote the surface area of the $(n - 1)$ -sphere S^{n-1} in \mathbf{R}^n ; then

$$\begin{aligned} \int_{\Omega} |\nabla u_{\varepsilon}|^{\alpha} &= O(1) + \int_{|x| \leq R} \left(\frac{n-p}{p-1}\right)^{\alpha} \frac{|x|^{\alpha}}{(\varepsilon + |x|^{\frac{p-s}{p-1}})^{\frac{\alpha(n-s)}{p-s}} |x|^{\frac{(p-2)\alpha}{p-1}}} dx \\ &= O(1) + \omega_n \int_0^R \left(\frac{n-p}{p-1}\right)^{\alpha} \frac{r^{\alpha} \cdot r^{n-1}}{(\varepsilon + r^{\frac{p-s}{p-1}})^{\frac{\alpha(n-s)}{p-s}} r^{\frac{(p-2)\alpha}{p-1}}} dr \\ &\leq O(1) + \omega_n \int_0^R \left(\frac{n-p}{p-1}\right)^{\alpha} r^{\alpha+n-1 - \frac{\alpha(n-s)}{p-1} - \frac{\alpha(p-2)}{p-1}} dr, \end{aligned}$$

and the order of r in the integrand is

$$\begin{aligned} \alpha + n - 1 - \frac{\alpha(n-s)}{p-1} - \frac{\alpha(p-2)}{p-1} &= \frac{pn - p - n + 1 - \alpha n + \alpha s + \alpha}{p-1} \\ &= \frac{pn - n - \alpha n + \alpha s + \alpha}{p-1} - 1 > -1 \end{aligned}$$

for $\alpha = 1, p - 2, p - 1$ and $\alpha = 2$ if $p \geq 3$. Thus

$$\int_{\Omega} |\nabla u_{\varepsilon}|^{\alpha} = O(1),$$

and we conclude that

$$\int_{\Omega} |\nabla v_{\varepsilon}|^{\alpha} = O(\varepsilon^{\frac{(n-p)\alpha}{p(p-s)}}).$$

For (3), (4) and (5),

$$\begin{aligned} \|u_{\varepsilon}\|_r^r &= O(1) + \omega_n \int_0^R (\varepsilon + x^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}} x^{n-1} dx \\ &= O(1) + \omega_n \varepsilon^{-\frac{n-p}{p-s} r + \frac{p-1}{p-s} n} \int_0^{R\varepsilon^{-\frac{p-1}{p-s}}} (1 + x^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}} x^{n-1} dx \end{aligned}$$

If $r = p^*(1 - \frac{1}{p})$, then $-\frac{n-p}{p-s}r + \frac{p-1}{p-s}n = 0$, and

$$\begin{aligned}\|u_\varepsilon\|_r^r &= O(1) + \omega_n \int_0^{R\varepsilon^{-\frac{p-1}{p-s}}} (1 + x^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}r} x^{n-1} dx \\ &= O(1) + \omega_n \int_0^{R\varepsilon^{-\frac{p-1}{p-s}}} \frac{1}{x} dx \\ &= O(1) + O(|\log \varepsilon|).\end{aligned}$$

So we get

$$\|v_\varepsilon\|_r^r = O(|\log \varepsilon| \varepsilon^{\frac{n-p}{p(p-s)}r}).$$

If $r < p^*(1 - \frac{1}{p})$, then $-\frac{n-p}{p-1}r + n - 1 > -1$. We conclude that

$$\begin{aligned}\|u_\varepsilon\|_r^r &= O(1) + \omega_n \int_0^R (\varepsilon + x^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}r} x^{n-1} dx \\ &\leq O(1) + \omega_n \int_0^R x^{-\frac{n-p}{p-1}r+n-1} dx = O(1)\end{aligned}$$

and

$$\|v_\varepsilon\|_r^r = O(\varepsilon^{\frac{n-p}{p(p-s)}r}).$$

If $r > p^*(1 - \frac{1}{p})$, then $-\frac{n-p}{p-1}r + n - 1 < -1$ and $-\frac{n-p}{p-s}r + \frac{p-1}{p-s}n < 0$. We have

$$\begin{aligned}\|u_\varepsilon\|_r^r &= O(1) + \omega_n \varepsilon^{-\frac{n-p}{p-s}r + \frac{p-1}{p-s}n} \int_1^\infty (1 + x^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}r} x^{n-1} dx \\ &= O(\varepsilon^{-\frac{n-p}{p-s}r + \frac{p-1}{p-s}n}),\end{aligned}$$

and

$$\|v_\varepsilon\|_r^r = O(\varepsilon^{-\frac{n-p}{p-s}r + \frac{p-1}{p-s}n + \frac{n-p}{p(p-s)}r}) = O(\varepsilon^{\frac{p-1}{p-s}(n - \frac{r(n-p)}{p})}).$$

(3), (4) and (5) are thus proved.

For (7) and (8), we have

$$\begin{aligned}\int_\Omega \frac{|u_\varepsilon|^\alpha}{|x|^s} dx &= O(1) + \int_{|x| \leq R} (\varepsilon + |x|^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}\alpha} |x|^{-s} dx \\ &= O(1) + \omega_n \int_0^R (\varepsilon + r^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}\alpha} r^{-s} \cdot r^{n-1} dr \\ &= O(1) + \omega_n \int_0^{R\varepsilon^{-\frac{p-1}{p-s}}} (1 + r^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}\alpha} r^{n-s-1} \varepsilon^{-\frac{n-p}{p-s}\alpha + (n-s)\frac{p-1}{p-s}} dr.\end{aligned}$$

If $\alpha = q - 1$, then $-\frac{n-p}{p-s}\alpha + (n-s)\frac{p-1}{p-s} = -1$. We have

$$\begin{aligned}\int_\Omega \frac{|u_\varepsilon|^\alpha}{|x|^s} dx &= O(1) + O(\varepsilon^{-1}) \omega_n \int_0^{R\varepsilon^{-\frac{p-1}{p-s}}} (1 + r^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}\alpha} r^{n-s-1} dr \\ &= O(1) + O(\varepsilon^{-1}) \omega_n \int_0^\infty (1 + r^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}\alpha} r^{n-s-1} dr = O(\varepsilon^{-1}),\end{aligned}$$

since $n - s - \frac{n-p}{p-1}(q - 1) = \frac{s-p}{p-1} < 0$. Then

$$\int_\Omega \frac{|v_\varepsilon|^\alpha}{|x|^s} dx = O(\varepsilon^{-1}) \cdot \varepsilon^{\frac{n-p}{p(p-s)}(q-1)} = O(\varepsilon^{\frac{p-1}{p}(\frac{n-p}{p-s})}).$$

If $\alpha = 1$, since $-\frac{n-p}{p-1} + n - s > -\frac{n-p}{p-1} + n - p = (n-p)(1 - \frac{1}{p-1}) \geq 0$ for $p \geq 2$, we have

$$\begin{aligned} \int_{\Omega} \frac{|u_{\varepsilon}|}{|x|^s} dx &= O(1) + \omega_n \int_0^R (\varepsilon + r^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}} r^{n-s-1} dr \\ &\leq O(1) + \omega \int_0^R r^{-\frac{n-p}{p-1} + n - s - 1} dr = O(1), \end{aligned}$$

and furthermore

$$\int_{\Omega} \frac{|v_{\varepsilon}|}{|x|^s} dx = O(\varepsilon^{\frac{n-p}{p(p-s)}}).$$

(7) and (8) are thus proved. □

Note that the above results are well known for the extremal functions associated to the Sobolev embedding, that is, when $s = 0$. In the following lemma, we prove additional properties in the case where $s > 0$.

Lemma 11.2. *For $0 \leq t < p$, we have*

$$\int_{\Omega} \frac{|v_{\varepsilon}|^{\alpha}}{|x|^t} = \begin{cases} O(\varepsilon^{\frac{n-p}{p^2}\alpha} |\log \varepsilon|), & \alpha = \frac{n-t}{n-p}(p-1), \\ O(\varepsilon^{\frac{(n-p)(p-1)}{p^2}(p^*(t)-\alpha)}), & \alpha > \frac{n-t}{n-p}(p-1), \\ O(\varepsilon^{\frac{n-p}{p^2}\alpha}), & \alpha < \frac{n-t}{n-p}(p-1). \end{cases}$$

Proof. As above, let ω_n denote the surface area of the $(n-1)$ sphere S^{n-1} in \mathbf{R}^n . Then

$$\int_{\Omega} \frac{|u_{\varepsilon}|^{\alpha}}{|x|^t} = O(1) + \omega_n \int_0^R (\varepsilon + r^{\frac{p}{p-1}})^{\frac{p-n}{p}\alpha} r^{n-t-1} dr.$$

Case 1. $\alpha = \frac{n-t}{n-p}(p-1)$ (then $\frac{p-n}{p-1}\alpha + n - t = 0$). Then

$$\begin{aligned} \int_{\Omega} \frac{|u_{\varepsilon}|^{\alpha}}{|x|^t} &= O(1) + \omega_n \int_0^{R\varepsilon^{-\frac{p-1}{p}}} (1 + r^{\frac{p}{p-1}})^{\frac{p-n}{p}\alpha} r^{n-t-1} dr \\ &= O(1) + \omega_n \int_0^{R\varepsilon^{-\frac{p-1}{p}}} \frac{1}{r} dr = O(|\log \varepsilon|). \end{aligned}$$

Case 2. $\alpha > \frac{n-t}{n-p}(p-1)$ (then $\frac{p-n}{p-1}\alpha + n - t - 1 < -1$). Then

$$\begin{aligned} \int_{\Omega} \frac{|u_{\varepsilon}|^{\alpha}}{|x|^t} &= O(1) + \omega_n \int_0^{R\varepsilon^{-\frac{p-1}{p}}} (1 + r^{\frac{p}{p-1}})^{\frac{p-n}{p}\alpha} r^{n-t-1} dr \varepsilon^{\frac{p-n}{p}\alpha + (n-s)\frac{p-1}{p}} \\ &= O(1) + O(\varepsilon^{\frac{p-n}{p}\alpha + (n-s)\frac{p-1}{p}}). \end{aligned}$$

Case 3. $\alpha < \frac{n-t}{n-p}(p-1)$ (i.e. $\frac{p-n}{p-1}\alpha + n - t - 1 > -1$). Then

$$\begin{aligned} \int_{\Omega} \frac{|u_{\varepsilon}|^{\alpha}}{|x|^t} &= O(1) + \omega_n \int_0^R (\varepsilon + r^{\frac{p}{p-1}})^{\frac{p-n}{p}\alpha} r^{n-t-1} dr \\ &= O(1) + \omega_n \int_0^R r^{\frac{p-n}{p-1}\alpha + n - t - 1} dr \\ &= O(1). \end{aligned}$$

Now, the conclusion follows immediately. □

REFERENCES

- [1] T. Aubin: *Nonlinear Analysis on Manifolds. Monge-Ampère Equations*, Springer-Verlag, 1982. MR **85j**:58002
- [2] G. Bliss: *An Integral Inequality*, J. London Math. Soc., 5 (1930) 40-46.
- [3] L. Boccardo and F. Murat: *Almost Everywhere Convergence of the Gradients of Solutions to Elliptic and Parabolic Equations*, Nonlinear Analysis, TMA 19, (1992) 581-597. MR **93h**:35061
- [4] H. Brezis, E. Lieb: *A Relation Between Point Convergence of Functions and Convergence of Functionals*, Proc. Amer. Math. Soc. 88(1983), 486-490. MR **84c**:28003
- [5] H. Brezis and L. Nirenberg: *Positive Solutions of Nonlinear Elliptic Equations involving Critical Exponents*, Communications on Pure and Applied Mathematics, 34 (1983), 437-477. MR **84h**:35059
- [6] H. Brezis and L. Nirenberg: *A Minimization Problem with Critical Exponent and Non-zero Data*, in "Symmetry in Nature", Scuola Norm. Sup. Pisa (1989), 129-140.
- [7] L. Caffarelli, B. Gidas and J. Spruck: *Asymptotic Symmetry and Local Behavior of Semilinear Elliptic Equations with Critical Sobolev Growth*, Comm. Pure Appl. Math. 42 (1989) 271-477. MR **90c**:35075
- [8] L. Caffarelli, R. Kohn and L. Nirenberg: *First Order Interpolation Inequality with Weights*, Compositio Math. 53 (1984) 259-275. MR **86c**:46028
- [9] G. Cerami, S. Solimini, and M. Struwe: *Some Existence Results for Super-linear Elliptic Boundary Value Problems Involving Critical Exponents*, J. Functional Analysis 69, 9 (1986) 289-306. MR **88b**:35074
- [10] K. S. Chou and C. W. Chu: *On the Best Constant for a Weighted Sobolev-Hardy Inequality*, J. London Math. Soc. (2) 48 (1993) 137-151. MR **94h**:46052
- [11] P. Drabek: *Strongly Nonlinear Degenerated and Singular Elliptic Problems*, Pitman Research Notes in Mathematics Series 343 (1996) 112-146. MR **97k**:35091
- [12] P. Drabek and S. I. Pohozaev: *Positive Solutions for the p -Laplacian: Application of the Fiberning Method*, Proceeding of the Royal Society of Edinburgh, 127A (1997) 703-726. MR **98h**:35061
- [13] J. P. Garcia Azorero and I. Peral Alonso: *Hardy Inequalities and Some Critical Elliptic and Parabolic Problems*, J. of Differential Equations 144 (1998) 441-476. MR **99f**:35099
- [14] J. Garcia Azorero and I. Peral Alonso: *Multiplicity of Solutions for Elliptic Problems with Critical Exponent or with a Non-symmetric Term*, Trans. Amer. Math. Soc. 323 (1991), 877-895. MR **99g**:35108
- [15] J. Garcia Azorero and I. Peral Alonso: *Some Results about the Existence of a Second Positive Solution in a Quasi-linear Critical Problem*, Indiana University Math. Journal 43 (1994), 941-957. MR **95i**:35093
- [16] J. Garcia Azorero and I. Peral Alonso: *Existence and Non-uniqueness for the p -Laplacian: Nonlinear Eigenvalues*. Commun. in PDE, 12 (1987), 1389-1430. MR **89e**:35058
- [17] N. Ghoussoub: *Duality and Perturbation Methods in Critical Point Theory*, Cambridge University Press (1993). MR **95a**:58021
- [18] D. Gilbarg, N. S. Trudinger: *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag (1983). MR **86c**:35035
- [19] M. Guedda, L. Veron: *Local and Global Properties of Solutions of Quasi-linear Elliptic Equations*, Journal of Differential Equations 76 (1988) 159-189. MR **89k**:35078
- [20] M. Guedda, L. Veron: *Quasi-linear Elliptic Equations Involving Critical Sobolev Exponents*, Nonlinear Anal TMA, 13 (8) (1989) 879-902. MR **90h**:35100
- [21] E. H. Lieb: *Sharp Constants in the Hardy-Littlewood-Sobolev and Related Inequalities*, Ann. of Math. 118 (1983) 349-374. MR **86i**:42010
- [22] E. H. Lieb and M. Loss: *Analysis*, Graduate Studies in Mathematics Vol. 14 Amer. Math. Soc., Providence, RI, (1997). MR **98b**:00004
- [23] M. Otani: *Existence and Nonexistence of Nontrivial Solutions of Some Nonlinear Degenerated Elliptic Equations*, Journal of Functional Analysis 76 (1988) 140-159. MR **89c**:35067
- [24] E. S. Noussair, C. A. Swanson, J. Yang: *Quasi-linear Elliptic Problems with Critical Exponents*, Nonlin. Anal. 20 (1993) 285-301. MR **94b**:35106
- [25] I. Peral: *Multiplicity of Solutions for the p -Laplacian*, Lecture Notes at the Second School on Nonlinear Functional Analysis and Applications to Differential Equations at ICTP of Trieste (April 21 - May 9, 1997).

- [26] M. Struwe: *Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Second Edition, Springer-Verlag (1996). MR **98f**:49002
- [27] G. Talenti: *Best Constant in Sobolev Inequalities*, Ann. Math. Pura Appl. 110 (1976), 353–372. MR **57**:3846
- [28] G. Tarantello: *Nodal Solutions of Semi-linear Elliptic Equations with Critical Exponent*, Differential and Integral Equations, 5 (1992) 25-42. MR **92k**:35109
- [29] P. Tolksdorf: *Regularity for a More General Class of Quasi-linear Elliptic Equations*, Journal of Differential Equations 51 (1984) 126-150. MR **85g**:35047
- [30] J. Yang: *Positive Solutions of Quasi-linear Elliptic Obstacle Problems with Critical Exponents*, Nonlinear Analysis, TMA 25 (1995) 1283-1306. MR **96m**:35120
- [31] C. Yuan: *On Non-homogeneous Quasi-linear PDEs Involving the p -Laplacian and the Critical Sobolev Exponent*, Ph.D. Dissertation, UBC 1998.

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