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THE RANGE OF TRACES ON QUANTUM HEISENBERG MANIFOLDS

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ABSTRACT. We embed the quantum Heisenberg manifold $D^c_{\mu\nu}$ in a crossed product C*-algebra. This enables us to show that all tracial states on $D^c_{\mu\nu}$ induce the same homomorphism on $K_0(D^c_{\mu\nu})$, whose range is the group $\mathbf{Z} + 2\mu\mathbf{Z} + 2\nu\mathbf{Z}$.

1. INTRODUCTION

For a positive integer c, let M_c denote the Heisenberg manifold consisting of the quotient G/H_c, where G is the Heisenberg group,

$$G = \left\{ \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in R \right\},$$

and H_c is the subgroup of G obtained when x, y, and cz are integers.

In [RF3] Rieffel constructed a quantization deformation $\{D^{c,\hbar}_{\mu\nu}\}_{\hbar \in R}$ of M_c in the direction of a given Poisson bracket $\Lambda_{\mu\nu}$ determined by two real parameters μ and ν. We drop from now on the Planck constant $\hbar$ from our notation, because the algebras $D^{c,\hbar}_{\mu\nu}$ and $D^{c,1}_{\hbar\mu,\hbar\nu}$ are isomorphic and we will denote either one by $D^c_{\hbar\mu,\hbar\nu}$. Also, since $D^c_{\mu\nu} \cong D^c_{\mu+n,\nu+m}$ for any integers n and m ([AB1]), we view the parameters μ and ν as running in the circle $\mathbf{T}$.

We discussed the K-theory of the quantum Heisenberg manifolds in [AB2] and found that $K_0(D^c_{\mu\nu}) = \mathbf{Z}^3 \oplus \mathbf{Z}_c$ and $K_1(D^c_{\mu\nu}) = \mathbf{Z}^3$, which shows that two algebras corresponding to deformations of different Heisenberg manifolds are not isomorphic. In [AB1] we constructed finitely generated projective modules over the algebra $D^c_{\mu\nu}$ with traces 2μ and 2ν respectively, where the trace considered was that defined in [RF3]. This suggests employing the range of traces on $K_0(D^c_{\mu\nu})$ as an invariant to discuss isomorphism and strong-Morita equivalence types of the family $\{D^c_{\mu\nu}\}$, as was done for non-commutative tori ([PV], [RF1]) and Heisenberg C*-algebras ([PA2], [PA1]).

This work is organized as follows. In Section 2 we embed the algebra $D^c_{\mu\nu}$ in a crossed product. This is done in a more general context, by viewing the quantum Heisenberg manifolds as generalized fixed-point algebras, as in [RF3]. In Section 3

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we show that all traces on $D^c_{\mu\nu}$ give rise to the same homomorphism on $K_0(D^c_{\mu\nu})$, whose range is the group $\mathbf{Z} + 2\mu\mathbf{Z} + 2\nu\mathbf{Z}$.

2. The embedding

The purpose of this section is to embed each quantum Heisenberg manifold in a crossed product algebra $A \rtimes \mathbf{Z}$, A being a C*-subalgebra of $L^\infty(\mathbf{T}^2)$. Our construction carries over into a somewhat more general context, which we next describe.

We first recall some facts established in [AB2]. Let λ and σ be two commuting automorphisms of a C*-algebra B. Let $u : \mathbf{Z} \times \mathbf{Z} \longrightarrow U\mathbf{Z}\mathcal{M}(B)$ be a λ-cocycle in the first variable and a σ-cocycle in the second one, and define the action $\gamma^{\sigma,u}$ of $\mathbf{Z}$ on $B \rtimes_\lambda \mathbf{Z}$ by $(\gamma^{\sigma,u}_k \Phi)(p) = u(p,k)\sigma_k[\Phi(p)]$. When the C*-algebra $B = C_0(M)$ is commutative and the actions λ and σ are free and proper, then $\gamma^{\sigma,u}$ is proper and the corresponding generalized fixed-point algebra $D^{\sigma,u}$, in the sense of Rieffel ([RF4]), is the closure in the multiplier algebra $\mathcal{M}(C_0(M) \rtimes_\lambda \mathbf{Z})$ of the *-subalgebra $C^{\sigma,u}$ consisting of functions $\Phi \in C_c(\beta M \times \mathbf{Z})$ such that the projection of $supp_M(\Phi)$ on M/σ is precompact and $\gamma^{\sigma,u}_k \Phi = \Phi$ for all $k \in \mathbf{Z}$, where $\gamma^{\sigma,u}$ has been extended to the multiplier algebra, and βM denotes the Stone-Čech compactification of M.

When the space M is taken to be $\mathbf{R} \times \mathbf{T}$, and $\sigma(x,y) = (x-1,y)$, $\lambda(x,y) = (x+2\mu, y+2\nu)$, and $u(p,k) = \exp(2\pi ickp(y-p\nu))$ for $(x,y) \in \mathbf{R} \times \mathbf{T}$, $k,p \in \mathbf{Z}$, then $D^{\sigma,u}$ is the quantum Heisenberg manifold denoted in [RF3] by $D^c_{\mu\nu}$, and we denote by $C^c_{\mu\nu}$ the dense *-subalgebra corresponding to $C^{\sigma,u}$.

In the general case, if F is a fundamental domain in M for the action σ (i.e. the canonical projection $\Pi : F \longrightarrow M/\sigma$ is a bijection), then any Φ in the dense subalgebra $C^{\sigma,u}$ is determined by the values $\Phi(m,p)$, for m running in F and $p \in \mathbf{Z}$. This suggests the idea of untwisting those functions so that they can be viewed as functions on the quotient space M/σ. A natural way of doing that is by multiplying them by a function H on M satisfying the opposite condition $\gamma^{\sigma,u^*} H = H$. Also, in order to get things to work from an algebraic point of view, it is necessary for H to satisfy

$$\overline{H}_{-p}(\lambda_{-p}m) = H_p(m) \quad \text{and} \quad H_{p+q}(m) = H_p(m)H_q(\lambda_{-p}m).$$

However, there might not be such a continuous function on M. This is the case for quantum Heisenberg manifolds. If a continuous map H as above existed, then multiplication by the function $\gamma \in C(\mathbf{R} \times \mathbf{T})$ defined by $\gamma(x,y) = H_1(x, y+\nu)$ would be a $C(\mathbf{T}^2)$-module isomorphism between $C(\mathbf{T}^2)$ and $X = \{\Phi \in C(\mathbf{R} \times \mathbf{T}) : \Phi(x+1,y) = \exp(2\pi icy)\Phi(x,y)\}$, in contradiction with [RF2, 3.9].

This is the reason why we are forced to get out of $C_0(M/\sigma)$ and consider a bigger subalgebra of $L^\infty(M/\sigma)$, as was done in [CU, 2.5] for the case of non-commutative tori.

Measurability considerations will impose some restrictions on the fundamental domain F. We next summarize the assumptions we will be making.

Assumptions and notation. In what follows, for a C*-algebra A we denote by $\mathcal{M}(A)$ its multiplier algebra, and by $\mathcal{U}(A)$ the group of unitary elements in A.

Throughout this section λ and σ denote free and proper commuting actions of $\mathbf{Z}$ on a locally compact Hausdorff space M, and $u : \mathbf{Z} \times \mathbf{Z} \longrightarrow \mathcal{U}M(C_0(M))$ denotes a map satisfying the cocycle conditions:

$$u(p+q,k) = u(p,k)\lambda_p[u(q,k)] \quad \text{and} \quad u(p,k+l) = u(p,k)\sigma_k[u(p,l)],$$

for any $k, l, p, q \in \mathbf{Z}$, where σ has been extended to the multiplier algebra. We also assume the existence of a Borel measurable fundamental domain F for σ in M such that the canonical projection $\Pi : F \longrightarrow M/\sigma$ has a Borel measurable inverse map. Thus a function f on M/σ is Borel measurable if and only if $f = \tilde{f} \circ \Pi$, for some Borel measurable function $\tilde{f}$ on M.

The generalized fixed-point algebra of $C_0(M) \rtimes_\lambda \mathbf{Z}$ under the action $\gamma^{\sigma,u}$ of $\mathbf{Z}$ defined by $(\gamma_k^{\sigma,u}\Phi)(m,p) = u(p,k)\Phi(\sigma_{-k}m,p)$, for $\Phi \in C_c(M \times \mathbf{Z})$ will be denoted by $D^{\sigma,u}$. We denote by $C^{\sigma,u}$ the dense *-subalgebra of $D^{\sigma,u}$ consisting of functions $\Phi \in C_c(\beta M \times \mathbf{Z})$ such that the projection of $supp_M(\Phi)$ on M/σ is precompact and that $\gamma_k^{\sigma,u}\Phi = \Phi$, for all $k \in \mathbf{Z}$.

Lemma 2.1. *Let $H : \mathbf{Z} \to \mathcal{U}L^\infty(M)$ be defined by: $H_1(m) = u^*(1,k)(m)$, for $m \in \sigma_k F$, and*

$$H_p(m) = \begin{cases} \prod_{q=0}^{p-1}(\lambda_q H_1)(m) & \text{if } p > 0, \\ 1 & \text{if } p = 0, \\ \prod_{q=p}^{-1}\overline{(\lambda_q H_1)}(m) & \text{if } p < 0. \end{cases}$$

Then:

i) H is a λ-cocycle (i.e. $H_{p+q}(m) = H_p(m)H_q(\lambda_{-p}m)$ for all $m \in M$, $p, q \in \mathbf{Z}$).
ii) $\overline{H}_{-p}(\lambda_{-p}m) = H_p(m)$, for all $m \in M$ and $p \in \mathbf{Z}$.
iii) $H_p(\sigma_{-k}m) = [u(p,k)H_p](m)$, for all $m \in M$ and $k, p \in \mathbf{Z}$.

Proof. i) For $q = 1$ and $p > 0$, we have

$$H_{p+1}(m) = \prod_{q=0}^{p}(\lambda_q H_1)(m) = H_p(m)(\lambda_{-p}H_1)(m) = H_p(m)H_1(\lambda_{-p}m).$$

An analogous computation shows that the equality holds for $p \leq 0$, and, once ii) is proven, the result follows by induction on q.

It suffices to prove ii) for $p > 0$, and in that case we have

$$\overline{H}_{-p}(\lambda_{-p}m) = \prod_{q=-p}^{q=-1}(\lambda_{p+q}H_1)(m) = \prod_{q=0}^{q=p-1}(\lambda_q H_1)(m) = H_p(m).$$

Finally, for $p > 0$, we have

$$\begin{aligned} H_p(\sigma_{-k}m) &= \prod_{q=0}^{p-1}(\lambda_q H_1)(\sigma_{-k}m) \\ &= \prod_{q=0}^{p-1}[\lambda_q(u(1,k))(\lambda_q H_1)](m) \\ &= u(p,k)H_p(m). \end{aligned}$$

This ends the proof in view of ii). □

Notation 2.2. Let H be as in Lemma 2.1. For $p \in \mathbf{Z}$ and $\Phi \in C^{\sigma,u}$ let $f_{\Phi,p} \in L^\infty(M/\sigma)$ be defined by $f_{\Phi,p}(\dot{m}) = H_p(m)\Phi(m,p)$, where $\dot{m}$ denotes the projection of m onto M/σ.

Theorem 2.3. *Let H be as in Lemma 2.1. Then the generalized fixed-point algebra $D^{\sigma,u}$ can be embedded in the crossed product $A \rtimes_\lambda \mathbf{Z}$, where A is any λ-invariant C^*-subalgebra of $L^\infty(M/\sigma)$ containing $\{f_{\Phi,p} : \Phi \in C^{\sigma,u}, p \in \mathbf{Z}\}$.*

Proof. Let $J : D^{\sigma,u} \longrightarrow A \rtimes_\lambda \mathbf{Z}$ be defined, at the level of functions $\Phi \in C^{\sigma,u}$, by $(J\Phi)(\dot{m}, p) = f_{\Phi,p}(\dot{m})$. Then, by properties i) and ii) in Lemma 2.1, J is a *-algebra homomorphism:

$$\begin{aligned}(J\Phi^*)(\dot{m}, p) &= H_p(m)\overline{\Phi}(\lambda_{-p}m, -p)\\ &= \overline{H}_{-p}(\lambda_{-p}m)\overline{\Phi}(\lambda_{-p}m, -p)\\ &= (J\Phi)^*(\dot{m}, p)\end{aligned}$$

and

$$\begin{aligned}J(\Phi * \Psi)(\dot{m}, p) &= \sum_{q\in\mathbf{Z}} H_q(m)H_{p-q}(\lambda_{-q}m)\Phi(m, q)\Psi(\lambda_{-q}m, p-q)\\ &= H_p(m)(\Phi * \Psi)(m, p)\\ &= [J(\Phi * \Psi)](\dot{m}, p).\end{aligned}$$

Let μ_0 be a Borel measure of full support on F and, for $\sigma_k : F \to \sigma_k F$ and $\Pi : F \to M/\sigma$, set $\mu_k = (\sigma_k)_*(\mu_0)$ and $\tilde{\mu} = \Pi_*(\mu_0)$. Then $\tilde{\mu}$ and μ_k have full support on M/σ and $\sigma_k F$ respectively, for all $k \in \mathbf{Z}$. In what follows we will also denote by μ_k the Borel measure on M obtained by setting $\mu_k(X) = \mu_k(X \cap \sigma_k F)$, for a Borel subset X of M. Now let $\tilde{\Theta}$ and Θ^k, for $k \in \mathbf{Z}$, denote the representations of $A \rtimes_\lambda \mathbf{Z}$ and $D^{\sigma,u}$ on $L^2(M/\sigma \times \mathbf{Z}, \tilde{\mu} \times \nu)$ and $L^2(M \times \mathbf{Z}, \mu_k \times \nu)$ (ν being counting measure on $\mathbf{Z}$), respectively, defined by

$$(\tilde{\Theta}_\Psi \xi)(\dot{m}, p) = \sum_{q\in\mathbf{Z}} \Psi(\lambda_p \dot{m}, q)\xi(\dot{m}, p-q)$$

and

$$(\Theta^k_\Phi \eta)(m, p) = \sum_{q\in\mathbf{Z}} \Phi(\lambda_p m, q)\eta(m, p-q),$$

where $\Phi \in C^{\sigma,u}$, $\Psi \in C_c(M/\sigma \times \mathbf{Z})$, $\xi \in L^2(M/\sigma \times \mathbf{Z}, \tilde{\mu} \times \nu)$, and moreover $\eta \in L^2(M \times \mathbf{Z}, \mu_k \times \nu)$. Let $U : L^2(M/\sigma \times \mathbf{Z}, \tilde{\mu} \times \nu) \to L^2(M \times \mathbf{Z}, \mu_k \times \nu)$ be the unitary operator defined by $(U\xi)(m, p) = \overline{H}_p(\lambda_p m)\xi(\dot{m}, p)$. Then, if $m \in \sigma_k F$, we have

$$\begin{aligned}|\tilde{\Theta}_{J\Phi}\xi(\dot{m}, p)| &= |\sum_{q\in\mathbf{Z}} (J\Phi)(\lambda_p \dot{m}, q)\xi(\dot{m}, p-q)|\\ &= |\sum_{q\in\mathbf{Z}} H_q(\lambda_p m)\Phi(\lambda_p m, q)(U\xi)(m, p-q)H_{p-q}(\lambda_{p-q}m)|\\ &= |\sum_{q\in\mathbf{Z}} H_p(\lambda_p m)\Phi(\lambda_p m, q)(U\xi)(m, p-q)|\\ &= |\Theta^k_\Phi(U\xi)(m, p)|,\end{aligned}$$

and it follows that $\|\tilde{\Theta}_{J\Phi}\xi\| = \|\Theta^k_\Phi(U\xi)\|$.

Now, the representation $\tilde{\Theta}$ is faithful ([PD, 7.7.5, 7.7.7]); therefore, for $\Phi \in C^{\sigma,u}$,

$$\|J\Phi\| = \|\tilde{\Theta}_{J\Phi}\| = \|\Theta^k_\Phi\| \le \|\Phi\|,$$

so J can be extended to a continuous map on $D^{\sigma,u}$.

We next show that, for $\Phi \in C^{\sigma,u}$, we have $\|\Phi\| = \sup_k \|\Theta^k_\Phi\| = \|J\Phi\|$, which takes care of the injectivity of J.

First notice that the representation $\bigoplus_k \Theta^k$ is unitarily equivalent to the representation Θ of $D^{\sigma,u}$ on $L^2(M \times \mathbf{Z}, \mu \times \nu)$ defined by the same formula as Θ^k, where, for a Borel subset X of M, we set $\mu(X) = \sum_k \mu_k(X \cap \sigma_k F)$.

Thus it suffices to prove that Θ is faithful. In order to do that, we show ([PD, 7.7.5, 7.7.7]) that μ has full support on M: Let $O \subset M$ be an open set such that $\mu(O) = 0$. Then, for all $k \in \mathbf{Z}$, we have that $O \cap \sigma_k F$ is an open subset of $\sigma_k F$ and $\mu_k(O \cap \sigma_k F) = 0$. Since μ_k has full support on $\sigma_k F$, it follows that $A = \bigcup A \cap \sigma_k F = \emptyset$, which ends the proof. ∎

From now on we will be dealing with the case of quantum Heisenberg manifolds. We specialize Theorem 2.3 to that case.

Corollary 2.4. *Let λ be the action of $\mathbf{Z}$ on $\mathbf{T}^2$ defined by*

$$\lambda_k(x, y) = (x + 2k\mu, y + 2k\nu),$$

and let A denote the smallest λ-invariant C^*-subalgebra of $L^\infty(\mathbf{T}^2)$ containing $C(\mathbf{T}^2)$ and the characteristic functions of the sets $[2k\mu, 2(k+1)\mu] \times \mathbf{T}$, for all $k \in \mathbf{Z}$. Then the quantum Heisenberg manifold $D^c_{\mu\nu}$ can be embedded in $A \rtimes_\lambda \mathbf{Z}$.

Proof. Let us take $F = [0, 1) \times \mathbf{T}$ as a fundamental domain for σ, and H as in Lemma 2.1. If $\Phi \in C^c_{\mu\nu}$ and $p \in \mathbf{Z}$, then $f_{\Phi,p}(x, y) = \Phi(x', y, p)$, where $x' \in [0, 1)$ and $\exp(2\pi i x') = \exp(2\pi i x)$. Therefore $f_{\Phi,p}$ belongs to the λ-invariant algebra A. Thus Theorem 2.3 applies to A. ∎

3. The range of traces on $K_0(D^c_{\mu\nu})$

In this section we discuss the range of traces on $K_0(D^c_{\mu\nu})$. We first give a description of tracial states on the algebra $D^c_{\mu\nu}$. The techniques involved are an adaptation of those usually employed (see [TO, 3.3]) to relate λ-invariant probability measures on a G-space X to tracial states on $C_0(X) \rtimes_\lambda G$. Then, by embedding $D^c_{\mu\nu}$ in a crossed product as in Section 2, we show that any tracial state τ on $D^c_{\mu\nu}$ induces the same homomorphism on $K_0(D^c_{\mu\nu})$, and that $\tau_*(K_0(D^c_{\mu\nu})) = \mathbf{Z} + 2\mu\mathbf{Z} + 2\nu\mathbf{Z}$.

Lemma 3.1. *For each $p \in \mathbf{Z}$ there exist $\Delta^p_1, \Delta^p_2 \in C^c_{\mu\nu}$ such that $\Delta^p_i(x, y, n) = 0$ if $n \neq p$, and*

i) $(\Delta^p_1)^* * \Delta^p_1 + (\Delta^p_2)^* * \Delta^p_2 = 1 = \Delta^p_1 * (\Delta^p_1)^* + \Delta^p_2 * (\Delta^p_2)^*$,

ii) $\Delta^p_1 * f * (\Delta^p_1)^* + \Delta^p_2 * f * (\Delta^p_2)^* = \lambda_p(f)$ for all $f \in C(\mathbf{T}^2)$.

Proof. Let $d \in C(\mathbf{T})$ be such that $0 \le d \le 1$, $d(0) = 0$, and $d(1/2) = 1$. For $p \in \mathbf{Z}$ let $\Delta^p_1(x, y, n) = d^{1/2}(x)\delta_p(n)$, for $x \in [0, 1], y \in \mathbf{T}$,

$$\Delta^p_2(x, y, n) = \begin{cases} (1 - d(x))^{1/2}\delta_p(n) & \forall x \in [0, 1/2], y \in \mathbf{T}, \\ (1 - d(x))^{1/2}\exp(-2\pi i c p(y - p\nu))\delta_p(n) & \forall x \in [1/2, 1], y \in \mathbf{T}, \end{cases}$$

and extend Δ^p_i, for $i = 1, 2$, to continuous functions on $\mathbf{R} \times \mathbf{T} \times \mathbf{Z}$ by setting $\Delta^p_i(x + 1, y, n) = \exp(-2\pi i c p(y - p\nu))\Delta^p_i(x, y)\delta_p(n)$, for all $(x, y) \in \mathbf{R} \times \mathbf{T}$. Then

$$[(\Delta^p_i)^* * \Delta^p_i](x, y, n) = |\Delta^p_i(x + 2p\mu, y + 2p\nu, p)|^2\delta_0(n),$$

so $(\Delta^p_1)^* * \Delta^p_1 + (\Delta^p_2)^* * \Delta^p_2 = (|\Delta^p_1|^2 + |\Delta^p_2|^2)\delta_0 = 1$.

Moreover, if $f \in C(\mathbf{T}^2)$, then

$$[\Delta^p_i * f * (\Delta^p_i)^*](x, y, n) = |\Delta^p_i(x, y, p)|^2 f(x - 2p\mu, y - 2p\nu)\delta_0(n),$$

so

$$\Delta_1^p * f * (\Delta_1^p) * + \Delta_2^p * f * (\Delta_2^p)^* = (|\Delta_1^p|^2 + |\Delta_2^p|^2)\lambda_p(f) = \lambda_p(f).$$

The second equality in i) now follows from taking $f = 1$ in ii). □

Notation 3.2. Throughout this section $e(a)$ denotes $\exp(2\pi i a)$, for a real number a.

Remark 3.3. It was shown in [AE, 2] that the C*-algebra $D^c_{\mu\nu}$ is the crossed product, in the sense of [AEE], of $C(\mathbf{T}^2)$ by the Hilbert C*-bimodule $M^c_{\mu\nu}$, where $M^c_{\mu\nu} = \{f \in C(\mathbf{R} \times \mathbf{T}) : f(x+1, y) = e(-cy)f(x, y)\}$ with the structure defined by

$$(f \cdot \Phi)(x, y) = f(x, y)\Phi(x - 2\mu, y - 2\nu), \quad (\Phi \cdot f)(x, y) = \Phi(x, y)f(x, y),$$

$$\langle f, g\rangle_R(x, y) = \overline{f}(x + 2\mu, y + 2\nu)g(x + 2\mu, y + 2\nu),$$

$$\langle f, g\rangle_L(x, y) = f(x, y)\overline{g}(x, y),$$

for $\Phi \in C(\mathbf{T}^2)$ and $f, g \in M^c_{\mu\nu}$.

Since the Hilbert C*-bimodules $M^c_{\mu\nu}$, $M^c_{\mu+\frac{1}{2},\nu}$, and $M^c_{\mu,\nu+\frac{1}{2}}$ are clearly isomorphic, it follows that so are the C*-algebras $D^c_{\mu\nu}$, $D^c_{\mu+\frac{1}{2},\nu}$, and $D^c_{\mu,\nu+\frac{1}{2}}$.

In [AE] the Picard group of $C(\mathbf{T}^2)$ was shown to be the semidirect product of $\mathrm{Aut}(C(\mathbf{T}^2))$ by $\{M^c_{00} : c \in \mathbf{Z}\} \cong \mathbf{Z}$. By using this description, it was proved ([AE, 2.2]) that $D^c_{\mu\nu}$ and $D^c_{\mu'\nu'}$ are isomorphic if (μ, ν) and (μ', ν') belong to the same orbit under the usual action of $GL_2(\mathbf{Z})$ on $\mathbf{T}^2$. This result carries over to the case when $(2\mu, 2\nu)$ and $(2\mu', 2\nu')$ belong to the same orbit because then, if $A \in GL_2(\mathbf{Z})$ is such that $A\binom{2\mu}{2\nu} = \binom{2\mu'}{2\nu'} + \binom{k}{l}$, for some $k, l \in \mathbf{Z}$, then

$$A\begin{pmatrix} \mu \\ \nu \end{pmatrix} = \begin{pmatrix} \mu' + k/2 \\ \nu' + l/2 \end{pmatrix},$$

so

$$D^c_{\mu\nu} \cong D^c_{\mu'+\frac{k}{2},\nu'+\frac{l}{2}} \cong D^c_{\mu',\nu'}.$$

Lemma 3.4. *Let a, b, p, q be non-zero integers such that $\gcd(a, p) = \gcd(b, q) = 1$, and let $m = \mathrm{lcm}(p, q)$. Then $(\frac{a}{p}, \frac{b}{q})$ and $(\frac{1}{m}, 0)$ are in the same orbit under the action of $GL_2(\mathbf{Z})$ on $\mathbf{T}^2$, so $D^c_{\frac{a}{p},\frac{b}{q}} \cong D^c_{\frac{1}{m},0}$. If $\gcd(a, p) = 1$, then $(\frac{a}{p}, 0)$, $(\frac{1}{p}, 0)$, and $(0, \frac{a}{p})$ belong to the same orbit under the action of $GL_2(\mathbf{Z})$, and $D^c_{\frac{a}{p},0} \cong D^c_{\frac{1}{p},0} \cong D^c_{0,\frac{a}{p}}$.*

Proof. Let us write $m = pp' = qq'$, so $\gcd(p', q') = 1$ and $\gcd(ap', bq', m) = 1$. Then it suffices to show that, if $\gcd(a, b, p) = 1$, then $A(\frac{a}{p}, \frac{b}{p}) = (\frac{1}{p}, 0)$ for some $A \in GL_2(\mathbf{Z})$, viewing $(\frac{a}{p}, \frac{b}{p})$ and $(\frac{1}{p}, 0)$ as elements of $\mathbf{T}^2$. This will also show our second statement, since, in $\mathbf{T}^2$, $(\frac{a}{p}, 0) = (\frac{a}{p}, \frac{p}{p})$ and $(0, \frac{a}{p}) = (\frac{p}{p}, \frac{a}{p})$. The isomorphisms between the corresponding quantum Heisenberg manifolds will then follow from [AE, 2.2].

For a, b, p as above, let $d = \gcd(a, b)$, so $\gcd(d, p) = 1$. Write $a = a'd$, $b = b'd$, and choose integers r, s such that $a'r + b's = 1$. Then

$$\begin{pmatrix} r & s \\ -b' & a' \end{pmatrix} \in GL_2(\mathbf{Z}) \quad \text{and} \quad \begin{pmatrix} r & s \\ -b' & a' \end{pmatrix}\begin{pmatrix} \frac{a}{p} \\ \frac{b}{p} \end{pmatrix} = \begin{pmatrix} \frac{d}{p} \\ 0 \end{pmatrix}.$$

Now, as elements of $\mathbf{T}^2$, $(\frac{d}{p},0) = (\frac{d}{p},\frac{p}{p})$, and $\gcd(d,p)=1$, so, by making use of the result we have just proved, we get that $(\frac{d}{p},0)$ and $(\frac{1}{p},0)$ belong to the same orbit under the action of $GL_2(\mathbf{Z})$ on $\mathbf{T}^2$. $\square$

Remark 3.5. Let $a,m,b,n \in \mathbf{Z}$ be such that $m,n \neq 0$, $\gcd(a,m) = \gcd(b,n) = 1$. Set $p = \frac{1}{2}\operatorname{lcm}(m,n)$ if either m or n is even, and $p = \operatorname{lcm}(m,n)$ otherwise. Then $(\frac{2a}{m},\frac{2b}{n})$ and $(\frac{1}{p},0)$ are in the same orbit under the action of $GL_2(\mathbf{Z})$ on $\mathbf{T}^2$, so $D^c_{\frac{a}{m},\frac{b}{n}}$ is isomorphic to $D^c_{\frac{1}{2p},0}$.

Proof. The statement follows from Remark 3.3 and Lemma 3.4. $\square$

Notation 3.6. For the remainder of this section, given a quantum Heisenberg manifold $D^c_{\mu\nu}$, if both μ and ν are rational we assume that $\mu = 1/2p$, for $p \in \mathbf{Z}$, $p > 0$, and that $\nu = 0$, as in Remark 3.5. If either μ or ν is irrational, we set $p = 0$.

Let B^c_p be the C*-subalgebra of $D^c_{\mu\nu}$ generated by $\{\phi \in C^c_{\mu\nu} : supp_{\mathbf{Z}}\phi \subset p\mathbf{Z}\}$, and denote by $E^c_p : D^c_{\mu\nu} \longrightarrow B^c_p$ the conditional expectation on B^c_p given by

$$(E^c_p\phi)(x,y,n) = \begin{cases} \phi(x,y,n) & \text{if } n \in p\mathbf{Z}, \\ 0 & \text{otherwise,} \end{cases}$$

for $\phi \in C^c_{\mu\nu}$.

Proposition 3.7. *If τ is a tracial state on $D^c_{\mu\nu}$, then $\tau = \tau \circ E^c_p$.*

Proof. We show that $\tau(\Phi\delta_n) = 0$, for $n \notin p\mathbf{Z}$. Since for Δ^n_i as in Lemma 3.1 we have that

$$\phi\delta_n = \phi\delta_n * (\Delta^n_1)^* * \Delta^n_1 + \phi\delta_n * (\Delta^n_2)^* * \Delta^n_2,$$

and $\phi\delta_n * (\Delta^n_i)^* \in C(\mathbf{T}^2)$, for $i = 1,2$, it suffices to show that $\tau(g * \Delta^n_i) = 0$, for all $g \in C(\mathbf{T}^2)$, $i = 1,2$, and $n \notin p\mathbf{Z}$. For a fixed $n \notin p\mathbf{Z}$, we can assume that $g = f^2$ for some positive function f satisfying $supp(f) \cap supp(\lambda_n f) = \emptyset$, because, since in this case $\lambda^n(x,y) \neq (x,y)$ for all $(x,y) \in \mathbf{T}^2$, any function $g \in C(\mathbf{T}^2)$ is the linear combination of functions satisfying those conditions. So let $g \in C(\mathbf{T}^2)$ be as above. Then

$$\tau(g * \Delta^n_i) = \tau(f^2 * \Delta^n_i) = \tau(f * f * \Delta^n_i) = \tau(f * \Delta^n_i * f) = 0,$$

because

$$f * \Delta^n_i * f = f\Delta^n_i(\lambda_n f) = 0.$$

This shows that $\tau = \tau \circ E^c_p$, since both sides are continuous and agree on $C^c_{\mu\nu}$. $\square$

Proposition 3.8. *Let $D^c_{\mu\nu}$, p, B^c_p, and E^c_p be as in Notation 3.6, and let $\gamma : B^c_p \longrightarrow B^c_p$ be given by*

$$\gamma\phi = \Delta^1_1 * \phi * (\Delta^1_1)^* + \Delta^1_2 * \phi * (\Delta^1_2)^*,$$

for $\phi \in B^c_p$ and Δ^1_i, $i = 1,2$, as in Lemma 3.1.

Then, for $\phi \in B^c_p$ compactly supported on $\mathbf{Z}$,

$$(\gamma\phi)(x,y,m) = \begin{cases} \phi(x - 2\mu, y - 2\nu, 0)\delta_0(m) & \text{if } p = 0, \\ e(-cny)\phi(x - 1/p, y, np)\delta_{np}(m) & \text{if } p \neq 0. \end{cases}$$

Also, the correspondence $\tau \mapsto \tau \circ E^c_p$ is a bijection between the set of γ-invariant tracial states on B^c_p and tracial states on $D^c_{\mu\nu}$.

Proof. If τ is a trace on $D^c_{\mu\nu}$ then, by Proposition 3.7, we have that $\tau = \tau \circ E^c_p$, and the restriction of τ to B^c_p is γ-invariant because

$$\tau(\gamma\phi) = \tau[(\Delta^1_1)^* * \Delta^1_1 * \phi + (\Delta^1_2)^* * \Delta^1_2 * \phi] = \tau(\phi).$$

Now, for $\phi \in B^c_p$ compactly supported on $\mathbf{Z}$, we have

$$\begin{aligned}&[\Delta^1_i * \phi * (\Delta^1_i)^*](x, y, np)\\ &\quad = \Delta^1_i(x, y, 1)\phi(x - 2\mu, y - 2\nu, np)\overline{\Delta^1_i(x - 2np\mu, y - 2np\nu, 1)},\end{aligned}$$

so

$$(\gamma\phi)(x, y, m) = \begin{cases}\phi(x - 2\mu, y - 2\nu, 0)\delta_0(m) & \text{if } p = 0,\\ e(-cny)\phi(x - 1/p, y, np)\delta_{np}(m) & \text{if } p \neq 0.\end{cases}$$

Now let τ be a γ-invariant tracial state on B^c_p. Since $\tau \circ E^c_p$ is a state, we only need to show that $\tau \circ E^c_p(\phi * \psi) = \tau \circ E^c_p(\psi * \phi)$, for $\phi = f\delta_k$, $\psi = g\delta_l$. We can assume that $k + l \in p\mathbf{Z}$, since otherwise $E^c_p(\phi * \psi) = 0 = E^c_p(\psi * \phi)$.

If $p \neq 0$, we take ϕ and ψ as above, with $k + l = np$, and we have

$$\begin{aligned}[\gamma^{-k}(\phi * \psi)](x, y, m) &= e(cnky)f(x + k/p, y)g(x, y)\delta_{np}(m)\\ &= g(x, y)f(x + (k - np)/p, y)\delta_{np}(m)\\ &= g(x, y)f(x - l/p, y)\delta_{np}(m)\\ &= (\psi * \phi)(x, y, m).\end{aligned}$$

So $(\tau \circ E^c_p)(\phi * \psi) = \tau(\phi * \psi) = \tau(\gamma^k(\psi * \phi)) = \tau(\psi * \phi) = (\tau \circ E^c_p)(\psi * \phi)$. Similar computations prove the case $p = 0$. $\square$

Proposition 3.9. *Given a quantum Heisenberg manifold $D^c_{\mu\nu}$, let p, B^c_p and E^c_p be as in Remark 3.6. Then $B^c_p \cong C(\mathbf{T}^2)$ if $p = 0$, and $B^c_p \cong D^{cp}_{0,0}$ if $p \neq 0$.*

Proof. It is clear that $B^c_p \cong C(\mathbf{T}^2)$ for $p = 0$. If $p \neq 0$, set $J : B^c_p \longrightarrow D^{cp}_{0,0}$,

$$J\phi(x, y, n) = u_p(n, y)\phi(x, y, np),$$

for $\phi \in B^c_p \cap C^c_{\frac{1}{2p},0}$, where $u_p(n, y) = e(-\frac{1}{2}cpn(n - 1)y)$.

Notice that

$$(J\phi)(x + 1, y, n) = u_p(n, y)e(-cnpy)\phi(x, y, np) = e(-cpny)(J\phi)(x, y, n),$$

so $J\phi \in D^{cp}_{0,0}$, for $\phi \in B^c_p \cap C^c_{\frac{1}{2p},0}$.

Let Π and σ denote, respectively, the faithful representations ([RF3]) of $D^c_{\frac{1}{2p},0}$ and $D^{cp}_{0,0}$ on $L^2(\mathbf{R}\times\mathbf{T}\times\mathbf{Z})$ given by

$$(\Pi_\phi\xi)(x, y, n) = \sum_q \phi(x + n/p, y, qp)\xi(x, y, n - qp),$$

$$(\sigma_\psi\eta)(x, y, n) = \sum_q \phi(x, y, q)\eta(x, y, n - q),$$

for $\phi \in C^c_{1/p,0}$, $\psi \in C^{cp}_{0,0}$, ξ, $\eta \in L^2(\mathbf{R}\times\mathbf{T}\times\mathbf{Z})$.

Let $U : L^2(\mathbf{R}\times\mathbf{T}\times\mathbf{Z}) \longrightarrow \bigoplus_0^{p-1} L^2(\mathbf{R}\times\mathbf{T}\times\mathbf{Z})$ be given by

$$(U\xi)_i(x, y, n) = \overline{u_p(-n, y)}\xi(x, y, np + i),$$

for $\xi \in L^2(\mathbf{R}\times\mathbf{T}\times\mathbf{Z})$. It is easily checked that U is unitary and that

$$[U^*((\eta_i))](x,y,n) = u_p(-k,y)\eta_i(x,y,k) \quad \text{ for } n = kp+i,\ 0 \le i < p.$$

Now,

$$\begin{aligned}
&[U\Pi_\phi U^*((\eta_i))]_j(x,y,n)\\
&\quad = \overline{u_p(-n,y)}(\Pi_\phi U^*((\eta_i)))(x,y,np+j)\\
&\quad = \sum_q \overline{u_p(-n,y)}\phi(x+(np+j)/p,y,qp)(U^*((\eta_i))(x,y,(n-q)p+j)\\
&\quad = \sum_q \overline{u_p(-n,y)}e(-cnpqy)\phi(x+j/p,y,qp)\eta_j(x,y,n-q)u_p(q-n,y)\\
&\quad = \sum_q u_p(q,y)\phi(x+j/p,y,qp)\eta_j(x,y,n-q)\\
&\quad = \sum_q (J\phi)(x+j/p,y,q)\eta_j(x,y,n-q)\\
&\quad = [\sigma_{(\delta^j(J\phi))}(\eta_j)](x,y,n-q),
\end{aligned}$$

where $(\delta^j\psi)(x,y,n) = \psi(x+j/p,y,n)$ for all $\psi \in C^{cp}_{0,0}$, and $0 \le j < p$. Notice that δ^j defines an automorphism of $D^{cp}_{0,0}$: apply [AB2, 1.1] to define δ^j on $C_b(\mathbf{R}\times\mathbf{T})\rtimes_{id}\mathbf{Z}$ and then check that $D^{cp}_{0,0}$ is invariant under it. Thus U intertwines Π_ϕ and $\bigoplus_j(\sigma\circ\delta^j)(J\phi)$, which shows that J extends to an isomorphism. ☐

Remark 3.10. Recall ([RF3]) that, for a positive integer c, the C*-algebra $D^c_{0,0}$ is isomorphic to the (commutative) Heisenberg manifold $C(M^c)$, where M^c is the quotient space of $\mathbf{R}\times\mathbf{T}^2$ under the equivalence relation given by

$$(x,y,z) \cong (x',y',z') \text{ if and only if } (x',y',z') = (x+k,y,z+cky)$$

for some $k \in \mathbf{Z}$, and (x,y,z), $(x',y',z') \in \mathbf{R}\times\mathbf{T}^2$ (viewing $\mathbf{T}$ as $\mathbf{R}/\mathbf{Z}$).

The isomorphism is obtained by taking Fourier transform in the third variable, that is, $F : C(M^c) \longrightarrow D^c_{0,0}$, $(Ff)(x,y,n) = \int_{\mathbf{T}} e(-nz)f(x,y,z)dz$.

Corollary 3.11. *Given a quantum Heisenberg manifold $D^c_{\mu\nu}$, let p, B^c_p, and E^c_p be as in Notation 3.6. There is a bijective correspondence between tracial states on $D^c_{\mu\nu}$ and γ-invariant probability measures on X, where*

$$X = \mathbf{T}^2, \qquad \gamma(x,y) = (x+2\mu, y+2\nu),$$

if either μ or ν is irrational, and

$$X = M^{cp}, \qquad \gamma(x,y,z) = (x+1/p, y, z+cy)$$

if $\mu = \frac{1}{2p}$, $\nu = 0$.

The correspondence is given by $m \mapsto \tau_m \circ E^c_p$, where $\tau_m(f) = \int_X f dm$, once B^c_p is identified with $C(X)$, according to Proposition 3.9 and Remark 3.10.

Proof. It is easily checked that the formula above is the formula for γ in Proposition 3.8, when one keeps track of the isomorphisms J and F in Proposition 3.9 and Remark 3.10, respectively. ☐

Corollary 3.12. *If $\{1,\mu,\nu\}$ is linearly independent over the field of rational numbers, then the trace corresponding to Haar measure on $\mathbf{T}^2$ is the only tracial state on $D^c_{\mu\nu}$.*

Proof. Under the conditions above, μ and ν are irrational, and the λ-orbits in $\mathbf{T}^2$ are dense. Therefore Haar measure is the only λ-invariant measure on $\mathbf{T}^2$. The uniqueness of the trace now follows from Corollary 3.11. $\square$

Remark 3.13. For $D^c_{\mu\nu}$ and p as in Notation 3.6, we can identify $C(\mathbf{T}^2)$ with the C*-algebra consisting of the δ_0-maps in B^c_p. It follows from Proposition 3.8 that, for any value of p, a trace on $D^c_{\mu\nu}$ induces a probability measure m_τ on $\mathbf{T}^2$, invariant under translation by $(2\mu, 2\nu)$, and such that $\tau(f) = \int_{\mathbf{T}^2} f dm_\tau$, for all $f \in C(\mathbf{T}^2)$.

Proposition 3.14. *Let $D^c_{\mu\nu}$ be a quantum Heisenberg manifold, where $(\mu,\nu) = (\frac{1}{2p}, 0)$ as in Remark 3.5 if μ and ν are rational. Then, in the notation of Corollary 2.4, all traces on $D^c_{\mu\nu}$ arise from restricting traces on $A \rtimes_\lambda \mathbf{Z}$, where $D^c_{\mu\nu}$ is embedded in $A \rtimes_\lambda \mathbf{Z}$ as in Theorem 2.3.*

Proof. Let A be as in Corollary 2.4. Notice that the embedding J in Theorem 2.3 maps the C*-algebra B^c_p defined in Notation 3.6 to the commutative C*-subalgebra B of $A \rtimes_\lambda \mathbf{Z}$ generated by $\{\phi \in C_c(\mathbf{Z}, A) : supp_{\mathbf{Z}}\phi \subset p\mathbf{Z}\}$, and that J is the identity when restricted to $C(\mathbf{T}^2) \subset B^c_p$ as in Corollary 3.13. So, if either μ or ν is irrational, then the statement follows from Proposition 3.7, Corollary 3.11, and [TO, 3.3.9].

If $(\mu,\nu) = (\frac{1}{2p}, 0)$, given a trace τ on $D^c_{\frac{1}{2p},0}$, let S denote the set of states on B extending $\tau_0 \circ J^{-1}$ on $J(B^c_p)$, where τ_0 denotes the restriction of τ to B^c_p.

Let $T : B \longrightarrow B$ be given by

$$T(a) = J(\Delta^1_1) * a * J(\Delta^1_1)^* + J(\Delta^1_2) * a * J(\Delta^1_2)^*,$$

with Δ^1_i, $i = 1, 2$, as in Lemma 3.1, and J as in Theorem 2.3, and set $T^* : B^* \longrightarrow B^*$, $T^*(\rho) = \rho \circ T$. If $\rho \in S$, then $T^*(\rho)$ is positive and $\|T^*(\rho)\| = [T^*(\rho)](1) = \rho(1) = 1$, by Lemma 3.1. Besides, the restriction of $T^*(\rho)$ to $J(B^c_p)$ is τ_0 by Proposition 3.8. Then $T^*(S) \subset S$, and S is a w^*-compact, convex, non-empty set, so it follows from Markov's fixed-point theorem that there exists $\tau_1 \in S$ such that $T^*(\tau_1) = \tau_1$.

We next show that if P denotes the conditional expectation $P : A \rtimes_\lambda \mathbf{Z} \longrightarrow B$ given by

$$(P\phi)(x,y,n) = \begin{cases} \phi(x,y,n) & \text{if } n \in p\mathbf{Z}, \\ 0 & \text{otherwise,} \end{cases}$$

for $\phi \in C_c(\mathbf{Z}, A)$, then $\tau_1 \circ P$ is a trace on $A \rtimes_\lambda \mathbf{Z}$. This will end the proof, because the diagram

$$\begin{array}{ccc} D^c_{\frac{1}{p},0} & \xrightarrow{J} & A \rtimes_\lambda \mathbf{Z} \\ {\scriptstyle E^c_p}\downarrow & & \downarrow{\scriptstyle P} \\ B^c_{\frac{1}{p}} & \xrightarrow{J} & B \end{array}$$

commutes, and, by Proposition 3.7, $\tau_0 = \tau \circ E^c_p$, so

$$\tau = \tau_0 \circ E^c_p = \tau_1 \circ J \circ E^c_p = \tau_1 \circ P \circ J.$$

Now, τ_1 is a state, so it suffices to show that $(\tau \circ P)(\phi * \psi) = (\tau \circ P)(\psi * \phi)$, for $\phi = F\delta_k$, $\psi = G\delta_l$, for some $F, G \in A$.

First notice that if $H \in A$, and $n \in \mathbf{Z}$, then $T(H\delta_{np}) = (\lambda H)\delta_{np}$. In fact,

$$\begin{aligned}
&T(H\delta_{np})(x,y,m)\\
&\quad= \sum_{i=1}^{2} [J(\Delta_i^1) * H\delta_{np} * J(\Delta_i^1)^*](x,y,m)\\
&\quad= \sum_{i=1}^{2} J(\Delta_i^1)(x,y,1)H(x-\frac{1}{p},y,np)J(\Delta_i^1)^*(x-\frac{1}{p}-n,y,-1)\delta_{np}(m)\\
&\quad= \sum_{i=1}^{2} |J(\Delta_i^1)(x,y,1)|^2(\lambda H)(x,y)\delta_{np}(m)\\
&\quad= \sum_{i=1}^{2} |(\Delta_i^1)(x,y,1)|^2(\lambda H)(x,y)\delta_{np}(m)\\
&\quad= [(\lambda H)\delta_{np}](x,y,m).
\end{aligned}$$

Now, for ϕ and ψ as above, we can assume that $k + l = np$ for some $n \in \mathbf{Z}$, since otherwise $P(\phi * \psi) = 0 = P(\psi * \phi)$. In this case

$$\begin{aligned}
[T^k(\psi * \phi)](x,y,m) &= (\psi * \phi)(x-\frac{k}{p},y,m)\\
&= G(x-\frac{k}{p},y)F(x-\frac{l}{p}-\frac{k}{p},y)\delta_{np}(m)\\
&= F(x,y)G(x-\frac{k}{p},y)\delta_{np}(m)\\
&= (\phi * \psi)(x,y,m).
\end{aligned}$$

Therefore

$$(\tau_1 \circ P)(\phi * \psi) = \tau_1(\phi * \psi) = \tau_1[T^k(\psi * \phi)] = \tau_1(\psi * \phi) = \tau_1 \circ P(\psi * \phi),$$

as we wanted to show. □

Lemma 3.15. *If $\mu \leq 1/2$ and m is a λ-invariant probability measure on $\mathbf{T}^2$, then $m([0,2\mu) \times \mathbf{T}) = 2\mu$.*

Proof. First notice that the analogous result holds for $\mathbf{T}$. Fix a real number $\alpha \in [0,1]$. If υ is a measure on $\mathbf{T}$ invariant under translation by α, then $\upsilon([0,\alpha)) = \alpha$. If α is irrational, then υ is Haar measure on $\mathbf{T}$, and the result is obviously true. If α is rational, $\alpha = p/q$, for $p,q \in \mathbf{Z}$, with $(p,q) = 1$, then $\mathbf{T}$ is the disjoint union of the intervals $I_i = [i/q,(i+1)/q)$, $i = 0,1,...,q-1$.

Now, for all i, I_i can be obtained by translating I_0 by α an appropriate number of times . Therefore $\upsilon(I_i) = \upsilon(I_0) = 1/q$, for all $i = 1,...,q-1$, and it follows that $\upsilon([0,\alpha)) = \upsilon([0,p/q)) = p/q = \alpha$.

Now let m be a λ-invariant probability measure on $\mathbf{T}^2$. Define a probability measure υ on $\mathbf{T}$ by setting $\upsilon(X) = m(X \times \mathbf{T})$.

Then $\upsilon(A + 2\mu) = m((A + 2\mu) \times \mathbf{T}) = m(\lambda(A \times \mathbf{T})) = m(A \times \mathbf{T}) = \upsilon(A)$.

It follows now that $m([0,2\mu) \times \mathbf{T}) = \upsilon([0,2\mu)) = 2\mu$. □

Theorem 3.16. *All tracial states τ on $D^c_{\mu\nu}$ induce the same homomorphism τ_* on $K_0(D^c_{\mu\nu})$. Moreover, $\tau_*(K_0(D^c_{\mu\nu})) = \mathbf{Z} + 2\mu\mathbf{Z} + 2\nu\mathbf{Z}$.*

Proof. For a tracial state τ on $D^c_{\mu\nu}$, we denote by τ' an extension of τ to $A \rtimes_\lambda \mathbf{Z}$, as in Proposition 3.14. We have the following short exact sequence ([PM, 3,4]):

$$0 \longrightarrow \tau_*(K_0(A)) \xrightarrow{i} \tau'_*(K_0(A \rtimes_\lambda \mathbf{Z})) \xrightarrow{q} \Delta^\lambda_\tau(K) \longrightarrow 0,$$

where $K = \{u \in \mathcal{U}_1(A) : [u]_{K_1} \in ker(1-\lambda_*)\}$, i and q are inclusion and projection on $\mathbf{R}/\tau_*(K_0(A))$ respectively, $\Delta^\lambda_\tau(u) = q[\Delta_\tau(u\lambda(u^{-1}))]$, and $\Delta_\tau : (\mathcal{U}_1)_0 \longrightarrow R$ is defined by $\Delta_\tau(e^{2\pi i y}) = \tau(y)$, for y self-adjoint.

Let us relabel the set $X = (2\mu\mathbf{Z} + \mathbf{Z}) \cap (0,1)$ so that $X = \{x_i : i \in N\}$. Let A_n be the smallest C*-subalgebra of $L^\infty(\mathbf{T}^2)$ generated by $C(\mathbf{T}^2)$ and $\chi_{[0,x_i]\times\mathbf{T}}$, for $i = 1, ..., n$. Then $A_1 \subseteq A_2 \subseteq ... \subseteq A_n \subseteq ...$, and A is the direct limit of $\{A_n\}$.

Now, $A_n \simeq \bigoplus_{j=0}^{j=n} C([x_{i_j}, x_{i_{j+1}}] \times \mathbf{T})$, where $\{x_{i_j}\}_{j=1}^n = \{x_i\}_{i=1}^n$, $x_{i_0} = 0$, $x_{i_{n+1}} = 1$, and $x_{i_j} < x_{i_{j+1}}$ for all $j = 0, 1, ..., n$.

Since $[a,b] \times \mathbf{T}$ can be deformed to $\mathbf{T}$, it follows that $K_j(A_n) = \mathbf{Z}^{n+1}$ $\forall n \in N$, $j = 1, 2$. The set

$$\{[\chi_{[x_i,x_j]\times\mathbf{T}}]_{K_0} : x_i, x_j \in X \cup \{0,1\}, x_i < x_j\}$$

is a generator of $K_0(A)$, and any arbitrary element of $K_1(A)$ has a representative u of the form

$$u(x,y) = e(n_i y) \quad \text{ if } x \in [t_i, t_{i+1})$$

for a partition $0 = t_0 < t_1 < ... < t_n = 1, \{t_i\}_{i=1}^{i=n-1} \subset X$, and integers $n_i, i = 0, ..., n-1$.

Now, by Lemma 3.15 and Remark 3.13, we have that $\tau_*(K_0(A)) \subseteq \mathbf{Z}+2\mu\mathbf{Z}$. Since id and $\chi_{[0,2\mu+k_0]\times\mathbf{T}} \in A$ for some k_0, the equality holds, and $\tau_*(K_0(A)) = \mathbf{Z}+2\mu\mathbf{Z}$.

Let us now find the elements $[u]_{K_1} \in K_1(A)$ that are left fixed by λ_*, where u is as above.

For $[u]_{K_1} \in K_1(A)$,

$$\lambda_k(u)(x,y) = u(x - 2k\mu, y - 2k\nu),$$

that is,

$$\lambda_k(u)(x,y) = e(n_i(y - 2k\nu)), \qquad \text{where } x - 2k\mu \in [x_i, x_{i+1}).$$

Fix $a \in [x_0, x_1)$. If μ is irrational, then for all $i = 0, 1, ..., n$ there exists $k_i \in \mathbf{Z}$ such that $a - 2k_i\mu \in [x_i, x_{i+1})$ and $(\lambda_{k_i}(u))(a,y) = e(n_i(y - 2k_i\nu))$. It is clear now that $[u]_{K_1} = [\lambda_k(u)]_{K_1}$ for all $k \in \mathbf{Z}$ if and only if $n_i = n_0$ for all $i = 0, 1, ..., n$. Therefore $\Delta_{\tau'}(u\lambda(u^{-1})) = \tau(2n_0\nu.Id) = 2n_0\nu$, and it follows that $\Delta^\lambda_\tau(K) = 2\nu\mathbf{Z}$.

If 2μ is rational, $2\mu = p/q$, where $p, q \in \mathbf{Z}$, $(p,q) = 1$, then $X = \{i/q : i = 0, ..., q\}$ and u is of the form

$$u(x,y) = e(n_k y) \quad \text{ for } x \in I_k = [k/q, (k+1)/q], k = 0, 1, ..., q-1.$$

Translation by p/q gives a transitive action of $\mathbf{Z}_q$ on the set $\{I_k\}$, since $(p,q) = 1$, so the same reasoning as for the irrational case applies, and $[u]_{K_1} = [\lambda u]_{K_1}$ if and only if $u(x,y) = e(ny)$ for all x, y. Then, as above, $\Delta^\lambda_\tau(K) = 2\nu\mathbf{Z}$.

Therefore the short exact sequence above splits, and $\tau'_*(K_0(A \rtimes_\lambda \mathbf{Z})) = \mathbf{Z} + 2\mu\mathbf{Z} + 2\nu\mathbf{Z}$, so $\tau_*(K_0(D^c_{\mu\nu})) \subseteq \mathbf{Z} + 2\mu\mathbf{Z} + 2\nu\mathbf{Z}$.

Now, it is shown in [PM, 2,3,4] that, for $[p] \in K_0(A \rtimes_\lambda \mathbf{Z})$, the choice of $u \in K$ such that $q(\tau'_*([p])) = \Delta^\lambda_\tau(u)$ does not depend on τ, and we just proved that $\Delta^\lambda_\tau(u)$ does not depend on τ either.

So we have $\tau'_*[p] = \Delta^\lambda_\tau(u) + \tau_*([p_0])$, for some $p_0 \in K_0(A)$. We next show that $\tau_*([p_0])$ is independent of τ as well. The preceding remarks show that $[p_0]$ has a representative $h \in \bigoplus C([x_{i_j}, x_{i_{j+1}}] \times \mathbf{T})$, so h is constant on $[x_{i_j}, x_{i_{j+1}}] \times \mathbf{T}$ for each j. Our claim then follows from Lemma 3.15, since $\tau_*([p_0]) = \int_{\mathbf{T}^2} h dm_\tau$. So τ_* does not depend on τ, and $\tau_*(K_0(D^c_{\mu\nu})) \subset \mathbf{Z} + 2\mu\mathbf{Z} + 2\nu\mathbf{Z}$. Finally, the equality holds because it is attained for the trace induced by Haar measure on $\mathbf{T}^2$ ([AB1]). ☐

Corollary 3.17. *Given a quantum Heisenberg manifold $D^c_{\mu\nu}$, let $G_{\mu\nu}$ denote the group $\mathbf{Z} + 2\mu\mathbf{Z} + 2\nu\mathbf{Z}$.*

If $G_{\mu\nu}$ has rank 1 or 3, then $D^c_{\mu\nu}$ and $D^c_{\mu'\nu'}$ are isomorphic if and only if $(2\mu, 2\nu)$ and $(2\mu', 2\nu')$ belong to the same orbit under the usual action of $GL_2(\mathbf{Z})$ on $\mathbf{T}^2$.

Proof. If $D^c_{\mu\nu} \cong D^c_{\mu'\nu'}$, then $G_{\mu\nu} = G_{\mu'\nu'}$, by Theorem 3.16. If $3 = \mathrm{rank}(G_{\mu\nu}) = \mathrm{rank}(G_{\mu'\nu'})$, then $G_{\mu\nu} = G_{\mu'\nu'}$ implies (see, for instance, [PA1, 2.13]) that $(2\mu, 2\nu)$ and $(2\mu', 2\nu')$ are in the same orbit under the action of $GL_2(\mathbf{Z})$.

If $1 = \mathrm{rank}(G_{\mu\nu}) = \mathrm{rank}(G_{\mu'\nu'})$, then μ, ν, μ', and ν' are rational numbers. By virtue of Remark 3.5 we can assume that $(\mu, \nu) = (\frac{1}{2p_1}, 0)$ and $(\mu', \nu') = (\frac{1}{2p_2}, 0)$ for some $p_1, p_2 \in \mathbf{Z}$, $p_1, p_2 \neq 0$. Now the equality

$$\mathbf{Z} + \frac{1}{p_1}\mathbf{Z} = G_{\mu,\nu} = G_{\mu',\nu'} = \mathbf{Z} + \frac{1}{p_2}\mathbf{Z}$$

implies that $\frac{1}{p_1}\mathbf{Z} = \frac{1}{p_2}\mathbf{Z}$, so $p_1 = \pm p_2$, and the result follows.

The converse statement was shown in [AE, Thm. 2.2] (see also Remark 3.3). ☐

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