

## MULTIPLE SOLUTIONS FOR QUASI-LINEAR PDES INVOLVING THE CRITICAL SOBOLEV AND HARDY EXPONENTS

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ABSTRACT. We use variational methods to study the existence and multiplicity of solutions for the following quasi-linear partial differential equation:

$$\begin{cases} -\Delta_p u = \lambda|u|^{r-2}u + \mu \frac{|u|^{q-2}}{|x|^s} u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where  $\lambda$  and  $\mu$  are two positive parameters and  $\Omega$  is a smooth bounded domain in  $\mathbf{R}^n$  containing 0 in its interior. The variational approach requires that  $1 < p < n$ ,  $p \leq q \leq p^*(s) \equiv \frac{n-s}{n-p}p$  and  $p \leq r \leq p^* \equiv p^*(0) = \frac{np}{n-p}$ , which we assume throughout. However, the situations differ widely with  $q$  and  $r$ , and the interesting cases occur either at the critical Sobolev exponent ( $r = p^*$ ) or in the Hardy-critical setting ( $s = p = q$ ) or in the more general Hardy-Sobolev setting when  $q = \frac{n-s}{n-p}p$ . In these cases some compactness can be restored by establishing Palais-Smale type conditions around appropriately chosen *dual sets*. Many of the results are new even in the case  $p = 2$ , especially those corresponding to singularities (i.e., when  $0 < s \leq p$ ).

### 1. INTRODUCTION

Consider the following quasi-linear partial differential equation:

$$(P_{\lambda,\mu}) \quad \begin{cases} -\Delta_p u = \lambda|u|^{r-2}u + \mu \frac{|u|^{q-2}}{|x|^s} u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where  $\lambda$  and  $\mu$  are two positive parameters and  $\Omega$  is a smooth bounded domain in  $\mathbf{R}^n$  containing 0 in its interior. We shall assume throughout that  $0 \leq s \leq p < n$ .

The starting point of the variational approach to these problems is the following *Sobolev-Hardy inequality*, which is essentially due to Caffarelli, Kohn and Nirenberg [8]. Assume that  $1 < p < n$  and that  $q \leq p^*(s) \equiv \frac{n-s}{n-p}p$ ; then there is a constant  $C > 0$  such that

$$C \left( \int_{\Omega} \frac{|u|^q}{|x|^s} dx \right)^{\frac{p}{q}} \leq \int_{\Omega} |\nabla u|^p dx \quad \text{for all } u \in H_0^{1,p}(\Omega).$$

We use  $\mu_{s,q}(\Omega)$  to denote the best *Sobolev-Hardy* constant, i.e. the largest constant  $C$  satisfying the above inequality for all  $u \in H_0^{1,p}(\Omega)$ ; that is,

$$\mu_{s,q}(\Omega) = \inf_{u \in H_0^{1,p}(\Omega), u \neq 0} \frac{\int_{\Omega} |\nabla u|^p dx}{\left( \int_{\Omega} \frac{|u|^q}{|x|^s} dx \right)^{\frac{p}{q}}}.$$

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In the important case where  $q = p^*(s)$ , we shall simply denote  $\mu_{s,p^*(s)}$  as  $\mu_s$ . Note that  $\mu_0$  is nothing but the best constant in the *Sobolev inequality* while  $\mu_p$  is the best constant in the *Hardy inequality*, i.e.,

$$\mu_p(\Omega) = \inf_{u \in H_0^{1,p}(\Omega), u \neq 0} \frac{\int_{\Omega} |\nabla u|^p dx}{\int_{\Omega} \frac{|u|^p}{|x|^p} dx}.$$

We shall always assume that  $p \leq r \leq p^* \equiv p^*(0) = \frac{np}{n-p}$  for the non-singular term in such a way that the functional

$$E_{\lambda,\mu}(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{\lambda}{r} \int_{\Omega} |u|^r dx - \frac{\mu}{q} \int_{\Omega} \frac{|u|^q}{|x|^s} dx$$

is then well defined on the Sobolev space  $H_0^{1,p}(\Omega)$ . The (weak) solutions of the problem  $(P_{\lambda,\mu})$  are then the critical points of the functional  $E_{\lambda,\mu}$ .

Another relevant parameter will be the first “eigenvalue” of the  $p$ -Laplacian  $-\Delta_p$ , defined as

$$\lambda_1(\Omega) \equiv \mu_{0,p}(\Omega) = \inf \left\{ \int_{\Omega} |\nabla w|^p dx : w \in H_0^{1,p}(\Omega), \int_{\Omega} |w|^p dx = 1 \right\}.$$

Here are the main results of this paper.

**Theorem 1.1** (Hardy-Sobolev subcritical singular and non-singular terms). *Suppose  $1 < p \leq q < p^*(s)$  and  $r < p^*$ . Assume one of the following conditions holds:*

- (1) (*High order singular term*)  $p < q$ ,  $p \leq r$ ,  $\lambda > 0$  and  $\mu > 0$ .
- (2) (*Low order singular term*)  $p = q$ ,  $p < r$ ,  $\lambda > 0$  and  $\mu_{s,p} > \mu > 0$ .

*Then  $(P_{\lambda,\mu})$  has infinitely many solutions. Moreover,  $(P_{\lambda,\mu})$  has an everywhere positive solution with least energy and another one that is sign-changing.*

**Theorem 1.2** (Hardy-critical singular term). *Suppose  $1 < p = q = p^*(s)$  (i.e.,  $s = p$ ).*

1. (*Subcritical non-singular term*) *If  $r < p^*$ , then  $(P_{\lambda,\mu})$  has infinitely many solutions –at least one of them being positive– for any  $\lambda > 0$  and  $0 < \mu < \mu_p$ .*
2. (*Critical non-singular term*) *If  $r = p^*$  and  $\Omega$  is star-shaped. Then  $(P_{\lambda,\mu})$  has no non-trivial solution for any  $\lambda > 0, \mu > 0$ .*

**Theorem 1.3** (Hardy-Sobolev critical singular term). *Suppose  $1 \leq p < q = p^*(s)$  (i.e.,  $s < p$ ).*

1. (*High order non-singular term*) *Assume  $p < r < p^*$  and  $\lambda > 0, \mu > 0$ .*
  - *If  $n > \frac{p(p-1)r+p^2}{p+(p-1)(r-p)}$  (in particular if  $n \geq p^2$ ), then  $(P_{\lambda,\mu})$  has a positive solution.*
  - *If  $n > \frac{p(p-1)r+p}{1+(p-1)(r-p)}$  (in particular if  $n > p^3 - p^2 + p$ ), then  $(P_{\lambda,\mu})$  has also a sign-changing solution.*
2. (*Low order non-singular term*) *Assume  $p = r < p^*$  and  $0 < \lambda < \lambda_1, \mu > 0$ .*
  - *If  $n \geq p^2$ , then  $(P_{\lambda,\mu})$  has a positive solution.*
  - *If  $n > p^3 - p^2 + p$ , then  $(P_{\lambda,\mu})$  has also a sign-changing solution.*
3. (*Sobolev-critical non-singular term*) *Assume  $r = p^*$  and  $\Omega$  is star-shaped, then  $(P_{\lambda,\mu})$  has no non-trivial solution for any  $\lambda > 0$  and any  $\mu > 0$ .*

**Theorem 1.4** (Sobolev-critical non-singular term and subcritical singular term). *Suppose  $1 < p \leq q < p^*(s)$  and  $r = p^*$ .*

1. (High order singular term) Assume that  $p < q$  and  $\lambda > 0, \mu > 0$ .
  - If  $n > \frac{p(p-1)(q-s)+p^2}{p+(p-1)(q-p)}$  (in particular if  $n \geq p^2 - (p-1)s$ ), then  $(P_{\lambda,\mu})$  has a positive solution.
  - If  $n > \frac{p(p-1)(q-s)+p}{1+(p-1)(q-p)}$  (in particular if  $n > p(p-1)(p-s) + p$ ), then  $(P_{\lambda,\mu})$  has also a sign-changing solution.
2. (Low order non-singular term) Assume  $p = q$  and  $\lambda > 0, \mu_{s,p} > \mu > 0$ .
  - If  $n \geq p^2 - (p-1)s$ , then  $(P_{\lambda,\mu})$  has a positive solution.
  - If  $n > p((p-1)(p-s) + 1)$ , then  $(P_{\lambda,\mu})$  has also a sign-changing solution.

The following tables summarize our results.

TABLE 1. Sobolev-subcritical non-singular term

Singular term	Parameters	Non-singular term	Dimension	# of solutions
(HS-subcritical) $p < q < p^*(s)$ $p = q < p^*(s)$	$\lambda > 0; \mu > 0$ $\mu_{s,p} > \mu > 0; \lambda > 0$	$1 \leq p \leq r < p^*$ $1 \leq p < r < p^*$	$n > p$ $n > p$	Infinite One positive One positive
(H-critical) $p = q = p^*(s)$	$\lambda > 0; \mu_p > \mu > 0$	$1 \leq p < r < p^*$	$n > p$	Infinite (One positive)
(HS-critical) $p < q = p^*(s)$	$\lambda > 0; \mu > 0$ — $\lambda_1 > \lambda > 0; \mu > 0$ —	$1 \leq p < r < p^*$ $2 \leq p < r < p^*$ $1 \leq p = r < p^*$ $2 \leq p = r < p^*$	$n > \frac{p(p-1)r+p^2}{p+(p-1)(r-p)}$ $n > \frac{p(p-1)r+p}{1+(p-1)(r-p)}$ $n \geq p^2$ $n > p^3 - p^2 + p$	One positive Two One positive Two

TABLE 2. Sobolev-critical non-singular term

Singular term	Parameters	Non-singular term	Dimension	# of solutions
$1 \leq p = q < p^*(s)$ $2 \leq p = q < p^*(s)$	$\mu_{s,p} > \mu > 0$ and $\lambda > 0$	$r = p^*$ —	$n > p^2 - (p-1)s$ $n > p((p-1)(p-s) + 1)$	One positive Two
$1 \leq p < q < p^*(s)$ $2 \leq p < q < p^*(s)$	$\lambda > 0; \mu > 0$ —	$r = p^*$ —	$n > \frac{p(p-1)(q-s)+p^2}{p+(p-1)(q-p)}$ $n > \frac{p(p-1)(q-s)+p}{1+(p-1)(q-p)}$	One positive Two
$p \leq q = p^*(s)$	$\lambda > 0, \mu > 0$	$r = p^*$	$n > p$	None

## 2. A POHOZAEV-TYPE IDENTITY

In this section, we start by identifying the constraints on the problem of existence of solutions for  $(P_{\lambda,\mu})$ . Here is the main result.

**Theorem 2.1.** *If  $\Omega$  is a star-shaped domain in  $\mathbf{R}^n$ , then problem  $(P_{\lambda,\mu})$  has no solution in the doubly critical case: That is, for  $r = p^*$  and  $q = p^*(s) = \frac{n-s}{n-p}p$ , the problem  $(P_{\lambda,\mu})$  has no non-trivial solution.*

Assume  $\Omega$  is a star-shaped domain. Then, if we let  $v$  denote the outwards normal to  $\partial\Omega$ , then  $\langle x, v \rangle > 0$  on  $\partial\Omega$ . We assume we have the necessary regularity in the following operations; otherwise, we can use an approximation argument as in Guedda and Veron [20].

Multiplying the equation  $(P_{\lambda,\mu})$  by  $\langle x, \nabla u \rangle$  on both sides and integrate by parts, we get

$$\frac{p-1}{p} \int_{\partial\Omega} |\nabla u|^p \langle x, v \rangle dx + \frac{n-p}{p} \int_{\Omega} |\nabla u|^p dx = \mu \frac{n-s}{q} \int_{\Omega} \frac{|u|^q}{|x|^s} dx + \lambda \frac{n}{r} \int_{\Omega} |u|^r dx.$$

On the other hand, multiplying the equation by  $u$  and integrating, we get

$$\int_{\Omega} |\nabla u|^p = \mu \int_{\Omega} \frac{|u|^q}{|x|^s} dx + \lambda \int_{\Omega} |u|^r dx.$$

Putting the two identities together, we have

$$\frac{p-1}{p} \int_{\partial\Omega} |\nabla u|^p \langle x, \nu \rangle d\sigma = \mu \left( \frac{n-s}{q} - \frac{n-p}{p} \right) \int_{\Omega} \frac{|u|^q}{|x|^s} dx + \lambda \left( \frac{n}{r} - \frac{n-p}{p} \right) \int_{\Omega} |u|^r.$$

So if  $r = \frac{np}{n-p} = p^*$  and  $q = \frac{n-s}{n-p}p$ , the problem has no non-trivial solution.

### 3. THE EXTREMAL FUNCTIONS IN THE HARDY-SOBOLEV INEQUALITIES

In this section, we summarize the needed results concerning the Hardy-Sobolev inequalities. We first recall the *Hardy* inequality.

**Lemma 3.1** ([13]). *Assume that  $1 < p < n$  and  $u \in H^{1,p}(\mathbf{R}^n)$ . Then:*

- (1)  $\frac{u}{|x|} \in L^p(\mathbf{R}^n)$ .
- (2) (*Hardy Inequality*)  $\int_{\mathbf{R}^n} \frac{|u|^p}{|x|^p} dx \leq C_{n,p} \int_{\mathbf{R}^n} |\nabla u|^p dx$ , where  $C_{n,p} = \left(\frac{p}{n-p}\right)^p$ .
- (3) *The constant  $C_{n,p}$  is optimal.*

The following extension of the Hardy and Sobolev inequalities is essentially due to Caffarelli, Kohn and Nirenberg[8].

**Lemma 3.2** (Sobolev-Hardy Inequality). *Assume that  $1 < p < n$  and that  $p \leq q \leq p^*(s) := \frac{n-s}{n-p}p$ . Then:*

- (1) *There exists a constant  $C > 0$  such that for any  $u \in H_0^p(\Omega)$ ,*

$$\left( \int_{\Omega} \frac{|u|^q}{|x|^s} dx \right)^p \leq C \left( \int_{\Omega} |\nabla u|^p dx \right)^q.$$

- (2) *The map  $u \rightarrow \frac{u}{|x|^{s/q}}$  from  $H_0^p(\Omega)$  into  $L^q(\Omega)$  is compact provided  $q < p^*(s)$ .*

*Proof.* (1) For  $s = 0$  or  $s = p$ , this is just the Sobolev (resp., the Hardy) inequality. Since  $p^*(s) \geq p$ , we have  $0 \leq s \leq p$ . We can therefore only consider the case where  $0 < s < p$ . By the Hardy, Sobolev and Hölder inequalities, we have

$$\begin{aligned} \int_{\mathbf{R}^n} \frac{|u|^{p^*(s)}}{|x|^s} &= \int_{\mathbf{R}^n} \frac{|u|^s}{|x|^s} \cdot |u|^{p^*(s)-s} \\ &\leq \left( \int_{\mathbf{R}^n} \frac{|u|^p}{|x|^p} \right)^{\frac{s}{p}} \left( \int_{\mathbf{R}^n} |u|^{(p^*(s)-s)\frac{p}{p-s}} \right)^{\frac{p-s}{p}} \\ &= \left( \int_{\mathbf{R}^n} \frac{|u|^p}{|x|^p} \right)^{\frac{s}{p}} \left( \int_{\mathbf{R}^n} |u|^{p^*} \right)^{\frac{p-s}{p}} \\ &\leq C_1 \left( \int_{\mathbf{R}^n} |\nabla u|^p \right)^{\frac{s}{p}} \left( \int_{\mathbf{R}^n} |\nabla u|^p \right)^{\frac{p-s}{p} \cdot \frac{p^*}{p}} \\ &= C_1 \left( \int_{\mathbf{R}^n} |\nabla u|^p \right)^{\frac{n-s}{n-p}}. \end{aligned}$$

□

*Remark 3.1.* If  $\Omega$  is the whole space, one can show that the conditions  $p \leq q = p^*(s) := \frac{n-s}{n-p}p$  are also necessary for the above inequality to hold. Indeed, a standard scaling argument shows that  $q$  must be equal to  $p^*(s)$ . On the other hand,

if we insert into the inequality the following function ( $\rho$  and  $\theta \in S^{n-1}$  being the polar coordinates),

$$u(x) = \begin{cases} 0 & \text{for } |x| \geq 1, \\ |x|^{\frac{p-n}{p}} \log \frac{1}{|x|} & \text{for } \varepsilon \leq |x| < 1, \\ \varepsilon^{\frac{p-n}{p}} \log \frac{1}{\varepsilon} & \text{for } |x| \leq \varepsilon, \end{cases}$$

and

$$\frac{du(x)}{d\rho} = \begin{cases} 0, & |x| \geq 1, \\ 0, & |x| \leq \varepsilon, \\ (1 - \frac{n}{p})\rho^{-\frac{n}{p}} \log \frac{1}{\rho} - \rho^{-\frac{n}{p}}, & \varepsilon \leq |x| < 1, \end{cases}$$

we get

$$\int_{\mathbf{R}^n} |\nabla u|^p \sim \int_{\varepsilon}^1 \rho^{-1} (1 + (\frac{n}{p} - 1) \log \frac{1}{\rho})^p d\rho.$$

By L'Hospital's rule, we have

$$\lim_{\varepsilon \rightarrow 0} \frac{\int_{\varepsilon}^1 \rho^{-1} (1 + (\frac{n}{p} - 1) \log \frac{1}{\rho})^p d\rho}{\log^{1+p} \frac{1}{\varepsilon}} = \frac{\frac{n}{p} - 1}{1 + p},$$

and also

$$\int_{\mathbf{R}^n} \frac{|u|^q}{|x|^s} \sim \int_{\varepsilon}^1 \rho^{-s} \log^q \frac{1}{\rho} \rho^{\frac{p-n}{p}q} \rho^{n-1} = \int_{\varepsilon}^1 \frac{1}{\rho} \log^q \frac{1}{\rho} \sim \log^{1+q} \frac{1}{\varepsilon}.$$

Thus from the inequality

$$\log^{1+\frac{1}{q}} \frac{1}{\varepsilon} \leq \log^{1+\frac{1}{p}} \frac{1}{\varepsilon},$$

we get that  $q \geq p$ .

The following is an extension of what is well known in the case  $p = 2$  and  $s = 0$ .

**Theorem 3.1.** *Suppose  $1 < p < n$ ,  $0 \leq s < p$  and  $q = p^*(s)$ . Then the following hold:*

- (1)  $\mu_s(\Omega)$  is independent of  $\Omega$  (and will henceforth be denoted by  $\mu_s$ ).
- (2)  $\mu_s$  is attained when  $\Omega = \mathbf{R}^n$  by the functions

$$y_a(x) = (a \cdot (n - s) (\frac{n-p}{p-1})^{p-1})^{\frac{n-p}{p(p-s)}} (a + |x|^{\frac{p-s}{p-1}})^{\frac{p-n}{p-s}}$$

for some  $a > 0$ . Moreover the functions  $y_a$  are the only positive radial solutions of

$$-\operatorname{div}(|\nabla u|^{p-2} \nabla u) = \frac{u^{p^*(s)-1}}{|x|^s}$$

in  $\mathbf{R}^n$ . Hence,

$$\mu_s \left( \int_{\mathbf{R}^n} \frac{|y_a|^q}{|x|^s} \right)^{\frac{p}{q}} = \|\nabla y_a\|_p^p = \int_{\mathbf{R}^n} \frac{|y_a|^q}{|x|^s} = \mu_s^{\frac{n-s}{p-s}}.$$

*Proof.* We prove (2). We show the best constants are attained at functions

$$u_s(x) = c(\lambda_0 + |x|^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}} \quad (0 \leq s < p), \text{ where } \lambda_0 > 0 \text{ is a constant.}$$

For any  $f$ , let  $f^*$  be its Schwarz symmetrization (or rearrangement) [21]. Then we have

$$\int_{\mathbf{R}^n} |\nabla f^*|^p \leq \int_{\mathbf{R}^n} |\nabla f|^p \quad \text{and} \quad \int_{\mathbf{R}^n} \frac{|f^*|^q}{|x|^t} \geq \int_{\mathbf{R}^n} \frac{|f|^q}{|x|^t},$$

assuming the above integrals are well defined (refer to Lieb [21], [22]). By these inequalities, we may restrict our discussion to radial symmetric functions. Thus we may consider the following variational problem:

$$\text{Maximize } I(g) = \int_0^\infty |g(r)|^q r^{n-s-1} dr, \quad \text{when } J(g) = \int_0^\infty |g'(r)|^p r^{n-1} dr = C.$$

where  $C$  is a given constant. The *Euler-Lagrange* equation is

$$(*) \quad (r^{n-1}|u'(r)|^{p-2}u'(r))_r + kr^{n-s-1}|u|^{q-1} = 0.$$

It can be easily verified that the functions

$$u_s(x) = (\lambda + |x|^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}} \quad (0 \leq s < p)$$

are solutions of  $(*)$ , where  $\lambda > 0$ . To continue, we need the following lemma of Bliss ([1], [2]).

**Lemma 3.3.** *Let  $h(x) \geq 0$  be a measurable, real-valued function defined on  $\mathbf{R}$  such that the integral  $J_0 = \int_0^\infty h^{p_0}(x)dx$  is finite and given. Set  $g(x) = \int_0^x h(t)dt$ . Then  $I_0 = \int_0^\infty g^{q_0}(x)x^{\alpha-q_0}dx$  attains its maximum value at the functions  $h(x) = (\lambda x^\alpha + 1)^{-\frac{\alpha+1}{\alpha}}$ , with  $p_0$  and  $q_0$  two constants satisfying  $q_0 > p_0 > 1$ ,  $\alpha = \frac{q_0}{p_0} - 1$ , and  $\lambda > 0$ , a real number.*

By this lemma and with the change of variables  $x = r^{\frac{p-n}{p-1}}$  we can deduce that  $I(\cdot)$  attains its maximum at the functions

$$u_s(x) = (\lambda + |x|^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}} \quad (0 \leq s < p).$$

Note that if

$$h(x) = (\lambda x^\alpha + 1)^{-\frac{\alpha+1}{\alpha}},$$

then

$$g(x) = \int_0^x h(t)dt = (\lambda + x^{-\alpha})^{-\frac{1}{\alpha}}.$$

And if  $q = \frac{n-s}{n-p}p$ , then  $\alpha = \frac{q}{p} - 1 = \frac{p-s}{n-p}$ . The theorem is thus proved.  $\square$

*Remark 3.2.* As expected, the compactness of the embedding  $u \rightarrow \frac{u}{x^{s/q}}$  in Lemma 3.2.(2) above does not hold when  $q = p^*(s)$ . Indeed, let

$$f_k(x) = \left(\frac{1}{k}\right)^{\frac{n-p}{p(p-s)}} \left(\frac{1}{k} + |x|^{\frac{p-s}{p-1}}\right)^{\frac{p-n}{p-s}},$$

and set  $\|\nabla f_k(x)\|_p^p = A$  and  $\int_\Omega \frac{|f_k(x)|^{p^*(s)}}{|x|^s} dx = C$ . Let

$$h_k(x) = f_k(x) - \left(\frac{1}{k}\right)^{\frac{n-p}{p(p-s)}} \left(\frac{1}{k} + 1\right)^{\frac{p-n}{p-s}}$$

for  $|x| \leq 1$ , so that  $h_k(x) \in H_0^{1,p}(B)$  and  $\|h_k\|_{H_0^{1,p}(\Omega)} \rightarrow A^{\frac{1}{p}}$ . Hence  $\{h_k\}$  is bounded in  $H_0^{1,p}(\Omega)$  and  $\left\|\frac{h_k}{|x|^{s/p^*(s)}}\right\|_{L^{p^*(s)}} \rightarrow C^{1/p^*(s)}$ . Now,  $h_k(x) \rightarrow 0$  for  $|x| \neq 0$  and 0 is the only possible cluster point of  $\left\{\frac{h_k}{|x|^{s/p^*(s)}}\right\}$  in  $L^{p^*(s)}(\Omega)$ , which is impossible since  $C \neq 0$ .

4. THE COMPACTNESS LEMMAS

This section deals with the compactness properties of the functional

$$E_{\lambda,\mu}(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{\lambda}{r} \int_{\Omega} |u|^r dx - \frac{\mu}{q} \int_{\Omega} \frac{|u|^q}{|x|^s} dx.$$

We recall the following standard definition.

**Definition 4.1.** A  $C^1$ -functional  $E$  on Banach space  $X$  satisfies the Palais-Smale condition at the level  $c$  (in short  $(PS)_c$ ), if every sequence  $(u_n)_n$  satisfying  $\lim_n E(u_n) = c$  and  $\lim_n \|E'(u_n)\| = 0$  has a convergent subsequence.

Define the following function:

$$L(\mu, s) = \begin{cases} \frac{p-s}{p(n-s)} \left(\frac{\mu_s^{n-s}}{\mu^{n-p}}\right)^{\frac{1}{p-s}} & \text{if } s < p, \\ +\infty & \text{if } s = p \text{ and } \mu < \mu_p, \\ 0 & \text{if } s = p \text{ and } \mu \geq \mu_p. \end{cases}$$

**Theorem 4.1.** Assume  $0 \leq s \leq p < n$ ,  $p \leq q \leq p^*(s)$  and  $p \leq r \leq p^*$ .

- (1) If  $p \leq q < p^*(s)$  and  $r < p^*$ , then for any  $\lambda > 0$  and any  $\mu > 0$ , the functional  $E_{\lambda,\mu}$  satisfies  $(PS)_c$  for all  $c$ .
- (2) If  $p \leq q = p^*(s)$  and  $r < p^*$ , then for any  $\lambda > 0$  and any  $\mu > 0$ , the functional  $E_{\lambda,\mu}$  satisfies  $(PS)_c$  for all  $c < L(\mu, s)$ .
- (3) If  $p \leq q < p^*(s)$  and  $r = p^*$ , then for any  $\lambda > 0$  and any  $\mu > 0$ , the functional  $E_{\lambda,\mu}$  satisfies  $(PS)_c$  for all  $c < L(\lambda, 0) = \frac{1}{n} \left(\frac{\mu_0^n}{\lambda^{n-p}}\right)^{\frac{1}{p}}$ .

Note that statement 2 above yields that  $E_{\lambda,\mu}$  also satisfies  $(PS)_c$  for all  $c$  when  $p = q = q^*(s)$  (i.e., when  $s = p$ ) as long as  $\mu < \mu_p$ . This solves a problem in [13] and [25], where only a certain singular Palais-Smale condition is established. On the other hand, when  $\mu = 1$ ,  $p = 2$  and  $s = 0$ , we recover the (by now well known) restricted compactness properties that appears in Yamabe-type problems, [4].

We first recall a few known results.

**Lemma 4.1** ([25]). Let  $x, y \in \mathbf{R}^n$ , and let  $\langle \cdot, \cdot \rangle_e$  be the standard scalar product in  $\mathbf{R}^n$ . Then

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y \rangle_e \geq \begin{cases} C_p |x - y|^p, & \text{if } p \geq 2, \\ C_p \frac{|x-y|^2}{(|x|+|y|)^{2-p}}, & \text{if } 1 < p < 2. \end{cases}$$

The following result of Brezis and Lieb ([4]) will be useful in the sequel.

**Lemma 4.2.** Suppose  $f_n \rightarrow f$  a.e. and  $\|f_n\|_p \leq C < \infty$  for all  $n$  and for some  $0 < p < \infty$ . Then

$$\lim_{n \rightarrow \infty} \{\|f_n\|_p^p - \|f_n - f\|_p^p\} = \|f\|_p^p.$$

**Lemma 4.3.** Let  $(u_n)_n$  be a bounded sequence in  $H_0^{1,p}(\Omega)$  and let  $(q_n)_n$  be a sequence such that  $p < q_n \leq p^*(s)$ ,  $q_n \rightarrow p^*(s)$  as  $n \rightarrow \infty$ . Then there exists a subsequence (without loss of generality still denoted by  $(u_n)_n$ ) such that:

- (1)  $u_n \rightarrow u$  weakly in  $H_0^{1,p}(\Omega)$ .
- (2)  $u_n \rightarrow u$  in  $L^r(\Omega)$  if  $1 < r < p^* = \frac{np}{n-p}$ .
- (3)  $u_n \rightarrow u$  almost everywhere.
- (4)  $\frac{u_n}{x} \rightarrow \frac{u}{x}$  weakly in  $L^p(\Omega)$ .

(5) For any  $f \in H_0^{1,p}(\Omega)$ ,

$$\int_{\Omega} \frac{|u_n|^{q_n-2} u_n}{|x|^{s_n}} f \rightarrow \int_{\Omega} \frac{|u|^{p^*(s)-2} u}{|x|^s} f.$$

(6) If  $p \geq 2$ , then

$$\int_{\Omega} |u_n|^{q_n} \leq \int_{\Omega} |u|^{q_n} + \int_{\Omega} |u_n - u|^{q_n} + o(1).$$

(7)

$$\int_{\Omega} \frac{|u_n - u|^{p^*(s)}}{|x|^s} = \int_{\Omega} \frac{|u_n|^{p^*(s)}}{|x|^s} - \int_{\Omega} \frac{|u|^{p^*(s)}}{|x|^s} + o(1).$$

*Proof.* These are standard applications of the Hardy-Sobolev embedding theorem and the Brezis-Lieb result. We just give the proofs of (5) and (6). Without loss of generality, we may assume that  $q_n < p^*(s)$ . For (5), it is clear that

$$\frac{|u_n|^{q_n-2} u_n}{|x|^{s \frac{q_n-1}{q_n}}} \rightarrow \frac{|u|^{p^*(s)-2} u}{|x|^{s \frac{p^*(s)-1}{p^*(s)}}} \text{ a.e.,}$$

and that the integral

$$\begin{aligned} \int_{\Omega} \left| \frac{|u_n|^{q_n-1}}{|x|^{s(1-\frac{1}{p^*(s)})}} \right|^{\frac{p^*(s)}{p^*(s)-1}} &= \int_{\Omega} \frac{|u_n|^{p^*(s) \frac{q_n-1}{p^*(s)-1}}}{|x|^{s \frac{q_n-1}{p^*(s)-1}}} \cdot \frac{1}{|x|^{s(1-\frac{q_n-1}{p^*(s)-1})}} \\ &\leq \left( \int_{\Omega} \frac{|u_n|^{p^*(s)}}{|x|^s} \right)^{\frac{q_n-1}{p^*(s)-1}} \cdot \left( \int_{\Omega} \frac{1}{|x|^s} \right)^{\frac{p^*(s)-q_n}{p^*(s)-1}} \end{aligned}$$

is uniformly bounded in  $n$ . Since  $f/|x|^{\frac{s}{p^*(s)}} \in L^{p^*(s)}(\Omega)$  for any  $f \in H_0^{1,p}(\Omega)$ , the conclusion follows.

In order to prove (6), we need the following easy lemma.

**Calculus Lemma.** For every  $1 \leq q \leq 3$ , there exists a constant  $C$  (depending on  $q$ ) such that for  $\alpha, \beta \in \mathbf{R}$  we have

$$| |\alpha + \beta|^q - |\alpha|^q - |\beta|^q - q\alpha\beta(|\alpha|^{q-2} + |\beta|^{q-2}) | \leq \begin{cases} C|\alpha||\beta|^{q-1} & \text{if } |\alpha| \geq |\beta|, \\ C|\alpha|^{q-1}|\beta| & \text{if } |\alpha| \leq |\beta|. \end{cases}$$

For  $q \geq 3$ , there exists a constant  $C$  (depending on  $q$ ) such that for  $\alpha, \beta \in \mathbf{R}$  we have

$$| |\alpha + \beta|^q - |\alpha|^q - |\beta|^q - q\alpha\beta(|\alpha|^{q-2} + |\beta|^{q-2}) | \leq C(|\alpha|^{q-2}\beta^2 + \alpha^2|\beta|^{q-2}).$$

From this inequality, we can actually deduce the following more convenient result for any  $q \geq 1$ :

$$| |\alpha + \beta|^q - |\alpha|^q - |\beta|^q - q\alpha\beta(|\alpha|^{q-2} + |\beta|^{q-2}) | \leq 2C(|\alpha|^{q-1}\beta + \alpha|\beta|^{q-1}).$$

Now, back to the proof of (6). Let  $w_n = u_n - u$ ; then  $w_n \rightarrow 0$  weakly in  $H_0^{1,p}(\Omega)$ . By the above calculus lemma,

$$\frac{|u_n|^{q_n}}{|x|^s} = \frac{|w_n + u|^{q_n}}{|x|^s} \leq \frac{|w_n|^{q_n}}{|x|^s} + \frac{|u|^{q_n}}{|x|^s} + C_1 \frac{|u||w_n|^{q_n-1}}{|x|^s} + C_2 \frac{|w_n||u|^{q_n-1}}{|x|^s}.$$

In view of (5), we only need to show that

$$\lim_n \int_{\Omega} \frac{|w_n||u|^{q_n-1}}{|x|^s} = \lim_n \int_{\Omega} \frac{|w_n||u|^{q_n-2}}{|x|^{s(1-\frac{1}{p^*(s)})}} \cdot \frac{|u|}{|x|^{\frac{s}{p^*(s)}}} = 0.$$



For that, we check that

$$\begin{aligned} \int_{\Omega} \left( \frac{|w_n||u|^{q_n-2}}{|x|^{s(1-\frac{1}{p^*(s)})}} \right)^{\frac{p^*(s)}{p^*(s)-1}} &= \int_{\Omega} \frac{|w_n|^{\frac{p^*(s)}{p^*(s)-1}}}{|x|^{s\frac{1}{p^*(s)-1}}} \cdot \frac{|u|^{\frac{q_n-2}{p^*(s)-1}p^*(s)}}{|x|^{s\frac{q_n-2}{p^*(s)-1}}} \cdot \frac{1}{|x|^{s\frac{p^*(s)-q_n}{p^*(s)-1}}} \\ &\leq \left( \int_{\Omega} \frac{|u_n|^{p^*(s)}}{|x|^s} \right)^{\frac{1}{p^*(s)-1}} \left( \int_{\Omega} \frac{|u|^{p^*(s)}}{|x|^s} \right)^{\frac{q_n-2}{p^*(s)-1}} \left( \int_{\Omega} \frac{1}{|x|^s} \right)^{\frac{p^*(s)-q_n}{p^*(s)-1}}. \end{aligned}$$

Hence it is uniformly bounded in  $n$ , and the claim follows. □

**Lemma 4.4.** *Let  $E_n(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{\mu}{q_n} \int_{\Omega} \frac{|u|^{q_n}}{|x|^s} dx - \frac{\lambda}{r} \int_{\Omega} |u|^r dx$  ( $\lambda > 0, \mu > 0$ ), where  $q_n$  satisfy the conditions in the previous lemma and  $1 < p \leq r < p^*$ . Assume the sequence  $\{u_n\}$  satisfies  $E_n(u_n) \rightarrow c, E'_n(u_n) \rightarrow 0$ . Then, there exists a subsequence, still denoted by  $\{u_n\}$ , such that for some  $u \in H_0^{1,p}(\Omega)$ :*

- (1)  $u_n \rightarrow u$  weakly in  $u \in H_0^{1,p}(\Omega)$ .
- (2)  $\nabla u_n \rightarrow \nabla u$  a.e.
- (3)  $\int_{\Omega} |\nabla u_n - \nabla u|^p = \int_{\Omega} |\nabla u_n|^p - \int_{\Omega} |\nabla u|^p + o(1)$ .
- (4)  $|\nabla u_n|^{p-2} \nabla u_n \rightarrow |\nabla u|^{p-2} \nabla u$  weakly in  $[L^{\frac{p}{p-1}}(\Omega)]^n$ .

*Proof.* Since

$$\lim_n E_n(u_n) = c \text{ and } \lim_n E'_n(u_n) = 0,$$

and

$$\langle E'_n(u_n), u_n \rangle = \int_{\Omega} |\nabla u_n|^p - \mu \int_{\Omega} \frac{|u_n|^{q_n}}{|x|^s} - \lambda \int_{\Omega} |u_n|^r,$$

we have

$$\begin{aligned} o(1)(1 + \|u_n\|) + p|c| &\geq pE_n(u_n) - \langle E'_n(u_n), u_n \rangle \\ &= \begin{cases} \mu(1 - \frac{p}{q_n}) \int_{\Omega} \frac{|u|^{q_n}}{|x|^s} dx + \lambda(1 - \frac{p}{r}) \int_{\Omega} |u_n|^r dx, & r > p, \\ (1 - \frac{p}{q_n}) \int_{\Omega} \frac{|u|^{q_n}}{|x|^s} dx, & r = p. \end{cases} \end{aligned}$$

Since  $\Omega$  is bounded, we have

$$\int_{\Omega} |u_n|^p = \int_{\Omega} \frac{|u_n|^p}{|x|^{ps/q_n}} \cdot |x|^{ps/q_n} dx \leq M \left( \int_{\Omega} \frac{|u_n|^{q_n}}{|x|^s} \right)^{p/q_n},$$

and

$$\|\nabla u_n\|_p^p = pE_n(u_n) + \mu \frac{p}{q_n} \int_{\Omega} \frac{|u_n|^{q_n}}{|x|^s} dx + \lambda \frac{p}{r} \int_{\Omega} |u_n|^r dx.$$

We conclude that  $\{u_n\}$  is a bounded sequence in  $H_0^{1,p}(\Omega)$ .

We therefore can assume that  $\{u_n\}$  satisfies all of the conclusions in Lemma 4.3. Now we use a technique initiated by Boccardo and Murat and already used by Garcia and Peral in a related context.

Define the functions

$$\tau_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ ks/|s| & \text{if } |s| \geq k. \end{cases}$$

We may assume also that  $\tau_k(u_n - u) \rightarrow 0$  weakly in  $H_0^{1,p}(\Omega)$  for any fixed positive  $k$ , since  $\tau_k(u_n - u) \rightarrow 0$  a.e. and it is bounded. Then from the assumption we get

$$\begin{aligned} o(1) &= \langle E'_n(u_n) - (E'_\lambda)^'(u), \tau_k(u_n - u) \rangle + o(1) \\ &= \int_{\Omega} \langle |\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u, \nabla \tau_k(u_n - u) \rangle_e \\ &\quad - \lambda \int_{\Omega} \left( \frac{|u_n|^{q_n-2}}{|x|^s} u_n - \frac{|u|^{p^*(s)-2}}{|x|^s} u \right) \tau_k(u_n - u). \end{aligned}$$

Since

$$\frac{|u_n|^{q_n-2}}{|x|^s} u_n \rightarrow \frac{|u|^{p^*(s)-2}}{|x|^s} u$$

in the weak star topology of  $H^{-1,p'}(\Omega)$  (by Lemma 4.3), we have

$$\left| \int_{\Omega} \left( \frac{|u_n|^{q_n-2}}{|x|^s} u_n - \frac{|u|^{p^*(s)-2}}{|x|^s} u \right) \tau_k(u_n - u) \right| \leq Ck,$$

and

$$\limsup_{n \rightarrow \infty} \int_{\Omega} \langle |\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u, \nabla \tau_k(u_n - u) \rangle_e dx \leq Ck.$$

Let  $e_n(x) = \langle |\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u, \nabla \tau_k(u_n - u) \rangle_e$ ; then  $e_n(x) \geq 0$  by Lemma 3.1, and is uniformly bounded in  $L^1(\Omega)$ . Take  $0 < \theta < 1$  and split  $\Omega$  into

$$S_n^k = \{x \in \Omega \mid |u_n - u| \leq k\}, \quad G_n^k = \{x \in \Omega \mid |u_n - u| > k\}.$$

Then

$$\begin{aligned} \int_{\Omega} e_n^\theta dx &= \int_{S_n^k} e_n^\theta dx + \int_{G_n^k} e_n^\theta dx \\ &\leq \left( \int_{S_n^k} e_n dx \right)^\theta |S_n^k|^{1-\theta} + \left( \int_{G_n^k} e_n dx \right)^\theta |G_n^k|^{1-\theta}. \end{aligned}$$

Now, for fixed  $k$ ,  $|G_n^k| \rightarrow 0$  as  $n \rightarrow \infty$ , and from the uniform boundedness in  $L^1$  we get

$$\limsup_n \int_{\Omega} e_n^\theta dx \leq (Ck)^\theta |\Omega|^{1-\theta}.$$

Letting  $k \rightarrow 0$ , we get that  $e_n^\theta \rightarrow 0$  strongly in  $L^1$ . By Lemma 4.1,

$$\nabla u_n \rightarrow \nabla u \quad \text{in } L^q$$

for  $1 < q < p$ . By passing to a subsequence, we have

$$\nabla u_n \rightarrow \nabla u \quad \text{a.e.}$$

Thus (1) holds. As for (2), just apply Lemma 4.2. The proof of this lemma is thus complete.  $\square$

*Proof of Theorem 4.1.(1):* If  $p \leq q < p^*(s)$  and  $r < p^*$ , it is standard to show that the compactness of the Hardy-Sobolev embedding and of the Sobolev embedding imply that for any  $\lambda > 0$  and any  $\mu > 0$ , the functional  $E_{\lambda,\mu}$  satisfies  $(PS)_c$  for all  $c$ .  $\square$

*Proof of Theorem 4.1.(2):* Recall that

$$L(\mu, s) = \begin{cases} \frac{p-s}{p(n-s)} \left(\frac{\mu_s^{n-s}}{\mu^{n-p}}\right)^{\frac{1}{p-s}} & \text{if } s < p, \\ +\infty & \text{if } s = p \text{ and } \mu < \mu_s, \\ 0 & \text{if } s = p \text{ and } \mu \geq \mu_s, \end{cases}$$

and assume that  $p \leq q = p^*(s)$  and  $r < p^*$ . We need to show that

$$E_{\lambda,\mu}(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{\mu}{p^*(s)} \int_{\Omega} \frac{|u|^{p^*(s)}}{|x|^s} - \frac{\lambda}{r} \int_{\Omega} |u|^r dx$$

satisfies the Palais-Smale condition at any energy level less than  $L(\mu, s)$ .

For that, assume  $\{u_n\}$  is a sequence in  $H_0^{1,p}(\Omega)$  satisfying

$$E_{\lambda,\mu}(u_n) \rightarrow c < L(\mu, s) \text{ and } E'_{\lambda,\mu}(u_n) \rightarrow 0.$$

By Lemma 4.4, we may assume that  $\{u_n\}$  satisfies the conclusions of both Lemma 4.2 and Lemma 4.3. For any  $v \in C_0^\infty(\Omega)$ ,

$$\langle E'_{\lambda,\mu}(u_n), v \rangle = \int_{\Omega} (\langle |\nabla u_n|^{p-2} \nabla u_n, \nabla v \rangle - \lambda |u_n|^{r-2} u_n v - \mu \frac{|u_n|^{p^*(s)-2} u_n}{|x|^s} v) dx,$$

which converges as  $n \rightarrow \infty$  to

$$0 = \int_{\Omega} (\langle |\nabla u|^{p-2} \nabla u, \nabla v \rangle - \lambda |u|^{r-2} uv - \mu \frac{|u|^{p^*(s)-2} u}{|x|^s} v) dx = \langle E'_{\lambda,\mu}(u), v \rangle.$$

Hence  $u \in H_0^{1,p}(\Omega)$  is a weak solution of  $(P_\lambda, \mu)$ . Choosing  $v = u$ , we have

$$0 = \langle E'_{\lambda,\mu}(u), u \rangle = \int_{\Omega} (|\nabla u|^p - \lambda |u|^r - \mu \frac{|u|^{p^*(s)}}{|x|^s}) dx,$$

and thus

$$E_{\lambda,\mu}(u) = \lambda \left(\frac{1}{p} - \frac{1}{r}\right) \int_{\Omega} |u|^r + \mu \left(\frac{1}{p} - \frac{1}{p^*(s)}\right) \int_{\Omega} \frac{|u|^{p^*(s)}}{|x|^s} dx \geq 0.$$

By Lemmas 4.3 and 4.4, we have

$$E_{\lambda,\mu}(u_n) = E_{\lambda,\mu}(u) + E_{0,\mu}(u_n - u) + o(1)$$

and

$$\begin{aligned} o(1) = \langle E'_{\lambda,\mu}(u_n), u_n - u \rangle &= \langle E'_{\lambda,\mu}(u_n) - E'_{\lambda,\mu}(u), u_n - u \rangle \\ &= \int_{\Omega} (|\nabla u_n - \nabla u|^p - \mu \frac{|u_n - u|^{p^*(s)}}{|x|^s}) + o(1). \end{aligned}$$

If  $s = p = p^*(s)$  and  $\mu < \mu_p$ , then

$$o(1) = \int_{\Omega} (|\nabla u_n - \nabla u|^p - \mu \frac{|u_n - u|^p}{|x|^p}) + o(1) \leq (1 - \frac{\mu}{\mu_p}) \int_{\Omega} (|\nabla u_n - \nabla u|^p + o(1));$$

that is,  $u_n \rightarrow u$  strongly.

If  $s < p$  (i.e.,  $p < p^*(s)$ ), we have, for large  $n$ ,

$$\begin{aligned} E_{0,\mu}(u_n - u) &= E_{\lambda,\mu}(u_n) - E_{\lambda,\mu}(u) + o(1) \\ &\leq E_{\lambda,\mu}(u_n) + o(1) \leq c < L(\mu, s). \end{aligned}$$

Thus, for such  $n$ ,

$$\left(\frac{1}{p} - \frac{1}{p^*(s)}\right) \|\nabla u_n - \nabla u\|^p \leq c < \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} \left(\frac{1}{\mu}\right)^{\frac{n-p}{p-s}}.$$

By the Sobolev-Hardy inequality, we finally get

$$\begin{aligned}
 o(1) &= \int_{\Omega} (|\nabla u_n - \nabla u|^p - \mu \frac{|u_n - u|^{p^*(s)}}{|x|^s}) dx \\
 &\geq \int_{\Omega} |\nabla u_n - \nabla u|^p - \mu \mu_s^{-\frac{p^*(s)}{p}} \left( \int_{\Omega} |\nabla u_n - \nabla u|^p \right)^{\frac{p^*(s)}{p}} \\
 &= \left( \int_{\Omega} |\nabla u_n - \nabla u|^p \right) [1 - \mu \mu_s^{-\frac{p^*(s)}{p}} \left( \int_{\Omega} |\nabla u_n - \nabla u|^p \right)^{\frac{p^*(s)-p}{p}}] \\
 &\geq C \int_{\Omega} |\nabla u_n - \nabla u|^p dx.
 \end{aligned}$$

So again  $u_n \rightarrow u$  in  $H_0^{1,p}(\Omega)$  strongly.  $\square$

*Proof of Theorem 4.1.(3):* Suppose now that  $p \leq q < p^*(s)$  and  $r = p^*$ ; then we have compactness in the singular term but we will be dealing with a non-singular term involving the critical Sobolev exponent. We have again

$$\begin{aligned}
 E_0(u_n - u) &= E_{\lambda,\mu}(u_n) - E_{\lambda,\mu}(u) + o(1) \\
 &\leq E_{\lambda,\mu}(u_n) + o(1) \leq c < L(\lambda, 0) = \frac{1}{n} \mu_0^{\frac{n}{p}} \left( \frac{1}{\lambda} \right)^{\frac{n-p}{p}}
 \end{aligned}$$

Thus, for such  $n$ ,

$$\left( \frac{1}{p} - \frac{1}{p^*} \right) \|\nabla u_n - \nabla u\|^p \leq c < \frac{1}{n} \mu_0^{\frac{n}{p}} \left( \frac{1}{\lambda} \right)^{\frac{n-p}{p}},$$

so that this time we get, from the Sobolev inequality,

$$\begin{aligned}
 o(1) &= \langle E'_{\lambda,\mu}(u_n), u_n - u \rangle = \langle E'_{\lambda,\mu}(u_n) - E'_{\lambda,\mu}(u), u_n - u \rangle \\
 &= \int_{\Omega} (|\nabla u_n - \nabla u|^p - \lambda \int_{\Omega} |u_n - u|^{p^*}) + o(1) \\
 &= \left( \int_{\Omega} |\nabla u_n - \nabla u|^p \right) [1 - \lambda \mu_0^{-\frac{p^*}{p}} \left( \int_{\Omega} |\nabla u_n - \nabla u|^p \right)^{\frac{p^*}{p}-1}] + o(1) \\
 &\geq C \int_{\Omega} |\nabla u_n - \nabla u|^p dx.
 \end{aligned}$$

So again  $u_n \rightarrow u$  in  $H_0^{1,p}(\Omega)$  strongly.  $\square$

## 5. MIN-MAX PRINCIPLES AND DUAL SETS ASSOCIATED TO $E_{\lambda,\mu}$

For Banach spaces  $X$  and  $Y$ , we use  $C(X, Y)$  to denote the space of all continuous maps from  $X$  to  $Y$ .

**Definition 5.1.** Let  $X$  be a Banach space and  $B$  be a closed subset of  $X$ . We say that a class  $\mathcal{F}$  of compact subsets of  $X$  is a *homotopy-stable family with boundary  $B$*  provided that

- (1) every set in  $\mathcal{F}$  contains  $B$ , and
- (2) for any set  $A$  in  $\mathcal{F}$  and any  $\eta \in C([0, 1] \times X; X)$  satisfying  $\eta(t, x) = x$  for all  $(t, x)$  in  $(\{0\} \times X) \cup ([0, 1] \times B)$  we have that  $\eta(\{1\} \times A) \in \mathcal{F}$ .

We say that the class  $\mathcal{F}$  is  $Z_2$ -homotopy stable if all sets in  $\mathcal{F}$  are symmetric and if we only require stability under odd homotopies  $\eta$  (i.e.,  $\eta(t, -x) = -\eta(t, x)$ ).

We say that a closed set  $M$  is dual to the family  $\mathcal{F}$  if

$$M \cap B = \emptyset \text{ and } M \cap A \neq \emptyset \text{ for all } A \in \mathcal{F}.$$

We shall need the following weakened version of the Palais-Smale condition.

**Definition 5.2.** A  $C^1$ -functional  $E$  on Banach space  $X$  satisfies the Palais-Smale condition at level  $c$  and around the set  $M$  (in short,  $(PS)_{M,c}$ ), if every sequence  $(u_n)_n$  satisfying  $\lim_n E(u_n) = c$ ,  $\lim_n \|E'(u_n)\| = 0$  and  $\lim_n \text{dist}(u_n, M) = 0$  has a convergent subsequence.

The following theorem of Ghoussoub [17] will be frequently used in the sequel.

**Theorem 5.1.** Let  $E$  be a  $C^1$ -functional on  $X$  and consider a homotopy stable family  $\mathcal{F}$  of compact subsets of  $X$  with a closed boundary  $B$ . Let  $M$  be a dual set to  $\mathcal{F}$  such that

$$\inf_{x \in M} E(x) = c := c(E, \mathcal{F}) = \inf_{A \in \mathcal{F}} \max_{x \in A} E(x).$$

If  $E$  satisfies  $(PS)_{M,c}$ , then  $M \cap K_c \neq \emptyset$ , where  $K_c$  is the set of all critical points of  $E$  at level  $c$ .

If  $\mathcal{F}$  is only  $Z_2$ -homotopy stable, then the result still holds true as long as the functional  $E$  is even and the dual set  $M$  is symmetric.

Note that the above theorem includes the classical min-max principle which holds under the assumption that  $\sup_{x \in B} E(x) < c$ . It is enough to notice that in that case  $M = \{x \in X; E(x) \geq c\}$  is a dual set.

Consider again the functional

$$E_{\lambda,\mu}(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{\lambda}{r} \int_{\Omega} |u|^r dx - \frac{\mu}{q} \int_{\Omega} \frac{|u|^q}{|x|^s} dx.$$

Recall that we assume that  $1 < p < n$ ,  $0 \leq s \leq p$ ,  $0 \leq q \leq p^*(s) \equiv \frac{n-s}{n-p}p$  and that  $p \leq r \leq p^* \equiv \frac{np}{n-p}$ , so that  $E$  is a  $C^1$ -functional on the Sobolev space  $H_0^{1,p}(\Omega)$ .

**A first dual set:** Define the Mountain Pass class to be

$$\mathcal{F}_1 = \{\gamma \in C([0, 1]; H_0^{1,p}(\Omega)); \gamma(0) = 0, \gamma(1) \neq 0 \text{ and } E(\gamma(1)) \leq 0\},$$

which is clearly homotopy-stable with boundary  $B = \{E \leq 0\}$ . Let

$$M_1 = \{u \in H_0^{1,p}(\Omega); u \neq 0, \langle E'(u), u \rangle = 0\}.$$

The following Nehari-type duality property is by now standard.

**Theorem 5.2.** Assume  $p \leq q \leq p^*(s)$  and one of the following cases:

- (1)  $p = r$ ,  $p < q$  and  $0 < \lambda < \lambda_1$ ,  $\mu > 0$ .
- (2)  $p < r \leq p^*$ ,  $p = q$  and  $0 < \mu < \mu_{s,p}$ ,  $\lambda > 0$ .
- (3)  $p < r \leq p^*$ ,  $p < q$  and  $\mu > 0$ ,  $\lambda > 0$ .

The set  $M_1$  is then closed, is dual to  $\mathcal{F}_1$  and satisfies

$$\inf_{M_1} E_{\lambda,\mu} = c_1 := c(E_{\lambda,\mu}, \mathcal{F}_1)$$

*Proof.* By definition,

$$\langle E'_{\lambda,\mu}(u), u \rangle = \int_{\Omega} (|\nabla u|^p - \lambda|u|^r - \mu \frac{|u|^q}{|x|^s}) dx.$$

Note that  $B \cap M_1 = \emptyset$ , since for every  $u \in M_1$  we have

$$E_{\lambda,\mu}(u) = \lambda\left(\frac{1}{p} - \frac{1}{r}\right) \int_{\Omega} |u|^r dx + \mu\left(\frac{1}{p} - \frac{1}{q}\right) \int_{\Omega} \frac{|u|^q}{|x|^s} dx > 0,$$

under the assumption that either  $r$  or  $q$  is different from  $p$ . We also show that under this assumption, we have the estimate  $c_1 \leq \inf_{u \in M_1} E_{\lambda,\mu}(u)$ .

Let  $u \neq 0$  in  $M_1$ , and consider the straight path  $\gamma(t) = tu$ . We have

$$E_{\lambda,\mu}(tu) = \frac{t^p}{p} \int_{\Omega} |\nabla u|^p - \frac{\lambda t^r}{r} \int_{\Omega} |u|^r - \mu \frac{t^q}{q} \int_{\Omega} \frac{|u|^q}{|x|^s}.$$

Since  $\lim_{t \rightarrow \infty} E_{\lambda,\mu}(tu) = -\infty$ , we have that  $c_1 \leq \sup_{0 \leq t < \infty} E_{\lambda,\mu}(tu) = E_{\lambda,\mu}(t_0 u)$ . From

$$\frac{dE_{\lambda,\mu}(tu)}{dt} = t^{p-1} \int_{\Omega} |\nabla u|^p - \lambda t^{r-1} \int_{\Omega} |u|^r - \mu t^{q-1} \int_{\Omega} \frac{|u|^q}{|x|^s}$$

and  $\frac{dE_{\lambda,\mu}(tu)}{dt}(t_0) = 0$ , we get

$$\int_{\Omega} |\nabla u|^p = t_0^{r-p} \cdot \lambda \int_{\Omega} |u|^r + \mu t_0^{q-p} \int_{\Omega} \frac{|u|^q}{|x|^s}.$$

Since  $u \in M_1$ , we should have

$$\int_{\Omega} |\nabla u|^p = \lambda \int_{\Omega} |u|^r + \mu \int_{\Omega} \frac{|u|^q}{|x|^s}.$$

Thus,  $t_0$  must be equal to 1 as long as either  $r$  or  $q$  is distinct from  $p$ . This clearly shows that under any of the 3 conditions above, we have

$$c_1 \leq \inf_{u \in M_1} E_{\lambda,\mu}(u).$$

For the rest, we have to distinguish the 3 cases.

*Case (1).*  $2 \leq p = r$ ,  $p < q$  and  $0 < \lambda < \lambda_1$ .

To prove that  $M_1$  is closed, use the Sobolev-Hardy inequality and the definition of  $\lambda_1$  to find a constant  $c > 0$  such that

$$\begin{aligned} \langle E'_{\lambda,\mu}(u), u \rangle &= \left(1 - \frac{\lambda}{\lambda_1}\right) \|u\|_{H_0^{1,p}(\Omega)}^p - c \|u\|_{H_0^{1,p}(\Omega)}^q \\ &= \|u\|_{H_0^{1,p}(\Omega)}^p \left(1 - \frac{\lambda}{\lambda_1} - c \|u\|_{H_0^{1,p}(\Omega)}^{q-p}\right). \end{aligned}$$

Choose some  $\beta > 0$  such that if  $\|u\| < \beta$ , then  $1 - \frac{\lambda}{\lambda_1} - c \|u\|_{H_0^{1,p}(\Omega)}^{q-p} > 0$ . This means that we can find some constant  $\beta > 0$  such that for any  $u \in M_1$ , we have  $\|u\| \geq \beta$ . So  $M_1$  is closed.

To prove the intersection property, fix  $\gamma \in \mathcal{F}_1$  joining 0 to  $v$ , where  $v \neq 0$  and  $E_{\lambda,\mu}(v) \leq 0$ . Note that since  $\lambda < \lambda_1$ , we have  $\langle E'_{\lambda,\mu}(\gamma(t)), \gamma(t) \rangle > 0$  for  $t$  close to 0 (same proof as for the closedness of  $M_1$ ). On the other hand, since  $v \neq 0$ , we have

$$\langle E'_{\lambda,\mu}(v), v \rangle < p E_{\lambda,\mu}(v) \leq 0.$$

It follows from the intermediate value theorem that there exists  $t_0$  such that  $\gamma(t_0) \in M_1$ . This proves the duality, and consequently  $c_1 \geq \inf\{E_{\lambda,\mu}(u) : u \in M_1\}$ .

*Case (2).*  $1 < p < r \leq p^*$ ,  $p = q \leq p^*(s)$  and  $0 < \mu < \mu_{s,q}$ .

To prove that  $M_1$  is closed, use the Sobolev-Hardy inequality with its best constant  $\mu_{s,p}$  and the Sobolev inequality to get

$$\begin{aligned} \langle E'_{\lambda,\mu}(u), u \rangle &= \int_{\Omega} |\nabla u|^p - \lambda \int_{\Omega} |u|^r - \mu \int_{\Omega} \frac{|u|^q}{|x|^s} \\ &\geq \int_{\Omega} |\nabla u|^p - c \left( \int_{\Omega} |\nabla u|^p \right)^{\frac{r}{p}} - \frac{\mu}{\mu_{s,q}} \int_{\Omega} |\nabla u|^p \\ &= \left( 1 - \frac{\mu}{\mu_{s,q}} \right) \|u\|_{H_0^{1,p}(\Omega)}^p - c \|u\|_{H_0^{1,p}(\Omega)}^r \\ &= \|u\|_{H_0^{1,p}(\Omega)}^p \left( 1 - \frac{\mu}{\mu_{s,q}} - c \|u\|_{H_0^{1,p}(\Omega)}^{r-p} \right). \end{aligned}$$

Choose some  $\beta > 0$  such that if  $\|u\| < \beta$ , then  $1 - \frac{\mu}{\mu_{s,q}} - c \|u\|_{H_0^{1,p}(\Omega)}^{r-p} > 0$ . This means that we can find some constant  $\beta > 0$  such that for any  $u \in M_1$ , we have  $\|u\| \geq \beta$ . So  $M_1$  is closed.

To prove the intersection property, fix  $\gamma \in \mathcal{F}_1$  joining 0 to  $v$ , where  $v \neq 0$  and  $E_{\lambda,\mu}(v) \leq 0$ . Note that since  $\mu < \mu_{s,q}$ , we have  $\langle E'_{\lambda,\mu}(\gamma(t)), \gamma(t) \rangle > 0$  for  $t$  close to 0 (same proof as for the closedness of  $M_1$ ). On the other hand, since  $v \neq 0$ , we have  $\langle E'_{\lambda,\mu}(v), v \rangle < pE_{\lambda,\mu}(v) \leq 0$ . It follows from the intermediate value theorem that there exists  $t_0$  such that  $\gamma(t_0) \in M_1$ . This proves the duality, and consequently  $c_1 \geq \inf\{E_{\lambda,\mu}(u) : u \in M_1\}$ .

Case (3).  $2 \leq p < r \leq p^*$  and  $\lambda > 0$ .

To prove that  $M_1$  is closed, again use the Sobolev-Hardy inequality and the Sobolev embedding to find constants  $c' > 0$ ,  $c'' > 0$  such that

$$\begin{aligned} \langle E'_{\lambda,\mu}(u), u \rangle &\geq \|u\|^p - c' \lambda \|u\|^r - c'' \|u\|^q \\ &= \|u\|^p (1 - c' \lambda \|u\|^{r-p} - c'' \|u\|^{q-p}). \end{aligned}$$

Since both  $r$  and  $q$  are distinct from  $p$ , we may choose  $\gamma > 0$  such that for any  $u \in H_0^{p-1}(\Omega)$  with  $\|u\| < \gamma$ , we have  $1 - c_1 \lambda \|u\|^{r-p} - c_2 \|u\|^{q-p} > 0$ . This means that  $\|u\| \geq \gamma$  for any  $u \in M_1$ ; hence  $M_1$  is closed.

For the intersection property, consider any  $\gamma \in \mathcal{F}_1$  joining 0 and  $v$ . Since  $p < r$ , again the proof above of the closedness of  $M_1$  yields that  $\langle E_{\lambda,\mu}(\gamma(t)), \gamma(t) \rangle > 0$  for  $t$  close to 0. Also since  $v \neq 0$ , we have

$$\langle E'_{\lambda,\mu}(v), v \rangle < pE_{\lambda,\mu}(v) \neq 0.$$

Then again, by the intermediate value theorem, we conclude that there exists  $t_0$  such that  $\gamma(t_0) \in M_1$ . This proves the duality and the inequality

$$c_1 \geq \inf\{E_{\lambda,\mu}(u), u \in M_1\}.$$

□

**Another dual set:** Denote by  $S_\rho$  the sphere  $S_\rho = \{u \in H_0^{1,p}(\Omega); \|u\|_{H_0^{1,p}(\Omega)} = \rho\}$  and by  $\mathcal{H}$  the set

$$\mathcal{H} = \{h : H_0^{1,p}(\Omega) \rightarrow H_0^{1,p}(\Omega) \text{ an odd homeomorphism}\}.$$

Let  $\gamma_{Z_2}$  denote the Krasnoselskii genus, defined for every closed symmetric subset  $D$  of  $H_0^{1,p}(\Omega)$  as

$$\gamma_{Z_2}(D) = \inf\{n; \text{there exists an odd and continuous map } h : D \rightarrow \mathbf{R}^n \setminus \{0\}\},$$

and consider the class

$$\mathcal{F}_2 = \{A; A \text{ closed symmetric with } \gamma_{Z_2}(h(A) \cap S_\rho) \geq 2, \forall h \in \mathcal{H}\}.$$

It is easy to verify that  $\mathcal{F}_2$  is a  $Z_2$ -homotopy stable class. Let

$$c_2 = \inf_{A \in \mathcal{F}_2} \sup_A E_{\lambda, \mu}.$$

We shall now consider an appropriate dual set to  $\mathcal{F}_2$ . First, we recall a few facts about the following weighted eigenvalue problem ( $1 < p < \infty$ ):

$$(*) \quad \begin{cases} -\Delta_p u = \lambda b(x)|u|^{p-2}u, \\ u \in H_0^{1,p}(\Omega), \quad u \neq 0. \end{cases}$$

We will say that  $\lambda \in \mathbf{R}$  is the eigenvalue and  $u \in H_0^{1,p}(\Omega)$ ,  $u \neq 0$ , is the corresponding eigenfunction of the above problem if the equality

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx = \lambda \int_{\Omega} b(x)|u|^{p-2} u \varphi \, dx$$

holds for any  $\varphi \in H_0^{1,p}(\Omega)$ . The following lemma is well known.

**Lemma 5.1** ([25] [11]). *Assume  $b(x) \geq 0$ ,  $b(x) \in L^t(\Omega)$ , and  $|\{x \in \Omega : b(x) > 0\}| \neq 0$ , where  $t \geq 1$  if  $p > n$ ,  $t > 1$  if  $p = n$  and  $t > \frac{n}{p} > 1$  otherwise. Let  $\lambda_0 = \inf\{\int_{\Omega} |\nabla v|^p; \int_{\Omega} b(x)|v|^p = 1\}$ . Then:*

- (1)  $\lambda_0 > 0$  is the first eigenvalue of the problem (\*).
- (2)  $\lambda_0$  is simple, and there exists precisely one pair of normalized eigenfunctions corresponding to  $\lambda_0$  which do not change sign in  $\Omega$ . Here,  $v$  being normalized means that  $\int_{\Omega} b(x)|v|^p = 1$ .

We use the lemma to prove the following fact:

**Lemma 5.2.** *For  $2 \leq p \leq r < p^*$ ,  $p \leq q < p^*(s)$ ,  $\lambda > 0, \mu > 0$  and any  $u \in H_0^{1,p}(\Omega)$ ,  $u \neq 0$ , there exists a unique  $v = v(u) \in H_0^{1,p}(\Omega)$  such that*

- (a)  $\int_{\Omega} (\lambda|u|^{r-p} + \mu \frac{|u|^{q-p}}{|x|^s}) v^p = 1, v \geq 0;$
- (b)  $\|\nabla v\|_p^p = \inf\{\|\nabla \omega\|_p^p : \int_{\Omega} (\lambda|u|^{r-p} + \mu \frac{|u|^{q-p}}{|x|^s}) |\omega|^p = 1\}.$

Furthermore, the map  $u \rightarrow v(u)$  is continuous from  $L^r(\Omega) \rightarrow H_0^{1,p}(\Omega)$ .

*Remark 5.1.* It is quite unfortunate that the above lemma is not applicable –unless  $p = 2$ – whenever  $r = p^*$  or when  $q = p^*(s)$ . This will create additional complications in the search for a second solution of the critical problems.

*Proof.* Since  $\frac{np^*}{sp^*+(q-p)n} > \frac{n}{p}$ , choose  $\frac{n}{p} < t < \frac{np^*}{sp^*+(q-p)n}$ ; then  $st \frac{p^*}{p^*-(q-p)t} < n$ . Because

$$\int_{\Omega} \frac{|u|^{(q-p)t}}{|x|^{st}} \leq \left( \int_{\Omega} |u|^{p^*} \right)^{\frac{t(q-p)}{p^*}} \left( \int_{\Omega} \frac{1}{|x|^{st \cdot \frac{p^*}{p^*-(q-p)t}}} \right)^{\frac{p^*-(q-p)t}{p^*}},$$

we get that  $|u|^{q-p}/|x|^s \in L^t(\Omega)$ .

The functional  $\psi(u) = \|u\|_p^p = \int_{\Omega} |\nabla u|^p \, dx$  is clearly weakly lower semicontinuous and coercive. Moreover, the constraint set

$$C = \{\omega \in H : \int_{\Omega} (\lambda|u|^{r-p} + \mu \frac{|u|^{q-p}}{|x|^s}) |\omega|^p \, dx = 1\}$$



is weakly closed in  $H_0^{1,p}(\Omega)$  and  $\psi(\cdot)$  is bounded below on  $C$ . Therefore, by the direct methods of the calculus of variations (Struwe [26], p. 4), the infimum in (b) is achieved and this infimum is the first eigenvalue of (\*) and thus is simple. Any function where such an infimum is achieved is the eigenfunction corresponding to the first eigenvalue of (\*). By Lemma 5.1, it cannot change sign in  $\Omega$ . This gives the uniqueness of  $v(u)$  and therefore its continuity for non-zero  $u$ .  $\square$

Note that  $(\nu_1(u), v(u))$  corresponds to the first eigenpair of the (weighted) eigenvalue problem

$$(**) \quad \begin{cases} -\Delta_p v = \nu(\lambda|u|^{r-p} + \mu \frac{|u|^{q-p}}{|x|^s})|v|^{p-2}v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Now let

$$M_2 = M_1 \cap \{u \in H_0^{1,p}(\Omega); \int_{\Omega} (\lambda|u|^{r-p} + \mu \frac{|u|^{q-p}}{|x|^s})v(u)^{p-1}u = 0\}.$$

The following duality result was first noticed by G. Tarantello [28] in the case when  $s = 0$  and  $p = 2$ .

**Theorem 5.3.** *Assume  $p \leq q < p^*(s)$  and  $r < p^*$ . Then  $M_2$  is a closed set that is dual to  $\mathcal{F}_2$ , and*

$$\inf_{M_2} E_{\lambda,\mu} = c_2 := c(E_{\lambda,\mu}, \mathcal{F}_2)$$

as long as we are in one of the following cases:

- (1)  $p = r, p < q$  and  $0 < \lambda < \lambda_1, 0 < \mu$ .
- (2)  $p < r, p = q$  and  $0 < \mu < \mu_{s,p}, 0 < \lambda$ .
- (3)  $p < r, p < q$  and  $0 < \mu, 0 < \lambda$ .

*Proof.* In the 3 cases, we get from Theorem 5.2 (and its proof) that  $M_1$  is closed and that for any  $u \neq 0$ , there exists a unique  $t(u) > 0$  such that  $t(u)u \in M_1$ . Clearly,  $t(u) = t(|u|) = t(-u)$  and

$$E_{\lambda,\mu}(t(u)u) = \max_{t \geq 0} E_{\lambda,\mu}(tu).$$

The uniqueness of  $t(u)$  and its properties tell us that the map  $u \rightarrow t(u)$  is continuous on  $H_0^{1,p}(\Omega)$  and that the map  $u \rightarrow t(u)u$  defines an odd homeomorphism between  $S_\rho$  and  $M_1$  which gives that  $\gamma_{Z_2}(A \cap M_1) \geq 2$  for all  $A \in \mathcal{F}_2$ .

On the other hand, the map  $h : A \cap M_1 \rightarrow \mathbf{R}$  given by

$$h(u) = \int_{\Omega} (\lambda|u|^{r-p} + \mu \frac{|u|^{q-p}}{|x|^s})v(u)^{p-1}u dx$$

defines an odd and continuous map. Since  $\gamma_{Z_2}(h(A \cap M_1)) \geq 2$ , we get that  $0 \in h(A \cap M_1)$  which means that  $A \cap M_2 \neq \emptyset$  and  $M_2$  is dual to  $\mathcal{F}_2$ . In particular,  $c_2 \geq \inf_{u \in M_2} E_{\lambda,\mu}(u)$ .

To prove the reverse inequality, take  $u \in M_2$  and let  $v(u)$  be such that

$$\int_{\Omega} (\lambda|u|^{r-p} + \mu \frac{|u|^{q-p}}{|x|^s})v(u)^{p-1}u dx = 0.$$

Let  $\omega(u)$  be a minimizer for the problem:

$$\mu_2 = \inf \left\{ \psi(\omega); \omega \in H_0^{1,p}, \int_{\Omega} (\lambda|u|^{r-p} + \mu \frac{|u|^{q-p}}{|x|^s}) v(u)^{p-1} \omega = 0, \right. \\ \left. \int_{\Omega} (\lambda|u|^{r-p} + \mu \frac{|u|^{q-p}}{|x|^s}) |\omega|^p = 1 \right\}.$$

Since  $u \in M_1$ , we obtain

$$\mu_2 \leq \frac{\|\nabla u\|_p^p}{\int_{\Omega} (\lambda|u|^r + \mu \frac{|u|^q}{|x|^s})} = 1.$$

Define  $A = \text{span}\{v(u), \omega(u)\} \in \mathcal{F}_2$ . Then, clearly,

$$1 \geq \mu_2 \geq \frac{\|\nabla \omega\|_p^p}{\int_{\Omega} (\lambda|u|^{r-p} + \mu \frac{|u|^{q-p}}{|x|^s}) |\omega|^p}, \quad \forall \omega \in A, \omega \neq 0.$$

For  $\omega_0 \in A$  satisfying  $E_{\lambda,\mu}(\omega_0) = \sup_A E_{\lambda,\mu} \geq c_2$ , we have  $\omega_0 \neq 0$  and  $\omega_0 \in M_1$ . From the above inequality, we derive

$$\int_{\Omega} (\lambda|u|^{r-p} + \mu \frac{|u|^{q-p}}{|x|^s}) |\omega_0|^p \geq \|\nabla \omega_0\|_p^p.$$

This implies

$$\frac{1}{p} \int_{\Omega} (\lambda|u|^{r-p} + \mu \frac{|u|^{q-p}}{|x|^s}) (|\omega_0|^p - |u|^p) \geq \frac{\|\nabla \omega_0\|_p^p}{p} - \frac{\|\nabla u\|_p^p}{p}.$$

Applying the inequality (valid for  $t \geq p$  and  $x, y \in \mathbf{R}$ )

$$\frac{1}{t} (|x|^t - |y|^t) \geq \frac{1}{p} (|x|^p - |y|^p) |y|^{t-p}$$

with  $t = r$  (resp.  $t = q$ ), we conclude that

$$\frac{\lambda}{r} \int_{\Omega} |\omega_0|^r + \mu \frac{1}{q} \int_{\Omega} \frac{|\omega_0|^q}{|x|^s} - \frac{\lambda}{r} \int_{\Omega} |u|^r - \mu \frac{1}{q} \int_{\Omega} \frac{|u|^q}{|x|^s} \geq \frac{\|\nabla \omega_0\|_p^p}{p} - \frac{\|\nabla u\|_p^p}{p},$$

that is,

$$E_{\lambda,\mu}(u) \geq E_{\lambda,\mu}(\omega_0) \geq c_2.$$

This finishes the proof of the theorem.  $\square$

## 6. THE SOLUTIONS IN THE CASE OF AN HS-SUBCRITICAL SINGULAR TERM

In this section, we consider the problem

$$(P_{\lambda,\mu}) \quad \begin{cases} -\Delta_p u = \lambda|u|^{r-2}u + \frac{|u|^{q-2}}{|x|^s}u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where  $0 \leq s \leq p < n$ , in the presence of a subcritical singular term ( $1 < p \leq q < p^*(s)$ ) and a subcritical non-singular term ( $1 < p \leq r < p^*$ ).

**Theorem 6.1** (Hardy-Sobolev subcritical singular term). *Suppose  $1 < p \leq q < p^*(s)$  and  $r < p^*$ . Also assume one of the following conditions:*

- (1)  $p < q$ ,  $p \leq r$  and  $\lambda > 0$ ,  $\mu > 0$ .
- (2)  $p = q$ ,  $p < r$  and  $\lambda > 0$ ,  $\mu_{s,p} > \mu > 0$ .

Then the equation  $(P_{\lambda,\mu})$  has infinitely many solutions. Moreover, it has an everywhere positive solution  $u_1$  with minimal energy and a sign-changing solution  $u_2$  that satisfies

$$\int_{\Omega} (\lambda|u_2|^{r-p} + \mu \frac{|u_2|^{q-p}}{|x|^s}) v(u_2)^{p-1} u_2 = 0,$$

where  $v(u_2)$  is the first eigenvector of the (weighted) eigenvalue problem

$$\begin{cases} -\Delta_p v = \nu(\lambda|u_2|^{r-p} + \mu \frac{|u_2|^{q-p}}{|x|^s}) |v|^{p-2} v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

*Proof.* Now that under these conditions the functional  $E_{\lambda,\mu}$  satisfies  $(PS)_c$  for any  $c$ . It is now enough to apply Theorem 5.1 to  $\mathcal{F}_1$  and its dual set  $M_1$  (resp., to  $\mathcal{F}_2$  and its dual set  $M_2$ ) to get a solution  $u_1$  (resp.  $u_2$ ) which minimizes the energy functional on  $M_1$  (resp.  $M_2$ ).

To obtain other solutions, we need the following result of Rabinowitz ([17]).

**Lemma 6.1.** *Let  $E$  be an even  $C^1$ -functional satisfying the Palais-Smale condition on a Banach space  $X = Y \oplus Z$  with  $\dim(Y) < \infty$ . Assume  $E(0) = 0$ , as well as the following conditions:*

- (1) *There is  $\rho > 0$  such that  $\inf_{S_\rho(Z)} E \geq 0$ .*
- (2) *There exists an increasing sequence  $\{Y_n\}_n$  of finite dimensional subspaces of  $X$ , all containing  $Y$ , such that  $\lim_n \dim(Y_n) = \infty$  and for each  $n$ ,  $\sup_{S_{R_n}(Y_n)} E \leq 0$  for some  $R_n > \rho$ .*

Then  $E$  has an unbounded sequence of critical values.

We now show that the functional

$$E(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p - \frac{\mu}{q} \int_{\Omega} \frac{|u|^q}{|x|^s} - \frac{\lambda}{r} \int_{\Omega} |u|^r$$

satisfies the hypothesis of the lemma.

Without loss of generality, we assume that  $\Omega = (0, 1)^n$ . Let  $Y_k$  be the  $k$ -dimensional subspace of  $X = H_0^{1,p}(\Omega)$ , generated by the first  $k$  functions of the basis

$$\{(\sin k_1 \pi x_1, \dots, \sin k_n \pi x_n), k_i \in \mathbf{N}, i = 1, \dots, n\}.$$

Let  $Z_k$  denote the complement of  $Y_k$  in  $X$ , that is, the set generated by the base vectors not in  $Y_k$ . For any  $u \in Y_{k-1}^c$ , the topological complement of  $Y_{k-1}$ ,

$$\|u\|_p \leq C \|\nabla u\|_p / k^{\frac{1}{n}} \quad (\text{Peral [25]}).$$

*Claim 1.* For  $k$  sufficiently large, there exists  $\rho > 0$  such that  $E(u) \geq 1$  for all  $u \in Z_{k-1}$  with  $\|u\|_{H_0^{1,p}} = \rho$

*Proof of Claim 1.* We first consider the case where  $p = q, p < r$  and  $\mu < \mu_{s,p}$ :

$$E(u) \geq (1 - \frac{\mu}{\mu_{s,p}}) \frac{1}{p} \int_{\Omega} |\nabla u|^p - C \int_{\Omega} |u|^r.$$

By the Gagliardo-Nirenberg inequality,

$$\left(\int_{\Omega} |u|^r\right)^{\frac{1}{r}} \leq C_1 \left(\int_{\Omega} |\nabla u|^p\right)^{\frac{\alpha}{p}} \left(\int_{\Omega} |u|^p\right)^{\frac{1-\alpha}{p}}$$

with  $a = \frac{n}{p}(1 - \frac{2}{r})$ . Hence, for  $u \in \partial B_\rho \cap Y_{k-1}^c$ ,

$$\begin{aligned} E(u) &\geq C_2 \int_\Omega |\nabla u|^p - C_1 \left( \int_\Omega |\nabla u|^p \right)^{\frac{ra}{p}} \left( \int_\Omega |u|^p \right)^{\frac{r(1-a)}{p}} \\ &= \rho^p (C_2 - C_1 \rho^{ra} \left( \frac{C_1 \rho}{k^{\frac{1}{n}}} \right)^{r(1-a)} \rho^{-p}) \\ &= \rho^p (C_2 - C_3 \rho^{r-p} \frac{1}{k^{r(1-a)/n}}). \end{aligned}$$

Choosing  $\rho = \left( \frac{C_2}{2C_3} k^{\frac{r(1-a)}{n}} \right)^{\frac{1}{r-p}}$ , we get that  $E(u) \geq \frac{1}{2} C_2 \rho^p = C_4 k^{\frac{pr(1-a)}{n(r-p)}} \geq 1$ , for  $k$  large enough. This completes the proof of Claim 1 in the first case.

We turn to the case where  $p < q$ : Since  $q < p^*(s)$ , choose  $\varepsilon > 0$  such that  $q < \frac{n-s-\varepsilon}{n-p} p$ ; then

$$\int_\Omega \frac{|u|^q}{|x|^s} \leq C_0 \left( \int_\Omega |u|^{q \frac{n-s-\varepsilon}{n-s-\varepsilon}} \right)^{\frac{n-s-\varepsilon}{n-s-\varepsilon}}.$$

If  $q \frac{n-s-\varepsilon}{n-s-\varepsilon} \leq r$ , then

$$\int_\Omega |u|^{q \frac{n-s-\varepsilon}{n-s-\varepsilon}} \leq C_1 \left( \int_\Omega |u|^r \right)^{\frac{q \frac{n-s-\varepsilon}{n-s-\varepsilon}}{r}}.$$

Thus

$$\int_\Omega \frac{|u|^q}{|x|^s} \leq C_2 \left( \int_\Omega |u|^r \right)^{\frac{q}{r}}.$$

If  $q \frac{n-s-\varepsilon}{n-s-\varepsilon} \geq r$ , then

$$\int_\Omega |u|^r \leq C_3 \left( \int_\Omega |u|^{q \frac{n-s-\varepsilon}{n-s-\varepsilon}} \right)^{\frac{n-s-\varepsilon}{n-s-\varepsilon} \cdot \frac{r}{q}}.$$

Because of these relationships, we could combine the last two terms of the functional  $E$  together. In this sense, we may assume that

$$E(u) \geq \frac{1}{p} \int_\Omega |\nabla u|^p - C \int_\Omega |u|^r,$$

and the rest is as in case (1).  $\square$

Let  $Y = Y_k$  with the  $k$  chosen in Claim 1. We now show the following.

*Claim 2.* In both cases, there exist for each finite dimensional subspace  $Y_k \subset H_0^{1,p}(\Omega)$ , positive constants  $C_1, C_2$  (depending on  $Y_k$ ) such that

$$\sup_{u \in \partial B_R(Y_k)} E(u) \leq C_1 R^p - C_2 R^r.$$

Indeed, for any  $u \in H_0^{1,p}(\Omega)$  and any  $R > 0$ , we have

$$E(Ru) \leq \frac{R^p}{p} \|u\|_{H_0^{1,p}(\Omega)}^p - \frac{R^r}{r} \|u\|_r^r.$$

Since  $Y_k$  is a finite dimensional space, it is closed and the two norms  $\|\cdot\|_r$  and  $\|\cdot\|_{H_0^{1,p}(\Omega)}$  on  $Y_k$  are equivalent. This implies the Claim.

Now we can apply Lemma 6.1 to conclude that  $E_{\lambda,\mu}$  has an unbounded sequence of critical values. Theorem 6.1 is proved.  $\square$

7. THE SOLUTIONS IN THE CASE OF  
A HARDY-CRITICAL SINGULAR TERM

**Theorem 7.1** (Hardy-critical singular term). *Suppose  $1 < p = q = p^*(s)$  (i.e.,  $s = p$ ).*

- (1) *If  $p < r < p^*$  (high order non-singular term), then  $(P_{\lambda,\mu})$  has infinitely many solutions –at least one of them being positive– for any  $\lambda > 0$  and any  $0 < \mu < \mu_p$ .*
- (2) *If  $r = p^*$  (critical non-singular term) and  $\Omega$  is star-shaped, then  $(P_{\lambda,\mu})$  has no non-trivial solution for any  $\lambda > 0, \mu > 0$ .*

*Proof.* If  $r < p^*$ , then by Theorem 4.1.2 the functional  $E_{\lambda,\mu}$  satisfies  $(PS)_c$  for any  $c$  as long as  $\mu < \mu_p$ . Since  $p < r$ , the proof is the same as in Theorem 6.1.(2), while the second case of the theorem is covered in section 2. □

*Remark 7.1.* The case when  $p = q = r$  is really an eigenvalue problem. There are solutions for  $(P_{\lambda,\mu})$  as long as  $\lambda$  is an eigenvalue of the problem  $-\Delta_p u - \frac{\mu|u|^{p-2}u}{|x|^p} = \lambda|u|^{p-2}u$  in  $H_0^{1,p}(\Omega)$ .

If  $0 \leq \mu < \mu_p$ , one can show that there is an infinite number of eigenvalues for the above problem. Indeed, these correspond to the critical levels of the restriction of the functional

$$\tilde{E}(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{\mu}{p} \int_{\Omega} \frac{|u|^p}{|x|^p} dx$$

to the submanifold  $\{u; \int_{\Omega} |u|^p dx = 1\}$ . But a slight variation of Theorem 4.1.(2) shows that in this case  $\tilde{E}$  has  $(PS)_c$  for any  $c$ , and therefore a standard application of Ljusternik-Schnirelmann theory applied to the genus  $\gamma_{\mathbb{Z}_2}$  will yield the result.

8. A POSITIVE SOLUTION IN THE CASE OF  
A HARDY-SOBOLEV CRITICAL SINGULAR TERM

In this section, we consider the first solution for the problem  $(P_{\lambda,\mu})$  with the critical *Sobolev-Hardy* exponent.

**Theorem 8.1** (Hardy-Sobolev critical singular term). *Suppose  $1 < p < q = p^*(s)$  (i.e.,  $s < p$ ) in the equation:*

$$(P_{\lambda,\mu}) \quad \begin{cases} -\Delta_p u = \lambda|u|^{r-2}u + \mu \frac{|u|^{p^*(s)-2}u}{|x|^s} & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

- *If  $r < p^*$ , then  $(P_{\lambda,\mu})$  has a solution that is strictly positive everywhere on  $\Omega$ , under any one of the following conditions:*
  - (1)  $p = r < p^*$  and  $n \geq p^2$ ,  $0 < \lambda < \lambda_1$  and  $\mu > 0$ .
  - (2)  $p < r < p^*$ ,  $\lambda$  is large enough and  $\mu > 0$ .
  - (3)  $p < r < p^*$  and  $n > \frac{p(p-1)r+p^2}{p+(p-1)(r-p)}$ ,  $\lambda > 0, \mu > 0$ .
- *If  $r = p^*$  and  $\Omega$  is star-shaped, then  $(P_{\lambda,\mu})$  has no non-trivial solution for any  $\lambda > 0, \mu > 0$ .*

*Proof.* Note that the last case ( $r = p^*$ ) was covered in section 2. Now if  $r < p^*$ , then Theorem 5.2 asserts that any one of the 3 conditions yields that the set

$$M_1 = \{u \in H_0^{1,p}(\Omega); u \neq 0, \langle E'_{\lambda,\mu}(u), u \rangle = 0\},$$

is closed, that it is dual to the Mountain Pass class

$$\mathcal{F}_1 = \{\gamma \in C([0, 1]; H_0^{1,p}(\Omega)); \gamma(0) = 0, \gamma(1) \neq 0 \text{ and } E_{\lambda,\mu}(\gamma(1)) \leq 0\},$$

and that

$$\inf_{M_1} E_{\lambda,\mu} = c_1 := c(E_{\lambda,\mu}, \mathcal{F}_1).$$

On the other hand, Theorem 4.1.(2) yields that  $E_{\lambda,\mu}$  satisfies the  $(PS)_c$  condition for any

$$c < \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}}.$$

Therefore, we should be able to apply Theorem 5.1 and obtain our desired assertion, if only we can prove the following case.  $\square$

**Lemma 8.1.** *In any one of the above three cases, we have*

$$c_1 < \frac{p-s}{p(n-s)} \left( \frac{\mu_s^{n-s}}{\mu^{n-p}} \right)^{\frac{1}{p-s}}.$$

*Proof.* We may assume without loss of generality that  $\mu = 1$ . We first consider the following case:

*Case (1).*  $p < r$  and  $\lambda$  is large.

In order to estimate the energy level  $c_1$ , we consider the functions

$$g(t) = E_{\lambda,\mu}(tv_\varepsilon) = \frac{t^p}{p} \int_{\Omega} |\nabla v_\varepsilon|^p - \frac{t^{p^*(s)}}{p^*(s)} - \frac{\lambda t^r}{r} \int_{\Omega} |v_\varepsilon|^r$$

and

$$\bar{g}(t) = \frac{t^p}{p} \int_{\Omega} |\nabla v_\varepsilon|^p - \frac{t^{p^*(s)}}{p^*(s)},$$

where  $v_\varepsilon$  is the extremal function defined in the appendix. Note that  $\lim_{t \rightarrow \infty} g(t) = -\infty$  and  $g(t) > 0$  when  $t$  is close to 0, so that  $\sup_{t \geq 0} g(t)$  is attained for some  $t_\varepsilon > 0$ . From

$$0 = g'(t_\varepsilon) = t_\varepsilon^{p-1} \left( \int_{\Omega} |\nabla v_\varepsilon|^p - t_\varepsilon^{p^*(s)-p} - \lambda t_\varepsilon^{r-p} \int_{\Omega} |v_\varepsilon|^r \right)$$

we have

$$\int_{\Omega} |\nabla v_\varepsilon|^p = t_\varepsilon^{p^*(s)-p} + \lambda t_\varepsilon^{r-p} \int_{\Omega} |v_\varepsilon|^r > t_\varepsilon^{p^*(s)-p},$$

and therefore

$$t_\varepsilon \leq \left( \int_{\Omega} |\nabla v_\varepsilon|^p \right)^{\frac{1}{p^*(s)-p}}.$$

Thus

$$\int_{\Omega} |\nabla v_\varepsilon|^p \leq t_\varepsilon^{p^*(s)-p} + \lambda \left( \int_{\Omega} |\nabla v_\varepsilon|^p \right)^{\frac{r-p}{p^*(s)-p}} \left( \int_{\Omega} |v_\varepsilon|^r \right).$$

Choose  $\varepsilon$  small enough so that by (1) and (6) of Lemma 11.1 we have  $t_\varepsilon^{p^*(s)-p} \geq \frac{\mu_s}{2}$ . That is, we get a lower bound for  $t_\varepsilon$ , which is independent of  $\varepsilon$ .

Now we estimate  $g(t_\varepsilon)$ . The function  $\bar{g}(t)$  attains its maximum at

$$t = \left( \int_{\Omega} |\nabla v_\varepsilon|^p \right)^{\frac{1}{p^*(s)-p}}$$

and is increasing in the interval

$$\left[ 0, \left( \int_{\Omega} |\nabla v_\varepsilon|^p \right)^{\frac{1}{p^*(s)-p}} \right].$$

By Lemma 11.1, we have

$$\begin{aligned} g(t_\varepsilon) &= \bar{g}(t_\varepsilon) - \frac{\lambda}{r} t_\varepsilon^r \int_\Omega |v_\varepsilon|^r \\ &\leq \bar{g}\left(\left(\int_\Omega |\nabla v_\varepsilon|^p\right)^{\frac{1}{p^*(s)-p}}\right) - \frac{\lambda}{r} t_\varepsilon^r \int_\Omega |v_\varepsilon|^r \\ &= \left(\frac{1}{p} - \frac{1}{p^*(s)}\right) \left(\int_\Omega |\nabla v_\varepsilon|^p\right)^{\frac{p^*(s)}{p^*(s)-p}} - \frac{\lambda}{r} t_\varepsilon^r \int_\Omega |v_\varepsilon|^r \\ &\leq \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} + O\left(\varepsilon^{\frac{n-p}{p-s}}\right) - \frac{\lambda}{r} \left(\frac{\mu_s}{2}\right)^{\frac{r}{p^*(s)-p}} \int_\Omega |v_\varepsilon|^r. \end{aligned}$$

So for  $\lambda$  large enough, we have

$$g(t_\varepsilon) < \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}}.$$

Case (2).  $p < r < p^*$  and  $n > \frac{p(p-1)r+p^2}{p+(p-1)(r-p)}$ ,  $\lambda > 0$ .

Note first that the above condition is equivalent to  $\max\{p, p^* - \frac{p}{p-1}\} < r < p^*$  and  $\lambda > 0$ .

For any  $\lambda > 0$ , the above estimate on  $g(t_\varepsilon)$  and Lemma 6.1 yield

$$g(t_\varepsilon) \leq \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} + O\left(\varepsilon^{\frac{n-p}{p-s}}\right) - O\left(\varepsilon^{\frac{p-1}{p-s}\left(n - \frac{r(n-p)}{p}\right)}\right),$$

so that if  $r$  is chosen in such a way that

$$\frac{n-p}{p-s} > \frac{p-1}{p-s} \left(n - \frac{r(n-p)}{p}\right),$$

i.e.  $r > p^* - \frac{p}{p-1}$ , then

$$g(t_\varepsilon) < \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}}.$$

Case (3).  $p = r$ ,  $0 < \lambda < \lambda_1$  and  $n \geq p^2$ .

We still use the function  $g(t)$ . Since  $\lambda < \lambda_1$ , we have  $g(t) > 0$  when  $t$  is close to 0, and  $\lim_{t \rightarrow \infty} g(t) = -\infty$ . So again  $g(t)$  attains its maximum at some  $t_\varepsilon > 0$ . From

$$g'(t) = t^{p-1} \left( \int_\Omega |\nabla v_\varepsilon|^p - t^{p^*(s)-p} - \lambda \int_\Omega |v_\varepsilon|^p \right) = 0$$

we get

$$t_\varepsilon = \left( \int_\Omega |\nabla v_\varepsilon|^p - \lambda \int_\Omega |v_\varepsilon|^p \right)^{\frac{1}{p^*(s)-p}}.$$

Thus

$$\begin{aligned} g(t_\varepsilon) &= \left(\frac{1}{p} - \frac{1}{p^*(s)}\right) \left(\int_\Omega |\nabla v_\varepsilon|^p - \lambda \int_\Omega |v_\varepsilon|^p\right)^{\frac{p^*(s)}{p^*(s)-p}} \\ &= \begin{cases} \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} + O\left(\varepsilon^{\frac{n-p}{p-s}}\right) - O\left(\varepsilon^{\frac{p(p-1)}{p-s}}\right), & p > p^* \left(1 - \frac{1}{p}\right), \\ \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} + O\left(\varepsilon^{\frac{n-p}{p-s}}\right) - O\left(\varepsilon^{\frac{n-p}{p-s}} |\log \varepsilon|\right), & p = p^* \left(1 - \frac{1}{p}\right). \end{cases} \end{aligned}$$

In the case where  $p > p^*(1 - \frac{1}{p})$ , we require that  $\frac{n-p}{p-s} > \frac{p(p-1)}{p-s}$ , but both are equivalent to  $p^2 < n$ . In the case where  $p = p^*(1 - \frac{1}{p})$  we have  $p^2 = n$ , and the proof the lemma is now complete.  $\square$

*Remark 8.1.* (1) If  $p^2 \leq n$ , then  $p^* - \frac{p}{p-1} \leq p^*(1 - \frac{1}{p}) \leq p$ , so that  $r > p^* - \frac{p}{p-1}$  whenever  $p < r$ . In this case,  $r$  can take any value between  $p$  and  $p^*$ .  
 (2) If  $p^2 > n$ , then  $p < p^*(1 - \frac{1}{p}) < p^* - \frac{p}{p-1}$ , and then we require that  $p^* - \frac{p}{p-1} < p < p^*$ .

### 9. A SIGN CHANGING SOLUTION IN THE HARDY-SOBOLEV CRITICAL CASE

In this section, we extend the arguments of Tarantello [28] to establish the following.

**Theorem 9.1** (Hardy-Sobolev critical singular term). *Suppose  $2 \leq p < q = p^*(s)$  and  $r < p^*$  in the equation*

$$(P_{\lambda,\mu}) \quad \begin{cases} -\Delta_p u = \lambda |u|^{r-2} u + \mu \frac{|u|^{p^*(s)-2}}{|x|^s} u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

Assume any one of the following conditions:

- (1)  $p = r < p^*$ ,  $n > p^3 - p^2 + p$ ,  $\mu > 0$  and  $0 < \lambda < \lambda_1$ .
- (2)  $p < r < p^*$ ,  $\mu > 0$  and  $\lambda$  large enough.
- (3)  $p < r < p^*$ ,  $n > \frac{p(p-1)r+p}{1+(p-1)(r-p)}$  and  $\mu > 0, \lambda > 0$ .

Then  $(P_{\lambda,\mu})$  has also a changing-sign solution  $u$  that satisfies

$$\int_{\Omega} (\lambda |u|^{r-p} + \mu \frac{|u|^{p^*(s)-p}}{|x|^s}) v(u)^{p-1} u = 0,$$

where  $v(u)$  is the first eigenvector of the (weighted) eigenvalue problem

$$\begin{cases} -\Delta_p v = \nu (\lambda |u|^{r-p} + \mu \frac{|u|^{p^*(s)-p}}{|x|^s}) |v|^{p-2} v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

*Proof.* We assume without loss of generality that  $\mu = 1$ . Theorem 5.2 asserts that for any  $q < p^*(s)$ , any one of the 3 conditions yields that the closed set

$$M_2^q = M_1^q \cap \{u \in H_0^{1,p}(\Omega); \int_{\Omega} (\lambda |u|^{r-p} + \frac{|u|^{q-p}}{|x|^s}) v(u)^{p-1} u = 0\}.$$

is dual to the class  $\mathcal{F}_2$ , and that

$$\inf_{M_2^q} E_{\lambda,q} = c_{2,q} := c(E_{\lambda,q}, \mathcal{F}_2),$$

where for each  $q < q^*(s)$ , the sets  $M_1^q$  (resp.  $M_2^q$ ) denote the dual sets associated to the functional

$$E_{\lambda,q} := \frac{1}{p} \int_{\Omega} |\nabla u|^p - \frac{1}{q} \int_{\Omega} \frac{|u|^q}{|x|^s} - \frac{\lambda}{r} \int_{\Omega} |u|^r.$$

$E_{\lambda}$  will denote  $E_{\lambda,p^*(s)}$  and  $M_1$  (resp.  $M_2$ ) will denote  $M_1^{p^*(s)}$  (resp.  $M_2^{p^*(s)}$ ). Note that by Theorem 5.2,  $M_1$  is dual to  $\mathcal{F}_1$ , but the same cannot be said about  $M_2$  and  $\mathcal{F}_2$  unless  $p = 2$ . Therefore, to establish the theorem above, we shall resort to a limiting argument as  $q \rightarrow p^*(s)$ .



**Lemma 9.1.** *Under any one of the 3 conditions in Theorem 9.1, we have:*

- (1)  $c_{i,q} \rightarrow c_i$  ( $i = 1, 2$ ) as  $q \rightarrow p^*(s)$ .
- (2) *There exist  $\sigma > 0$  and  $\delta_0 > 0$  such that for  $0 < |q - p^*(s)| < \delta_0$ , we have  $c_{2,q} \leq c_{1,q} + \frac{1}{n}S^{\frac{n}{p}} - \sigma$ .*

*Proof.* (1) First we prove that  $(c_{1,q})_q$  and  $(c_{2,q})_q$  are uniformly bounded in  $q$ . We shall only show it for  $c_{2,q}$ . For any  $u \in M_2^q$ ,

$$\int_{\Omega} |\nabla u|^p - \int_{\Omega} \frac{|u|^q}{|x|^s} - \lambda \int_{\Omega} |u|^r = 0.$$

Thus

$$E_{\lambda,q}(u) = \lambda \left(\frac{1}{p} - \frac{1}{r}\right) \int_{\Omega} |u|^r + \left(\frac{1}{p} - \frac{1}{q}\right) \int_{\Omega} \frac{|u|^q}{|x|^s} dx \geq 0,$$

i.e.,  $c_{2,q} \geq \inf_{M_2^q} E_{\lambda,q} \geq 0$ . Since now

$$\begin{aligned} E_{\lambda,q}(u) &= \frac{1}{p} \int_{\Omega} |\nabla u|^p - \frac{1}{q} \int_{\Omega} \frac{|u|^q}{|x|^s} - \frac{\lambda}{r} \int_{\Omega} |u|^r \\ &\leq \frac{1}{p} \int_{\Omega} |\nabla u|^p - \frac{\lambda}{r} \int_{\Omega} |u|^r \equiv E(u), \end{aligned}$$

and for any  $u_0, v_0 \in H_0^{1,p}(\Omega)$ ,

$$\begin{aligned} &\lim_{\alpha \rightarrow \infty, \beta \rightarrow \infty} E(\alpha u_0 + \beta v_0) \\ &= \lim_{\alpha \rightarrow \infty, \beta \rightarrow \infty} \left( \frac{1}{p} \int_{\Omega} |\nabla(\alpha u_0 + \beta v_0)|^p - \frac{\lambda}{r} \int_{\Omega} |\alpha u_0 + \beta v_0|^r \right) = -\infty, \end{aligned}$$

$E(\alpha u_0 + \beta v_0)$  attains its maximum at some finite  $\alpha_0$  and  $\beta_0$ . This means

$$0 \leq c_{2,q} \leq E(\alpha_0 u_0 + \beta_0 v_0),$$

which is independent of  $q$ .

This implies the existence of constants  $C_1 > 0$  and  $C_2 > 0$  such that

$$C_1 \leq \int_{\Omega} \frac{|u_{1,q}|^q}{|x|^s} \leq C_2.$$

Similar estimates also hold for  $\|\nabla u_{1,q}\|_p$  and  $\|u_{1,q}\|_r$ . Notice that for every  $u \neq 0$ , there exist unique  $t_q(u) > 0$  and  $t(u) > 0$  such that

$$t(u)u \in M_1 \quad \text{and} \quad t_q(u)u \in M_1^q.$$

Furthermore,  $t_q(u) \rightarrow t(u)$  as  $q \rightarrow p^*(s)$ . Set  $s_q = t(u_{1,q})$  so that  $s_q u_{1,q} \in M_1$ . We have

$$\begin{aligned} c_1 &\leq E_{\lambda}(s_q u_{1,q}) \\ &= \frac{1}{p} \int_{\Omega} |\nabla s_q u_{1,q}|^p - \frac{\lambda}{r} \int_{\Omega} |s_q u_{1,q}|^r - \frac{1}{p^*(s)} \int_{\Omega} \frac{|s_q u_{1,q}|^{p^*(s)}}{|x|^s} \\ &= \left(\frac{1}{p} - \frac{1}{p^*(s)}\right) \int_{\Omega} |\nabla s_q u_{1,q}|^p + \lambda \left(\frac{1}{p^*(s)} - \frac{1}{r}\right) \int_{\Omega} |s_q u_{1,q}|^r. \end{aligned}$$

Since  $u_{1,q} \in M_1^q$ , we have

$$\begin{aligned} E_{\lambda,q}(u_{1,q}) &= \frac{1}{p} \int_{\Omega} |\nabla u_{1,q}|^p - \frac{\lambda}{r} \int_{\Omega} |u_{1,q}|^r - \frac{1}{q} \int_{\Omega} \frac{|u_{1,q}|^q}{|x|^s} \\ &= \left(\frac{1}{p} - \frac{1}{q}\right) \int_{\Omega} |\nabla u_{1,q}|^p + \lambda \left(\frac{1}{q} - \frac{1}{r}\right) \int_{\Omega} |u_{1,q}|^r. \end{aligned}$$

Thus

$$\begin{aligned} c_1 &\leq E_{\lambda}(s_q u_{1,q}) \\ &= \left(\frac{1}{p} - \frac{1}{q}\right) \int_{\Omega} s_q^p |\nabla u_{1,q}|^p + \left(\frac{1}{q} - \frac{1}{p^*(s)}\right) \int_{\Omega} s_q^p |\nabla u_{1,q}|^p \\ &\quad + \lambda \left(\frac{1}{p^*(s)} - \frac{1}{r}\right) \int_{\Omega} |s_q u_{1,q}|^r \\ &= \left(\frac{1}{p} - \frac{1}{q}\right) \int_{\Omega} |\nabla u_{1,q}|^p + \left(\frac{1}{p} - \frac{1}{q}\right) (s_q^p - 1) \int_{\Omega} |\nabla u_{1,q}|^p \\ &\quad + \left(\frac{1}{q} - \frac{1}{p^*(s)}\right) \int_{\Omega} s_q^p |\nabla u_{1,q}|^p + \lambda \left(\frac{1}{p^*(s)} - \frac{1}{r}\right) \int_{\Omega} |s_q u_{1,q}|^r \\ &= E_{\lambda,q}(u_{1,q}) + \left(\frac{1}{p} - \frac{1}{q}\right) (s_q^p - 1) \int_{\Omega} |\nabla u_{1,q}|^p \\ &\quad + \left(\frac{1}{q} - \frac{1}{p^*(s)}\right) \int_{\Omega} s_q^p |\nabla u_{1,q}|^p \\ &\quad + \lambda \left(\frac{1}{q} - \frac{1}{r}\right) (s_q^r - 1) \int_{\Omega} |u_{1,q}|^r + \lambda \left(\frac{1}{p^*(s)} - \frac{1}{q}\right) \int_{\Omega} s_q^r |u_{1,q}|^r. \end{aligned}$$

Note that  $s_q \rightarrow 1$  as  $q \rightarrow p^*(s)$ ; therefore

$$c_1 \leq c_{1,q} + o(1).$$

To obtain the reverse inequality, set  $t_q = t_q(u_1) > 0$ . Thus,  $t_q u_1 \in M_1^q$ ,  $t_q \rightarrow 1$  as  $q \rightarrow p^*(s)$ , and

$$\begin{aligned} c_{1,q} \leq E_{\lambda,q}(t_q u_1) &= E_{\lambda}(t_q u_1) + \frac{1}{p^*(s)} \int_{\Omega} \frac{|u|^{p^*(s)}}{|x|^s} - \frac{1}{q} \int_{\Omega} \frac{|u|^q}{|x|^s} \\ &= E_{\lambda}(u_1) + o(1) = c_1 + o(1). \end{aligned}$$

This completes part (1) of the lemma.

We prove part (2) by estimating  $\sup_{A_{\varepsilon}} E_{\lambda,q}$ , where  $A_{\varepsilon} = \text{span}\{u_1, v_{\varepsilon}\} \in \mathcal{F}_2$  and  $u_1$  is the first solution of the critical problem. For that, we need the smoothness of  $u_1$ ; but this cannot be guaranteed unless  $p = 2$ . However, an easy approximation argument, and the fact that  $\sup_{t \geq 0} E_{\lambda}(tu) \rightarrow \sup_{t \geq 0} E_{\lambda}(tu_1)$  as  $u \rightarrow u_1$  strongly, allow us to assume that  $u_1$  has the required smoothness.

Therefore we may suppose that  $u_1, \nabla u_1 \in L^{\infty}(\Omega)$ . We shall consider the case where  $r > p$  first.

*Case (1).* Assume  $p < r < p^*$  and  $\lambda > 0$ .

For  $\varepsilon > 0$  and  $q$  sufficiently close to  $p^*(s)$ , and by the calculus lemma,

$$\begin{aligned}
 E_{\lambda,q}(\alpha u_1 + \beta v_\varepsilon) &= \frac{1}{p} \int_{\Omega} |\nabla(\alpha u_1 + \beta v_\varepsilon)|^p - \frac{\lambda}{r} \int_{\Omega} |\alpha u_1 + \beta v_\varepsilon|^r - \frac{1}{q} \int_{\Omega} \frac{|\alpha u_1 + \beta v_\varepsilon|^q}{|x|^s} \\
 &\leq \frac{1}{p} \int_{\Omega} |\nabla(\alpha u_1)|^p - \frac{\lambda}{r} \int_{\Omega} |\alpha u_1|^r - \frac{1}{q} \int_{\Omega} \frac{|\alpha u_1|^q}{|x|^s} \\
 &\quad + \frac{1}{p} \int_{\Omega} |\nabla(\beta v_\varepsilon)|^p - \frac{\lambda}{r} \int_{\Omega} |\beta v_\varepsilon|^r - \frac{1}{q} \int_{\Omega} \frac{|\beta v_\varepsilon|^q}{|x|^s} \\
 &\quad + A_1 \left[ \int_{\Omega} |\nabla \alpha u_1| |\nabla \beta v_\varepsilon|^{p-1} + |\nabla \alpha u_1|^{p-1} |\nabla \beta v_\varepsilon| \right] \\
 &\quad + B_1 \left[ \int_{\Omega} |\alpha u_1| |\beta v_\varepsilon|^{r-1} + |\alpha u_1|^{r-1} |\beta v_\varepsilon| \right] \\
 &\quad + C_1 \left[ \int_{\Omega} |\alpha u_1| \frac{|\beta v_\varepsilon|^{q-1}}{|x|^s} + |\alpha u_1|^{q-1} \frac{|\beta v_\varepsilon|}{|x|^s} \right] \\
 &\leq \frac{1}{p} \int_{\Omega} |\nabla(\alpha u_1)|^p - \frac{\lambda}{r} \int_{\Omega} |\alpha u_1|^r - \frac{1}{q} \int_{\Omega} \frac{|\alpha u_1|^q}{|x|^s} \\
 &\quad + \frac{1}{p} \int_{\Omega} |\nabla(\beta v_\varepsilon)|^p - \frac{\lambda}{r} \int_{\Omega} |\beta v_\varepsilon|^r - \frac{1}{q} \int_{\Omega} \frac{|\beta v_\varepsilon|^q}{|x|^s} \\
 &\quad + A_2 (|\alpha|^p + |\beta|^p) \varepsilon^{\frac{n-p}{p(p-s)}} + B_2 (|\alpha|^r + |\beta|^r) \varepsilon^{\frac{p-1}{p-s} (n - \frac{(r-1)(n-p)}{p})} \\
 &\quad + C_2 (|\alpha|^q + |\beta|^q) \varepsilon^{\frac{n-p}{p(p-s)}}.
 \end{aligned}$$

Therefore, for  $\varepsilon$  sufficiently small,

$$\lim_{\alpha, \beta \rightarrow \infty} E_{\lambda,q}(\alpha u_1 + \beta v_\varepsilon) = -\infty.$$

So we may assume that  $\alpha$  and  $\beta$  are in a bounded set.

As in the study of the first solution, let us consider the function

$$g(t) = E_\lambda(t v_\varepsilon) = \frac{t^p}{p} \int_{\Omega} |\nabla v_\varepsilon|^p - \frac{t^{p^*(s)}}{p^*(s)} - \frac{\lambda t^r}{r} \int_{\Omega} |v_\varepsilon|^r$$

again. As in the previous section, we have

$$g(t_\varepsilon) \leq \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} + C \varepsilon^{\frac{n-p}{p-s}} - \frac{\lambda}{r} \left( \frac{\mu_s}{2} \right)^{\frac{r}{p^*(s)-p}} \int_{\Omega} |v_\varepsilon|^r.$$

If now  $r > p^* - \frac{1}{p-1} > p^* - 1$ , we have

$$\begin{aligned}
E_{\lambda,q}(\alpha u_1 + \beta v_\varepsilon) &\leq E_{\lambda,q}(\alpha u_1) + E_\lambda(\beta v_\varepsilon) + \frac{|\beta|^{p^*(s)}}{p^*(s)} - \frac{|\beta|^q}{q} \int_\Omega \frac{|v_\varepsilon|^q}{|x|^s} \\
&\quad + A_3 \varepsilon^{\frac{n-p}{p(p-s)}} + B_3 \varepsilon^{\frac{p-1}{p-s} (n - \frac{(r-1)(n-p)}{p})} \\
&\leq c_{1,q} + \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} + C \varepsilon^{\frac{n-p}{p-s}} + \frac{|\beta|^{p^*(s)}}{p^*(s)} \\
&\quad - \frac{|\beta|^q}{q} \int_\Omega \frac{|v_\varepsilon|^q}{|x|^s} - \frac{\lambda}{r} \left(\frac{\mu_s}{2}\right)^{\frac{r}{p^*(s)-p}} \int_\Omega |v_\varepsilon|^r \\
&\quad + A_3 \varepsilon^{\frac{n-p}{p(p-s)}} + B_3 \varepsilon^{\frac{p-1}{p-s} (n - \frac{(r-1)(n-p)}{p})} \\
&\leq c_{1,q} + \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} + C \varepsilon^{\frac{n-p}{p-s}} + A_3 \varepsilon^{\frac{n-p}{p(p-s)}} \\
&\quad + B_3 \varepsilon^{\frac{p-1}{p-s} (n - \frac{(r-1)(n-p)}{p})} - C_4 \varepsilon^{\frac{p-1}{p-s} (n - \frac{r(n-p)}{p})} \\
&\quad + \frac{|\beta|^{p^*(s)}}{p^*(s)} - \frac{|\beta|^q}{q} \int_\Omega \frac{|v_\varepsilon|^q}{|x|^s}.
\end{aligned}$$

Choose  $\varepsilon$  small enough so that

$$C \varepsilon^{\frac{n-p}{p-s}} + A_3 \varepsilon^{\frac{n-p}{p(p-s)}} + B_3 \varepsilon^{\frac{p-1}{p-s} (n - \frac{(r-1)(n-p)}{p})} - C_4 \varepsilon^{\frac{p-1}{p-s} (n - \frac{r(n-p)}{p})} \leq -2\sigma$$

for some constant  $\sigma > 0$ . Now choose  $\delta_0 > 0$  small enough so that

$$\frac{|\beta|^{p^*(s)}}{p^*(s)} - \frac{|\beta|^q}{q} \int_\Omega \frac{|v_\varepsilon|^q}{|x|^s} < \sigma \quad \text{for } 0 < |q - p^*(s)| < \delta_0.$$

Thus the case when  $r > p$  is established.

*Case (2).*  $r = p$ ,  $p^3 - p^2 + p < n$  and  $0 < \lambda < \lambda_1$ .

The assumption  $p^3 - p^2 + p < n$  implies that  $p^2 < n$ ,  $p > p^*(1 - \frac{1}{p})$  and  $p-1 < p^*(1 - \frac{1}{p})$ . We assume that  $\alpha$  and  $\beta$  are in a bounded set, and we estimate  $E_\lambda(\alpha u_1 + \beta v_\varepsilon)$ . Again let

$$g(t) = E_\lambda(t v_\varepsilon) = \frac{t^p}{p} \int_\Omega |\nabla v_\varepsilon|^p - \frac{t^{p^*(s)}}{p^*(s)} - \frac{\lambda t^p}{p} \int_\Omega |v_\varepsilon|^p;$$

then the maximum  $g(t_\varepsilon)$  of  $g(t)$  satisfies

$$g(t_\varepsilon) = \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} + O(\varepsilon^{\frac{n-p}{p-s}}) - O(\varepsilon^{\frac{p(p-1)}{p-s}}).$$

Thus we have

$$\begin{aligned}
 & E_{\lambda,q}(\alpha u_1 + \beta v_\varepsilon) \\
 &= \frac{1}{p} \int_{\Omega} |\nabla(\alpha u_1 + \beta v_\varepsilon)|^p - \frac{\lambda}{p} \int_{\Omega} |\alpha u_1 + \beta v_\varepsilon|^p - \frac{1}{q} \int_{\Omega} \frac{|\alpha u_1 + \beta v_\varepsilon|^q}{|x|^s} \\
 &\leq \frac{1}{p} \int_{\Omega} |\nabla(\alpha u_1)|^p - \frac{\lambda}{p} \int_{\Omega} |\alpha u_1|^p - \frac{1}{q} \int_{\Omega} \frac{|\alpha u_1|^q}{|x|^s} \\
 &\quad + \frac{1}{p} \int_{\Omega} |\nabla(\beta v_\varepsilon)|^p - \frac{\lambda}{p} \int_{\Omega} |\beta v_\varepsilon|^p - \frac{1}{q} \int_{\Omega} \frac{|\beta v_\varepsilon|^q}{|x|^s} \\
 &\quad + A_1 \left[ \int_{\Omega} |\nabla \alpha u_1| |\nabla \beta v_\varepsilon|^{p-1} + |\nabla \alpha u_1|^{p-1} |\nabla \beta v_\varepsilon| \right] \\
 &\quad + B_1 \left[ \int_{\Omega} |\alpha u_1| |\beta v_\varepsilon|^{p-1} + |\alpha u_1|^{p-1} |\beta v_\varepsilon| \right] \\
 &\quad + C_1 \left[ \int_{\Omega} |\alpha u_1| \frac{|\beta v_\varepsilon|^{q-1}}{|x|^s} + |\alpha u_1|^{q-1} \frac{|\beta v_\varepsilon|}{|x|^s} \right] \\
 &\leq \frac{1}{p} \int_{\Omega} |\nabla(\alpha u_1)|^p - \frac{\lambda}{p} \int_{\Omega} |\alpha u_1|^p - \frac{1}{q} \int_{\Omega} \frac{|\alpha u_1|^q}{|x|^s} \\
 &\quad + \frac{1}{p} \int_{\Omega} |\nabla(\beta v_\varepsilon)|^p - \frac{\lambda}{p} \int_{\Omega} |\beta v_\varepsilon|^p - \frac{1}{q} \int_{\Omega} \frac{|\beta v_\varepsilon|^q}{|x|^s} \\
 &\quad + A_2 (|\alpha|^p + |\beta|^p) \varepsilon^{\frac{n-p}{p(p-s)}} + B_2 (|\alpha|^p + |\beta|^p) \varepsilon^{\frac{(n-p)(p-1)}{p(p-s)}} \\
 &\quad + C_2 (|\alpha|^q + |\beta|^q) \varepsilon^{\frac{n-p}{p(p-s)}} \\
 &\leq E_{\lambda,q}(\alpha u_1) + E_{\lambda}(\beta v_\varepsilon) + \frac{|\beta|^{p^*(s)}}{p^*(s)} - \frac{|\beta|^q}{q} \int_{\Omega} \frac{|v_\varepsilon|^q}{|x|^s} \\
 &\quad + A_3 \varepsilon^{\frac{n-p}{p(p-s)}} + B_3 \varepsilon^{\frac{(n-p)(p-1)}{p(p-s)}} \\
 &\leq c_{1,q} + \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} + C \varepsilon^{\frac{n-p}{p-s}} + \frac{|\beta|^{p^*(s)}}{p^*(s)} - \frac{|\beta|^q}{q} \int_{\Omega} \frac{|v_\varepsilon|^q}{|x|^s} - O\left(\varepsilon^{\frac{p(p-1)}{p-s}}\right) \\
 &\quad + A_3 \varepsilon^{\frac{n-p}{p(n-s)}} + B_3 \varepsilon^{\frac{(n-p)(p-1)}{p(p-s)}} \\
 &\leq c_{1,q} + \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} + C \varepsilon^{\frac{n-p}{p-s}} + A_3 \varepsilon^{\frac{n-p}{p(p-s)}} + B_3 \varepsilon^{\frac{(n-p)(p-1)}{p(p-s)}} \\
 &\quad - C_4 \varepsilon^{\frac{p(p-1)}{p-s}} + \frac{|\beta|^{p^*(s)}}{p^*(s)} - \frac{|\beta|^q}{q} \int_{\Omega} \frac{|v_\varepsilon|^q}{|x|^s}.
 \end{aligned}$$

Since  $p^3 - p^2 + p < n$ , we may choose  $\varepsilon$  small enough so that

$$C \varepsilon^{\frac{n-p}{p-s}} + A_3 \varepsilon^{\frac{n-p}{p(p-s)}} + B_3 \varepsilon^{\frac{(n-p)(p-1)}{p(p-s)}} - C_4 \varepsilon^{\frac{p(p-1)}{p-s}} \leq -2\sigma$$

for some constant  $\sigma > 0$ . Now choose  $\delta_0 > 0$  small enough so that

$$\frac{|\beta|^{p^*(s)}}{p^*(s)} - \frac{|\beta|^q}{q} \int_{\Omega} \frac{|v_\varepsilon|^q}{|x|^s} < \sigma \text{ for } 0 < |q - p^*(s)| < \delta_0.$$

The proof of the lemma is now complete. □

*Proof of Theorem 9.1.* In order to get the second solution, we shall consider the second solutions  $u_{2,q}$  of the problems corresponding to  $q < p^*(s)$  and we will find a

limit as  $q \rightarrow p^*(s)$ . The location of  $u_{2,q}$  on the dual sets  $M_2^q$  will be crucial for the compactness.

Since  $c_{2,q}$  is bounded uniformly in  $q$ , there is  $K > 0$  such that

$$\|\nabla u_{2,q}\|_p \leq K \text{ whenever } 0 < |q - p^*(s)| < \delta_0.$$

For  $x \in \Omega$ , define  $(u_{2,q})^+(x) = \max\{u_{2,q}(x), 0\}$  and  $(u_{2,q})^-(x) = \max\{-u_{2,q}(x), 0\}$ . Since  $u_{2,q} \in M_2^q$ , both  $(u_{2,q})^+$  and  $(u_{2,q})^-$  are non-zero and belong to  $H_0^{1,p}(\Omega)$ . In addition,

$$\|\nabla(u_{2,q})^\pm\| \leq K \text{ whenever } 0 < |q - p^*(s)| < \delta_0.$$

Thus, we can find  $q_n$  such that  $q_n \rightarrow p^*(s)$  as  $n \rightarrow +\infty$ ,  $u^+, u^- \in H_0^{1,p}$  and

$$(u_{2,q_n})^\pm \rightharpoonup u^\pm \text{ weakly in } H_0^{1,p} \text{ as } n \rightarrow +\infty.$$

We claim that  $u^+ \neq 0$  and  $u^- \neq 0$ . To shorten notation, set  $u_n^\pm = (u_{2,q_n})^\pm$ ,  $c_{1,n} = c_{1,q_n}$ ,  $E_n = E_{\lambda,q_n}$  and  $\Gamma_n = M_1^{q_n}$ . Since  $u_n$  is the solution of the corresponding sub-critical problem, we have that  $u_n^\pm \in \Gamma_n$ . In particular,

$$E_n(u_n^\pm) \geq c_{1,n}.$$

From Lemma 9.1, we also know that

$$E_n(u_n^+) + E_n(u_n^-) = E_n(u_n) = c_{2,q_n} \leq c_{1,n} + \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} - \sigma$$

for  $n$  large. Necessarily,

$$E_n(u_n^\pm) \leq \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} - \sigma$$

for  $n$  large. From the fact that  $u_n^\pm \in \Gamma_n$  and  $c_1^n \rightarrow c_1 > 0$ , we derive

$$K_1 \leq \int_\Omega \frac{|u_n^\pm|^{q_n}}{|x|^s} \leq K_2$$

with suitable positive constants  $K_1$  and  $K_2$ .

Arguing by contradiction, assume, for example, that  $u^+ = 0$ . From the above and the fact that  $u_n^\pm \in \Gamma_n$ , we obtain

$$\frac{1}{p} \|\nabla u_n^+\|_p^p - \frac{1}{q_n} \int_\Omega \frac{|u_n^+|^{q_n}}{|x|^s} \leq \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} - \sigma + o(1)$$

and

$$\|\nabla u_n^+\|_p^p - \int_\Omega \frac{|u_n^+|^{q_n}}{|x|^s} = o(1).$$

Consequently,

$$\begin{aligned} & \mu_s \left( \int_\Omega \frac{|u_n^+|^{p^*(s)}}{|x|^s} \right)^{\frac{p}{p^*(s)}} \\ & \leq \|\nabla u_n^+\|_p^p = \int_\Omega \frac{|u_n^+|^{q_n}}{|x|^s} + o(1) = \int_\Omega \frac{|u_n^+|^{q_n}}{|x|^{\frac{q_n}{p^*(s)}s}} \cdot \frac{1}{|x|^{s(1-\frac{q_n}{p^*(s)})}} \\ & \leq \left( \int_\Omega \frac{|u_n^+|^{q_n}}{|x|^s} \right)^{\frac{q_n}{p^*(s)}} \left( \int_\Omega \frac{1}{|x|^s} \right)^{\frac{p^*(s)-q_n}{p^*(s)}} + o(1). \end{aligned}$$

Since  $\int_\Omega \frac{|u_n^+|^{q_n}}{|x|^s}$  is bounded away from zero, we conclude that

$$\left( \int_\Omega \frac{|u_n^+|^{q_n}}{|x|^s} \right)^{\frac{q_n-p}{p^*(s)}} \geq \left( \int_\Omega \frac{1}{|x|^s} \right)^{\frac{q_n-p^*(s)}{p^*(s)}} \mu_s + o(1).$$

That is,

$$\int_{\Omega} \frac{|u_n^+|^{q_n}}{|x|^s} \geq \mu_s^{\frac{n-s}{p-s}} + o(1).$$

Thus, we have

$$\begin{aligned} \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} + o(1) &\leq \frac{p-s}{p(n-s)} \int_{\Omega} \frac{|u_n^+|^{q_n}}{|x|^s} \\ &= \frac{1}{p} \|\nabla u_n^+\|_p^p - \frac{1}{q_n} \int_{\Omega} \frac{|u_n^+|^{q_n}}{|x|^s} + o(1) \leq \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} - \sigma + o(1). \end{aligned}$$

This is a contradiction, and we conclude that  $u^+ \neq 0$ . Similarly,  $u^- \neq 0$ .

Set  $u = u^+ - u^-$ ; that is,  $u$  changes sign in  $\Omega$  and

$$u_n := u_{q_n} \rightharpoonup u \text{ weakly in } H_0^{1,p}(\Omega).$$

So,  $\langle E'_\lambda(u), w \rangle = 0$  for any  $w \in H_0^{1,p}(\Omega)$ , i.e.  $u$  is a weak solution of  $(P_\lambda)$ . Now, we prove that a subsequence of  $\{u_n\}$  converges to  $u$  strongly in  $H_0^{1,p}(\Omega)$  and conclude that  $u$  is a solution of  $(P_{\lambda,p^*(s)})$  that is located on  $M_2$ .

Since  $\{E(u_n)\}$  is bounded and  $E'_n(u_n) \rightarrow 0$ , we may assume that the conclusions of Lemma 4.4 hold for the sequence  $(u_n)_n$ .

Note that  $u \in M_1$ ; hence  $E(u) \geq c_1$ . Set  $u_n = u + w_n$ , with  $w_n \rightharpoonup 0$  weakly in  $H_0^{1,p}$ . We have

$$\begin{aligned} c_{1,n} + \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} - \sigma &\geq E_n(u + w_n) \\ &\geq \frac{1}{p} \|\nabla u\|_p^p - \frac{1}{q_n} \int_{\Omega} \frac{|u|^{q_n}}{|x|^s} - \frac{\lambda}{r} \|u\|_r^r + \frac{1}{p} \|\nabla w_n\|_p^p - \frac{1}{q_n} \int_{\Omega} \frac{|w_n|^{q_n}}{|x|^s} + o(1) \\ &\geq c_1 + \frac{1}{p} \|\nabla w_n\|_p^p - \frac{1}{q_n} \int_{\Omega} \frac{|w_n|^{q_n}}{|x|^s} + o(1). \end{aligned}$$

Since  $|c_{1,n} - c_1| = o(1)$ , we derive

$$\frac{1}{p} \|\nabla w_n\|_p^p - \frac{1}{q_n} \int_{\Omega} \frac{|w_n|^{q_n}}{|x|^s} \leq \frac{p-s}{p(n-s)} \mu_s^{\frac{n-s}{p-s}} - \sigma + o(1).$$

Furthermore,

$$0 = \langle E'_n(u_n), u_n \rangle = \langle E'(u), u \rangle + \|\nabla w_n\|_p^p - \int_{\Omega} \frac{|w_n|^{q_n}}{|x|^s} + o(1);$$

i.e.

$$\|\nabla w_n\|_p^p - \int_{\Omega} \frac{|w_n|^{q_n}}{|x|^s} = o(1).$$

The last two relations show that the sequence  $\|\nabla w_n\|_p$  cannot be bounded away from zero, and therefore a subsequence of  $\{w_n\}$  converges strongly to zero.  $\square$

### 10. SOBOLEV CRITICAL NON-SINGULAR TERM

In this section, we prove Theorem 1.4. We reformulate it as follows.

**Theorem 10.1.** *Suppose  $1 < p \leq q < p^*(s)$  and  $r = p^*$  in the equation*

$$(P_{\lambda,\mu}) \quad \begin{cases} -\Delta_p u = \lambda |u|^{p^*-2} u + \mu \frac{|u|^{q-2}}{|x|^s} u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

*Then  $(P_{\lambda,\mu})$  has a positive solution if any one of the following conditions holds:*

- (1)  $p < q, n > \frac{p(p-1)(q-s)+p^2}{p+(p-1)(q-p)}$  and  $\lambda > 0, \mu > 0$ .
- (2)  $p = q, n \geq p^2 - (p-1)s$  and  $\lambda > 0, \mu_{s,p} > \mu > 0$ .

If one of the following conditions holds:

- (1')  $p < q, n > \frac{p(p-1)(q-s)+p}{1+(p-1)(q-p)}$  and  $\lambda > 0, \mu > 0$ ,
- (2')  $p = q, n > p((p-1)(p-s) + 1)$  and  $\lambda > 0, \mu_{s,p} > \mu > 0$ ,

then  $(P_{\lambda,\mu})$  has also a sign-changing solution.

*Remark 10.1.* The existence of a positive solution under condition (2) above has already been noticed in [13] in the case where  $p = q$ .

By scaling, we can always assume that  $\lambda = 1$ . The corresponding functional is again

$$E_\mu(u) = \frac{1}{p} \int_\Omega |\nabla u|^p - \frac{1}{p^*} \int_\Omega |u|^{p^*} - \frac{\mu}{q} \int_\Omega \frac{|u|^q}{|x|^s}.$$

Recall that under any one of the above conditions, the set

$$M_1 = \{u \in H_0^{1,p}(\Omega); u \neq 0, \langle E'_\mu(u), u \rangle = 0\}$$

is dual to the class

$$\mathcal{F}_1 = \{\gamma \in C([0, 1]; H_0^{1,p}(\Omega)); \gamma(0) = 0, \gamma(1) \neq 0 \text{ and } E_\mu(\gamma(1)) \leq 0\}.$$

Moreover, the energy level  $c_1 = \inf_{A \in \mathcal{F}_1} \sup_{u \in A} E_\mu(u)$  is equal to  $\inf_{u \in M_1} E_\mu(u)$ .

By Theorem 4.1.(3),  $E_\mu$  satisfies  $(PS)_c$  for any  $c < \frac{1}{n} \mu_0^{\frac{n}{p}}$ . So, the existence of the first positive solution follows immediately from the following estimates.

**Lemma 10.1.** *If  $r = p^*$ , then  $c_1 < \frac{1}{n} \mu_0^{\frac{n}{p}}$  in any one of the following three cases:*

- (1)  $q = p, 0 < \mu < \mu_{s,p}$  and  $n \geq p^2 - (p-1)s$ .
- (2)  $p < q < p^*(s)$  and  $\mu$  large enough.
- (3)  $p < q < p^*(s)$  and  $n > \frac{p(p-1)(q-s)+p^2}{p+(p-1)(q-p)}$ .

*Proof.* Take  $v_\varepsilon$  to be the function as in Lemma 11.2 of the appendix. Then, as in the proof of Lemma 9.1, we consider:

*Case  $q > p$ .* We have

$$\begin{aligned} \max_{0 \leq t < \infty} E_\mu(tv_\varepsilon) &\leq \frac{1}{n} \mu_0^{\frac{n}{p}} + O(\varepsilon^{\frac{n-p}{p}}) - \frac{\mu}{q} \left(\frac{\mu_0}{s}\right)^{\frac{q}{p^*-p}} \int_\Omega \frac{|v_\varepsilon|^q}{|x|^s} \\ &= \frac{1}{n} \mu_0^{\frac{n}{p}} + O(\varepsilon^{\frac{n-p}{p}}) - O(\varepsilon^{\frac{(n-p)(p-1)}{p^2} (p^*(s)-q)}). \end{aligned}$$

where we require that  $q > \frac{n-s}{n-p}(p-1)$ . The estimate then follows.

*Case  $q = p$ .* We have

$$\begin{aligned} \max_{0 \leq t < \infty} E_\mu(tv_\varepsilon) &= \frac{1}{n} \left( \int_\Omega |\nabla v_\varepsilon|^p - \frac{|v_\varepsilon|^p}{|x|^s} \right)^{\frac{n}{p}} \\ &= \begin{cases} \frac{1}{n} \mu_0^{\frac{n}{p}} + O(\varepsilon^{\frac{n-p}{p}}) - O(\varepsilon^{\frac{(n-p)(p-1)}{p^2} (p^*(s)-p)}), & p > \frac{n-s}{n-p}(p-1), \\ \frac{1}{n} \mu_0^{\frac{n}{p}} + O(\varepsilon^{\frac{n-p}{p}}) - O(\varepsilon^{\frac{n-p}{p}} |\log \varepsilon|), & p = \frac{n-s}{n-p}(p-1). \end{cases} \end{aligned}$$

Since

$$\frac{(n-p)(p-1)}{p^2} (p^*(s) - p) = \frac{(p-s)(p-1)}{p},$$

the conclusions now follow immediately.



For the sign changing solution, we shall proceed as in the case of the Hardy-Sobolev critical exponent. First we find appropriate sign changing solutions for the sub-critical problem, i.e. when  $r < p^*$ , and then we pass to the limit as  $r \rightarrow p^*$ .

Write again

$$E_{\mu,r}(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p - \frac{1}{r} \int_{\Omega} |u|^r - \frac{\mu}{q} \int_{\Omega} \frac{|u|^q}{|x|^s},$$

$$\mathcal{F}_1^r = \{ \gamma \in C([0, 1]; H_0^{1,p}(\Omega)); \gamma(0) = 0, \gamma(1) \neq 0 \text{ and } E_{\mu,r}(\gamma(1)) \leq 0 \},$$

$$M_1^r = \{ u \in H_0^{1,p}(\Omega); u \neq 0, \langle (E_{\mu,r})'(u), u \rangle = 0 \},$$

and

$$c_{1,r} = \inf_{A \in \mathcal{F}_1} \sup_{u \in A} E_{\mu,r}(u).$$

Then, as previously, we know that  $M_1^r$  is dual to  $\mathcal{F}_1^r$  and  $c_{1,r} = \inf_{u \in M_1} E_{\mu}(u)$ . Also define

$$c_{2,r} = \inf_{A \in \mathcal{F}_2} \sup_{u \in A} E_{\mu,r}(u),$$

where  $\mathcal{F}_2$  is defined in section 5. We write  $c_2$  (resp.  $E_{\mu}$ ) for  $c_{2,p^*}$  (resp.  $E_{\mu,p^*}$ ).  $\square$

**Lemma 10.2.** *Under either one of the following conditions,*

(1)  $p = q, 0 < \mu < \mu_{s,p}$  and  $n > p(p-1)(p-s) + p,$

(2)  $p < q, \mu > 0$  and  $n > \frac{p(p-1)(q-s)+p}{1+(p-1)(q-p)},$

we have

(i)  $c_{i,r} \rightarrow c_i$  ( $i = 1, 2$ ) as  $r \rightarrow p^*,$

(ii)  $c_{2,r} \leq c_{1,r} + \frac{1}{n} \mu_0^{\frac{n}{p}} - \sigma$  for some  $\sigma > 0$  and  $r$  sufficiently close to  $p^*.$

*Proof.* For the first conclusion, the proof is exactly the same as in the last section. For the second one, we can assume that the first solution  $u_1$  is smooth and  $\nabla u_1 \in L^\infty(\Omega).$

For  $\varepsilon > 0$  and  $q$  sufficiently close to  $p^*,$  apply the calculus lemma to obtain

$$\begin{aligned} E_{\mu,r}(\alpha u_1 + \beta v_\varepsilon) &\leq E_{\mu,r}(\alpha u_1) + E_{\mu,r}(\beta v_\varepsilon) \\ &\quad + A_1 \left[ \int_{\Omega} |\nabla \alpha u_1| |\nabla \beta v_\varepsilon|^{p-1} + |\nabla \alpha u_1|^{p-1} |\nabla \beta v_\varepsilon| \right] \\ &\quad + B_1 \left[ \int_{\Omega} |\alpha u_1| |\beta v_\varepsilon|^{r-1} + |\alpha u_1|^{r-1} |\beta v_\varepsilon| \right] \\ &\quad + C_1 \left[ \int_{\Omega} |\alpha u_1| \frac{|\beta v_\varepsilon|^{q-1}}{|x|^s} + |\alpha u_1|^{q-1} \frac{|\beta v_\varepsilon|}{|x|^s} \right] \\ &\leq E_{\mu,r}(\alpha u_1) + E_{\mu,r}(\beta v_\varepsilon) + A_2 (|\alpha|^p + |\beta|^p) \varepsilon^{\frac{n-p}{p^2}} \\ &\quad + B_2 (|\alpha|^r + |\beta|^r) \varepsilon^{\frac{p-1}{p} (n - \frac{(r-1)(n-p)}{p})} \\ &\quad + C_2 (|\alpha|^q + |\beta|^q) \varepsilon^{\frac{n-p}{p^2}}. \end{aligned}$$

Again, we note that for  $\varepsilon$  sufficiently small,

$$\lim_{\alpha, \beta \rightarrow \infty} E_{\mu,r}(\alpha u_1 + \beta v_\varepsilon) = -\infty.$$

So we may assume that  $\alpha$  and  $\beta$  are in a bounded set.

*Case 1.* Assume  $p < q, 0 < \mu$  and  $n > \frac{p(p-1)(q-s)+p}{1+(p-1)(q-p)}.$

Then, by the calculus lemma,

$$\begin{aligned}
& E_{\mu,r}(\alpha u_1 + \beta v_\varepsilon) \\
& \leq E_{\mu,r}(\alpha u_1) + E_\mu(\beta v_\varepsilon) + \frac{|\beta|^{p^*}}{p^*} - \frac{|\beta|^r}{r} \int_\Omega |v_\varepsilon|^r \\
& \quad + A_3 \varepsilon^{\frac{n-p}{p^2}} + B_3 \varepsilon^{\frac{p-1}{p}(n-\frac{(r-1)(n-p)}{p})} \\
& \leq c_{1,r} + \frac{1}{n} S^{\frac{n}{p}} + C \varepsilon^{\frac{n-p}{p}} + \frac{|\beta|^{p^*}}{p^*} - \frac{|\beta|^r}{r} \int_\Omega |v_\varepsilon|^r - \frac{\mu}{q} \left(\frac{S}{2}\right)^{\frac{q}{p^*-p}} \int_\Omega \frac{|v_\varepsilon|^q}{|x|^s} \\
& \quad + A_3 \varepsilon^{\frac{n-p}{p^2}} + B_3 \varepsilon^{\frac{p-1}{p}(n-\frac{(r-1)(n-p)}{p})} \\
& \leq c_{1,r} + \frac{1}{n} S^{\frac{n}{p}} + C \varepsilon^{\frac{n-p}{p}} + A_3 \varepsilon^{\frac{n-p}{p^2}} + B_3 \varepsilon^{\frac{p-1}{p}(n-\frac{(r-1)(n-p)}{p})} \\
& \quad - C_4 \varepsilon^{\frac{(n-p)(p-1)}{p^2}(p^*(s)-q)} + \frac{|\beta|^{p^*}}{p^*} - \frac{|\beta|^r}{r} \int_\Omega |v_\varepsilon|^r,
\end{aligned}$$

where we require  $q > \frac{n-s}{n-p}(p-1)$ . From

$$\frac{p-1}{p} \left(n - \frac{(r-1)(n-p)}{p}\right) > \frac{(n-p)(p-1)}{p^2} (p^*(s) - q)$$

we get  $q > r - 1 - \frac{ns}{n-p}$ . From

$$\frac{n-p}{p^2} > \frac{(n-p)(p-1)}{p^2} (p^*(s) - q),$$

we get that  $q > p^*(s) - \frac{1}{p-1}$ . Since

$$p^*(s) - \frac{1}{p-1} \geq p^*(s) - 1 = p^* - 1 - \frac{ps}{n-p} > r - 1 - \frac{ns}{n-p},$$

the hypothesis  $q > p^*(s) - \frac{1}{p-1}$  (i.e.  $n > \frac{p(p-1)(q-s)+p}{1+(p-1)(q-p)}$ ) is sufficient to allow us to choose  $\varepsilon$  small enough so that

$$C \varepsilon^{\frac{n-p}{p}} + A_3 \varepsilon^{\frac{n-p}{p^2}} + B_3 \varepsilon^{\frac{p-1}{p}(n-\frac{(r-1)(n-p)}{p})} - C_4 \varepsilon^{\frac{(n-p)(p-1)}{p^2}(p^*(s)-q)} \leq -2\sigma$$

for some constant  $\sigma > 0$ . Now choose  $\delta_0 > 0$  small enough so that

$$\frac{|\beta|^{p^*}}{p^*} - \frac{|\beta|^r}{r} \int_\Omega |v_\varepsilon|^r < \sigma \quad \text{for } 0 < |r - p^*| < \delta_0.$$

Thus we have proved the case for  $q > p$ .

*Case 2.*  $q = p$ ,  $n > p(p-1)(p-s) + p$  and  $0 < \mu < \mu_{s,p}$ .

We assume that  $\alpha$  and  $\beta$  are in a bounded set, and we estimate  $E_\lambda(\alpha u_1 + \beta v_\varepsilon)$ :

$$\begin{aligned} & E_{\mu,r}(\alpha u_1 + \beta v_\varepsilon) \\ & \leq E_{\mu,r}(\alpha u_1) + E_{\mu,r}(\beta v_\varepsilon) \\ & \quad + A_1 \left[ \int_\Omega |\nabla \alpha u_1| |\nabla \beta v_\varepsilon|^{p-1} + |\nabla \alpha u_1|^{p-1} |\nabla \beta v_\varepsilon| \right] \\ & \quad + B_1 \left[ \int_\Omega |\alpha u_1| \frac{|\beta v_\varepsilon|^{p-1}}{|x|^s} + |\alpha u_1|^{p-1} \frac{|\beta v_\varepsilon|}{|x|^s} \right] \\ & \quad + C_1 \left[ \int_\Omega |\alpha u_1| |\beta v_\varepsilon|^{r-1} + |\alpha u_1|^{r-1} |\beta v_\varepsilon| \right] \\ & \leq E_{\mu,r}(\alpha u_1) + E_\mu(\beta v_\varepsilon) + \frac{|\beta|^{p^*}}{p^*} - \frac{|\beta|^r}{r} \int_\Omega |v_\varepsilon|^r \\ & \quad + A_3 \varepsilon^{\frac{n-p}{p^2}} + B_3 \varepsilon^{\frac{p-1}{p} (n - \frac{(r-1)(n-p)}{p})} \\ & \leq c_{1,r} + \frac{1}{n} S^{\frac{n}{p}} + C \varepsilon^{\frac{n-p}{p}} + \frac{|\beta|^{p^*}}{p^*} - \frac{|\beta|^r}{r} \int_\Omega |v_\varepsilon|^r - O(\varepsilon^{\frac{(p-s)(p-1)}{p}}) \\ & \quad + A_3 \varepsilon^{\frac{n-p}{p^2}} + B_3 \varepsilon^{\frac{p-1}{p} (n - \frac{(r-1)(n-p)}{p})} \\ & \leq c_{1,r} + \frac{1}{n} S^{\frac{n}{p}} + C \varepsilon^{\frac{n-p}{p}} + A_3 \varepsilon^{\frac{n-p}{p^2}} + B_3 \varepsilon^{\frac{p-1}{p} (n - \frac{(r-1)(n-p)}{p})} \\ & \quad - C_4 \varepsilon^{\frac{(p-s)(p-1)}{p}} + \frac{|\beta|^{p^*}}{p^*} - \frac{|\beta|^r}{r} \int_\Omega |v_\varepsilon|^r. \end{aligned}$$

Note that we have required that  $p > \frac{n-s}{n-p}(p-1)$ . By the assumption, we can choose  $\varepsilon$  small enough so that

$$C \varepsilon^{\frac{n-p}{p}} + A_3 \varepsilon^{\frac{n-p}{p^2}} + B_3 \varepsilon^{\frac{p-1}{p} (n - \frac{(r-1)(n-p)}{p})} - C_4 \varepsilon^{\frac{(p-s)(p-1)}{p}} \leq -2\sigma$$

for some constant  $\sigma > 0$ . Now choose  $\delta_0 > 0$  small enough so that

$$\frac{|\beta|^{p^*}}{p^*} - \frac{|\beta|^r}{r} \int_\Omega |v_\varepsilon|^r < \sigma \quad \text{for } 0 < |q - p^*| < \delta_0.$$

The proof of this lemma is now complete. □

The rest of the proof of the theorem is now very similar to Theorem 9.1. The details are left for the interested reader. □

11. APPENDIX: ESTIMATES ON THE EXTREMAL SOBOLEV-HARDY FUNCTIONS

Assume, without loss of generality, that  $0 \in \Omega$ , and let

$$U_\varepsilon(x) = (\varepsilon + |x|^{\frac{p-s}{p-1}})^{\frac{p-n}{p-s}}.$$

$U_\varepsilon(x)$  is a function in  $H^{1,p}(\mathbf{R}^n)$  where the best constant in the Sobolev-Hardy inequality is attained. They are, modulo translation and dilations, the unique positive ones where the best constant is achieved. (See section 2.)

Let  $0 \leq \phi(x) \leq 1$  be a function in  $C_0^\infty(\Omega)$  defined as

$$\phi(x) = \begin{cases} 1 & \text{if } |x| \leq R, \\ 0 & \text{if } |x| \geq 2R, \end{cases}$$

where  $B_{2R}(0) \subset \Omega$ . Set  $u_\varepsilon(x) = \phi(x)U_\varepsilon(x)$ . For  $\varepsilon \rightarrow 0$ , the behavior of  $u_\varepsilon$  has to be the same as that of  $U_\varepsilon$  but we need precise estimates of the error terms.

**Lemma 11.1.** *Assume  $0 \leq s < p$ ,  $p \geq 2$  and  $q = \frac{n-s}{n-p}p$ . By taking*

$$v_\varepsilon = \frac{u_\varepsilon}{\left(\int_\Omega \frac{|u_\varepsilon|^q}{|x|^s}\right)^{\frac{1}{q}}}$$

so that  $\int_\Omega \frac{|v_\varepsilon|^q}{|x|^s} = 1$ , we have the following estimates:

- (1)  $\|\nabla v_\varepsilon\|_p^p = \mu_s + O(\varepsilon^{\frac{n-p}{p-s}})$ ,
- (2)  $\int_\Omega |\nabla v_\varepsilon|^\alpha = O(\varepsilon^{\frac{(n-p)\alpha}{p(p-s)}})$ , for  $\alpha = 1, 2, p-2, p-1$ .
- (3) if  $r > p^*(1 - \frac{1}{p})$ , then

$$C_1 \varepsilon^{\frac{(p-1)(n-\frac{r(n-p)}{p})}{p-s}} \leq \|v_\varepsilon\|_r^r \leq C_2 \varepsilon^{\frac{(p-1)(n-\frac{r(n-p)}{p})}{p-s}},$$

- (4) if  $r = p^*(1 - \frac{1}{p})$ , then

$$C_1 \varepsilon^{\frac{(n-p)r}{p(p-s)}} |\log \varepsilon| \leq \|v_\varepsilon\|_r^r \leq C_2 \varepsilon^{\frac{(n-p)r}{p(p-s)}} |\log \varepsilon|,$$

- (5) if  $r < p^*(1 - \frac{1}{p})$ , then

$$C_1 \varepsilon^{\frac{(n-p)r}{p(p-s)}} \leq \|v_\varepsilon\|_r^r \leq C_2 \varepsilon^{\frac{(n-p)r}{p(p-s)}},$$

- (6) if  $p < r < p^*$ , then  $\|v_\varepsilon\|_r^r \rightarrow 0$  (as  $\varepsilon \rightarrow 0$ ),

(7)

$$\int_\Omega \frac{|v_\varepsilon|^{q-1}}{|x|^s} = O(\varepsilon^{\frac{p-1}{p}(\frac{n-p}{p-s})}).$$

(8)

$$\int_\Omega \frac{|v_\varepsilon|}{|x|^s} = O(\varepsilon^{\frac{n-p}{p(p-s)}}).$$

where  $C_1, C_2 > 0$  are constants.

*Proof.* Let

$$k(\varepsilon) = (\varepsilon \cdot (n-s) \left(\frac{n-p}{p-1}\right)^{p-1})^{\frac{n-p}{p(p-s)}}.$$

Then  $y_\varepsilon(x) = k(\varepsilon)U_\varepsilon(x)$  is the extremal function in the Sobolev-Hardy inequality. Furthermore,

$$k(\varepsilon)^p \|\nabla U_\varepsilon(x)\|_p^p = \|\nabla y_\varepsilon(x)\|_p^p = \mu_s^{\frac{n-s}{p-s}}$$

The gradient of  $u_\varepsilon(x)$  is given by

$$\begin{aligned} \nabla u_\varepsilon(x) &= (\varepsilon + |x|^{\frac{p-s}{p-1}})^{\frac{p-n}{p-s}} \nabla \phi(x) + \frac{p-n}{p-1} \cdot \frac{x\phi(x)}{(\varepsilon + |x|^{\frac{p-s}{p-1}})^{\frac{n-s}{p-s}} |x|^{\frac{p-2}{p-1}}} \\ &= \begin{cases} \frac{p-n}{p-1} \cdot \frac{x}{(\varepsilon + |x|^{\frac{p-s}{p-1}})^{\frac{n-s}{p-s}} |x|^{\frac{p-2}{p-1}}} & \text{if } |x| \leq R, \\ 0 & \text{if } |x| \geq 2R. \end{cases} \end{aligned}$$

Thus we have

$$\begin{aligned} \int_\Omega |\nabla u_\varepsilon|^p &= O(1) + \int_{|x| \leq R} |\nabla U_\varepsilon(x)|^p dx = O(1) + \int_{\mathbf{R}^n} |\nabla U_\varepsilon(x)|^p dx \\ &= O(1) + \|\nabla U_\varepsilon(x)\|_p^p, \end{aligned}$$

and

$$\int_{\Omega} \frac{|u_{\varepsilon}|^p}{|x|^s} = O(1) + \int_{\mathbf{R}^n} \frac{|U_{\varepsilon}|^q}{|x|^s} = O(1) + \int_{\mathbf{R}^n} \frac{|y_{\varepsilon}|^q}{|x|^s} \cdot k^{-q}(\varepsilon) = O(k^{-q}(\varepsilon)).$$

From this we further get

$$\begin{aligned} \|\nabla v_{\varepsilon}\|_p^p &= \frac{\|\nabla u_{\varepsilon}\|_p^p}{\left(\int_{\Omega} \frac{|u_{\varepsilon}|^q}{|x|^s}\right)^{\frac{p}{q}}} = \frac{O(1) + \|\nabla U_{\varepsilon}\|_p^p}{\left(\int_{\Omega} \frac{|u_{\varepsilon}|^q}{|x|^s}\right)^{\frac{p}{q}}} \\ &= \frac{O(1) + \mu_s^{\frac{n-s}{p-s}} k(\varepsilon)^{-p}}{\left(\int_{\Omega} \frac{|u_{\varepsilon}|^q}{|x|^s}\right)^{\frac{p}{q}}} = \frac{O(1) + \mu_s^{\frac{n-s}{p-s}} k(\varepsilon)^{-p}}{O(1) + k(\varepsilon)^{-p} \mu_s^{\frac{p(n-s)}{(p-s)q}}} \\ &= O(k(\varepsilon)^p) + \mu_s^{\frac{n-s}{p-s} - \frac{p(n-s)}{(p-s)q}} = \mu_s + O(\varepsilon^{\frac{n-p}{p-s}}). \end{aligned}$$

(1) is thus proved. For (2), let  $\omega_n$  denote the surface area of the  $(n - 1)$ -sphere  $S^{n-1}$  in  $\mathbf{R}^n$ ; then

$$\begin{aligned} \int_{\Omega} |\nabla u_{\varepsilon}|^{\alpha} &= O(1) + \int_{|x| \leq R} \left(\frac{n-p}{p-1}\right)^{\alpha} \frac{|x|^{\alpha}}{(\varepsilon + |x|^{\frac{p-s}{p-1}})^{\frac{\alpha(n-s)}{p-s}} |x|^{\frac{(p-2)\alpha}{p-1}}} dx \\ &= O(1) + \omega_n \int_0^R \left(\frac{n-p}{p-1}\right)^{\alpha} \frac{r^{\alpha} \cdot r^{n-1}}{(\varepsilon + r^{\frac{p-s}{p-1}})^{\frac{\alpha(n-s)}{p-s}} r^{\frac{(p-2)\alpha}{p-1}}} dr \\ &\leq O(1) + \omega_n \int_0^R \left(\frac{n-p}{p-1}\right)^{\alpha} r^{\alpha+n-1 - \frac{\alpha(n-s)}{p-1} - \frac{\alpha(p-2)}{p-1}} dr, \end{aligned}$$

and the order of  $r$  in the integrand is

$$\begin{aligned} \alpha + n - 1 - \frac{\alpha(n-s)}{p-1} - \frac{\alpha(p-2)}{p-1} &= \frac{pn - p - n + 1 - \alpha n + \alpha s + \alpha}{p-1} \\ &= \frac{pn - n - \alpha n + \alpha s + \alpha}{p-1} - 1 > -1 \end{aligned}$$

for  $\alpha = 1, p - 2, p - 1$  and  $\alpha = 2$  if  $p \geq 3$ . Thus

$$\int_{\Omega} |\nabla u_{\varepsilon}|^{\alpha} = O(1),$$

and we conclude that

$$\int_{\Omega} |\nabla v_{\varepsilon}|^{\alpha} = O(\varepsilon^{\frac{(n-p)\alpha}{p(p-s)}}).$$

For (3), (4) and (5),

$$\begin{aligned} \|u_{\varepsilon}\|_r^r &= O(1) + \omega_n \int_0^R (\varepsilon + x^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}} x^{n-1} dx \\ &= O(1) + \omega_n \varepsilon^{-\frac{n-p}{p-s} r + \frac{p-1}{p-s} n} \int_0^{R\varepsilon^{-\frac{p-1}{p-s}}} (1 + x^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}} x^{n-1} dx \end{aligned}$$

If  $r = p^*(1 - \frac{1}{p})$ , then  $-\frac{n-p}{p-s}r + \frac{p-1}{p-s}n = 0$ , and

$$\begin{aligned}\|u_\varepsilon\|_r^r &= O(1) + \omega_n \int_0^{R\varepsilon^{-\frac{p-1}{p-s}}} (1 + x^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}r} x^{n-1} dx \\ &= O(1) + \omega_n \int_0^{R\varepsilon^{-\frac{p-1}{p-s}}} \frac{1}{x} dx \\ &= O(1) + O(|\log \varepsilon|).\end{aligned}$$

So we get

$$\|v_\varepsilon\|_r^r = O(|\log \varepsilon| \varepsilon^{\frac{n-p}{p(p-s)}r}).$$

If  $r < p^*(1 - \frac{1}{p})$ , then  $-\frac{n-p}{p-1}r + n - 1 > -1$ . We conclude that

$$\begin{aligned}\|u_\varepsilon\|_r^r &= O(1) + \omega_n \int_0^R (\varepsilon + x^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}r} x^{n-1} dx \\ &\leq O(1) + \omega_n \int_0^R x^{-\frac{n-p}{p-1}r+n-1} dx = O(1)\end{aligned}$$

and

$$\|v_\varepsilon\|_r^r = O(\varepsilon^{\frac{n-p}{p(p-s)}r}).$$

If  $r > p^*(1 - \frac{1}{p})$ , then  $-\frac{n-p}{p-1}r + n - 1 < -1$  and  $-\frac{n-p}{p-s}r + \frac{p-1}{p-s}n < 0$ . We have

$$\begin{aligned}\|u_\varepsilon\|_r^r &= O(1) + \omega_n \varepsilon^{-\frac{n-p}{p-s}r + \frac{p-1}{p-s}n} \int_1^\infty (1 + x^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}r} x^{n-1} dx \\ &= O(\varepsilon^{-\frac{n-p}{p-s}r + \frac{p-1}{p-s}n}),\end{aligned}$$

and

$$\|v_\varepsilon\|_r^r = O(\varepsilon^{-\frac{n-p}{p-s}r + \frac{p-1}{p-s}n + \frac{n-p}{p(p-s)}r}) = O(\varepsilon^{\frac{p-1}{p-s}(n - \frac{r(n-p)}{p})}).$$

(3), (4) and (5) are thus proved.

For (7) and (8), we have

$$\begin{aligned}\int_\Omega \frac{|u_\varepsilon|^\alpha}{|x|^s} dx &= O(1) + \int_{|x| \leq R} (\varepsilon + |x|^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}\alpha} |x|^{-s} dx \\ &= O(1) + \omega_n \int_0^R (\varepsilon + r^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}\alpha} r^{-s} \cdot r^{n-1} dr \\ &= O(1) + \omega_n \int_0^{R\varepsilon^{-\frac{p-1}{p-s}}} (1 + r^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}\alpha} r^{n-s-1} \varepsilon^{-\frac{n-p}{p-s}\alpha + (n-s)\frac{p-1}{p-s}} dr.\end{aligned}$$

If  $\alpha = q - 1$ , then  $-\frac{n-p}{p-s}\alpha + (n-s)\frac{p-1}{p-s} = -1$ . We have

$$\begin{aligned}\int_\Omega \frac{|u_\varepsilon|^\alpha}{|x|^s} dx &= O(1) + O(\varepsilon^{-1}) \omega_n \int_0^{R\varepsilon^{-\frac{p-1}{p-s}}} (1 + r^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}\alpha} r^{n-s-1} dr \\ &= O(1) + O(\varepsilon^{-1}) \omega_n \int_0^\infty (1 + r^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}\alpha} r^{n-s-1} dr = O(\varepsilon^{-1}),\end{aligned}$$

since  $n - s - \frac{n-p}{p-1}(q - 1) = \frac{s-p}{p-1} < 0$ . Then

$$\int_\Omega \frac{|v_\varepsilon|^\alpha}{|x|^s} dx = O(\varepsilon^{-1}) \cdot \varepsilon^{\frac{n-p}{p(p-s)}(q-1)} = O(\varepsilon^{\frac{p-1}{p}(\frac{n-p}{p-s})}).$$

If  $\alpha = 1$ , since  $-\frac{n-p}{p-1} + n - s > -\frac{n-p}{p-1} + n - p = (n - p)(1 - \frac{1}{p-1}) \geq 0$  for  $p \geq 2$ , we have

$$\begin{aligned} \int_{\Omega} \frac{|u_{\varepsilon}|}{|x|^s} dx &= O(1) + \omega_n \int_0^R (\varepsilon + r^{\frac{p-s}{p-1}})^{-\frac{n-p}{p-s}} r^{n-s-1} dr \\ &\leq O(1) + \omega \int_0^R r^{-\frac{n-p}{p-1} + n - s - 1} dr = O(1), \end{aligned}$$

and furthermore

$$\int_{\Omega} \frac{|v_{\varepsilon}|}{|x|^s} dx = O(\varepsilon^{\frac{n-p}{p(p-s)}}).$$

(7) and (8) are thus proved. □

Note that the above results are well known for the extremal functions associated to the Sobolev embedding, that is, when  $s = 0$ . In the following lemma, we prove additional properties in the case where  $s > 0$ .

**Lemma 11.2.** *For  $0 \leq t < p$ , we have*

$$\int_{\Omega} \frac{|v_{\varepsilon}|^{\alpha}}{|x|^t} = \begin{cases} O(\varepsilon^{\frac{n-p}{p^2}\alpha} |\log \varepsilon|), & \alpha = \frac{n-t}{n-p}(p-1), \\ O(\varepsilon^{\frac{(n-p)(p-1)}{p^2}(p^*(t)-\alpha)}), & \alpha > \frac{n-t}{n-p}(p-1), \\ O(\varepsilon^{\frac{n-p}{p^2}\alpha}), & \alpha < \frac{n-t}{n-p}(p-1). \end{cases}$$

*Proof.* As above, let  $\omega_n$  denote the surface area of the  $(n - 1)$  sphere  $S^{n-1}$  in  $\mathbf{R}^n$ . Then

$$\int_{\Omega} \frac{|u_{\varepsilon}|^{\alpha}}{|x|^t} = O(1) + \omega_n \int_0^R (\varepsilon + r^{\frac{p}{p-1}})^{\frac{p-n}{p}\alpha} r^{n-t-1} dr.$$

*Case 1.*  $\alpha = \frac{n-t}{n-p}(p-1)$  (then  $\frac{p-n}{p-1}\alpha + n - t = 0$ ). Then

$$\begin{aligned} \int_{\Omega} \frac{|u_{\varepsilon}|^{\alpha}}{|x|^t} &= O(1) + \omega_n \int_0^{R\varepsilon^{-\frac{p-1}{p}}} (1 + r^{\frac{p}{p-1}})^{\frac{p-n}{p}\alpha} r^{n-t-1} dr \\ &= O(1) + \omega_n \int_0^{R\varepsilon^{-\frac{p-1}{p}}} \frac{1}{r} dr = O(|\log \varepsilon|). \end{aligned}$$

*Case 2.*  $\alpha > \frac{n-t}{n-p}(p-1)$  (then  $\frac{p-n}{p-1}\alpha + n - t - 1 < -1$ ). Then

$$\begin{aligned} \int_{\Omega} \frac{|u_{\varepsilon}|^{\alpha}}{|x|^t} &= O(1) + \omega_n \int_0^{R\varepsilon^{-\frac{p-1}{p}}} (1 + r^{\frac{p}{p-1}})^{\frac{p-n}{p}\alpha} r^{n-t-1} dr \varepsilon^{\frac{p-n}{p}\alpha + (n-s)\frac{p-1}{p}} \\ &= O(1) + O(\varepsilon^{\frac{p-n}{p}\alpha + (n-s)\frac{p-1}{p}}). \end{aligned}$$

*Case 3.*  $\alpha < \frac{n-t}{n-p}(p-1)$  (i.e.  $\frac{p-n}{p-1}\alpha + n - t - 1 > -1$ ). Then

$$\begin{aligned} \int_{\Omega} \frac{|u_{\varepsilon}|^{\alpha}}{|x|^t} &= O(1) + \omega_n \int_0^R (\varepsilon + r^{\frac{p}{p-1}})^{\frac{p-n}{p}\alpha} r^{n-t-1} dr \\ &= O(1) + \omega_n \int_0^R r^{\frac{p-n}{p-1}\alpha + n - t - 1} dr \\ &= O(1). \end{aligned}$$

Now, the conclusion follows immediately. □

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