THE RANGE OF TRACES
ON QUANTUM HEISENBerg MANIFOLDS

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Abstract. We embed the quantum Heisenberg manifold $D^c_{\mu,\nu}$ in a crossed product $C^*$-algebra. This enables us to show that all tracial states on $D^c_{\mu,\nu}$ induce the same homomorphism on $K_0(D^c_{\mu,\nu})$, whose range is the group $\mathbb{Z} + 2\mu \mathbb{Z} + 2\nu \mathbb{Z}$.

1. Introduction

For a positive integer $c$, let $M_c$ denote the Heisenberg manifold consisting of the quotient $G/H_c$, where $G$ is the Heisenberg group,

$$G = \{ \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \},$$

and $H_c$ is the subgroup of $G$ obtained when $x$, $y$, and $cz$ are integers.

In [RF3] Rieffel constructed a quantization deformation $\{D^c_{\mu,\nu}\}_{h \in \mathbb{R}}$ of $M_c$ in the direction of a given Poisson bracket $\lambda_{\mu,\nu}$ determined by two real parameters $\mu$ and $\nu$. We drop from now on the Planck constant $\hbar$ from our notation, because the algebras $D^c_{\mu,\nu}$ and $D^c_{\mu,\nu'}$ are isomorphic and we will denote either one by $D^{c}_{\mu,\nu}$. Also, since $D^c_{\mu,\nu} \cong D^c_{\mu+n,\nu+m}$ for any integers $n$ and $m$ (AB1), we view the parameters $\mu$ and $\nu$ as running in the circle $\mathbb{T}$.

We discussed the K-theory of the quantum Heisenberg manifolds in [AB2] and found that $K_0(D^c_{\mu,\nu}) = \mathbb{Z}^3 \oplus \mathbb{Z}_c$ and $K_1(D^c_{\mu,\nu}) = \mathbb{Z}^3$, which shows that two algebras corresponding to deformations of different Heisenberg manifolds are not isomorphic.

In [AB1] we constructed finitely generated projective modules over the algebra $D^c_{\mu,\nu}$ with traces $2\mu$ and $2\nu$ respectively, where the trace considered was that defined in [RF3]. This suggests employing the range of traces on $K_0(D^c_{\mu,\nu})$ as an invariant to discuss isomorphism and strong-Morita equivalence types of the family $\{D^c_{\mu,\nu}\}$, as was done for non-commutative tori ([PV], [RF4]) and Heisenberg $C^*$-algebras ([PA2], [PA1]).

This work is organized as follows. In Section 2 we embed the algebra $D^c_{\mu,\nu}$ in a crossed product. This is done in a more general context, by viewing the quantum Heisenberg manifolds as generalized fixed-point algebras, as in [RF3]. In Section 3
we show that all traces on $D^c_{\mu\nu}$ give rise to the same homomorphism on $K_0(D^c_{\mu\nu})$, whose range is the group $\mathbb{Z} + 2\mu\mathbb{Z} + 2\nu\mathbb{Z}$.

2. The embedding

The purpose of this section is to embed each quantum Heisenberg manifold in a crossed product algebra $A \times \mathbb{Z}$, $A$ being a $C^*$-subalgebra of $L^\infty(\mathbb{T}^2)$. Our construction carries over into a somewhat more general context, which we next describe.

We first recall some facts established in [AB2]. Let $\lambda$ and $\sigma$ be two commuting automorphisms of a $C^*$-algebra $B$. Let $u : \mathbb{Z} \times \mathbb{Z} \rightarrow U\mathcal{M}(B)$ be a $\lambda$-cocycle in the first variable and a $\sigma$-cocycle in the second one, and define the action $\gamma_{\sigma,\mu}$ of $\mathbb{Z}$ on $B \rtimes \lambda \mathbb{Z}$ by $(\gamma_{\sigma,\mu}^*\Phi)(p) = u(p, k)\sigma_k[\Phi(p)]$. When the $C^*$-algebra $B = C_0(M)$ is commutative and the actions $\lambda$ and $\sigma$ are free and proper, then $\gamma_{\sigma,\mu}$ is proper and the corresponding generalized fixed-point algebra $D_{\sigma,\mu}$, in the sense of Rieffel ([RF1]), is the closure in the multiplier algebra $\mathcal{M}(C_0(M) \rtimes \lambda \mathbb{Z})$ of the $\ast$-subalgebra $C_{\sigma,\mu}$ consisting of functions $\Phi \in C_c(\beta M \times \mathbb{Z})$ such that the projection of $\text{supp}\mathcal{M}(\Phi)$ on $M/\sigma$ is precompact and $\gamma_{\sigma,\mu}^k[\Phi] = \Phi$ for all $k \in \mathbb{Z}$, where $\gamma_{\sigma,\mu}$ has been extended to the multiplier algebra, and $\beta M$ denotes the Stone–Čech compactification of $M$.

When the space $M$ is taken to be $\mathbb{R} \times \mathbb{T}$, and $\sigma(x, y) = (x - 1, y)$, $\lambda(x, y) = (x + 2\mu, y + 2\nu)$, and $u(p, k) = \exp(2\pi i cp(y - \nu v))$ for $(x, y) \in \mathbb{R} \times \mathbb{T}$, $p, k \in \mathbb{Z}$, then $D_{\sigma,\mu}$ is the quantum Heisenberg manifold denoted in [RF3] by $D^c_{\mu\nu}$, and we denote by $C^\ast_{\mu\nu}$ the dense $\ast$-subalgebra corresponding to $C_{\sigma,\mu}$.

In the general case, if $F$ is a fundamental domain in $M$ for the action $\sigma$ (i.e. the canonical projection $\Pi : F \rightarrow M/\sigma$ is a bijection), then any $\Phi$ in the dense subalgebra $C^\ast_{\sigma,\mu}$ is determined by the values $\Phi(m, p)$, for $m$ running in $F$ and $p \in \mathbb{Z}$. This suggests the idea of untwisting those functions so that they can be viewed as functions on the quotient space $M/\sigma$. A natural way of doing that is by multiplying them by a function $H$ on $M$ satisfying the opposite condition $\gamma_{\sigma,\mu}^*H = H$. Also, in order to get things to work from an algebraic point of view, it is necessary for $H$ to satisfy

$$\Pi_{-p}(\lambda_{-p}m) = H_p(m) \quad \text{and} \quad H_{p+q}(m) = H_p(m)H_q(\lambda_{-p}m).$$

However, there might not be such a continuous function on $M$. This is the case for quantum Heisenberg manifolds. If a continuous map $H$ as above existed, then multiplication by the function $\gamma \in C(\mathbb{R} \times \mathbb{T})$ defined by $\gamma(x, y) = H_1(x, y + \nu)$ would be a $C(\mathbb{T}^2)$-module isomorphism between $C(\mathbb{T}^2)$ and $X = \{ \Phi \in C(\mathbb{R} \times \mathbb{T}) : \Phi(x + 1, y) = \exp(2\pi i cy)\Phi(x, y) \}$, in contradiction with [RF2, 3.9].

This is the reason why we are forced to get out of $C_0(M/\sigma)$ and consider a bigger subalgebra of $L^\infty(M/\sigma)$, as was done in [CU, 2.5] for the case of non-commutative tori.

Measurability considerations will impose some restrictions on the fundamental domain $F$. We next summarize the assumptions we will be making.

**Assumptions and notation.** In what follows, for a $C^*$-algebra $A$ we denote by $\mathcal{M}(A)$ its multiplier algebra, and by $U(A)$ the group of unitary elements in $A$.

Throughout this section $\lambda$ and $\sigma$ denote free and proper commuting actions of $\mathbb{Z}$ on a locally compact Hausdorff space $M$, and $u : \mathbb{Z} \times \mathbb{Z} \rightarrow U\mathcal{M}(C_0(M))$ denotes a map satisfying the cocycle conditions:

$$u(p + q, k) = u(p, k)\lambda_p[u(q, k)] \quad \text{and} \quad u(p, k + l) = u(p, k)\sigma_k[u(p, l)],$$
for any $k, l, p, q \in \mathbb{Z}$, where $\sigma$ has been extended to the multiplier algebra. We also assume the existence of a Borel measurable fundamental domain $F$ for $\sigma$ in $M$ such that the canonical projection $\Pi : F \to M/\sigma$ has a Borel measurable inverse map. Thus a function $f$ on $M/\sigma$ is Borel measurable if and only if $f = \hat{f} \circ \Pi$, for some Borel measurable function $\hat{f}$ on $M$.

The generalized fixed-point algebra of $C_0(M) \times \Lambda \mathbb{Z}$ under the action $\gamma^\sigma_{-k}$ of $\mathbb{Z}$ defined by $(\gamma^\sigma_{-k} \Phi)(m, p) = u(p, k)\Phi(\sigma^{-k} m, p)$, for $\Phi \in C_c(M \times \mathbb{Z})$ will be denoted by $D^\sigma_{-k}$. We denote by $C^\sigma_{-k}$ the dense $\ast$-subalgebra of $D^\sigma_{-k}$ consisting of functions $\Phi \in C_c(\beta M \times \mathbb{Z})$ such that the projection of $\text{supp}_M(\Phi)$ on $M/\sigma$ is precompact and that $\gamma^\sigma_{-k} \Phi = \Phi$, for all $k \in \mathbb{Z}$.

**Lemma 2.1.** Let $H : \mathbb{Z} \to \mathcal{U}L^\infty(M)$ be defined by: $H_1(m) = u^*(1, k)(m)$, for $m \in \sigma_k F$, and

$$H_p(m) = \begin{cases} \prod_{q=0}^{p-1} (\lambda_q H_1)(m) & \text{if } p > 0, \\ 1 & \text{if } p = 0, \\ \prod_{q=p}^{p-1} (\lambda_q H_1)(m) & \text{if } p < 0. \end{cases}$$

Then:

i) $H$ is a $\lambda$-cocycle (i.e. $H_{p+q}(m) = H_p(m)H_q(\lambda^{-p} m)$ for all $m \in M$, $p, q \in \mathbb{Z}$).

ii) $\prod_{p}^{q} (\lambda^{-p} m) = H_p(m)$, for all $m \in M$ and $p \in \mathbb{Z}$.

iii) $H_p(\sigma^{-m} m) = [u(p, k)H_p](m)$, for all $m \in M$ and $k, p \in \mathbb{Z}$.

**Proof.**

i) For $q = 1$ and $p > 0$, we have

$$H_{p+1}(m) = \prod_{q=0}^{p} (\lambda_q H_1)(m) = H_p(m)(\lambda^{-p} H_1)(m) = H_p(m)H_1(\lambda^{-p} m).$$

An analogous computation shows that the equality holds for $p \leq 0$, and, once ii) is proven, the result follows by induction on $q$.

It suffices to prove ii) for $p > 0$, and in that case we have

$$\prod_{p}^{q} (\lambda^{-p} m) = \prod_{q=0}^{p} (\lambda_{p+q} H_1)(m) = \prod_{q=0}^{q-1} (\lambda_q H_1)(m) = H_p(m).$$

Finally, for $p > 0$, we have

$$H_p(\sigma^{-k} m) = \prod_{q=0}^{p-1} (\lambda_q H_1)(\sigma^{-k} m)$$

$$= \prod_{q=0}^{p-1} (\lambda_q (u(1, k))(\lambda_q H_1))(m)$$

$$= u(p, k)H_p(m).$$

This ends the proof in view of ii). \qed

**Notation 2.2.** Let $H$ be as in Lemma 2.1. For $p \in \mathbb{Z}$ and $\Phi \in C^\sigma_{-k}$ let $f_{\Phi, p} \in L^\infty(M/\sigma)$ be defined by $f_{\Phi, p}(m) = H_p(m)\Phi(m, p)$, where $m$ denotes the projection of $m$ onto $M/\sigma$. 

Theorem 2.3. Let $H$ be as in Lemma [Z.4]. Then the generalized fixed-point algebra $D^{\sigma, u}$ can be embedded in the crossed product $A \rtimes_\lambda \mathbb{Z}$, where $A$ is any $\lambda$-invariant $C^*$-subalgebra of $L^\infty(M/\sigma)$ containing $\{f_{\Phi, p}: \Phi \in C^{\sigma, u}, p \in \mathbb{Z}\}$.

Proof. Let $J: D^{\sigma, u} \rightarrow A \rtimes_\lambda \mathbb{Z}$ be defined, at the level of functions $\Phi \in C^{\sigma, u}$, by $(J\Phi)(\hat{m}, p) = f_{\Phi, p}(\hat{m})$. Then, by properties i) and ii) in Lemma [Z.1] $J$ is a $*$-algebra homomorphism:

$$J(\Phi^*)(\hat{m}, p) = H_p(m)\overline{\Phi}(\lambda_p m, -p) = \overline{H_{-p}(\lambda_p m)}\Phi(\lambda_p m, -p) = (J\Phi)^*(\hat{m}, p)$$

and

$$J(\Phi \ast \Psi)(\hat{m}, p) = \sum_{q \in \mathbb{Z}} H_q(m)H_{p-q}(\lambda_r m)\Phi(m, q)\Psi(\lambda_r m, p - q) = H_p(m)(\Phi \ast \Psi)(m, p) = |J(\Phi \ast \Psi)|(|\hat{m}, p)|.$$

Let $\mu_0$ be a Borel measure of full support on $F$ and, for $\sigma_k : F \rightarrow \sigma_k F$ and $\Pi : F \rightarrow M/\sigma$, set $\mu_k = (\sigma_k)_*(\mu_0)$ and $\bar{\mu} = \Pi_* (\mu_0)$. Then $\bar{\mu}$ and $\mu_k$ have full support on $M/\sigma$ and $\sigma_k F$ respectively, for all $k \in \mathbb{Z}$. In what follows we will also denote by $\mu_k$ the Borel measure on $M$ obtained by setting $\mu_k(X) = \mu_k(X \cap \sigma_k F)$, for a Borel subset $X$ of $M$. Now let $\hat{\Theta}$ and $\Theta^k$, for $k \in \mathbb{Z}$, denote the representations of $A \rtimes_\lambda \mathbb{Z}$ and $D^{\sigma, u}$ on $L^2(M/\sigma \times \mathbb{Z}, \tilde{\mu} \times \nu)$ and $L^2(M \times \mathbb{Z}, \mu_k \times \nu)$ ($\nu$ being counting measure on $\mathbb{Z}$), respectively, defined by

$$(\hat{\Theta}_\psi \xi)(\hat{m}, p) = \sum_{q \in \mathbb{Z}} \psi(\lambda_p \hat{m}, q)\xi(\hat{m}, p - q)$$

and

$$(\Theta^k_\eta)(m, p) = \sum_{q \in \mathbb{Z}} \Phi(\lambda_p m, q)\eta(m, p - q),$$

where $\Phi \in C^{\sigma, u}$, $\Psi \in C_c(M/\sigma \times \mathbb{Z})$, $\xi \in L^2(M/\sigma \times \mathbb{Z}, \tilde{\mu} \times \nu)$, and moreover $\eta \in L^2(M \times \mathbb{Z}, \mu_k \times \nu)$. Let $U : L^2(M/\sigma \times \mathbb{Z}, \tilde{\mu} \times \nu) \rightarrow L^2(M \times \mathbb{Z}, \mu_k \times \nu)$ be the unitary operator defined by $(U\xi)(m, p) = \overline{H_p}(\lambda_p m)\xi(\hat{m}, p)$. Then, if $m \in \sigma_k F$, we have

$$|\hat{\Theta}_{j\Phi}\xi(\hat{m}, p)| = |\sum_{q \in \mathbb{Z}} (J\Phi)(\lambda_p \hat{m}, q)\xi(\hat{m}, p - q)| = |\sum_{q \in \mathbb{Z}} H_q(\lambda_p m)\Phi(\lambda_p m, q)(U\xi)(m, p - q)H_{p-q}(\lambda_{-p} m)| = |\sum_{q \in \mathbb{Z}} H_p(\lambda_p m)\Phi(\lambda_p m, q)(U\xi)(m, p - q)| = |\Theta^k_\Phi(U\xi)(m, p)|,$$

and it follows that $||\hat{\Theta}_{j\Phi}\xi|| = ||\Theta^k_\Phi(U\xi)||$.

Now, the representation $\hat{\Theta}$ is faithful ([PD 7.7.5, 7.7.7]); therefore, for $\Phi \in C^{\sigma, u}$,

$$||J\Phi|| = ||\hat{\Theta}_{j\Phi}|| = ||\Theta^k_\Phi|| \leq ||\Phi||,$$

so $J$ can be extended to a continuous map on $D^{\sigma, u}$. 


We next show that, for $\Phi \in C^0$, we have $\|\Phi\| = \sup_k \|\Theta_k\| = \|J\Phi\|$, which takes care of the injectivity of $J$.

First notice that the representation $\bigoplus_k \Theta_k$ is unitarily equivalent to the representation $\Theta$ of $D^0$ on $L^2(M \times \mathbb{Z}, \mu \times \nu)$ defined by the same formula as $\Theta_k$, where, for a Borel subset $X$ of $M$, we set $\mu(X) = \sum_k \mu_k(X \cap \sigma_k F)$.

Thus it suffices to prove that $\Theta$ is faithful. In order to do that, we show ([PD 7.7.5, 7.7.7]) that $\mu$ has full support on $M$: Let $O \subset M$ be an open set such that $\mu(O) = 0$. Then, for all $k \in \mathbb{Z}$, we have that $O \cap \sigma_k F$ is an open subset of $\sigma_k F$ and $\mu_k(O \cap \sigma_k F) = 0$. Since $\mu_k$ has full support on $\sigma_k F$, it follows that $A = \bigcup A \cap \sigma_k F = \emptyset$, which ends the proof.

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From now on we will be dealing with the case of quantum Heisenberg manifolds. We specialize Theorem 2.3 to that case.

**Corollary 2.4.** Let $\lambda$ be the action of $\mathbb{Z}$ on $\mathbb{T}^2$ defined by

$$\lambda_k(x, y) = (x + 2k\mu, y + 2k\nu),$$

and let $A$ denote the smallest $\lambda$-invariant $C^*$-subalgebra of $L^\infty(\mathbb{T}^2)$ containing $C(\mathbb{T}^2)$ and the characteristic functions of the sets $[2k\underline{\mu}, 2(k+1)\underline{\mu}] \times \mathbb{T}$, for all $k \in \mathbb{Z}$. Then the quantum Heisenberg manifold $D^0$ can be embedded in $A \times \lambda \mathbb{Z}$.

**Proof.** Let us take $F = [0, 1) \times \mathbb{T}$ as a fundamental domain for $\sigma$, and $H$ as in Lemma 3.1. If $\Phi \in C^c_\nu$ and $p \in \mathbb{Z}$, then $f_{\Phi, p}(x, y) = \Phi(x', y, p)$, where $x' \in [0, 1)$ and $\exp(2\pi i x') = \exp(2\pi i x)$. Therefore $f_{\Phi, p}$ belongs to the $\lambda$-invariant algebra $A$. Thus Theorem 2.3 applies to $A$.

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3. THE RANGE OF TRACES ON $K_0(D^0)$

In this section we discuss the range of traces on $K_0(D^0)$. We first give a description of tracial states on the algebra $D^0$. The techniques involved are an adaptation of those usually employed (see [TO, 3.3]) to relate $\lambda$-invariant probability measures on a $G$-space $X$ to tracial states on $C_0(X) \rtimes \lambda G$. Then, by embedding $D^0$ in a crossed product as in Section 2 we show that any tracial state $\tau$ on $D^0$ induces the same homomorphism on $K_0(D^0)$, and that $\tau_0(K_0(D^0)) = \mathbb{Z} + 2\mu\mathbb{Z} + 2\nu\mathbb{Z}$.

**Lemma 3.1.** For each $p \in \mathbb{Z}$ there exist $\Delta_p^\mu, \Delta_p^\nu \in C^c_\nu$ such that $\Delta_p^\mu(x, y, n) = 0$ if $n \neq p$, and

i) $(\Delta_p^\mu)^* \star \Delta_p^\mu + (\Delta_p^\nu)^* \star \Delta_p^\nu = 1 = (\Delta_p^\mu)^* + (\Delta_p^\nu)^* \star (\Delta_p^\nu)^*$.

ii) $\Delta_p^\mu \star f \star (\Delta_p^\nu)^* \star \Delta_p^\nu \star f \star (\Delta_p^\nu)^* = \lambda_p(f)$ for all $f \in C(\mathbb{T}^2)$.

**Proof.** Let $d \in C(\mathbb{T})$ be such that $0 \leq d \leq 1$, $d(0) = 0$, and $d(1/2) = 1$. For $p \in \mathbb{Z}$ let $\Delta_p^\mu(x, y, n) = d^{1/2}(x)\delta_p(n)$, for $x \in [0, 1], y \in \mathbb{T}$,

$$\Delta_p^\mu(x, y, n) = \begin{cases} (1 - d(x))^{1/2}\delta_p(n) & \forall x \in [0, 1/2], y \in \mathbb{T}, \\ (1 - d(x))^{1/2}\exp(-2\pi icp(y - \nu))\delta_p(n) & \forall x \in [1/2, 1], y \in \mathbb{T}, \end{cases}$$

and extend $\Delta_p^\mu$, for $i = 1, 2$, to continuous functions on $\mathbb{R} \times \mathbb{T} \times \mathbb{Z}$ by setting $\Delta_p^\nu(x+1, y, n) = \exp(-2\pi icp(y - \nu))\Delta_p^\nu(x, y)\delta_p(n)$, for all $(x, y, n)$ in $\mathbb{R} \times \mathbb{T}$. Then

$$[\Delta_p^\nu \star (\Delta_p^\nu)^*]\delta_0 = (|\Delta_p^\nu(x+2p\mu, y + 2\nu, p)|^2 \delta_0(n),$$

so $(\Delta_p^\mu)^* \star \Delta_p^\mu + (\Delta_p^\nu)^* \star \Delta_p^\nu = (|\Delta_p^\mu|^2 + |\Delta_p^\nu|^2)\delta_0 = 1.$

Moreover, if $f \in C(\mathbb{T}^2)$, then

$$[\Delta_p^\mu \star f \star (\Delta_p^\nu)^*]\delta_0 = |\Delta_p^\nu(x, y, p)|^2 f(x - 2p\mu, y - 2\nu)\delta_0(n),$$

Thus Theorem 2.3 applies to $A \times \lambda \mathbb{Z}$. Therefore $f_{\Phi, p}$ belongs to the $\lambda$-invariant algebra $A$. Thus Theorem 2.3 applies to $A$.
\[ \Delta_1^p \ast f \ast (\Delta_1^p)^* + \Delta_2^p \ast f \ast (\Delta_2^p)^* = (\vert \Delta_1^p \vert^2 + \vert \Delta_2^p \vert^2) \lambda_p(f) = \lambda_p(f). \]

The second equality in i) now follows from taking \( f = 1 \) in ii). \( \square \)

**Notation 3.2.** Throughout this section \( e(a) \) denotes \( \exp(2\pi ia) \), for a real number \( a \).

**Remark 3.3.** It was shown in [AE 2] that the \( C^* \)-algebra \( D_{\mu,\nu}^c \) is the crossed product, in the sense of [AEE], of \( C(T^2) \) by the Hilbert \( C^* \)-bimodule \( M_{\mu,\nu}^c \), where \( M_{\mu,\nu}^c = \{ f \in C(\mathbb{R} \times T) : f(x+1,y) = e(-cy)f(x,y) \} \) with the structure defined by

\[ (f \cdot \Phi)(x,y) = f(x,y)\Phi(x - 2\mu, y - 2\nu), \quad (\Phi \cdot f)(x,y) = \Phi(x,y)f(x,y), \]

\[ \langle f,g \rangle_R(x,y) = \overline{f(x+2\mu,y+2\nu)}g(x+2\mu,y+2\nu), \]

for \( \Phi \in C(T^2) \) and \( f,g \in M_{\mu,\nu}^c \).

Since the Hilbert \( C^* \)-bimodules \( M_{\mu,\nu}^c, M_{\mu+\frac{1}{2},\nu}^c, \) and \( M_{\mu,\nu+\frac{1}{2}}^c \) are clearly isomorphic, it follows that so are the \( C^* \)-algebras \( D_{\mu,\nu}^c, D_{\mu+\frac{1}{2},\nu}^c, \) and \( D_{\mu,\nu+\frac{1}{2}}^c \).

In [AE] the Picard group of \( C(T^2) \) was shown to be the semidirect product of \( \text{Aut}(C(T^2)) \) by \( \{ M_{0,0}^c : c \in \mathbb{Z} \} \cong \mathbb{Z} \). By using this description, it was proved (AE 2.2) that \( D_{\mu,\nu}^c \) and \( D_{\mu',\nu'}^c \) are isomorphic if \( (\mu,\nu) \) and \( (\mu',\nu') \) belong to the same orbit under the usual action of \( GL_2(\mathbb{Z}) \) on \( T^2 \). This result carries over to the case when \( (2\mu, 2\nu) \) and \( (2\mu', 2\nu') \) belong to the same orbit because then, if \( A \in GL_2(\mathbb{Z}) \) is such that \( A(2\mu, 2\nu) = (2\mu', 2\nu') \), for some \( k,l \in \mathbb{Z} \), then

\[ A \begin{pmatrix} \mu \\ \nu \end{pmatrix} = \begin{pmatrix} \mu' + k/2 \\ \nu' + l/2 \end{pmatrix}, \]

so

\[ D_{\mu,\nu}^c \cong D_{\mu'+\frac{k}{2},\nu'+\frac{l}{2}}^c \cong D_{\mu',\nu'}. \]

**Lemma 3.4.** Let \( a, b, p, q \) be non-zero integers such that \( \gcd(a,p) = \gcd(b,q) = 1 \), and let \( m = \text{lcm}(p,q) \). Then \( (\frac{a}{p}, \frac{b}{q}) \) and \( (\frac{1}{m},0) \) are in the same orbit under the action of \( GL_2(\mathbb{Z}) \) on \( T^2 \), so \( D_{\frac{a}{p},\frac{b}{q}}^c \cong D_{\frac{1}{m},0}^c \). If \( \gcd(a,p) = 1 \), then \( (\frac{a}{p},0), (\frac{1}{p},0), \) and \( (0,\frac{a}{p}) \) belong to the same orbit under the action of \( GL_2(\mathbb{Z}) \), and \( D_{\frac{a}{p},0} \cong D_{\frac{1}{p},0} \cong D_{\frac{a}{p},\frac{a}{p}}^c \).

**Proof.** Let us write \( m = pp' = qq' \), so \( \gcd(p',q') = 1 \) and \( \gcd(ap', bq', m) = 1 \). Then it suffices to show that, if \( \gcd(a,b,p) = 1 \), then \( A(\frac{a}{p}, \frac{b}{p}) = (\frac{1}{p},0) \) for some \( A \in GL_2(\mathbb{Z}) \), viewing \( (\frac{a}{p}, \frac{b}{p}) \) and \( (\frac{1}{p},0) \) as elements of \( T^2 \). This will also show our second statement, since, in \( T^2 \), \( (\frac{a}{p},0) = (\frac{a}{p}, \frac{a}{p}) \) and \( (0,\frac{a}{p}) = (\frac{p}{p}, \frac{a}{p}) \). The isomorphisms between the corresponding quantum Heisenberg manifolds will then follow from [AE] 2.2.

For \( a, b, p \) as above, let \( d = \gcd(a,b) \), so \( \gcd(d,p) = 1 \). Write \( a = a'd, b = b'd, \) and choose integers \( r, s \) such that \( a'r + b's = 1 \). Then

\[ \begin{pmatrix} r & s \\ -b'r & a' \end{pmatrix} \in GL_2(\mathbb{Z}) \quad \text{and} \quad \begin{pmatrix} r & s \\ -b'r & a' \end{pmatrix} \begin{pmatrix} \frac{p}{p} \\ \frac{a}{p} \end{pmatrix} = \begin{pmatrix} \frac{d}{p} \\ 0 \end{pmatrix}. \]
We show that

Proof. Also, the correspondence for have that

and because

This shows that for .

Now, as elements of , we have that and belong to the same orbit under the action of on .

\[ (\frac{2a}{m}, \frac{2b}{n}) \] and \[ (\frac{1}{p}, 0) \] are in the same orbit under the action of on , so is isomorphic to .

\[ D^c_{\frac{m}{n}, \frac{p}{q}} \]

Proof. The statement follows from Remark 3.3 and Lemma 3.4.

For the remainder of this section, given a quantum Heisenberg manifold .

Let be the subalgebra of , and let .

For a fixed we set . Since for as in Lemma 3.1 we have that

\[ \phi \delta_n = \phi \delta_n \ast (\Delta_1^n)^* \ast \Delta_1^n + \phi \delta_n \ast (\Delta_2^n)^* \ast \Delta_2^n, \]

and \( \phi \delta_n \ast (\Delta_i^n)^* \in C(T^2) \), for \( i = 1, 2 \), it suffices to show that \( \tau(\phi \ast \Delta_i^n) = 0 \), for all \( g \in C(T^2) \), \( i = 1, 2 \), and \( n \not\in p\mathbb{Z} \).

For a fixed \( n \not\in p\mathbb{Z} \), we can assume that \( g = f^2 \) for some positive function \( f \) satisfying \( \text{supp}(f) \cap \text{supp}(\lambda_n f) = \emptyset \), because, since in this case \( \lambda^n(x, y) \neq (x, y) \) for all \( (x, y) \in T^2 \), any function \( g \in C(T^2) \) is the linear combination of functions satisfying those conditions. So let \( g \in C(T^2) \) be as above.

Then

\[ \tau(g \ast \Delta_i^n) = \tau(f^2 \ast \Delta_i^n) = \tau(f \ast f \ast \Delta_i^n) = \tau(f \ast \Delta_i^n \ast f) = 0, \]

because

\[ f \ast \Delta_i^n \ast f = f \Delta_i^n(\lambda_n f) = 0. \]

This shows that \( \tau = \tau \circ E_p^c \), since both sides are continuous and agree on .

\[ \gamma \phi = \Delta_1^1 \ast \phi \ast (\Delta_1^1)^* + \Delta_2^1 \ast \phi \ast (\Delta_2^1)^*, \]

for \( \phi \in B_p^c \) and \( \Delta_i^1 \), \( i = 1, 2 \), as in Lemma 3.5.

Then, for \( \phi \in B_p^c \) compactly supported on ,

\[ (\gamma \phi)(x, y, m) = \begin{cases} \phi(x - 2\mu, y - 2\nu, 0)\delta_0(m) & \text{if } p = 0, \\ e^{-cm} \phi(x - 1/p, y, np)\delta_{np}(m) & \text{if } p \neq 0. \end{cases} \]

Also, the correspondence \( \tau \mapsto \tau \circ E_p^c \) is a bijection between the set of \( \gamma \)-invariant tracial states on \( B_p^c \) and tracial states on \( D^c_{\mu, \nu} \).
Proof. If \( \tau \) is a trace on \( D^c_{\mu,\nu} \), then, by Proposition 3.7, we have that \( \tau = \tau \circ E^c_p \), and the restriction of \( \tau \) to \( B^c_p \) is \( \gamma \)-invariant because
\[
\tau(\gamma \phi) = \tau[(\Delta^1_1)^* \Delta^1_1 \ast \phi + (\Delta^1_2)^* \Delta^1_2 \ast \phi] = \tau(\phi).
\]
Now, for \( \phi \in B^c_p \) compactly supported on \( \mathbb{Z} \), we have
\[
[\Delta^1_1 \ast \phi \ast (\Delta^1_1)^*](x, y, np) = \Delta^1_1(x, y, 1) \phi(x - 2\mu, y - 2\nu, np) \Delta^1_1(x, 2n\mu, y - 2n\nu, 1),
\]
so
\[
(\gamma \phi)(x, y, m) = \begin{cases} 
\phi(x - 2\mu, y - 2\nu, 0) \delta_0(m) & \text{if } p = 0, \\
e^{-cny} \phi(x - 1/p, y, np) \delta_{np}(m) & \text{if } p \neq 0.
\end{cases}
\]
Now let \( \phi \) be a \( \gamma \)-invariant tracial state on \( B^c_p \). Since \( \tau \circ E^c_p \) is a state, we only need to show that \( \tau \circ E^c_p(\phi \ast \psi) = \tau \circ E^c_p(\psi \ast \phi) \), for \( \phi = f \delta_k, \psi = g \delta_l \). We can assume that \( k + l \in \mathbb{Z} \), since otherwise \( E^c_p(\phi \ast \psi) = 0 = E^c_p(\psi \ast \phi) \).

If \( p \neq 0 \), we take \( \phi \) and \( \psi \) as above, with \( k + l = np \), and we have
\[
[\gamma^{-k}(\phi \ast \psi)](x, y, m) = e(cny)f(x + k/p, y)g(x, y)\delta_{np}(m) = g(x, y)f(x + k/p, y)\delta_{np}(m) = (\psi \ast \phi)(x, y, m).
\]
So \( (\tau \circ E^c_p)(\phi \ast \psi) = \tau(\psi \ast \phi) = \tau(\gamma^k(\psi \ast \phi)) = \tau(\psi \ast \phi) = (\tau \circ E^c_p)(\psi \ast \phi) \). Similar computations prove the case \( p = 0 \). \( \square \)

**Proposition 3.9.** Given a quantum Heisenberg manifold \( D^c_{\mu,\nu} \), let \( p, B^c_p \) and \( E^c_p \) be as in Remark 3.6. Then \( B^c_p \cong C(\mathbb{T}^2) \) if \( p = 0 \), and \( B^c_p \cong D^c_{0,0} \) if \( p \neq 0 \).

Proof. It is clear that \( B^c_p \cong C(\mathbb{T}^2) \) for \( p = 0 \). If \( p \neq 0 \), set \( J : B^c_p \to D^c_{0,0} \),
\[
J \phi(x, y, n) = u_p(n, y) \phi(x, y, np),
\]
for \( \phi \in B^c_p \cap C^c_{0,0} \), where \( u_p(n, y) = e(-\frac{1}{2}cn\mu(n - 1)y) \).

Notice that
\[
(J \phi)(x + 1, y, n) = u_p(n, y)e(-cnpy)\phi(x, y, np) = e(-cnpy)(J \phi)(x, y, n),
\]
so \( J \phi \in D^c_{0,0} \) for \( \phi \in B^c_p \cap C^c_{0,0} \).

Let \( \Pi \) and \( \sigma \) denote, respectively, the faithful representations (Ref 3) of \( D^c_{0,0} \) and \( D^c_{0,0} \) on \( L^2(\mathbb{R} \times \mathbb{T} \times \mathbb{Z}) \) given by
\[
(\Pi_\phi \xi)(x, y, n) = \sum_q \phi(x + n/p, y, qp) \xi(x, y, n - qp),
\]
\[
(\sigma_\psi \eta)(x, y, n) = \sum_q \phi(x, y, q) \eta(x, y, n - q),
\]
for \( \phi \in C^c_{1/p,0}, \psi \in C^c_{0,0}, \xi, \eta \in L^2(\mathbb{R} \times \mathbb{T} \times \mathbb{Z}). \)

Let \( U : L^2(\mathbb{R} \times \mathbb{T} \times \mathbb{Z}) \to \bigoplus_{0}^{p-1} L^2(\mathbb{R} \times \mathbb{T} \times \mathbb{Z}) \) be given by
\[
(U \xi)_i(x, y, n) = u_p(-n, y) \xi(x, y, np + i),
\]
for $\xi \in L^2(\mathbb{R} \times \mathbb{T} \times \mathbb{Z})$. It is easily checked that $U$ is unitary and that

$$[U^*((\eta_i))](x, y, n) = u_p(-k, y)\eta_i(x, y, k) \quad \text{for } n = kp + i, 0 \leq i < p.$$  

Now,

$$[U\Pi_\phi U^*(\eta_i)]_j(x, y, n) = u_p(-n, y)(\Pi_\phi U^*(\eta_i))(x, y, np + j)$$

$$= \sum_q u_p(-n, y)\phi(x + (np + j)/p, y, qp)(U^*(\eta_i))(x, y, (n - q)p + j)$$

$$= \sum_q u_p(q, y)\phi(x + j/p, y, qp)\eta_j(x, y, n - q)$$

$$= \sum_q (J\phi)(x + j/p, y, q)\eta_j(x, y, n - q)$$

$$= [\sigma(J\phi)(\eta_j)]_j(x, y, n - q),$$

where $(\delta^j\psi)(x, y, n) = \psi(x + j/p, y, n)$ for all $\psi \in C^{cp}_{0,0}$ and $0 \leq j < p$. Notice that $\delta^j$ defines an automorphism of $D^{cp}_{0,0}$; apply [AB2, 1.1] to define $\delta^j$ on $C_b(\mathbb{R} \times \mathbb{T}) \rtimes_{id} \mathbb{Z}$ and then check that $D^{cp}_{0,0}$ is invariant under it. Thus $U$ intertwines $\Pi_\phi$ and $\bigoplus_j (\sigma \circ \delta^j)(J\phi)$, which shows that $J$ extends to an isomorphism.

**Remark 3.10.** Recall ([RF3]) that, for a positive integer $c$, the C*-algebra $D^{cp}_{0,0}$ is isomorphic to the (commutative) Heisenberg manifold $C(M^c)$, where $M^c$ is the quotient space of $\mathbb{R} \times \mathbb{T}^2$ under the equivalence relation given by

$$(x, y, z) \cong (x', y', z')$$

if and only if

$$(x', y', z') = (x + k, y, z + cky)$$

for some $k \in \mathbb{Z}$, and $(x, y, z)$, $(x', y', z') \in \mathbb{R} \times \mathbb{T}^2$ (viewing $\mathbb{T}$ as $\mathbb{R}/\mathbb{Z}$).

The isomorphism is obtained by taking Fourier transform in the third variable, that is, $F : C(M^c) \rightarrow D^{cp}_{0,0}$. $(F)f(x, y, n) = \int_{\mathbb{T}} e(-nz)f(x, y, z)dz$.

**Corollary 3.11.** Given a quantum Heisenberg manifold $D^{cp}_{\mu\nu}$, let $p$, $B^c_p$, and $E^c_\mu$ be as in Notation 3.8. There is a bijective correspondence between tracial states on $D^{cp}_{\mu\nu}$ and $\gamma$-invariant probability measures on $X$, where

$$X = \mathbb{T}^2, \quad \gamma(x, y) = (x + 2\mu, y + 2\nu),$$

if either $\mu$ or $\nu$ is irrational, and

$$X = M^{cp}, \quad \gamma(x, y, z) = (x + 1/p, y, z + cy)$$

if $\mu = \frac{1}{2p}$, $\nu = 0$.

The correspondence is given by $m \mapsto \tau_m \circ E^c_\mu$, where $\tau_m(f) = \int_X f dm$, once $B^c_p$ is identified with $C(X)$, according to Proposition 3.9 and Remark 3.10.

**Proof.** It is easily checked that the formula above is the formula for $\gamma$ in Proposition 3.8 when one keeps track of the isomorphisms $J$ and $F$ in Proposition 3.9 and Remark 3.10, respectively.

**Corollary 3.12.** If $\{1, \mu, \nu\}$ is linearly independent over the field of rational numbers, then the trace corresponding to Haar measure on $\mathbb{T}^2$ is the only tracial state on $D^{cp}_{\mu\nu}$. 

Proposition 3.8. Then $T_B$.

Let $T_B$ under translation by $(2\epsilon)\in\mathbb{C}$ in $A$ with $C_{\epsilon}$ when restricted to $\Delta$. Commuting, and by Proposition 3.7, then the statement follows from Proposition 3.7, Corollary 3.11, and [TO, 3.3.9].

Proof. Let $A$ be as in Corollary 2.4. Notice that the embedding $J$ in Theorem 2.3 maps the $C^*$-algebra $B^c_{\epsilon}$ defined in Notation 3.6 to the commutative $C^*$-subalgebra $B$ of $A\times_{\lambda}\mathbb{Z}$ generated by $\{\phi \in C_c(\mathbb{Z}, A) : \text{supp} \phi \subset p\mathbb{Z}\}$, and that $J$ is the identity when restricted to $C(\mathbb{T}^2) \subset B^c_{\epsilon}$ as in Corollary 3.13. So, if either $\mu$ or $\nu$ is irrational, then the statement follows from Proposition 3.7, Corollary 3.13, and [TO, 3.3.9].

If $(\mu, \nu) = \left(\frac{1}{2p}, 0\right)$, given a trace $\tau$ on $D^c_{\epsilon, \mu}$, let $S$ denote the set of states on $B$ extending $\tau_0 \circ J^{-1}$ on $J(B^c_{\epsilon, \mu})$, where $\tau_0$ denotes the restriction of $\tau$ to $B^c_{\epsilon, \mu}$.

Let $T : B \longrightarrow B$ be given by

$$T(a) = J(\Delta^1_i) \ast a \ast J(\Delta^2_i) \ast a \ast J(\Delta^2_i)^*,$$

with $\Delta^1_i, i = 1, 2$, as in Lemma 3.1 and $J$ as in Theorem 2.3, and set $T^* : B^* \longrightarrow B^*$, $T^*(\rho) = \rho \circ T$. If $\rho \in S$, then $T^*(\rho)$ is positive and $||T^*(\rho)|| = ||T^*(\rho)||_1 = \rho(1) = 1$, by Lemma 3.1. Besides, the restriction of $T^*(\rho)$ to $J(B^c_{\epsilon, \mu})$ is $\tau_0$ by Proposition 3.8. Then $T^*(S) \subset S$, and $S$ is a $w^*$-compact, convex, non-empty set, so it follows from Markov’s fixed-point theorem that there exists $\tau_1 \in S$ such that $T^*(\tau_1) = \tau_1$.

We next show that if $P$ denotes the conditional expectation $P : A\times_{\lambda}\mathbb{Z} \longrightarrow B$ given by

$$(P\phi)(x, y, n) = \begin{cases} \phi(x, y, n) & \text{if } n \in p\mathbb{Z}, \\ 0 & \text{otherwise}, \end{cases}$$

for $\phi \in C_c(\mathbb{Z}, A)$, then $\tau_1 \circ P$ is a trace on $A\times_{\lambda}\mathbb{Z}$. This will end the proof, because the diagram

$$
\begin{array}{ccc}
D^c_{\epsilon, 0} & \overset{J}{\longrightarrow} & A\times_{\lambda}\mathbb{Z} \\
E^c_{\epsilon, 1} & \overset{P}{\longrightarrow} & B^c_{\epsilon} \\
B^c_{\epsilon} & \overset{J}{\longrightarrow} & B
\end{array}
$$

commutes, and, by Proposition 3.7, $\tau_0 = \tau \circ \epsilon E^c_{\epsilon, \mu}$, so

$$\tau = \tau_0 \circ \epsilon E^c_{\epsilon, \mu} = \tau_1 \circ \epsilon J \circ \epsilon E^c_{\epsilon, \mu} = \tau_1 \circ P \circ J.$$
First notice that if \( H \in A \), and \( n \in \mathbb{Z} \), then \( T(H\delta_{np}) = (\lambda H)\delta_{np} \). In fact,

\[
T(H\delta_{np})(x, y, m) = \sum_{i=1}^{2} [J(\Delta^1_i) \ast H\delta_{np} \ast J(\Delta^1_i)^\ast](x, y, m)
\]

\[
= \sum_{i=1}^{2} J(\Delta^1_i)(x, y, 1)H(x - \frac{1}{p}, y, np)J(\Delta^1_i)^\ast(x - \frac{1}{p}, -n, y, -1)\delta_{np}(m)
\]

\[
= \sum_{i=1}^{2} |(\Delta^1_i)(x, y, 1)|^2(\lambda H)(x, y)\delta_{np}(m)
\]

\[
= \sum_{i=1}^{2} |(\Delta^1_i)(x, y, 1)|^2(\lambda H)(x, y)\delta_{np}(m)
\]

\[
= (\lambda H)\delta_{np}(x, y, m).
\]

Now, for \( \phi \) and \( \psi \) as above, we can assume that \( k + l = np \) for some \( n \in \mathbb{Z} \), since otherwise \( P(\phi \ast \psi) = 0 = P(\psi \ast \phi) \). In this case

\[
[T^k(\psi \ast \phi)](x, y, m) = (\psi \ast \phi)(x - \frac{k}{p}, y, m)
\]

\[
= G(x - \frac{k}{p}, y)F(x - \frac{1}{p} - \frac{k}{p}, y)\delta_{np}(m)
\]

\[
= F(x, y)G(x - \frac{k}{p}, y)\delta_{np}(m)
\]

\[
= (\phi \ast \psi)(x, y, m).
\]

Therefore

\[
(\tau_1 \circ P)(\phi \ast \psi) = \tau_1(\phi \ast \psi) = \tau_1(T^k(\psi \ast \phi)) = \tau_1(\psi \ast \phi) = \tau_1 \circ P(\psi \ast \phi),
\]

as we wanted to show. \qed

**Lemma 3.15.** If \( \mu \leq 1/2 \) and \( m \) is a \( \lambda \)-invariant probability measure on \( T^2 \), then \( m([0, 2\mu] \times T) = 2\mu \).

**Proof.** First notice that the analogous result holds for \( T \). Fix a real number \( \alpha \in [0, 1] \). If \( v \) is a measure on \( T \) invariant under translation by \( \alpha \), then \( v([0, \alpha]) = \alpha \). If \( \alpha \) is irrational, then \( v \) is Haar measure on \( T \), and the result is obviously true. If \( \alpha \) is rational, \( \alpha = \frac{p}{q} \), for \( p, q \in \mathbb{Z} \), with \( (p, q) = 1 \), then \( T \) is the disjoint union of the intervals \( I_i = [i/q, (i + 1)/q) \), \( i = 0, 1, \ldots, q - 1 \).

Now, for all \( i, I_i \) can be obtained by translating \( I_0 \) by \( \alpha \) an appropriate number of times. Therefore \( v(I_i) = v(I_0) = 1/q \), for all \( i = 1, \ldots, q - 1 \), and it follows that \( v([0, \alpha]) = v([0, p/q)) = p/q = \alpha \).

Now let \( m \) be a \( \lambda \)-invariant probability measure on \( T^2 \). Define a probability measure \( v \) on \( T \) by setting \( v(X) = m(X \times T) \).

Then \( v(A + 2\mu) = m((A + 2\mu) \times T) = m(\lambda(A \times T)) = m(A \times T) = v(A) \).

It follows now that \( m([0, 2\mu] \times T) = v([0, 2\mu]) = 2\mu \). \qed

**Theorem 3.16.** All tracial states \( \tau \) on \( D_{\mu v}^r \) induce the same homomorphism \( \tau_* \) on \( K_0(D_{\mu v}^r) \). Moreover, \( \tau_*(K_0(D_{\mu v}^r)) = \mathbb{Z} + 2\mu \mathbb{Z} + 2\nu \mathbb{Z} \).
Proof. For a tracial state $\tau$ on $D_{\mu\nu}'$, we denote by $\tau'$ an extension of $\tau$ to $A \times_\lambda \mathbb{Z}$, as in Proposition 3.14. We have the following short exact sequence ([PM, 3.4]):

$$0 \longrightarrow \tau_\ast(K_0(A)) \overset{i}{\longrightarrow} \tau'_\ast(K_0(A \times_\lambda \mathbb{Z})) \overset{q}{\longrightarrow} \Delta^\lambda(K) \longrightarrow 0,$$

where $K = \{u \in \mathcal{U}(A) : [u]_{K_1} \in \ker(1 - \lambda_x)\}$, $i$ and $q$ are inclusion and projection on $\mathbb{R}/\tau_\ast(K_0(A))$ respectively, $\Delta^\lambda(u) = q[\Delta_\ast(u\lambda(u^{-1}))]$, and $\Delta_\ast : (\mathcal{U}_i)_0 \longrightarrow \mathbb{R}$ is defined by $\Delta_\ast(e^{2\pi i y}) = \tau(y)$, for $y$ self-adjoint.

Let us relabel the set $X = (2\mu\mathbb{Z} + \mathbb{Z}) \cap (0, 1)$ so that $X = \{x_i : i \in \mathbb{N}\}$. Let $A_n$ be the smallest $C^*$-subalgebra of $L^\infty(\mathbb{T}^2)$ generated by $C(\mathbb{T}^2)$ and $\chi_{[0,x_i] \times \mathbb{T}}$, for $i = 1, ... , n$. Then $A_1 \subseteq A_2 \subseteq ... \subseteq A_n \subseteq ...$, and $A$ is the direct limit of $\{A_n\}$.

Now, $A_n \cong \bigoplus_{j=0}^n C([x_i, x_{i+1}] \times \mathbb{T})$, where $\{x_i\}_{i=1}^n$, $x_i = 0$, $x_{i+1} = 1$, and $x_j < x_j + 1$ for all $j = 0, 1, ... , n$.

Since $[a, b] \times \mathbb{T}$ can be deformed to $\mathbb{T}$, it follows that $K_j(A_n) = \mathbb{Z}^{n+1}$ for all $j = 1, 2$. The set

$$\{(\chi_{[x_i, x_j] \times \mathbb{T}}, K_0 : x_i, x_j \in X \cup \{0, 1\}, x_i < x_j\}$$

is a generator of $K_0(A)$, and any arbitrary element of $K_1(A)$ has a representative $u$ of the form

$$u(x, y) = e(n_i y) \quad \text{if } x \in [t_i, t_{i+1})$$

for a partition $0 = t_0 < t_1 < ... < t_n = 1, \{t_i\}_{i=1}^{n+1} \subset X$, and integers $n_i, i = 0, ... , n - 1$.

Now, by Lemma 3.13 and Remark 3.8 we have that $\tau_\ast(K_0(A)) \subseteq \mathbb{Z} + 2\mu \mathbb{Z}$. Since $id$ and $\chi_{[0, 2\mu \mathbb{Z} + 3\mathbb{T}] \times \mathbb{T}} \in A$ for some $K_0$, the equality holds, and $\tau_\ast(K_0(A)) = \mathbb{Z} + 2\mu \mathbb{Z}$.

Let us now find the elements $[u]_{K_1} \in K_1(A)$ that are left fixed by $\lambda_x$, where $u$ is as above.

For $[u]_{K_1} \in K_1(A)$,

$$\lambda_k(u)(x, y) = u(x - 2k\mu, y - 2k\nu),$$

that is,

$$\lambda_k(u)(x, y) = e(n_i(y - 2k\nu)), \quad \text{where } x - 2k\mu \in [x_i, x_{i+1}).$$

Fix $a \in [x_0, x_1]$. If $\mu$ is irrational, then for all $i = 0, 1, ... , n$ there exists $k_i \in \mathbb{Z}$ such that $a - 2k_i \mu \in [x_i, x_{i+1})$ and $\lambda_k(u)(a, y) = e(n_i(y - 2k_i\nu))$. It is clear now that $[u]_{K_1} = [\lambda_k(u)]_{K_1}$ for $\lambda_k \in \mathbb{Z}$ and only if $n_i = n_0$ for all $i = 0, 1, ... , n$. Therefore $\Delta_\ast(u\lambda(u^{-1})) = \tau(2n_0\nu \cdot Id) = 2n_0\nu$, and it follows that $\Delta^\lambda(K) = 2\nu \mathbb{Z}$.

If $2\mu$ is rational, $2\mu = p/q$, where $p, q \in \mathbb{Z}, (p, q) = 1$, then $X = \{i/q : i = 0, ... , q\}$ and $u$ is of the form

$$u(x, y) = e(n_{k} y) \quad \text{for } x \in I_k = [k/q, (k+1)/q], \quad k = 0, 1, ... , q - 1.$$  

Translation by $p/q$ gives a transitive action of $\mathbb{Z}_p$ on the set $\{I_k\}$, since $(p, q) = 1$, so the same reasoning as for the irrational case applies, and $[u]_{K_1} = [\lambda u]_{K_1}$ if and only if $u(x, y) = e(n_y)$ for all $x, y$. Then, as above, $\Delta^\lambda(K) = 2\nu \mathbb{Z}$.

Therefore the short exact sequence above splits, and $\tau'_\ast(K_0(A \times_\lambda \mathbb{Z})) = \mathbb{Z} + 2p\mathbb{Z} + 2q\mathbb{Z}$, so $\tau'_\ast(K_0(D_{\mu\nu}')) \subseteq \mathbb{Z} + 2p\mathbb{Z} + 2q\mathbb{Z}$.

Now, it is shown in [PM, 2.3.4] that, for $[p] \in K_0(A \times_\lambda \mathbb{Z})$, the choice of $u \in K$ such that $q(\tau'_\ast([p])) = \Delta^\lambda(K)$ does not depend on $\tau$, and we just proved that $\Delta^\lambda(K)$ does not depend on $\tau$ either.

So we have $\tau'_\ast([p]) = \Delta^\lambda(u) + \tau'_\ast([p])$, for some $p_0 \in K_0(A)$. We next show that $\tau'_\ast([p_0])$ is independent of $\tau$ as well. The preceding remarks show that $[p_0]$ has a
representative \( h \in \bigoplus C([x_{ij}, x_{ij+1}] \times T) \), so \( h \) is constant on \([x_{ij}, x_{ij+1}] \times T\) for each \( j \). Our claim then follows from Lemma 3.15 since \( \tau_1([p_0]) = \int_{T^2} h d\mu_\tau \). So \( \tau_1 \) does not depend on \( \tau \), and \( \tau_1(K_0(D_{\mu\nu}^c)) \subset \mathbb{Z} + 2\mu\mathbb{Z} + 2\nu\mathbb{Z} \). Finally, the equality holds because it is attained for the trace induced by Haar measure on \( T^2 \). [AB1]

Corollary 3.17. Given a quantum Heisenberg manifold \( D_{\mu\nu}^c \), let \( G_{\mu\nu} \) denote the group \( \mathbb{Z} + 2\mu\mathbb{Z} + 2\nu\mathbb{Z} \).

If \( G_{\mu\nu} \) has rank 1 or 3, then \( D_{\mu\nu}^c \) and \( D_{\mu'\nu'}^c \) are isomorphic if and only if \((2\mu, 2\nu) \) and \((2\mu', 2\nu') \) belong to the same orbit under the usual action of \( GL_2(\mathbb{Z}) \) on \( T^2 \).

Proof. If \( D_{\mu\nu}^c \cong D_{\mu'\nu'}^c \), then \( G_{\mu\nu} = G_{\mu'\nu'} \), by Theorem 3.16. If 3 = \( \text{rank}(G_{\mu\nu}) = \text{rank}(G_{\mu'\nu'}) \), then \( G_{\mu\nu} = G_{\mu'\nu'} \) implies (see, for instance, [PA1, 2.13]) that \((2\mu, 2\nu) \) and \((2\mu', 2\nu') \) are in the same orbit under the action of \( GL_2(\mathbb{Z}) \).

If 1 = \( \text{rank}(G_{\mu\nu}) = \text{rank}(G_{\mu'\nu'}) \), then \( \mu, \nu, \mu' \), and \( \nu' \) are rational numbers. By virtue of Remark 3.5 we can assume that \((\mu, \nu) = (\frac{1}{2p_1}, 0) \) and \((\mu', \nu') = (\frac{1}{2p_2}, 0) \) for some \( p_1, p_2 \in \mathbb{Z} \), \( p_1 \neq p_2 \). Now the equality

\[
\mathbb{Z} + \frac{1}{p_1} \mathbb{Z} = G_{\mu,\nu} = G_{\mu',\nu'} = \mathbb{Z} + \frac{1}{p_2} \mathbb{Z}
\]

implies that \( \frac{1}{p_1} \mathbb{Z} = \frac{1}{p_2} \mathbb{Z} \), so \( p_1 = \pm p_2 \), and the result follows.

The converse statement was shown in [AE Thm. 2.2] (see also Remark 3.3). [AB1]

References


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