TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 352, Number 12, Pages 5599–5618
S 0002-9947(00)02684-2
Article electronically published on August 8, 2000

ON THE DIMENSION OF THE PRODUCT OF TWO COMPACTA AND THE DIMENSION OF THEIR INTERSECTION IN GENERAL POSITION IN EUCLIDEAN SPACE

A. N. DRANISHNIKOV

Abstract. For every two compact metric spaces X and Y, both with dimension at most $n-3$, there are dense G_δ-subsets of mappings $f : X \to \mathbb{R}^n$ and $g : Y \to \mathbb{R}^n$ with $dim f(X) \cap g(Y) \leq dim(X \times Y) - n$.

1. Introduction

We know the formula for the dimension of the intersection of two hyperplanes α and β in general position in euclidean space $\mathbb{R}^n$: $dim(\alpha \cap \beta) = dim\alpha + dim\beta - n$, and the formula is valid for the estimation of the dimension of the intersection of two polyhedra in general position in $\mathbb{R}^n$: $dim(K \cap L) \leq dimK + dimL - n$. Using approximations of compacta by polyhedra, one can obtain the similar estimate for compacta:

$$dim(X \cap Y) \leq dimX + dimY - n.$$

The main result of this paper is strengthening of that inequality to the following: $dim(X \cap Y) \leq dim(X \times Y) - n$. Since for compact metric spaces the dimension of the product can be much smaller than the sum of the dimensions, the improvement is significant.

This paper can be considered as sequel to a series of papers on the mapping intersection problem. The series was initiated by two papers of D. McCullough and L. Rubin [1], [2] and then it was continued by J. Krasinkiewicz, K. Lorentz, S. Spież, J. Segal and H. Toruńczyk from one side and by E.V. Ščepin, D. Repovš, J. West and the author from the other [3], [4], [5], [12], [14], [13], [6], [8], [15], [17], [9], [10], [11], [22], [7], [16]. Under investigation was the following conjecture.

Conjecture. *There exists a pair of maps $f : X \to \mathbb{R}^n$, $g : Y \to \mathbb{R}^n$ of two compacta with stable intersection if and only if $dim(X \times Y) \geq n$.*

We say that two maps $f : X \to \mathbb{R}^n$, $g : Y \to \mathbb{R}^n$ have stable intersection in $\mathbb{R}^n$ if there is an $\epsilon > 0$ such that, for any ϵ-perturbations f' and g' of f and g, $Imf' \cap Img' \neq \emptyset$. Otherwise we say that f and g have unstable intersection.

In this introduction we first consider the history of work on this conjecture. Then we give a precise formulation of the main result (Theorem A) and a summary of its proof.

Received by the editors January 30, 1995 and, in revised form, January 27, 1999.
2000 *Mathematics Subject Classification.* Primary 55M10, 55N45.

The conjecture was first proved in the complementary case: $dimX + dimY \leq n$ [4],[12],[6],[8]. The main algebraic tool needed for that case was Alexander duality. Later the conjecture was proved in the so-called metastable case: $2dimX + dimY < 2n-1$ [9],[10],[14],[15]. In symmetric form the metastable case applies to compacta with $dimX + dimY \leq 4n/3 - 1$. This case turned out to be more difficult, and it required a more serious technique, such as the Spanier-Whitehead duality or Weber's imbedding theorem. In analogy to the Freudenthal suspension theorem, in the metastable range the conjecture breaks into the cases $2dimX + dimY < 2n-2$ and $2dimX + dimY = 2n-2$. The latter case required the development, mainly due to E. Ščepin [15],[14], of a higher dimensional version of the *Casson finger move*.

Looking for different solutions for the metastable case, I proved the following version of the conjecture: *For an imbedded compactum $X \subset \mathbb{R}^n$ there is a map $g : Y \to \mathbb{R}^n$ of a compactum Y with a stable intersection if and only if $dim(X \times Y) \geq n$* [10],[11],[22]. This led to the proof of one direction of the original conjecture [16]. Namely, if $dim(X \times Y) \geq n$, then there is a pair of maps $f : X \to \mathbb{R}^n$, $g : Y \to \mathbb{R}^n$ with stable intersection. Recently Y. Sternfeld [31] found a short proof of that. Consequently, the original conjecture was reduced to the following statement.

The remaining part of the conjecture. *If $dim(X \times Y) < n$ for two compacta X and Y, then every pair of maps $f : X \to \mathbb{R}^n$, $g : Y \to \mathbb{R}^n$ has unstable intersection.*

If one of the compacta X and Y is 0-dimensional, then the conjecture holds. If both compacta have dimensions higher than zero and $dim(X \times Y) < n$, then the upper bound for the sum of the dimensions is $dimX + dimY \leq 2n - 4$.

The next achievement was made in [17], when we realized that the conjecture depends only on the cohomological dimension types of X and Y. It was well-known [19] that the dimension of the product depends only on the cohomological dimensions of the factors with respect to the groups of the Bockstein family. If the conjecture were true, then the other part of the statement would depend only on the cd-types of the compacta. So, first we proved that the existence of a pair of maps of two compacta with stable intersection depends only on the cd-types of compacta. Since the conjecture was already proven for the case when one of the factors is imbedded in $\mathbb{R}^n$, it is not strange that we managed to reduce the conjecture to the cohomological dimension type imbedding problem: *Given a compactum X of dimension $\leq n - 2$, does there exist a compactum $X' \subset \mathbb{R}^n$ with the same set of cohomological dimensions with respect to all abelian groups?* A positive answer to the imbedding problem implies a positive answer to the conjecture. The modern state of the art in cohomological dimension theory allowed us to prove an imbedding theorem which in its turn gave the proof of the conjecture for the case when $dimX + dimY \leq 2n - 2\sqrt{n}$.

Further success in the area was due to the birth of a new discipline, called "extension theory". The main purpose of that theory is to study the *absolute extension property* of a space X and maps to a given complex (or ANR) M. The absolute extension property for X means that every map $\phi : A \to M$ of a closed subset $A \subset X$ can be extended over X. In that case M is called an absolute extensor for X (formally for the class of spaces $\{X\}$ consisting of one space X), and the notation for that is $M \in AE(X)$. When we want to emphasize that X has the absolute extension property, we use Kuratowski's notation $X\tau M$, which is not self explanatory but it puts X on the first place. The name 'extension theory' probably first appeared in [11], where I proved that $X\tau M$ implies $X\tau\Omega\Sigma M$. Among the other new results in the area we mention briefly the following three, since they are used in this paper. First, there is the description of the property $X\tau M$ for finite-dimensional compacta X and simply-connected complexes M in terms of the cohomological dimension of X with respect to homology groups of M as coefficients. Precisely, $X\tau M$ is equivalent to the system of inequalities $dim_{H_i(M)}X \leq i$, $i > 0$ (Theorem 2). Second, there is Dydak's union theorem: $X\tau K$, $Y\tau L$ imply $(X \cup Y)\tau(K * L)$, where $*$ means the join product (Theorem 5). Finally, there is Olszewski's completion theorem: For every separable metric space W and every countable complex K with the property $W\tau K$ there exists a completion $\bar{W}$ with $\bar{W}\tau K$ (Theorem 6).

Since $X\tau S^n$ means precisely $dimX \leq n$, extension theory generalizes dimension theory. It is quite natural to expect that all theorems of dimension theory are just the visible part of the iceberg (see [30]). In [7] I found a version of the Eilenberg-Borsuk theorem about extension of mappings to spheres. The classic version says that if X is an n-dimensional compactum and $\phi : A \to S^k$ is a partial map to the k-sphere defined on a closed subset of X, then the map ϕ can be extended over the complement $X - Z$ of an $n - k - 1$-dimensional compactum Z. The generalization says that if $X\tau M * N$, where $M * N$ means the join of M and N, and $\phi : A \to N$ is a partial map, then ϕ can be extended over the complement $X - Z$ of a compactum Z with $Z\tau M$. If we put $N = S^k$ and $M = S^{n-k-1}$, we will get the Eilenberg-Borsuk theorem. The generalization turns out to be so powerful that it gives the solution of the realization problem and the cohomological dimension type imbedding problem simultaneously [7]. Thus in view of the reduction in [17] the conjecture was proved except for the codimension two case.

There is still an open case when $dimX = n - 2$ or $dimY = n - 2$, and $n > 4$. The case $dimX = dimY = 2$ and $n = 4$ is covered by the complementary case. The proof in that case is different (see for example [8] or [27]) from the general case. The main problem for $n > 4$ appears in the version of the conjecture where one of the compacta is imbedded in $\mathbb{R}^n$. The difficulties there look enormous, and they are basically due to the presence of the fundamental groups. The problem with the fundamental group is that basically the extension theory for non-simple spaces is not constructed. There is some activity around cohomological dimension with non-abelian coefficients [29],[28], and perhaps that will grow into a theory which might help to treat the last case of the conjecture.

The main result of this paper extends the conjecture by giving a general estimate for the dimension of the intersection of compacta X and Y (when minimized over all nearby maps). Precisely, we have

Theorem A. *Let $f : X \to \mathbb{R}^n$ and $g : Y \to \mathbb{R}^n$ be two continuous maps of compact metric spaces to n-dimensional Euclidean space, and let $dimX < n - 2$, $dimY < n - 2$. Then for any $\epsilon > 0$ there are ϵ-approximations $f' : X \to \mathbb{R}^n$ and $g' : Y \to \mathbb{R}^n$ of f and g with $dimf'(X) \cap g'(Y) \leq dim(X \times Y) - n$.*

Notice that when $dim(X \times Y) < n$, then Theorem A becomes the remaining part of the conjecture (except for the codimension two case). The inequalities $dimX, dimY \leq n - 2$ are necessary. For example, if X and Y are Pontryagin's surfaces Π_p and Π_q with different primes p and q, then their intersection in $\mathbb{R}^3$ is 1-dimensional; but it would have to be 0-dimensional if the formula were true.

The proof of Theorem A is based on previous results in the area and on computations in a certain 'algebra' representing the algebra of cohomological dimension types.

If we know the cohomological dimensions of the factors X and Y with respect to all abelian groups, then in the case of compact spaces X and Y there are Bockstein formulas for the computation of cohomological dimensions of $X \times Y$. The formulas are rather elaborate. They are simple only if the coefficient group is the additive group of a field F; then the standard logarithmic law holds, $dim_F(X \times Y) = dim_F X + dim_F Y$. The set of cohomological dimensions of a compactum with respect to various groups leads to the notion of cohomological dimension type (cd-type). In this paper in Section 2 we describe a cd-type as a set of four functions $(d, \epsilon, \delta; c)$ on the set of all prime numbers $\mathcal{P}$. Two of these functions, d and c, are integral-valued, and the other two are $\mathbb{Z}_2$-valued. Moreover, the function c is a constant. We define these functions so that the operation on them generated by the product of compacta is a sum-product operation, denoted by $[+]$. It is sum for d and c and product for ϵ and δ.

Thus in our approach it is easy to compute the cd-type of the product of two spaces. We have to pay for that ease by some complications in the comparison problem. It is easy to compare two cd-types in the original definition [17]: $D_X \leq D_Y$ if $dim_G X \leq dim_G Y$ for all abelian groups G. In the language of 4-tuples the comparison is more complicated. By the definition of the 4-tuple $(d, \epsilon, \delta; c)$, the function $c(p) = c$ corresponds to the cohomological dimension $dim_{\mathbb{Q}}$ with respect to the rationals, and the function $d(p)$ corresponds to $dim_{\mathbb{Z}_p}$. It turns out to be convenient to consider 'cd-types' with negative c and $d(p)$. This idea is one of the basic ingredients of the proof of Theorem A.

The proof of Theorem A is contained in Section 5. It relies on preliminary work with the cohomological dimension type and the extension theory in Sections 3-4.

Having successfully replaced the sum of the dimensions $dimX + dimY$ by the dimension of the product $dim(X \times Y)$ in some dimensional inequality, it is natural to try the same in the other inequalities. In Section 6 we consider the classic Menger-Urysohn formula $dim(X \cup Y) \leq dimX + dimY + 1$. We prove a better inequality $dim(X \cup Y) \leq dim(X \times Y) + 1$ when the union $Z = X \cup Y$ is compact and satisfies $dim(Z \times Z) = 2dimZ$.

2. Bockstein algebra

In this section we introduce an abstract algebraic object which we call the Bockstein algebra. Elements of that algebra encode dimensional information on compacta. A connection between this and the classical Bockstein theory is made in the next section.

Let $\mathcal{P}$ denote the set of all prime numbers. A family $F = (d, \epsilon, \delta; c)$ consisting of three functions $d : \mathcal{P} \to \mathbb{Z}$, $\epsilon : \mathcal{P} \to \mathbb{Z}_2$, $\delta : \mathcal{P} \to \mathbb{Z}_2$ and a constant $c \in \mathbb{Z}$ is called a Bockstein function if $(d(p)-c)\delta(p) = 0$ and $(1-\epsilon(p))\delta(p) = 0$ for all $p \in \mathcal{P}$. The first product here can be interpreted as the product in $\mathbb{Z}$ after a multiplicative imbedding of $\mathbb{Z}_2$ in $\mathbb{Z}$ as $\{0, 1\}$. The second equation is in $\mathbb{Z}_2$. Thus, $F : \mathcal{P} \to \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}$ is a function on prime numbers, and $F(p) = (d(p), \epsilon(p), \delta(p); c)$.

For every two Bockstein functions $F_1 = (d_1, \epsilon_1, \delta_1; c_1)$ and $F_2 = (d_2, \epsilon_2, \delta_2; c_2)$ we define an operation $[+]$, called sum-product, by the formula

$$F_1[+]F_2 = (d_1 + d_2, \epsilon_1\epsilon_2, \delta_1\delta_2; c_1 + c_2).$$

Assertion 2.1. *$F_1[+]F_2$ is a Bockstein function.*

Proof. 1) $((d_1+d_2)-(c_1+c_2))\delta_1\delta_2 = ((d_1-c_1)+(d_2-c_2))\delta_1\delta_2 = ((d_1-c_1)\delta_1)\delta_2 + ((d_2-c_2)\delta_2)\delta_1 = 0$.

2) $(1-\epsilon_1\epsilon_2)\delta_1\delta_2 = (1-\epsilon_1+\epsilon_1(1-\epsilon_2))\delta_1\delta_2 = ((1-\epsilon_1)\delta_1)\delta_2+\epsilon_1((1-\epsilon_2)\delta_2)\delta_1 = 0$.

Note that the operation $[+]$ is associative. For every Bockstein function $F = (d,\epsilon,\delta;c)$ we define a conjugate function $\bar{F} = (\bar{d},\bar{\epsilon},\bar{\delta};\bar{c})$ by the formulae: $\bar{d} = -d$, $\bar{c} = -c$, $\bar{\delta} = \delta$ and $\bar{\epsilon} = 1-\epsilon+\delta$.

Assertion 2.2. *$\bar{F}$ is a Bockstein function.*

Proof. 1) $(\bar{d}-\bar{c})\bar{\delta} = -(d-c)\delta = 0$.

2) $(1-\bar{\epsilon})\bar{\delta} = (1-1+\epsilon-\delta)\delta = (\epsilon-\delta)\delta = (\epsilon-1)\delta+(1-\delta)\delta = (1-\delta)\delta = 0$.

Proposition 2.1. *$\bar{\bar{F}} = F$ for every Bockstein function F.*

Proof. We check that $\bar{\bar{\epsilon}} = 1-\bar{\epsilon}+\delta = 1-(1-\epsilon+\delta)+\delta = \epsilon$.

For any $n \in \mathbb{Z}$ we denote by the same letter n the Bockstein function $(n,1,1;n)$. Thus we have the natural monomorphism of the integers $\mathbb{Z}$ into the monoid $\mathcal{F}$ of Bockstein functions. We note that for every integer m and any Bockstein function $F = (d,\epsilon,\delta;c)$ the product $mF = (md, m\epsilon, m\delta; mc)$ is also a Bockstein function. Then we define $-F = (-1)F$. Note that $\bar{n} = -n$.

Generally we don't have the formula $\overline{(F_1[+]F_2)} = \bar{F_1}[+]\bar{F_2}$, but the following proposition holds:

Proposition 2.2. *$\overline{(F[+]n)} = \bar{F}[+](-n)$ for every Bockstein function F and every $n \in \mathbb{Z}$.*

Proof. $\overline{(F_1[+]n)} = \overline{(d+n,\epsilon,\delta;c+n)} = (\overline{d+n},\bar{\epsilon},\bar{\delta};\overline{c+n}) = (\bar{d}+\bar{n},\bar{\epsilon},\bar{\delta};\bar{c}+\bar{n}) = \bar{F}[+]\bar{n} = \bar{F}[+](-n)$.

We call a Bockstein function $F = (d,\epsilon,\delta;c)$ p-regular if $\delta(p) = 1$. In that case it follows that $d(p) = c$ and $\epsilon(p) = 1$. If a function F is not p-regular, we call it p-singular.

Proposition 2.3. *$F[+]\bar{F} = (0,\delta,\delta;0)$.*

Proof. $F[+]\bar{F} = (d-d, \epsilon(1-\epsilon+\delta), \epsilon\delta; c-c) = (0, \epsilon-\epsilon^2+\epsilon\delta, \epsilon\delta; 0) = (0,\epsilon\delta,\epsilon\delta;0) = (0,\delta,\delta;0)$. The last equality is due to the condition $(1-\epsilon)\delta = 0$.

There is a natural distributive product operation $[\times]$ on $\mathcal{F}$ defined by the formula $F_1[\times]F_2 = (d_1d_2, \epsilon_1\epsilon_2, \delta_1\delta_2; c_1c_2)$.

Assertion 2.3. *$F_1[\times]F_2$ is a Bockstein function for every pair of Bockstein functions F_1 and F_2.*

Proof. 1) $(d_1d_2-c_1c_2)\delta_1\delta_2 = (d_1d_2-c_1d_2+c_1d_2-c_1c_2)\delta_1\delta_2 = (d_1-c_1)d_2\delta_1\delta_2 + c_1(d_2-c_2)\delta_1\delta_2 = 0$.

2) $(1-\epsilon_1\epsilon_2)\delta_1\delta_2 = 0$, as was shown in Assertion 2.1.

The topological meaning of the operation $[\times]$ is yet to be understood.

3. Cohomological dimension type

In this section we give a standard definition of the cohomological dimension type (cd-type) of a compact space in terms of dimensional functions. Also we briefly review the Bockstein theory, and we connect the algebra of cd-types with the Bockstein algebra of Section 2.

We recall that the cohomological dimension of a compactum X over an abelian group G is $dim_G X = max\{n \mid \check{H}^n(X, A; G) \neq 0$ for some closed subset $A \subset X\}$. If there is no such maximum, then we let $dim_G X = \infty$. There is the Bockstein family of abelian groups $\sigma = \bigcup_{p \in P}\{\mathbb{Z}_p, \mathbb{Z}_{p^\infty}, \mathbb{Z}_{(p)}\} \cup \{\mathbb{Q}\}$, where $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at p and $\mathbb{Z}_{p^\infty} = DirLim\{\mathbb{Z}_{p^k}\}$ and $\mathbb{Q}$ is the group of rationals. Every finite-dimensional compactum X defines a function $D_X : \sigma \to \mathbb{Z}_+$ by the formula $D_X(G) = dim_G X$. A function from σ to $\mathbb{Z}_+$ is called a dimensional function if it is D_X for some X. In this paper we consider only finite-dimensional compacta, although it is known that Bockstein theory works for infinite dimensional compacta as well if one adds infinity to the set of values of dimensional functions [20].

We say two compacta X and Y have the same cd-type if $D_X = D_Y$, and we will often regard the cd-type as the dimension function itself. If a topological space Z can be presented as a countable union of compacta $\bigcup X_i$, then the countable union theorem states that $dim_G Z = max\{dim_G X_i \mid i\}$ for any group G [19]. By virtue of the countable union theorem we obtain that every σ-compact space (countable union of compacta) Z defines a cd-type of a compactum. Precisely, if Z is a countable union of compacta $Z = \bigcup X_i$, then $dim_G Z = dim_G \alpha(\coprod X_i)$, where $\alpha(\coprod X_i)$ is the one-point compactification of the disjoint union $\coprod X_i$.

On the set of all cd-types $\mathcal{D}$ one has the natural partial order $\leq$ and two operations: $D_1 \vee D_2$ and $D_1 \uplus D_2$ which correspond to taking the wedge and the product of compacta. By definition $(D_1 \vee D_2)(G) = max\{D_1(G), D_2(G)\}$ for every $G \in \sigma$. The $\uplus$ operation is defined in formulas (1)-(4S) below.

For every abelian group G we form the family $\sigma(G)$ by the following rule:

(B1) $\mathbb{Q} \in \sigma(G) \Leftrightarrow \mathbb{Q} \otimes G \neq 0$,

(B2) $\mathbb{Z}_p \in \sigma(G) \Leftrightarrow \mathbb{Z}_p \otimes G \neq 0$,

(B3) $\mathbb{Z}_{(p)} \in \sigma(G) \Leftrightarrow \mathbb{Z}_{p^\infty} \otimes G \neq 0$,

(B4) $\mathbb{Z}_{p^\infty} \in \sigma(G) \Leftrightarrow Tor(\mathbb{Z}_p, G) \neq 0$, where Tor means the torsion product.

We note that $\sigma(\mathbb{Z}) = \{\mathbb{Q}, \mathbb{Z}_{(p)}, \mathbb{Z}_p \mid p \in P\}$ and $\sigma(\mathbb{Q}) = \{\mathbb{Q}\}$, $\sigma(\mathbb{Z}_p) = \{\mathbb{Z}_p, \mathbb{Z}_{p^\infty}\}$, $\sigma(\mathbb{Z}_{p^\infty}) = \{\mathbb{Z}_{p^\infty}\}$, and $\sigma(\mathbb{Z}_{(p)}) = \{\mathbb{Q}, \mathbb{Z}_{(p)}, \mathbb{Z}_p\}$. Bockstein's basis theorem says that $dim_G X = max\{dim_H X \mid H \in \sigma(G)\}$. Consequently, the equality $D_X = D_Y$ implies the equality $dim_G X = dim_G Y$ for all abelian groups G.

There are Bockstein inequalities for cd-types:

BI1-2 $D(\mathbb{Z}_{p^\infty}) \leq D(\mathbb{Z}_p) \leq D(\mathbb{Z}_{p^\infty}) + 1$,

BI3-4 $D(\mathbb{Q}) \leq D(\mathbb{Z}_{(p)}) \geq D(\mathbb{Z}_p)$,

BI5 $D(\mathbb{Z}_{(p)}) \leq max\{D(\mathbb{Q}), D(\mathbb{Z}_{p^\infty}) + 1\}$,

BI6 $D(\mathbb{Z}_{p^\infty}) \leq max\{D(\mathbb{Q}), D(\mathbb{Z}_{(p)}) - 1\}$.

It is known that every function from σ to $\mathbb{Z}_+$ satisfying the Bockstein inequalities coincides with D_X for some finite-dimensional compactum X [20].

Proposition 3.1. *For every group $G \in \sigma$ with $G \neq \mathbb{Z}_{p^\infty}$, and for every compactum X, the inequality $dim_{H_i(K(G,n))} X \leq dim_G X$ holds for all integers $i \geq 1$ and $n \geq 1$.*

Proof. There are three cases.

1) $G = \mathbb{Q}$. In that case $H_i(K(\mathbb{Q}, n))$ is a vector space over $\mathbb{Q}$ [32]. Hence only $\mathbb{Q}$ from σ can be in $\sigma(H_i(K(\mathbb{Q}, n)))$. Hence $\sigma(H_i(K(\mathbb{Q}, n))) \subset \sigma(\mathbb{Q})$, and Bockstein's basis theorem implies the result.

2) $G = \mathbb{Z}_p$. Then $H_i(K(\mathbb{Z}_p, n))$ is a p-torsion group (for $n \geq 2$, this uses the generalized relative Hurewicz theorem, while for $n = 1$ it is by direct calculation). Hence only $\mathbb{Z}_p$ and $\mathbb{Z}_{p^\infty}$ could be in $\sigma(H_i(K(\mathbb{Z}_p, n)))$. Bockstein's basis theorem again applies.

3) $G = \mathbb{Z}_{(p)}$. Since $\mathbb{Z}_{(p)}$ is a localization at p of $\mathbb{Z}$, a map $f : K(\mathbb{Z}, n) \to K(\mathbb{Z}_{(p)}, n)$, generated by the inclusion $\mathbb{Z} \to \mathbb{Z}_{(p)}$ is the localization at p of $K(\mathbb{Z}, n)$ [32]. Then the homology groups $H_i(K(\mathbb{Z}_{(p)}, n)) = H_i(K(\mathbb{Z}, n)) \otimes \mathbb{Z}_{(p)}$ have the structure of a $\mathbb{Z}_{(p)}$-module. Hence these groups have no q-torsion elements for q relatively prime to p. Hence the groups $\mathbb{Z}_{(q)}$, $\mathbb{Z}_q$ and $\mathbb{Z}_{q^\infty}$ with q relatively prime to p cannot be in $\sigma(H_i(K(\mathbb{Z}_{(p)}, n)))$. Thus we can conclude that $\sigma(H_i(K(\mathbb{Z}_{(p)}, n))) \subset \{\mathbb{Q}, \mathbb{Z}_{(p)}, \mathbb{Z}_p, \mathbb{Z}_{p^\infty}\}$. Since $\sigma(\mathbb{Z}_{(p)}) = \{\mathbb{Q}, \mathbb{Z}_{(p)}, \mathbb{Z}_p\}$, the inequality BI1 shows that $dim_H X \leq dim_{\mathbb{Z}_{(p)}} X$ for any group $H \in \sigma(H_i(K(\mathbb{Z}_{(p)}, n)))$. Hence, by Bockstein's basis theorem, $dim_{H_i(K(\mathbb{Z}_{(p)}, n))} X \leq dim_{\mathbb{Z}_{(p)}} X$.

We call a cd-type $D : \sigma \to \mathbb{Z}_+$ p-regular if $D(\mathbb{Z}_{p^\infty}) = D(\mathbb{Z}_{(p)})$; otherwise we call D p-singular. Note that in the case of p-regular D we have $D(\mathbb{Z}_{(p)}) = D(\mathbb{Z}_p) = D(\mathbb{Z}_{p^\infty}) = D(\mathbb{Q})$.

We call a compactum X p-regular (p-singular) if its cd-type D_X is p-regular (p-singular).

Assertion 3.1. *Let D be a p-singular cd-type. Then*

$$D(\mathbb{Z}_{(p)}) = max\{D(\mathbb{Q}), D(\mathbb{Z}_{p^\infty}) + 1\}.$$

Proof. First we consider the case when $D(\mathbb{Q}) \geq D(\mathbb{Z}_{p^\infty}) + 1$. In this case $D(\mathbb{Z}_{(p)}) \leq D(\mathbb{Q})$ by BI5. Then BI3 implies that $D(\mathbb{Z}_{(p)}) = D(\mathbb{Q})$, and the formula holds.

If $D(\mathbb{Q}) < D(\mathbb{Z}_{p^\infty}) + 1$, then $D(\mathbb{Z}_{(p)}) \leq D(\mathbb{Z}_{p^\infty}) + 1$ by BI5. Since D is p-singular, $D(\mathbb{Z}_{(p)}) - 1 \geq D(\mathbb{Z}_{p^\infty}) \geq D(\mathbb{Q})$. The last inequality is due to the assumption. Therefore BI6 implies $D(\mathbb{Z}_{p^\infty}) \leq D(\mathbb{Z}_{(p)}) - 1$. Hence $D(\mathbb{Z}_{(p)}) = D(\mathbb{Z}_{p^\infty}) + 1$, and the formula holds.

There are Bockstein formulas for cohomological dimension of the product of two compacta [19]. We give them in terms of cd-types:

(1) $(D_1 \uplus D_2)(\mathbb{Q}) = D_1(\mathbb{Q}) + D_2(\mathbb{Q})$,

(2) $(D_1 \uplus D_2)(\mathbb{Z}_p) = D_1(\mathbb{Z}_p) + D_2(\mathbb{Z}_p)$,

(3) $(D_1 \uplus D_2)(\mathbb{Z}_{p^\infty}) = max\{D_1(\mathbb{Z}_{p^\infty}) + D_2(\mathbb{Z}_{p^\infty}), (D_1 \uplus D_2)(\mathbb{Z}_p) - 1\}$,

(4R) $(D_1 \uplus D_2)(\mathbb{Z}_{(p)}) = D_1(\mathbb{Z}_{(p)}) + D_2(\mathbb{Z}_{(p)})$ if D_1 or D_2 is p-regular,

(4S) $(D_1 \uplus D_2)(\mathbb{Z}_{(p)}) = max\{D_1(\mathbb{Z}_{p^\infty}) + D_2(\mathbb{Z}_{p^\infty}) + 1, (D_1 \uplus D_2)(\mathbb{Z}_p), (D_1 \uplus D_2)(\mathbb{Q})\}$ if both D_i are p-singular.

Proposition 3.2. *The product $D_1 \uplus D_2$ of two cd-types is p-regular if and only if both factors D_1 and D_2 are p-regular.*

Proof. If both factors are p-regular, then by the Bockstein formulas (3) and (4R) the product $D_1 \uplus D_2$ is p-regular.

If both factors are p-singular, then the Bockstein formulas (3) and (4S) imply that $(D_1 \uplus D_2)(\mathbb{Z}_{(p)}) \geq max\{D_1(\mathbb{Z}_{p^\infty}) + D_2(\mathbb{Z}_{p^\infty}) + 1, (D_1 \uplus D_2)(\mathbb{Z}_p)\} \geq max\{D_1(\mathbb{Z}_{p^\infty}) + D_2(\mathbb{Z}_{p^\infty}), (D_1 \uplus D_2)(\mathbb{Z}_p) - 1\} + 1 = (D_1 \uplus D_2)(\mathbb{Z}_{p^\infty}) + 1$. Hence $D_1 \uplus D_2$ is p-singular.

If only one of the factors, say D_2, is p-singular, then $(D_1 \uplus D_2)(\mathbb{Z}_{(p)}) = D_1(\mathbb{Z}_{(p)}) + D_2(\mathbb{Z}_{(p)})$ by (4R) and $D_2(\mathbb{Z}_{(p)}) \geq D_2(\mathbb{Z}_{p^\infty}) + 1$ by Assertion 3.1. Since $D_1(\mathbb{Z}_{(p)}) = D_1(\mathbb{Z}_{p^\infty})$, we have that $(D_1 \uplus D_2)(\mathbb{Z}_{(p)}) \geq D_1(\mathbb{Z}_{p^\infty}) + D_2(\mathbb{Z}_{p^\infty}) + 1$. On the other hand, $(D_1 \uplus D_2)(\mathbb{Z}_{(p)}) \geq (D_1 \uplus D_2)(\mathbb{Z}_p)$ by the inequality BI4. Hence, by the Bockstein formula (3), $(D_1 \uplus D_2)(\mathbb{Z}_{(p)}) \geq (D_1 \uplus D_2)(\mathbb{Z}_{p^\infty}) + 1$. Therefore $D_1 \uplus D_2$ is p-singular.

Proposition 3.3. *There is a morphism $\Phi : (\mathcal{D}, \uplus) \to (\mathcal{F}, [+])$ of monoids defined by the formulas $\Phi(D) = (d_D, \epsilon_D, \delta_D; c_D)$: $d_D(p) = D(\mathbb{Z}_p)$, $c_D = D(\mathbb{Q})$, $\epsilon_D(p) = 1 + D(\mathbb{Z}_{p^\infty}) - D(\mathbb{Z}_p)$ and $\delta_D(p) = 1$ if and only if D is p-regular. The morphism Φ is injective with the image*

$$Im\,\Phi = \mathcal{F}_+ = \{(d, \epsilon, \delta;\ c) \in \mathcal{F} \mid c > 0, (d(p) + \epsilon(p) - 1) > 0\} \cup \{(0, 1, 1; 0)\};$$

it takes p-regular cd-types to p-regular functions and p-singular to p-singular.

Proof. If D is p-singular, then $\delta(p) = 0$, so the formulas hold. If D is p-regular, we have $D(\mathbb{Q}) = D(\mathbb{Z}_p) = D(\mathbb{Z}_{p^\infty})$. Hence $d_D(p) = c_D$ and $\epsilon_D(p) = 1$. Then it follows that $\Phi(D) \in \mathcal{F}$.

We show that $\Phi(D_1 \uplus D_2) = \Phi(D_1)[+]\Phi(D_2)$. By Proposition 3.2, $\delta_{D_1 \uplus D_2}(p) = \delta_{D_1}(p)\delta_{D_2}(p)$. Now the only possible problem could be with ϵ. By the product formulae we have

$$\begin{aligned}\epsilon_{D_1 \uplus D_2}(p) = 1 + max\{D_1(\mathbb{Z}_{p^\infty}) + D_2(\mathbb{Z}_{p^\infty}), D_1(\mathbb{Z}_p) + D_2(\mathbb{Z}_p) - 1\} \\ - D_1(\mathbb{Z}_p) - D_2(\mathbb{Z}_p) = 0\end{aligned}$$

if $\epsilon_{D_1}(p) = \epsilon_{D_2}(p) = 0$, and $= \epsilon_{D_1}(p) + \epsilon_{D_2}(p) - 1$ otherwise. In both cases modulo 2 it equals $\epsilon_{D_1}(p)\epsilon_{D_2}(p)$.

The homomorphism Φ is a monomorphism because one can recover D uniquely from $\Phi(D)$. The formulas for the inverse map Φ^{-1} are:

$$\begin{aligned}&D(\mathbb{Q}) = c,\\ &D(\mathbb{Z}_p) = d(p),\\ &D(\mathbb{Z}_{p^\infty}) = d(p) + \epsilon(p) - 1,\\ &D(\mathbb{Z}_{(p)}) = max\{c, d(p) + \epsilon(p) - \delta(p)\}.\end{aligned}$$

In the case of p-regular D the last equality turns into the equality $D(\mathbb{Z}_{(p)}) = max\{D(\mathbb{Q}), D(\mathbb{Z}_{p^\infty})\}$, which is correct. In the case of p-singular D the last equality follows from Assertion 3.1.

If $D = D_X \in \mathcal{D}$ and $D(G) = 0$ for some $G \in \sigma$, then X is 0-dimensional, so $D(G) = 0$ for all $G \in \sigma$ and $\Phi(D) = (0, 1, 1, 0)$. If $D = D_X$ for a positive-dimensional X, then clearly c and $d + \epsilon - 1$ are positive. Conversely, one can check that if $c, d + \epsilon - 1 > 0$ then $D(G) > 0$ for all $G \in \sigma$. It easy to check that every function D defined as $\Phi^{-1}(F)$ for a given Bockstein function $F \in \mathcal{F}$ satisfies the Bockstein inequalities BI1-6. Hence by the realization theorem [20] there is a compactum X such that $D = D_X$ in the case of positive D.

It remains unclear how to define a compactum X with the dimensional function $D_X = D_Y[\times]D_Z$ in terms of compacta Y and Z.

Remark 3.1. The formulas for Φ^{-1} define an injective map μ from $\mathcal{F}$ to the set of integral-valued functions on σ. Using μ, one can define a partial order on $\mathcal{F}$ by setting $F_1 \leq F_2$ if and only if $\mu(F_1)(G) \leq \mu(F_2)(G)$ for all G.

From this point we are going to identify dimensional functions with their images in $\mathcal{F}$ under the imbedding Φ. Now we will use the same symbol $[+]$ for the product of cd-types. We denote the cd-type of the n-cube by n. Note that the function $D+n$ defined as $(D+n)(G)=D(G)+n$ is also a cd-type which corresponds to $D[+]n$.

Proposition 3.3 and the formulas for Φ^{-1} easily imply the next statement.

Proposition 3.4. *$D[+]\bar{D}(G)=0$ if $G\neq\mathbb{Z}_{p^\infty}$, and*

$$D[+]\bar{D}(\mathbb{Z}_{p^\infty})=\begin{cases}0, & \textit{if } D \textit{ is } p\textit{-regular};\\ -1, & \textit{if } D \textit{ is } p\textit{-singular}.\end{cases}$$

For every function $D:\sigma\to\mathbb{Z}$ we define the norm $\|D\|=max\{D(\mathbb{Z}_{(p)})\mid p\in\mathcal{P}\}$. We let $\|D\|=\infty$ if D is unbounded. By virtue of Bockstein's basis theorem and BI3-4, we have the equality $\|D_X\|=dim_{\mathbb{Z}}X$. For finite-dimensional compacta, by the Alexandroff theorem [19], $\|D_X\|=dimX$. Note that the norm of $D=(d,\epsilon,\delta;c)$ can be computed by the formula $\|D\|=max\{c,d(p)+\epsilon(p)-\delta(p)\ \mid p\in\mathcal{P}\}$.

Assertion 3.2. *Let D,D' and D_1 be cd-types such that $D\leq D'$. Then $D[+]D_1\leq D'[+]D_1$.*

Proof. Let X,X' and X_1 be compacta such that $D=D_X$, $D'=D_{X'}$ and $D_1=D_{X_1}$. Let $Y=X\coprod X'$. Then $D_{X'}=D_Y$. Therefore,

$$\begin{aligned}(D[+]D_1)(G)&=(D_X[+]D_{X_1})(G)=dim_G(X\times X_1)\leq dim_G(Y\times X_1)\\&=(D_Y[+]D_{X_1})(G)=(D'[+]D_1)(G)\end{aligned}$$

for any G.

We need the following:

Theorem 1. *Let X be a compactum with $dimX<n-2$. Then the set of maps $f:X\to\mathbb{R}^n$ for which the cd-type of the image $D_{f(X)}$ satisfies $D_X\leq D_{f(X)}\leq D_X\vee 2$ is a dense G_δ set in the space of all continuous maps $C(X,\mathbb{R}^n)$.*

Theorem 1 formally follows from the solution of the cohomological dimension type imbedding problem [7] and the Reduction theorem (Theorem 1.12 in [17]) asserting that the following three conditions are equivalent for any compactum X of dimension $\leq n-3$:

(1) $D_X=D_{X'}$ for some compactum $X'\subset\mathbb{R}^n$.

(2) For any compactum Y with $dim(X\times Y)<n$, every pair of maps $f:X\to\mathbb{R}^n$, $g:Y\to\mathbb{R}^n$ has an unstable intersection.

(3) The set of maps $f:X\to\mathbb{R}^n$ with $D_X=D_{f(X)}$ is a dense G_δ set in $C(X,\mathbb{R}^n)$.

Unfortunately the proof of (1) $\Rightarrow$ (2) in [17] contains a gap. The case of $dimY=n-2$ is not covered there. The correct version of the reduction theorem is the following:

Reduction Theorem. *For any compactum X of dimension $\leq n-3$ the following statements are equivalent:*

(1) There is a compactum $X'\subset\mathbb{R}^n$ with $D_X\leq D_{X'}\leq D_X\vee 2$.

(2) For any compactum Y of dimension $\leq n-3$ with $dim(X\times Y)<n$, all maps $f:X\to\mathbb{R}^n$, $g:Y\to\mathbb{R}^n$ have unstable intersections.

(3) The set of maps $f:X\to\mathbb{R}^n$ with the cd-type of the image $D_{f(X)}$ such that $D_X\leq D_{f(X)}\leq D_X\vee 2$ is a dense G_δ set in the space of all continuous maps $C(X,\mathbb{R}^n)$.

Proof. (1) $\Rightarrow$ (2). Suppose that $X' \subset \mathbb{R}^n$ with $D_{X'} \leq D_X \vee 2$. Let Y be a compactum with $dimY \leq n-3$ and $dim(X \times Y) < n$. Since $D_{X'} \leq D_{X \vee I^2}$, we have $dim(X' \times Y) \leq dim((X \vee I^2) \times Y)$. Hence

$$dim(X' \times Y) \leq max\{dim(X \times Y), dim(I^2 \times Y)\} < n.$$

The rest of the argument is the same as in Theorem 1.12 of [17].

(2) $\Rightarrow$ (3). Let $G \in \sigma$ and let $m = min\{n-3, (n-1) - dim_G X\}$. Consider a test space $Y = T_m(G)$. Since $dimX - dim_G X < m$, the G-testing equality implies $dim(X \times Y) = dim_G X + m < n$ (see Theorem 1.1 of [17]). Note that $dimY = m \leq n-3$. Then by (2) all maps $f : X \to \mathbb{R}^n$, $g : Y \to \mathbb{R}^n$ have unstable intersections. By Lemma 3.1 of [17] there is a dense G_δ-set $C_G \subset C(X, \mathbb{R}^n)$ such that for every $f' \in C_G$ the image $f'(X)$ is Y-negligible. Additionally we may assume that C_G consists of light maps which do not raise the covering dimension. Y-negligibility of $f'(X)$ means that every map $g : Y \to \mathbb{R}^n$ can be approximated by maps missing $f'(X)$. Then, by the main result of [11], $dim(f'(X) \times Y) < n$. Since a light map cannot lower the cohomological dimension, $dimf'(X) - dim_G f'(X) \leq dimX - dim_G X < m$, and hence the G-testing equality holds: $dim(f'(X) \times T_m(G)) = dim_G f'(X) + m$. Hence $dim_G f'(X) + m \leq n-1$. If $dim_G X = 1$, then $m = n-3$ and hence $dim_G f'(X) \leq 2$. If $dim_G X > 1$, then $m = n-1-dim_G X$, and hence $dim_G f'(X) \leq dim_G X$. Thus $D_X(G) \leq D_{f'(X)}(G) \leq (D_X \vee 2)(G) = max\{dim_G X, 2\}$ for all $f' \in C_G$. Note that the intersection $C = \bigcap_{G \in \sigma} C_G$ is a dense G_δ set. Then for every $f' \in C$ we have that $D_X \leq D_{f'(X)} \leq D_X \vee 2$.

(3) $\Rightarrow$ (1) is obvious.

Now Theorem 1 follows from the solution of the cd-type imbedding problem [7].

We note that Corollary 6 of [7] still is not proved, because the argument for that relied on the unproved version of the reduction theorem.

4. Extension type

An extension problem is the problem of extending a map $f : A \to M$ from a closed subset $A \subset X$ over the whole space X. The situation when every extension problem has a solution for given X and M we denote by the symbol $X\tau M$. Note that $X\tau S^n$ means $dimX \leq n$ and $X\tau K(G,n)$ means $dim_G X \leq n$. It makes sense to consider sufficiently nice spaces M, for example, CW-complexes. Let $\mathcal{C}$ be a class of topological spaces; we define a partial order on the set of all (countable) CW-complexes as follows: $N \leq M$ if $X\tau M$ implies $X\tau N$ for every space $X \in \mathcal{C}$. We say N and M define the same extension type if $N \leq M$ and $M \leq N$. This is an equivalence relation on the set of (countable) CW-complexes; we call it e-equivalence for the case when $\mathcal{C}$ is the class of all finite-dimensional metrizable compacta. Perhaps it is more natural to consider the class of all (metric) compacta, but for the purpose of this paper the class of finite dimensional compacta is more appropriate. Equivalence classes are called extension types. We note that homotopy equivalent complexes define the same extension type, but the converse is not necessarily true: the extension type of the n-sphere S^n is the same as the extension type of $S^n \vee S^m$ if $m \geq n$. We note that the partial order on CW-complexes induces a partial order on extension types. It is remarkable that a one-point space gives the minimal element with respect to that order and a two-point space gives the maximal element. It turns out that extension types of one-connected complexes are dual to cd-types (see [30] for more details).

Everywhere in the rest of this paper a compactum means a compact metric space.

Theorem 2 ([22]). *For any simply connected complex M and any finite-dimensional compactum X, the following are equivalent:*

1) $X\tau M$,

2) $dim_{H_i(M)}X \leq i$ for all i,

3) $dim_{\pi_i(M)}X \leq i$ for all i.

Here $H_i(M)$ is the i-dimensional homology group of M and $\pi_i(M)$ is the i-dimensional homotopy group of M.

The proof of Theorem 2 uses the Dold-Thom theorem on the infinite symmetric power SP^∞ [23].

Let M be a CW-complex and $G \in \sigma$ a group of the Bockstein family. We define $n_M(G) = min\{i \mid G \in \sigma(H_i(M))\}$ or infinity. It is convenient for us to define $K(G,\infty)$ to be a one-point space.

Note that $n_{\Sigma L}(G) = n_L(G)+1$, where ΣL is the suspension on L.

Proposition 4.1. *Every 1-connected CW-complex M is e-equivalent to the countable wedge $\bigvee_{G\in\sigma} K(G, n_M(G))$.*

Proof. Assume that $X\tau M$; then by Theorem 2 $dim_{H_i(M)}X \leq i$. For each $G \in \sigma$ with $n_M(G) < \infty$, let $i = n_M(G)$. Then $G \in \sigma(H_i(M))$, and hence

$$dim_G X \leq max\{dim_H X \mid H \in \sigma(H_i(M))\} = dim_{H_i(M)}X \leq n_M(G).$$

Therefore $X\tau K(G, n_M(G))$ and then $X\tau(\bigvee_\sigma K(G, n_M(G)))$, i.e.

$$\bigvee_\sigma K(G, n_M(G)) \leq M.$$

Now assume that $X\tau \bigvee_\sigma K(G, n_M(G))$. Then $X\tau K(G, n_M(G))$ for every $G \in \sigma$. Hence $dim_G X \leq n_M(G)$. By the Bockstein theorem $dim_{H_i(M)}X = dim_G X$ for some $G \in \sigma(H_i(M))$. By the definition, $i \geq n_M(G)$. Hence $dim_{H_i(M)}X \leq i$. Since i is arbitrary, Theorem 2 implies that $X\tau M$.

Assertion 4.1. *Let L, N be CW complexes and $L \leq N$. Then $\Sigma L \leq \Sigma N$.*

Proof. The inequality $L \leq N$ is equivalent the system of inequalities $n_L(G) \geq n_N(G)$, $G \in \sigma$. Since $n_{\Sigma L}(G) = n_L(G)+1$, we have that $n_{\Sigma L}(G) \geq n_{\Sigma N}(G)$ for all G. Hence $\Sigma L \leq \Sigma N$.

For every non-negative function $D : \sigma \to \mathbb{Z}_+$ we define

$$K(D) = \bigvee_{G\in\sigma} K(G, D(G)).$$

Note that for every compact space C with $C\tau K(D)$ we have the inequality $D_C \leq D$. Indeed, $C\tau K(D)$ implies that $C\tau K(G, D(G))$ for all G, so $\dim_G C \leq D(G)$, which means $D_C(G) \leq D(G)$.

Assertion 4.2. *Let D be a positive cd-type. Then $\Sigma K(D) \leq K(D+1)$.*

Proof. First we show that $\Sigma K(G,n) \leq K(G, n+1)$ for all $G \in \sigma$ and $n \geq 1$. If we have the property $Y\tau K(G, n+1)$, it means that $dim_G Y \leq n+1$. We are going to apply Theorem 2 to obtain the property $Y\tau\Sigma K(G,n)$. For $i \leq n$ we have

$$H_i(\Sigma K(G,n)) = 0,$$

and the inequality $dim_{H_i(\Sigma K(G,n))}Y \le i$ holds automatically. For $i > n$ by virtue of Proposition 3.1 we have $dim_{H_i(\Sigma K(G,n))}Y = dim_{H_{i-1}(K(G,n))}Y \le dim_G Y \le n+1 \le i$ for $G \neq \mathbb{Z}_{p^\infty}$. Since the groups $H_k(K(\mathbb{Z}_{p^\infty}, n)) = H_k(DirLim(K(\mathbb{Z}_{p^l}, n))) = DirLim(H_k(K(\mathbb{Z}_{p^l}, n)))$ are p-torsion groups [35], by the Bockstein basis theorem and the inequality BI2 we have $dim_{H_{i-1}(K(\mathbb{Z}_{p^\infty},n))}Y \le dim_{\mathbb{Z}_p} Y \le dim_{\mathbb{Z}_{p^\infty}} Y + 1 \le n+2 \le i$ for $i > n+1$. If $i = n+1$ then $H_n(K(\mathbb{Z}_{p^\infty}, n)) = \mathbb{Z}_{p^\infty}$ and again $dim_{H_{i-1}(K(\mathbb{Z}_{p^\infty},n))}Y = dim_{\mathbb{Z}_{p^\infty}} Y \le n+1 \le i$. Thus, we have $dim_{H_i(\Sigma K(G,n))}Y \le i$ for all i. Since $\Sigma K(G,n)$ is simply connected for $n \ge 1$, we can apply Theorem 2.

Next, in the general case we have equivalences

$$\begin{aligned} Y\tau K(D+1) \ &\Leftrightarrow\ Y\tau \bigvee_{G\in\sigma} K(G, D(G)+1) \\ &\Leftrightarrow\ Y\tau K(G, D(G)+1) \text{ for all } G \in \sigma. \end{aligned}$$

By the above argument we have $Y\tau\Sigma K(G, D(G))$ for all $G \in \sigma$. Hence

$$Y\tau \bigvee_{G\in\sigma} \Sigma K(G, D(G)),$$

which is equivalent to

$$Y\tau\Sigma(\bigvee_{G\in\sigma} K(G, D(G))) \ \Leftrightarrow\ Y\tau\Sigma K(D).$$

Let $K * L$ denote the join product of K and L in the category of CW-complexes, i.e. we consider a finer topology on the ordinary join $K \times L \times [-1,1]/(x, y, \pm 1) = (x', y', \pm 1)$. In other words, $K * L$ is the direct limit $DirLim\{K_\alpha * L_\beta\}$ over the partially ordered family of products of finite subcomplexes $K_\alpha \subset K$ and $L_\beta \subset L$. If we fix base points $x_0 \in K$ and $y_0 \in L$, then a contractible subcomplex $C \subset K * L$ can be defined as the union of two cones: $C(L, x_0)$ and $C(K, y_0)$. Then the quotient map $q : K * L \to K * L/C$ is a homotopy equivalence. We note that the quotient space is homeomorphic to the reduced suspension $\Sigma(K \wedge L)$ over the smash product in the category of CW-complexes.

The following theorem is proved in [7].

Theorem 3. *Suppose that K and L are countable CW-complexes. If the property $X\tau K * L$ holds for some compactum X, then there is an F_σ-set $Z \subset X$ such that $Z\tau L$ and $(X - Z)\tau K$.*

Proposition 4.2. *Let $M = K(D_1) \wedge K(D_2)$ for cd-types D_1 and D_2. Then:*
1) $n_M(H) = D_1(H) + D_2(H)$ for every group $H \in \sigma$ and $H \neq \mathbb{Z}_{p^\infty}$.
2)

$$n_M(\mathbb{Z}_{p^\infty}) = \begin{cases} D_1(\mathbb{Z}_{p^\infty}) + D_2(\mathbb{Z}_{p^\infty}), & \textit{if } D_1 \textit{ or } D_2 \textit{ is } p\textit{-regular}, \\ \min\{D_1(\mathbb{Z}_p) + D_2(\mathbb{Z}_p),\ D_1(\mathbb{Z}_{p^\infty}) + D_2(\mathbb{Z}_{p^\infty}) + 1\}, & \textit{otherwise}. \end{cases}$$

Proof. Denote $L_1(G) = K(G, D_1(G))$ and $L_2(G') = K(G', D_2(G'))$.

Since $M = \bigvee_{G,G'\in\sigma}(L_1(G) \wedge L_2(G'))$, we have

$$H_i(M) = \bigoplus_{G,G'} H_i(L_1(G) \wedge L_2(G')).$$

Denote $n_{G,G'}(H) = n_{L_1(G)\wedge L_2(G')}(H)$. Since $\sigma(\bigoplus G_i) = \bigcup \sigma(G_i)$, we have

$$n_M(H) = min\{n_{G,G'}(H)\}.$$

Since $H_i(K_1 \wedge K_2) \oplus H_i(K_1 \vee K_2) = H_i(K_1 \times K_2)$, the Künneth formula gives

$$0 \to \bigoplus_{k+l=i;k,l>0} (H_k(L_1(G)) \otimes H_l(L_2(G'))) \to H_i(L_1(G) \wedge L_2(G'))$$
$$\to \bigoplus_{k+l=i-1} Tor(\ ,\ ) \to 0.$$

Thus $H_i(L_1(G) \wedge L_2(G')) = 0$ when $i < D_1(G) + D_2(G')$, and

$$H_{D_1(G)+D_2(G')}(L_1(G) \wedge D_2(G')) = G \otimes G'.$$

Then $n_{G,G'}(H) \geq D_1(G) + D_2(G')$ for any group H and any G, G'.

We consider four cases.

1) $H = \mathbb{Q}$. By the definition, $n_{G,G}(\mathbb{Q}) = min\{i \mid H_i(L_1(G) \wedge L_2(G')) \otimes \mathbb{Q} \neq 0\}$. Since the tensor product of $\mathbb{Q}$ with a torsion group is zero, the Künneth formula implies that $n_{G,G'}(\mathbb{Q}) = min\{i \mid H_k(L_1(G)) \otimes H_l(L_2(G')) \otimes \mathbb{Q} \neq 0;\ k+l = i\}$. Since homology groups $H_*(K(G,n))$ of a torsion group G are torsion groups, $n_{G,G'}(\mathbb{Q}) < \infty$ only if $G, G' \in \{\mathbb{Q}, \mathbb{Z}_{(p)}\}$. Easy computations show that $n_{G,G'}(\mathbb{Q}) = D_1(G) + D_2(G')$ in that case. The Bockstein inequality BI3 implies that $D_1(\mathbb{Q}) + D_2(\mathbb{Q}) = min\{D_1(G) + D_2(G') \mid G, G' \in \{\mathbb{Q}, \mathbb{Z}_{(p)}\}\} = min\{n_{G,G'}(\mathbb{Q})\} = n_M(\mathbb{Q})$.

2) $H = \mathbb{Z}_p$. Note that in this case

$$n_{G,G'}(\mathbb{Z}_p) = min\{i \mid H_i(L_1(G) \wedge L_2(G')) \otimes \mathbb{Z}_p \neq 0\}.$$

By the Künneth formula, $H_i(L_1(G)) \wedge L_2(G')) \otimes \mathbb{Z}_p \neq 0$ if and only if

$$H_k(L_1(G)) \otimes H_l(L_2(G')) \otimes \mathbb{Z}_p \neq 0$$

for some k, l with $k + l = i$ or

$$Tor(H_k(L_1(G)), H_l(L_2(G')) \otimes \mathbb{Z}_p) \neq 0$$

for some k, l with $k + l = i - 1$. Since tensoring with $\mathbb{Z}_p$ does not preserve short exact sequences, we have to add here that if

$$(\bigoplus_{k+l=i-1} Tor(H_k(L_1(G)), H_l(L_2(G')))) \otimes \mathbb{Z}_p = 0,$$

then

$$Tor(\bigoplus_{k+l=i-1} Tor(H_k(L_1(G)), H_l(L_2(G'))), \mathbb{Z}_p) = 0$$

and hence

$$(\bigoplus_{k+l=i;k,l>0} (H_k(L_1(G)) \otimes H_l(L_2(G')))) \otimes \mathbb{Z}_p = H_i(L_1(G) \wedge L_2(G')) \otimes \mathbb{Z}_p.$$

Therefore

$$\begin{aligned} n_{G,G'}(\mathbb{Z}_p) = min\{i \mid\ & H_k(L_1(G)) \otimes H_l(L_2(G')) \otimes \mathbb{Z}_p \neq 0 \\ & \text{for some } k, l \text{ with } k + l = i \\ & \text{or } Tor(H_k(L_1(G)), H_l(L_2(G'))) \otimes \mathbb{Z}_p \neq 0 \text{ for some } k + l = i - 1\}. \end{aligned}$$

Since $H_k(K(\mathbb{Z}_{(q)}, n))$ is a p-divisible group (see the proof of Proposition 3.1), $n_{G,G'}(\mathbb{Z}_p) = \infty$ if one of the groups G, G' equals $\mathbb{Q}$ or $\mathbb{Z}_{(q)}$ for a prime $q \neq p$. Since $H_k(K(G,n))$ is a q-torsion group for a q-torsion group G, we may consider only $G, G' \in \{\mathbb{Z}_{(p)}, \mathbb{Z}_p, \mathbb{Z}_{p^\infty}\}$. Note that $n_{G,G'}(\mathbb{Z}_p) = D_1(G) + D_2(G')$ if $G, G' \in \{\mathbb{Z}_{(p)}, \mathbb{Z}_p\}$. If only one of the groups G, G' equals $\mathbb{Z}_{p^\infty}$, say G', then $G \otimes G' \otimes \mathbb{Z}_p = G \otimes \mathbb{Z}_{p^\infty} \otimes \mathbb{Z}_p = 0$ and hence $n_{G,G'}(\mathbb{Z}_p) > D_1(G) + D_2(\mathbb{Z}_{p^\infty})$. By the Bockstein inequalities, $n_{G,G'}(\mathbb{Z}_p) \geq D_1(G) + D_2(\mathbb{Z}_p) \geq D_1(\mathbb{Z}_p) + D_2(\mathbb{Z}_p)$. To complete the proof in this case we show that $n_{\mathbb{Z}_{p^\infty},\mathbb{Z}_{p^\infty}}(\mathbb{Z}_p) > D_1(\mathbb{Z}_{p^\infty}) + D_2(\mathbb{Z}_{p^\infty}) + 1$. Since $\mathbb{Z}_{p^\infty} \otimes \mathbb{Z}_{p^\infty} = 0$, it follows that $n_{\mathbb{Z}_{p^\infty},\mathbb{Z}_{p^\infty}}(\mathbb{Z}_p) > D_1(\mathbb{Z}_{p^\infty}) + D_2(\mathbb{Z}_{p^\infty})$. Denote $D_1(\mathbb{Z}_{p^\infty}) = k$ and $D_2(\mathbb{Z}_{p^\infty}) = l$. Note that $H_k(L_1(\mathbb{Z}_{p^\infty})) \otimes H_{l+1}(L_2(\mathbb{Z}_{p^\infty})) \otimes \mathbb{Z}_p = 0 = H_{k+1}(L_1(\mathbb{Z}_{p^\infty})) \otimes H_l(L_2(\mathbb{Z}_{p^\infty})) \otimes \mathbb{Z}_p$. Since $Tor(\mathbb{Z}_{p^\infty}, \mathbb{Z}_{p^\infty}) = \mathbb{Z}_{p^\infty}$, we have that $Tor(H_k(L_1(\mathbb{Z}_{p^\infty})), H_l(L_2(\mathbb{Z}_{p^\infty}))) \otimes \mathbb{Z}_p = 0$. Therefore

$$H_{k+l+1}(L_1(\mathbb{Z}_{p^\infty}) \wedge L_2(\mathbb{Z}_{p^\infty})) \otimes \mathbb{Z}_p = 0.$$

Hence $n_{\mathbb{Z}_{p^\infty},\mathbb{Z}_{p^\infty}}(\mathbb{Z}_p) > k + l + 1$. By the inequality B2 we have

$$n_{\mathbb{Z}_{p^\infty},\mathbb{Z}_{p^\infty}}(\mathbb{Z}_p) \geq (k+1) + (l+1) \geq D_1(\mathbb{Z}_p) + D_2(\mathbb{Z}_p).$$

Hence $n_M(\mathbb{Z}_p) = D_1(\mathbb{Z}_p) + D_2(\mathbb{Z}_p)$.

3) $H = \mathbb{Z}_{(p)}$. By the definition,

$$\begin{aligned} n_{G,G'}(\mathbb{Z}_{(p)}) &= min\{i \mid H_i(L_1(G) \wedge L_2(G')) \otimes \mathbb{Z}_{p^\infty} \neq 0\} \\ &= min\{i \mid H_k(L_1(G)) \otimes H_l(L_2(G')) \otimes \mathbb{Z}_{p^\infty} \neq 0; k + l = i\}. \end{aligned}$$

Note that only for $G = G' = \mathbb{Z}_{(p)}$ can we have $n_{G,G'}(\mathbb{Z}_{(p)}) \neq \infty$. Hence,

$$n_M(\mathbb{Z}_{(p)}) = n_{\mathbb{Z}_{(p)},\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)}) = D_1(\mathbb{Z}_{(p)}) + D_2(\mathbb{Z}_{(p)}).$$

4) $H = \mathbb{Z}_{p^\infty}$. By the definition

$$n_{G,G'}(\mathbb{Z}_{p^\infty}) = min\{i \mid Tor(H_i(L_1(G) \wedge L_2(G')), \mathbb{Z}_p) \neq 0\}.$$

Only when $G, G' \in \{\mathbb{Z}_{(p)}, \mathbb{Z}_p, \mathbb{Z}_{p^\infty}\}$ can the group $H_i(L_1(G) \wedge L_2(G'))$ contain p-torsion. Hence $n_M(\mathbb{Z}_{p^\infty}) = min\{n_{G,G'}(\mathbb{Z}_{p^\infty}) \mid G, G' \in \{\mathbb{Z}_{(p)}, \mathbb{Z}_p, \mathbb{Z}_{p^\infty}\}\}$. The computations show that

$$\begin{aligned} n_{\mathbb{Z}_{(p)},\mathbb{Z}_p}(\mathbb{Z}_{p^\infty}) &= D_1(\mathbb{Z}_{(p)}) + D_2(\mathbb{Z}_p), \\ n_{\mathbb{Z}_p,\mathbb{Z}_{(p)}}(\mathbb{Z}_{p^\infty}) &= D_1(\mathbb{Z}_p) + D_2(\mathbb{Z}_{(p)}), \\ n_{\mathbb{Z}_p,\mathbb{Z}_p}(\mathbb{Z}_{p^\infty}) &= D_1(\mathbb{Z}_p) + D_2(\mathbb{Z}_p), \\ n_{\mathbb{Z}_{p^\infty},\mathbb{Z}_{p^\infty}}(\mathbb{Z}_{p^\infty}) &= D_1(\mathbb{Z}_{p^\infty}) + D_2(\mathbb{Z}_{p^\infty}) + 1, \\ n_{\mathbb{Z}_p,\mathbb{Z}_{p^\infty}}(\mathbb{Z}_{p^\infty}) &= D_1(\mathbb{Z}_p) + D_2(\mathbb{Z}_{p^\infty}) + 1, \\ n_{\mathbb{Z}_{p^\infty},\mathbb{Z}_p}(\mathbb{Z}_{p^\infty}) &= D_1(\mathbb{Z}_{p^\infty}) + D_2(\mathbb{Z}_p) + 1, \\ n_{\mathbb{Z}_{(p)},\mathbb{Z}_{p^\infty}}(\mathbb{Z}_{p^\infty}) &= D_1(\mathbb{Z}_{(p)}) + D_2(\mathbb{Z}_{p^\infty}), \\ n_{\mathbb{Z}_{p^\infty},\mathbb{Z}_{(p)}}(\mathbb{Z}_{p^\infty}) &= D_1(\mathbb{Z}_{p^\infty}) + D_2(\mathbb{Z}_{(p)}). \end{aligned}$$

According to the general inequality $n_{G,G'}(H) \geq D_1(G) + D_2(G')$ we have that $n_{\mathbb{Z}_{(p)},\mathbb{Z}_{(p)}}(\mathbb{Z}_{p^\infty}) \geq D_1(\mathbb{Z}_{(p)}) + D_2(\mathbb{Z}_{(p)})$.

If D_1 is p-regular, then $n_M(\mathbb{Z}_{p^\infty}) = D_1(\mathbb{Z}_{p^\infty}) + D_2(\mathbb{Z}_{p^\infty}) = D_1(\mathbb{Z}_{(p)}) + D_2(\mathbb{Z}_{p^\infty})$ by Bockstein's inequalities. Similarly, $n_M(\mathbb{Z}_{p^\infty}) = D_1(\mathbb{Z}_{p^\infty}) + D_2(\mathbb{Z}_{p^\infty})$ if D_2 is p-regular.

If both D_1 and D_2 are p-singular, then $D_1(\mathbb{Z}_{(p)}) + D_2(\mathbb{Z}_{p^\infty}) \geq D_1(\mathbb{Z}_{p^\infty}) + D_2(\mathbb{Z}_{p^\infty}) + 1$ and $D_1(\mathbb{Z}_{p^\infty}) + D_2(\mathbb{Z}_{(p)}) \geq D_1(\mathbb{Z}_{p^\infty}) + D_2(\mathbb{Z}_{p^\infty}) + 1$. By virtue of this and the Bockstein inequalities,

$$n_M(\mathbb{Z}_{p^\infty}) = min\{D_1(\mathbb{Z}_p) + D_2(\mathbb{Z}_p), D_1(\mathbb{Z}_{p^\infty}) + D_2(\mathbb{Z}_{p^\infty}) + 1\}.$$

Proposition 4.3. *Let D_1 and D_2 be cd-types. Then*

$$K(D_1) \wedge K(D_2) \leq K(D_1[+]D_2).$$

Proof. Let M be as in Proposition 4.2. We may assume that M has the form specified in Proposition 4.1, and thus it suffices to show that $D_1[+]D_2(G) \leq n_M(G)$ for every $G \in \sigma$. The result follows from Proposition 4.2, Bockstein inequalities BI1-6, Bockstein formulas 1-4S, and Assertion 3.1.

5. Proof of Theorem A

We need the following theorem, which was proven first in [10], and later with a better proof, in [22].

Theorem 4. *Let $X \subset \mathbb{R}^n$ be a tame compact subset of dimension $dimX < n-2$. Assume that $dim(X \times Y) < n$ for some compact space Y. Then every map $g : Y \to \mathbb{R}^n$ can be approximated arbitrarily closely by maps $g' : Y \to \mathbb{R}^n - X$.*

We recall that a compactum $X \subset \mathbb{R}^n$ is tame if it has the 1-ULC property: for every $x \in X$ and for every open neighborhood U in $\mathbb{R}^n$ there is a smaller neighborhood V such that the inclusion $V - X \hookrightarrow U - X$ induces the zero homomorphism between the fundamental groups. For a codimension three tame compactum $X \subset \mathbb{R}^n$ every map of a 1-dimensional polyhedron $f : K^1 \to \mathbb{R}^n$ can be approximated arbitrarily closely by maps f' with $f'(K^1) \cap X = \emptyset$ [33]. This implies that a compact subset of a codimension three tame compactum is tame itself.

Remark 5.1. The compactum X in Theorem 4 can be replaced by a countable union $\bigcup X_i$ of tame compacta.

Proof of Theorem A. Let $f : X \to \mathbb{R}^n, g : Y \to \mathbb{R}^n$ be given maps and let $\epsilon > 0$. By Theorem 1 we can find an ϵ-approximation $f' : X \to \mathbb{R}^n$ with $D_X \leq D_{f'(X)} \leq D_X \vee 2$. By Štanko's reimbedding theorem [24] we may assume that $f'(X)$ is a tame subset in $\mathbb{R}^n$. Let us denote $X' = f'(X)$ and let $dim(X \times Y) = m$. If $m < n$ then the theorem is proven by [7]. Consider the cd-type $D' = D_{X'} \vee (\bar{D}_Y[+]m)$. We will show that $D'(G) \geq m - n + 2$ for all $G \in \sigma$. By virtue of the Bockstein inequalities it suffices to check this inequality for $G = \mathbb{Q}$ and $G = \mathbb{Z}_{p^\infty}$. First we note that $D'(\mathbb{Q}) = \max\{D_{X'}(\mathbb{Q}), \bar{D}_Y(\mathbb{Q}) + m\} \geq \bar{D}_Y(\mathbb{Q}) + m = m - D_Y(\mathbb{Q}) \geq m-(n-3) \geq m-n+2$. Here we used the inequality $D_Y(\mathbb{Q}) \leq dimY \leq n-3$. Next, $D'(\mathbb{Z}_{p^\infty}) = \max\{D_{X'}(\mathbb{Z}_{p^\infty}), (\bar{D}_Y[+]m)(\mathbb{Z}_{p^\infty})\} \geq (\bar{D}_Y[+]m)(\mathbb{Z}_{p^\infty}) = m + \bar{d}_Y + \bar{\epsilon}_Y - 1 = m - d_Y + (1 - \epsilon_Y + \delta_Y) - 1 = m - d_Y + \delta_Y - \epsilon_Y \geq m - d_Y - 1 \geq m-(n-3)-1 = m-n+2$. Let $k = m-n+1$; then $D' - k$ is a positive dimensional function. Since $D' - k = \Phi^{-1}(D'[+]-k)$, where Φ is the imbedding of Proposition 3.3, $D' - k$ is a positive cd-type. Assertions 4.1 and 4.2 imply the chain of inequalities of cd-types:

$$\Sigma^k K(D'-k) \leq \Sigma^{k-1} K(D'-k+1) \leq \cdots \leq \Sigma K(D'-1) \leq K(D').$$

Since $D_{X'} \leq D'$, it follows that $K(D_{X'}) \geq K(D')$. Since $X' \tau K(D_{X'})$ and $K(D_{X'}) \geq \Sigma^k K(D'-k)$, we have $X' \tau \Sigma^k K(D'-k)$. Note that $\Sigma^k K(D'-k)$ is homotopy equivalent to the join product $S^{k-1} * K(D'-k)$, and hence

$$X' \tau (S^{k-1} * K(D'-k)).$$

Theorem 3 implies that there exists an F_σ-set $Z \subset X'$ with

$$Z \tau K(D'-k) \quad \text{and} \quad (X'-Z)\tau S^{k-1},$$

i.e. $dim(X'-Z) \leq k-1$. Note that for every compact subset $C \subset Z$,

$$\begin{aligned} dim(C \times Y) = \|D_C[+]D_Y\| \ &\leq (\text{Assertion 3.2}) \|(D'-k)[+]D_Y\| \\ &= \ \max\{\|D_{X'}[+]D_Y\|, \|m + \bar{D}_Y[+]D_Y\|\} - k. \end{aligned}$$

Since $D_{X'} \leq D_X \vee 2$, Assertion 3.2 implies

$$\|D_{X'}[+]D_Y\| \leq \max\{\|D_X[+]D_Y\|, \|2[+]D_Y\|\}$$
$$= \max\{dim(X \times Y), 2 + dimY\} = m.$$

By Proposition 3.4 $\|\bar{D}_Y[+]D_Y\| = 0$ and hence $\|m[+]\bar{D}_Y[+]D_Y\| = m$. Then

$$dim(C \times Y) \leq m - k = n - 1.$$

Hence by the countable union theorem $dim(Z \times Y) \leq n-1$. Using Remark 5.1 and the fact that Z is a σ-compactum, there is an ϵ-approximation g' of $g : Y \to \mathbb{R}^n$ with $g'(Y) \cap Z = \emptyset$. Therefore $g'(Y) \cap f'(X) \subset X' - Z$, and hence

$$dim(g'(Y) \cap f'(X)) \leq k - 1 = m - n = dim(X \times Y) - n.$$

6. Dimension of the union

It is known that there are two possibilities for the dimension of the square of an n-dimensional compactum X: either $dimX^2 = 2n$ or $dimX^2 = 2n - 1$ [19]. This dichotomy defines a partition of the class of compacta into two types: the first (with $dimX^2 = 2dimX$) and the second.

Theorem B. *Assume that a compactum Z is expressed as the union $X \cup Y$. Then:*
1) $dim(X \cup Y) \leq dim(X \times Y) + 2$,
2) $dim(X \cup Y) \leq dim(X \times Y) + 1$ provided Z is of the first type.
If X is an F_σ-set in Z, then there exist compacta $X' \subset X$ and $Y' \subset Y$ such that
3) $dim(X \cup Y) \leq dim(X' \times Y') + 2$ and
4) $dim(X \cup Y) \leq dim(X' \times Y') + 1$ provided Z is of the first type.

For the second type of compacta Z statement 4) of Theorem B generally is not true.

Theorem C. *There exists a compactum $Z = X \cup Y$ with $X \in F_\sigma$ such that*

$$dim(X \cup Y) > dim(X' \times Y') + 1$$

for any compacta $X' \subset X$ and $Y' \subset Y$.

The following proposition is well-known.

Proposition 6.1. *For every finite-dimensional compactum Z there is a field $F \in \sigma$ such that $dimZ \leq dim_F Z + 1$. A compactum Z is of the first type if and only if there is a field $F \in \sigma$ such that $dim_F Z = dimZ$.*

Proof. Since Z is finite-dimensional, by the Alexandroff theorem $dimZ = dim_{\mathbb{Z}} Z$. By the Bockstein basis theorem, $dimZ = dim_{\mathbb{Z}_{(p)}} Z$ for some prime number p. By the inequalities BI5 and BI1 we have $dimZ \leq max\{dim_{\mathbb{Q}} Z, dim_{\mathbb{Z}_p} Z + 1\}$. Hence $\mathbb{Z}_p$ or $\mathbb{Q}$ is that field $F \in \sigma$.

If $dim_F Z = dimZ$ for some field $F \in \sigma$, then $dimZ^2 \geq dim_F Z^2 = dim_F Z + dim_F Z = 2dimZ$. The inequality $dimZ^2 \leq 2dimZ$ is well-known; hence $dimZ^2 = 2dimZ$. Assume that $dimZ^2 = 2dimZ$; then (see the formulas for Φ^{-1} in Proposition 5)

$$dimZ^2 = \|D_Z[+]D_Z\| = max\{2c, 2d(p) + \epsilon^2(p) - \delta^2(p)\}$$
$$= max\{2c, 2d(p) + \epsilon(p) - \delta(p)\} = 2max\{c, d(p) + 1/2(\epsilon(p) - \delta(p))\},$$

while

$$2dimZ = 2\|D_Z\| = 2max\{c, d(p) + \epsilon(p) - \delta(p)\}.$$

It follows that either this maximum is equal to c or $\epsilon(p) - \delta(p) = 0$. In the first case $F = \mathbb{Q}$, in the second $F = \mathbb{Z}_p$.

We need the following two theorems.

Theorem 5 (Dydak [21]). *If $X\tau K_1$ and $Y\tau K_2$, then $(X \cup Y)\tau(K_1 * K_2)$.*

Corollary 6.1 ([21]). *$dim_R(X \cup Y) \leq dim_R X + dim_R Y + 1$ for every ring with unity R.*

Theorem 6 (Olszewski [18]). *For every separable metric space W and every countable complex K with $W\tau K$ there is a completion $\bar{W}$ with the same property $\bar{W}\tau K$.*

Proposition 6.2. *For every separable metric space W and every compactum C with $dim(W \times C) \leq n$ there is a completion $\bar{W}$ with $dim(\bar{W} \times C) \leq n$.*

Proof. We assume that W lies in the Hilbert cube Q, and we find a G_δ extension $\bar{W}$ with the required property. Let $\pi : Q \times C \to Q$ be the projection. By the completion theorem for the covering dimension there is a G_δ-extension H of $W \times C$ in $Q \times C$ with $dimH \leq n$. Then H is the intersection $\bigcap O_i$ of open sets O_i. Define $O'_i = Q - \pi((Q \times C) - O_i)$. Then we define $\bar{W} = \bigcap O'_i$.

Proposition 6.3. *For every subset $Y \subset Z$ of a compact metric space Z there is a G_δ-set $\bar{Y} \supset Y$ such that $dim(\bar{Y} \times (Z - \bar{Y})) \leq dim(Y \times (Z - Y))$.*

Proof. Let $\bar{Y}_0 \supset Y$ be a G_δ-extension. Then $Z - \bar{Y}_0 = \bigcup F_0^i$ is a countable union of compacta. Using Proposition 12, we define a G_δ-extension $\bar{Y}_1 \supset Y$ of Y such that $dim(\bar{Y}_1 \times F_0^i) = dim(Y \times F_0^i)$ for all i. Then consider a countable union of compacta $\bigcup F_1^i = Z - \bar{Y}_1$ and define $\bar{Y}_2$ with $dim(\bar{Y}_2 \times F_1^i) = dim(Y \times F_1^i)$, and so on. We define $\bar{Y} = \bigcap \bar{Y}_k$. Let $C \subset Z - \bar{Y}$ be a compactum. Note that $C \subset \bigcup(Z - \bar{Y}_k) = \bigcup F_k^i$. Then

$$\begin{aligned} dim(\bar{Y} \times C) &\leq max\{dim(\bar{Y} \times (C \cap F_k^i))\} \leq max\{dim(\bar{Y} \times F_k^i)\} \\ &\leq max\{dim(\bar{Y}_{k+1} \times F_k^i)\} = \max\{dim(Y \times F_k^i)\} \leq dim(Y \times (Z - Y)). \end{aligned}$$

Then by the countable union theorem [34]

$$dim(\bar{Y} \times (Z - \bar{Y})) \leq max\{dim(\bar{Y} \times C) \mid C \subset Z - \bar{Y} \text{ is compact}\} \leq dim(Y \times (Z - Y)).$$

Proof of Theorem B. 1) In view of Proposition 6.3 we may assume that $X \in F_\sigma$. By Proposition 6.1 there is a field $F \in \sigma$ such that $dim(X \cup Y) \leq dim_F(X \cup Y) + 1$. By Corollary 6.1

$$dim_F(X \cup Y) + 1 \leq dim_F X + dim_F Y + 2.$$

The equality $dim_F X + dim_F Y = dim_F(X \times Y)$ holds if one of the factors is σ-compact. The inequality $dim(X \times Y) \geq dim_F(X \times Y)$ completes the proof.

2) By Proposition 11 we may assume that $dim(X \cup Y) = dim_F(X \cup Y)$; then the inequality follows by the same argument.

3-4) Assume that X is an F_σ. By Theorem 6 there is a G_δ-set $\bar{X} \supset X$ of $dim_G \bar{X} = dim_G X$ for every $G \in \sigma$. Apply the argument of the proof of 1) and 2) to the union $Z = (Z - \bar{X}) \cup \bar{X}$ to obtain inequalities

$$dimZ \leq dim_F(Z - \bar{X}) + dim_F \bar{X} + 2$$

and, in the case of the first type Z:

$$dimZ \leq dim_F(Z - \bar{X}) + dim_F \bar{X} + 1.$$

Note that $dim_F(Z-\bar{X})+dim_F\bar{X}=dim_F(Z-\bar{X})+dim_F X$. Since $Z-\bar{X}$ is an F_σ, the countable union theorem implies that there exists a compactum $Y'\subset Z-\bar{X}$ with $dim_F(Z-\bar{X})=\dim_F Y'$. Similarly there is a compact subset $X'\subset X$ with $dim_F X'=dim_F X$. Then

$$\begin{aligned} dim_F(Z-\bar{X})+dim_F X &= dim_F Y'+dim_F X' \\ &= dim_F(Y'\times X')\le dim(Y'\times X'). \end{aligned}$$

Then the inequalities 3-4) follow.

Proof of Theorem C. We fix a prime p and define three dimensional functions D_1, D_2 and D by setting $D_1(\mathbb{Z}_p)=D_1(\mathbb{Z}_{(p)})=D_1(\mathbb{Q})=2$, $D_1(\mathbb{Z}_{p^\infty})=D_1(G)=1$ for all other $G\in\sigma$; $D_2(\mathbb{Z}_{(p)})=2$, $D_2(\mathbb{Z}_p)=D_2(\mathbb{Z}_{p^\infty})=D_2(\mathbb{Q})=D_2(G)=1$ for all other $G\in\sigma$; and $D(\mathbb{Z}_{(p)})=5$, $D(\mathbb{Z}_p)=D(\mathbb{Z}_{p^\infty})=4$, $D(\mathbb{Q})=D(G)=3$. By the realization theorem [20] there is a compactum Z with $D_Z=D$. Using the formulas of Section 3 we can compute that $D_1[+]D_2(\mathbb{Z}_{(p)})=3$. Let us consider the join $M=K(D_1)*K(D_2)$ of complexes $K(D_1)$ and $K(D_2)$ defined in Section 4. By Proposition 4.1 the complex M is e-equivalent to a complex $\bigvee_{G\in\sigma}K(G,n_M(G))$. Since $K(D_1)*K(D_2)$ is itself homotopy equivalent to $\Sigma(K(D_1)\wedge K(D_2))$, we have

$$n_M=n_{K(D_1)\wedge K(D_2)}(G)+1.$$

The computations of Proposition 4.2 show that $n_M(G)\ge D(G)$ for all $G\in\sigma$. Therefore the property $Z\tau M$ holds. By Theorem 3 there are subsets $X,Y\subset Z$ such that $X\in F_\sigma$, $Y=Z-X$, $X\tau K(D_1)$ and $Y\tau K(D_2)$. For any compacta $X'\subset X$ and $Y'\subset Y$ we have the inequalities $D_{X'}\le D_1$ and $D_{Y'}\le D_2$. Hence $dim(X'\times Y')\le 3$. Since Z is 5-dimensional, we have $5=dimZ>dim(X'\times Y')+1$.

Remark. The proof of Theorem C shows that the extension type inequality $K(D_1)\wedge K(D_2)\le K(D_1[+]D_2)$ of Proposition 4.3 can be strict.

Still it is not clear whether the inequality $dim(X\cup Y)\le dim(X\times Y)+1$ holds for the second type of compacta $Z=X\cup Y$. Theorem C demonstrates that even if it does, its proof cannot be based only on the dimension theory of the product of compacta. For non-compact factors a corresponding dimension theory of the product does not exist (see an example in [26]). A related question is due to E.V. Ščepin: Let a topological space $Z=X\cup Y$ be a union of two subspaces X and Y, is always the dimension of the join $X*Y$ greater than or equal to the dimension of the union $X\cup Y=Z$?

Acknowledgments

I am very thankful to the referee of this paper for valuable comments.

References

1. D. McCullough and L. Rubin *Intersection of separators and essential submanifolds of* I^N, Fund. Math. vol 116 (1983), 131-142. MR **85j:**54055
2. D. McCullough and L. Rubin *Some m-dimensional compacta admitting a dense set of imbedding into* $\mathbb{R}^{2m}$, Fund. Math. vol 133 (1989), 237-245. MR **91h:**54055
3. J. Krasinkiewicz and K. Lorentz *Disjoint membranes in cubes*, Bull. Polish Acad. Sci. Math. 36 (1988), 397-402. MR **92d:**54045
4. J. Krasinkiewicz *Imbeddings into* $\mathbb{R}^n$ *and dimension of products*, Fund. Math. 133 (1989), 247-253. MR **91h:**54053
5. S. Spież *Imbeddings in* $\mathbb{R}^n$ *of m-dimensional compacta with* $\dim(X\times X)<2m$, Fund. Math. 134 (1990), 105-115. MR **91j:**54062

6. A.N. Dranishnikov and E.V. Ščepin *Stability of intersections of compact spaces in euclidean space*, Russian Mathematical Surveys 44:5 (1989), 194-195. MR **91k:**54057
7. A.N. Dranishnikov *On the mapping intersection problem*, Pacif. J. Math. 173 (1996), 403-412. MR **97e:**54030
8. A.N. Dranishnikov, D. Repovš, E.V. Ščepin *On intersections of compacta of complementary dimensions in Euclidean space*, Topol. Appl. 38 (1991), 237–253. MR **92g:**57032
9. A.N. Dranishnikov *Spanier-Whitehead duality and stability of intersections of compacta*, Trudy Mat. Inst. Steklov. 196 (1991), 47–50; English transl., Proc. Steklov Inst. Math. 1992, no. 4 (196), 53–56. MR **92h:**55003
10. A.N. Dranishnikov *On intersection of compacta in Euclidean space*, Proc. Amer. Math. Soc. 112 (1991), 267–275. MR **91h:**54024
11. A.N. Dranishnikov *On intersection of compacta in Euclidean space. II*, Proc. Amer. Math. Soc. 113 (1991), 1149-1154. MR **92c:**54015
12. S. Spież *On pairs of compacta with* $\dim(X \times Y) < \dim X + \dim Y$, Fund. Math. 135 (1990), 213–222. MR **91j:**54064
13. J. Segal and S. Spież *On transversaly trivial maps*, Q and A in Gen. Topol. 8 (1990), 91-100. MR **91i:**54010
14. S. Spiez and H. Toruńczyk *Moving compacta in* $\mathbb{R}^n$ *apart* , Topol. Appl. 41 (1991), 193–204. MR **92m:**57030
15. A. Dranishnikov, D. Repovš, E. Ščepin *On intersection of compacta in Euclidean space: the metastable case*, Tsukuba J. Math. 17 (1993), 549-564. MR **95a:**55002
16. A. Dranishnikov and J. West *On compacta that intersect unstably in Euclidean space*, Topol. Appl. 43 (1992), 181-187. MR **92m:**54063
17. A.N. Dranishnikov, D. Repovš, E.V.Ščepin *On approximation and embedding problems for cohomological dimension*, Topol. Appl. 55 (1994), 67-86. MR **94m:**55001
18. W. Olszewski *Completion theorem for cohomological dimensions*, Proc. Amer. Math. Soc. 123 (1995), 2261-2264. MR **95k:**54064
19. V.I. Kuzminov *Homological dimension theory*, Russian Math. Surveys 23, issue 5 (1968), 1–45. MR **39:**2158
20. A. Dranishnikov *Homological dimension theory*, Russian Math. Survey 43, issue 4 (1988), 11–63. MR **90e:**55003
21. J. Dydak *Cohomological dimension and metrizable spaces II*, Trans. Amer. Math. Soc. 348 (1996). MR **96h:**55001
22. A.N. Dranishnikov *An extension of mappings into CW-complexes*, Mat. Sb. 182 (1991), 1300–1310; English transl., Math. USSR Sb. 74 (1993), 47–56. MR **93a:**55002.
23. A. Dold and R. Thom *Quasifaserungen und unendliche symmetrische Produkte*, Ann. of Math. 67 (1958), 239–281. MR **20:**3542
24. M.A. Štanko *Embeddings of compacta in Euclidean space*, Math. USSR-Sbornik 10 (1970), 234–254. MR **42:**6804
25. A.N. Dranishnikov *Eilenberg-Borsuk theorem for maps in arbitrary complexes*, Russian Acad. Sci. Sb. Math. 81 (1995), 467-475. MR **95j:**54028
26. A.N. Dranishnikov, D. Repovš and E.V. Ščepin *Dimension of product with continua*, Topology Proceedings 18 (1993), 57-73. MR **96b:**54054
27. A. Dranishnikov and D. Repovš *On unstable intersection of 2-dimensional compacta in Euclidean 4-space*, Topol. Appl. 54 (1993), 3-11. MR **95a:**55001
28. A. Dranishnikov and D. Repovš *Cohomological dimension with respect to perfect groups*, Topol. Appl. 74 (1996), 123-140. MR **98i:**55002
29. J. Dydak and K. Yokoi *Hereditarily aspherical compacta*, Proc. Amer. Math. Soc. 124 (1996), 1933–1940. MR **96h:**57019
30. A. Dranishnikov and J. Dydak *Extension dimension and extension types*, Proc. Steklov Inst. Math. 212 (1996), 55–88. MR **99h:**54049
31. Y. Sternfeld *A short elementary proof of the Dranishnikov-West theorem on stable intersection of compacta in Euclidean space*, Topology Appl. 74 (1996), 177–178. MR **97m:**54058
32. D. Sullivan *Geometric topology, Part I: Localization, Periodicity, and Galois Symmetry*, M.I.T. Press (1970). MR **58:**13006a
33. R.D. Edwards *Demension theory I*, Lecture Notes in Mathematics 438 (1975), 195-211. MR **52:**15477

34. R. Engelking *Theory of Dimensions Finite and Infinite*, Heldermann Verlag (1995). MR **97j:**54033
35. R. E. Mosher and M.C. Tangora *Cohomology operations and applications in homotopy theory*, Harper & Row (1968). MR **37:**2223

Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania 16802

E-mail address: dranish@math.psu.edu

Current address: Department of Mathematics, University of Florida, Gainesville, Florida 32611

E-mail address: dranish@math.ufl.edu