

BLOW UP AND INSTABILITY OF SOLITARY-WAVE SOLUTIONS TO A GENERALIZED KADOMTSEV-PETVIASHVILI EQUATION

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ABSTRACT. In this paper we consider a generalized Kadomtsev-Petviashvili equation in the form

$$(u_t + u_{xxx} + u^p u_x)_x = u_{yy} \quad (x, y) \in \mathbb{R}^2, t \geq 0.$$

It is shown that the solutions blow up in finite time for the supercritical power of nonlinearity $p \geq 4/3$ with p the ratio of an even to an odd integer. Moreover, it is shown that the solitary waves are strongly unstable if $2 < p < 4$; that is, the solutions blow up in finite time provided they start near an unstable solitary wave.

1. INTRODUCTION

We are concerned in this paper with a generalized Kadomtsev-Petviashvili (KP) equation; that is,

$$(1.1) \quad \begin{cases} u_t + u_{xxx} + u^p u_x = v_y \\ v_x = u_y \end{cases} \quad (x, y) \in \mathbb{R}^2, t \geq 0.$$

When $p = 1$, the equation (1.1) is known as the KP-I equation. It is a model for the propagation of weakly nonlinear dispersive long waves on the surface of a fluid, when the wave motion is essentially one-directional with weak transverse effects along the y -axis [KaPe, PeYa]. The KP-I equation has been analyzed in great detail. It can be solved by the Inverse Scattering Transformation (IST) [AbSe, FoSu, Zh]. When $p = 2$, the equation (1.1) is also a model to describe the evolution of sound waves in antiferromagnetics [TuFa]. To some extent, the KP equation can be viewed as a two-dimensional analog of the generalized Korteweg-de Vries (GKdV) equation.

Many rigorous results have recently appeared concerning the Cauchy problem for the KP equation [Bou, FoSu, IsMeSt, Sa1, Sa2, Uk, Zh, To]. A general local existence theory has been established by Ukai [Uk] and developed by Saut [Sa1] in the classical Sobolev space $H^s(\mathbb{R}^2)$ for $s \geq 3$. As we know, one of the more challenging issues concerning the Cauchy problem for nonlinear dispersive equations is the possible blow-up of solutions in finite time. For instance, consider the GKdV equation

$$(1.2) \quad u_t + u_{xxx} + u^p u_x = 0.$$

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Though numerical simulations [BoDoKa] do show that blow-up can occur for $p \geq 4$, a rigorous proof of the blow-up for $p \geq 4$ is still missing. Compared with GKdV, the KP equation does have a blow-up solution if the energy of the initial data is negative and the power of nonlinearity $p \geq 4$. This has been proved by using a virial identity [TuFa, Sa1]. On the other hand, by using the anisotropic Sobolev embedding

$$(1.3) \quad \int_{\mathbb{R}^2} |u|^{p+2} dx dy \leq c \|u\|_{L^2}^{\frac{4-p}{2}} \|\partial_x u\|_{L^2}^p \|v\|_{L^2}^{\frac{p}{2}}$$

which is valid only for $0 \leq p \leq 4$, we can deduce that $\int ((u_x)^2 + v^2) dx dy$ is dominated by the conserved momentum $V(u) = \frac{1}{2} \int_{\mathbb{R}^2} u^2 dx dy$ and the Hamiltonian

$$E(u) = \int_{\mathbb{R}^2} \left(\frac{1}{2} u_x^2 + \frac{1}{2} v^2 - \frac{1}{(p+1)(p+2)} u^{p+2} \right) dx dy$$

if and only if $p < \frac{4}{3}$. It is also shown by some numerical simulations [Wi, WaAbSe] that there are blow-up solutions for the KP equation (1.1) in finite time for $p \geq \frac{4}{3}$. Therefore, it is natural to conjecture that the solutions of the KP equation (1.1) should blow up in finite time for $p \geq \frac{4}{3}$. The purpose of this paper is to investigate questions about blow-up solutions and instability of solutions for (1.1). We are able to show that the solution of (1.1) blows up in finite time for $p \geq \frac{4}{3}$ (Theorems 4.1, 4.2). It turns out that the power of $p = \frac{4}{3}$ is the critical power of the nonlinearity for the existence of solutions. The critical power $p = \frac{4}{3}$ is also observed in the study of stability and instability of solitary waves. As we know, in the case of the GKdV equation, the power $p = 4$ is the critical value for the stability and instability of the solitary waves $\varphi_c(x - ct)$ [We, BoSoSt, SoSt]. As in the case of the GKdV equation, it is found that the critical power $p = \frac{4}{3}$ is also linked to the stability and instability of the solitary-wave solutions $\varphi_c(x - ct, y)$ for (1.1) (see Definition 2.5). Indeed, the solitary-wave solutions vanishing in the infinity exist if and only if $1 \leq p < 4$ [WaAbSe], [Sa1], [Sa2]. It is shown that the set of the solitary waves φ_c with $c > 0$ is nonlinearly stable for $1 \leq p < \frac{4}{3}$ [LiuWa, BoSa3] and it is nonlinearly unstable for $\frac{4}{3} < p < 4$ [WaAbSe, BoSa3]. Therefore, it is reasonable to conjecture that the solution of the KP equation (1.1) blows up in finite time for $p \geq \frac{4}{3}$ provided it begins near the solitary-wave solution φ_c (i.e., strong instability). In this paper, we also give a satisfactory result to solve this strong instability problem (Theorem 5.1). More interestingly, it is proved in Theorem 5.1 that for the power $2 < p < 4$, solutions with initial data close to unstable solitary waves are actually bounded for all time in energy space. However, they do blow up in finite time in some sense ($\|\partial_y u(\cdot, t)\|_{L^2} \rightarrow \infty$ in finite time), due to the weak transverse dispersion along the y -axis.

In general, solutions of (1.1) satisfy the following conserved functionals.

$$E(u) = \int_{\mathbb{R}^2} \left(\frac{1}{2} u_x^2 + \frac{1}{2} v^2 - \frac{1}{(p+1)(p+2)} u^{p+2} \right) dx dy \quad (\text{Energy})$$

$$V(u) = \frac{1}{2} \int_{\mathbb{R}^2} u^2 dx dy \quad (\text{Momentum}),$$

$$\Phi_1(u) = \int_{\mathbb{R}^2} u dx dy \quad \text{and} \quad \Phi_2(u) = \int_{\mathbb{R}^2} v dx dy.$$

Also, in the case of $p = 1$,

$$\Phi_3(u) = \int_{\mathbb{R}^2} \left(\frac{3}{2}u_{xx}^2 + 5u_y^2 + \frac{5}{6}(\partial_x^{-1}v_y)^2 - \frac{5}{2}uv_x^2 - \frac{5}{9}uv^2 \right) dx dy$$

is conserved. In fact, for $p = 1$, there are infinitely many conservation laws.

In this paper, we shall employ the following notations. $|\cdot|_p$ and $\|\cdot\|_s$ will denote the norms in $L^p(\mathbb{R}^2)$ and Sobolev space $H^s(\mathbb{R}^2)$ respectively. And we will borrow the notations in [Sa1, Sa2]. Let Y be the closure of $\partial_x(C_0^\infty(\mathbb{R}^2))$ with the norm

$$\|u\|_Y = \|\partial_x \psi\|_Y = (\|\nabla \psi\|_{L^2}^2 + \|\partial_x^2 \psi\|_{L^2}^2)^{\frac{1}{2}}$$

for $u \in Y$ and $u = \partial_x \psi$, where $\psi \in L_{loc}^q(\mathbb{R}^2)$, $\forall 2 \leq q < \infty$. We also have $v = \partial_y \psi \in L^2$ by a choice of $\psi \in L_{loc}^q$. Let

$$X = \{u \in H^1(\mathbb{R}^2); \partial_x^2 u \in L^2\},$$

$$X_s = \{u \in H^1(\mathbb{R}^2); (\xi_1^{-1} \hat{u})^\vee \in H^s(\mathbb{R}^2)\}$$

with the norm $\|u\|_{X_s} = \|u\|_s + \|(\xi_1^{-1} \hat{u})^\vee\|_s$ and

$$\dot{H}^{-k} = \{u \in S'(\mathbb{R}^2), \xi_1^{-k} \hat{u} \in L^2(\mathbb{R}^2)\}$$

equipped with the norm $\|u\|_{-k,x} = |\xi_1^{-k} \hat{u}|_2$, where “ \wedge ” is the Fourier transform, “ \vee ” is the Fourier inverse transform and $\xi = (\xi_1, \xi_2)$ is the dual variable of (x, y) .

Throughout this paper, we only consider the case when $p = \frac{n_1}{n_2}$ where n_1 is any even integer and n_2 any odd integer so that $\int_{\mathbb{R}^2} u^{p+2} dx dy = |u|_{\frac{p+2}{p+2}}^{p+2}$.

The main aim of the paper is to investigate properties of blow-up and instability of the solitary-wave solutions φ_c for the critical value $p \geq \frac{4}{3}$. The main theorems in the paper are Theorem 4.1 and Theorem 5.1. The key ingredients used here are to construct some virial identities, and study the minimization problems which are related to the solitary waves.

The plan of this paper is as follows. In Section 2, we study the properties of the solitary-wave solutions φ_c . In particular, we consider the associated minimization problem and employ a refined Fatou lemma due to Lieb and Brézis [BrLi1] to obtain the suitable minimizer, which is also known as “ground state.” In Section 3, some invariant sets for the flow of the KP equation are constructed. With the help of the invariant sets, we are able to establish the blow-up results in the super critical case $p > \frac{4}{3}$, and a detailed proof is given in Section 4. In the last section, a strong instability of solitary-wave solution is investigated. It is proved that for $2 < p < 4$ the solution of (1.1) blows up in finite time provided it starts close to a solitary-wave solution.

2. SOLITARY-WAVE SOLUTIONS

In this section, we study some important properties of the solitary-wave solution φ_c to the KP equation (1.1). By a solitary-wave solution of (1.1), we mean a traveling-wave solution of (1.1) in the form $u(t, x, y) = \varphi_c(x - ct, y)$ with $u \rightarrow 0$ as $x^2 + y^2 \rightarrow \infty$. And φ_c is a ground state of the equation

$$(2.1) \quad \begin{cases} -c\varphi_c + \partial_x^3 \varphi_c - \partial_y \psi_c + \frac{1}{p+1} \varphi_c^{p+1} = 0, \\ \partial_x \psi_c = \partial_y \varphi_c \quad \varphi_c \in Y, \varphi_c \neq 0. \end{cases}$$

Let $L_c(u) = E(u) + cV(u)$. We define the ground states of (2.1) as follows.

Definition 2.1. Let Γ_c be the set of the solutions of (2.1); namely,

$$\Gamma_c = \{\phi \in Y \mid L'_c(\phi) = 0, \phi \neq 0\},$$

and let G_c be the set of the ground states of (2.1); that is

$$G_c = \{\varphi \in \Gamma_c \mid L_c(\varphi) \leq L_c(\phi), \forall \phi \in \Gamma_c\}.$$

Remark 2.2. The existence of ground states for (2.1) was proved in [WaAbSe], [BoSa]. It is shown that G_c is not empty for any $c > 0$ and $1 \leq p < 4$. But the uniqueness of the ground state is still unknown.

Remark 2.3. For $p = 1$, there exists an explicit traveling-wave solution of the KP equation (1.1), which is called lump soliton $\phi_c(x - ct, y)$ with $c > 0$ [AbSe], where

$$(2.2) \quad \phi_c(x, y) = \frac{8c \left(1 - \frac{c}{3}x^2 + \frac{c^2}{3}y^2\right)}{\left(1 + \frac{c}{3}x^2 + \frac{c^2}{3}y^2\right)^2}.$$

However, to our knowledge, it is still an open problem whether or not the lump is a ground state.

There are many results on local existence to the KP equation (1.1). For our purpose, we state here the local existence result by Saut [Sa1].

Lemma 2.4. *Let $u_0 \in X_s, s \geq 3$, and $\partial_{yy}^2 u_0 \in \dot{H}_x^{-2}$. Then there exists $T > 0$ such that the equation (1.1) has a unique solution $u(t)$ with $u(0) = u_0$ satisfying*

$$u \in C([0, T]; H^s(\mathbb{R}^2)) \cap C^1([0, T]; H^{s-3}(\mathbb{R}^2))$$

and

$$v \in C([0, T]; H^{s-1}(\mathbb{R}^2)).$$

Moreover, the energy $E(u)$ and the momentum $V(u)$ are well defined and independent of time t .

Next we state the definition of stability of the solitary waves in the following.

Definition 2.5. We say that a set $S \subset X$ is X -stable, if for any $\epsilon > 0$, there exists $\delta > 0$ with $\inf_{w \in S} \|u_0 - w\|_X < \delta$ for any $u_0 \in X \cap X_s$ and $\partial_{yy}^2 u_0 \in \dot{H}_x^{-2}, s \geq 3$; the solution $u(t)$ of (1.1) with $u(0) = u_0$ can be extended to a global solution in $C([0, \infty); X \cap X_3)$ and satisfies

$$\sup_{0 \leq t < \infty} \inf_{w \in S} \|u(t) - w\|_X < \epsilon.$$

Otherwise, S is called X -unstable. Also we say that the solitary wave φ_c is unstable if $O_c = \{\tau_{s,r} u \mid s, r \in \mathbb{R}^2, \tau_{s,r} u(x, y) = u(x + s, y + r)\}$ is O_c -unstable. Furthermore, we say that φ_c is strongly unstable if for any $\epsilon > 0$, there exists $u_0 \in Y$ such that the solution $u(t)$ of (1.1) with $u(0) = u_0$ blows up in finite time.

Remark 2.6. The set of ground states G_c with $c > 0$ for (1.1) is Y -stable provided $1 \leq p < \frac{4}{3}$ is proved in [LiuWa, BoSa3] and G_c is Y -unstable is also proved in [WaAbSe, BoSa3] if $\frac{4}{3} < p < 4$.

To show blow up and strong instability of solutions of (1.1), we need a series of lemmas.

The first lemma is called refined Fatou lemma due to Brézis and Lieb [BrLi1] which is crucial to solve a constrained minimization problem (Lemma 2.11).

Lemma 2.7. *Let $\{f_j\}$ be a bounded sequence in $L^r(\mathbb{R}^2)$ for $0 < r < \infty$. If $f_j \rightarrow f$ a.e. in \mathbb{R}^2 , then*

$$|f_j|_r^r - |f_j - f|_r^r - |f|_r^r \rightarrow 0.$$

When $r = 2$, the assumption that $f_j \rightarrow f$ a.e. is not necessary.

The following lemma is given in [BoSa1], Lemma 3.3.

Lemma 2.8. *Let u_n be a bounded sequence in Y , and let $R > 0$. Then there is a subsequence u_{n_k} which converges strongly to some u in $L^2(B_R)$, where B_R is a ball with radius R in \mathbb{R}^2 .*

Lemma 2.9. *Let $u \in Y$ such that $\|u\|_Y \leq C$ and $\mu(|u| > \epsilon) \geq \delta > 0$. Then there exists a shift $\tau_{s,r}u(x, y) = u(x + s, y + r)$, such that for some constant $\delta_0 = \delta_0(c, \delta, \epsilon) > 0$,*

$$(2.3) \quad \mu\left(B \cap (|\tau_{s,r}u| > \frac{\epsilon}{2})\right) > \delta_0$$

where B is the unit ball in \mathbb{R}^2 .

Proof. See Lemma 4 in [LiuWa] with the space Y .

Lemma 2.10. *Suppose $1 \leq p < 4$ and $c > 0$. Let φ_c be a ground state. Then*

- a) $0 = K(\varphi_c) = \inf\{K(u) \mid u \in Y, \int_{\mathbb{R}^2} (\partial_x u)^2 = \int_{\mathbb{R}^2} (\partial_x \varphi_c)^2\}$;
- b) $I_c(\varphi_c) = \inf\{I(u) \mid u \in Y, \int_{\mathbb{R}^2} (u)^{p+2} = \int_{\mathbb{R}^2} (\varphi_c)^{p+2}\}$,

where

$$K(u) = \frac{1}{2} \int_{\mathbb{R}^2} (cu^2 + v^2) - \frac{1}{(p+1)(p+2)} \int_{\mathbb{R}^2} u^{p+2},$$

$$K(\varphi_c) = 0 \text{ and } I_c(u) = \int_{\mathbb{R}^2} (cu^2 + v^2 + (\partial_x u)^2) dx dy.$$

Proof. See Lemma 2.1 in [BoSa1] or Theorem 1 in [LiuWa]. □

Lemma 2.11. *Let $\sigma = \inf\{L_c(u) \mid u \in M\}$, where*

$$M = \{u \in Y \mid u \neq 0, Q(u) = 0\}$$

and

$$(2.4) \quad Q(u) = \int_{\mathbb{R}^2} \left(v^2 + (\partial_x u)^2 - \frac{3p}{2(p+1)(p+2)} u^{p+2} \right) dx dy.$$

Assume $\frac{4}{3} < p < 4$. Then φ_c is a ground state for (2.1) if and only if $\varphi_c \in M$ and $L_c(\varphi_c) = \sigma$.

Proof. Let φ_c be a ground state and define $\phi_c^\lambda(x, y) = \lambda^{\frac{3}{2}} \varphi_c(\lambda x, \lambda^2 y)$. It is easy to see that

$$(2.5) \quad Q(\varphi_c) = \frac{d}{d\lambda} L_c(\phi_c^\lambda)|_{\lambda=1} = \left\langle L'_c(\varphi_c), \frac{d\phi_c^\lambda}{d\lambda}|_{\lambda=1} \right\rangle = 0.$$

This implies that $\varphi_c \in M$ for any $\varphi_c \in G_c$. Now we define

$$(2.6) \quad L_c^1(u) = L_c(u) - \frac{2}{3p} Q(u) = \frac{c}{2} \int_{\mathbb{R}^2} u^2 + \frac{3p-4}{6p} \int_{\mathbb{R}^2} (v^2 + u_x^2).$$

We claim $\sigma = m$, where

$$(2.7) \quad m = \inf\{L_c^1(u) \mid u \in Y, u \neq 0, Q(u) \leq 0\}.$$

In fact, suppose $Q(u) < 0$. Since

$$(2.8) \quad Q(\lambda u) = \lambda^2 \int_{\mathbb{R}^2} (v^2 + u_x^2) - \frac{3p}{2(p+1)(p+2)} \lambda^{p+2} \int_{\mathbb{R}^2} u^{p+2} > 0$$

for some sufficiently small $\lambda > 0$, there exists $\lambda_0 \in (0, 1)$ such that $Q(\lambda_0 u) = 0$. Hence we have

$$\begin{aligned} \sigma \leq L_c(\lambda_0 u) &= \lambda_0^2 \left(\frac{c}{2} \int_{\mathbb{R}^2} u^2 + \frac{3p-4}{6p} \int_{\mathbb{R}^2} (v^2 + u_x^2) \right) \\ &< \frac{c}{2} \int_{\mathbb{R}^2} u^2 + \frac{3p-4}{6p} \int_{\mathbb{R}^2} (v^2 + u_x^2) = L_c^1(u). \end{aligned}$$

Consequently, $m = \sigma$. To show that $\sigma = L_c(\varphi_c)$ for some $\varphi_c \in G_c$, it suffices to show that $m = L^1(\varphi_c)$ for some $\varphi_c \in G_c$. In fact, we have $L_c^1(u) > 0$ for $\frac{4}{3} < p < 4$. Hence, there exists a minimizing sequence $\{u_j\}$ of (2.7) satisfying that $\{u_j\}$ is bounded in Y , $L_c^1(u_j) \rightarrow m$ and $Q(u_j) \leq 0$. By the anisotropic Sobolev embedding [BellNi, p. 323],

$$\|u\|_{p+2} \leq c\|u\|_Y \quad \text{for } 0 \leq p \leq 4.$$

It turns out that $\{u_j\}$ is bounded in L^{p+2} for $0 \leq p < 4$. We then have some subsequences, still denoted by $\{u_j\}$ and $u_0 \in Y \cap L^{p+2}$ such that u_j weakly converges to u_0 in Y and in L^{p+2} for $0 < p < 4$. It follows from Lemma 2.8 that $u_j \rightarrow u_0$ a.e. in \mathbb{R}^2 . Now we are able to show $L_c^1(u_0) = m$ and $Q(u_0) = 0$. Toward this end, we split the proof into the following five steps.

Step 1. $\inf_j \|u_j\|_{p+2}^{p+2} > 0$.

Proof of Step 1. Suppose there exists a subsequence of $\{u_j\}$ such that $\|u_j\|_{p+2}^{p+2} \rightarrow 0$. Then from $Q(u_j) \leq 0$ we obtain

$$\|v_j\|_2^2 + \|\partial_x u_j\|_2^2 \rightarrow 0.$$

On the other hand, by the facts that $Q(u_j) \leq 0$ and the anisotropic Sobolev embedding [To], we have

$$(2.9) \quad \|u\|_{p+2}^{p+2} \leq C \|u\|_2^{\frac{4-p}{2}} \|v\|_2^{\frac{p}{2}} \|\partial_x u\|_2^p \quad \text{for } 0 \leq p \leq 4,$$

with a constant C . It follows that

$$\|v_j\|_2^2 + \|\partial_x u_j\|_2^2 \leq C \|u_j\|_2^{\frac{4-p}{2}} (\|v_j\|_2^2 + \|\partial_x u_j\|_2^2)^{\frac{3p}{4}} \leq C (\|v_j\|_2^2 + \|\partial_x u_j\|_2^2)^{\frac{3p}{4}}$$

or

$$(\|v_j\|_2^2 + \|\partial_x u_j\|_2^2) \left(1 - C (\|v_j\|_2^2 + \|\partial_x u_j\|_2^2)^{\frac{3p-4}{4}} \right) \leq 0.$$

This implies that

$$(2.10) \quad 1 \leq C (\|v_j\|_2^2 + \|\partial_x u_j\|_2^2)^{\frac{3p-4}{4}}.$$

But it contradicts $\|v_j\|_2^2 + \|\partial_x u_j\|_2^2 \rightarrow 0$, because $p > \frac{4}{3}$. Consequently, $\inf_j \|u_j\|_{p+2}^{p+2} > 0$.

Step 2. $u_0 \neq 0$ a.e.

Proof of Step 2. Let $\inf_j |u_j|_{p+2}^{p+2} = \alpha > 0$. We estimate

$$\begin{aligned}
 \alpha &\leq |u_j|_{p+2}^{p+2} = \int_{|u_j| \geq \frac{1}{\epsilon}} |u_j|^{p+2} + \int_{|u_j| \leq \epsilon} |u_j|^{p+2} + \int_{\epsilon < |u_j| < \frac{1}{\epsilon}} |u_j|^{p+2} \\
 (2.11) \quad &\leq \int_{|u_j| \geq \frac{1}{\epsilon}} \frac{|u_j|^{p+2+\gamma}}{|u_j|^\gamma} + \epsilon^p \int_{|u_j| \leq \epsilon} |u_j|^2 + \left(\frac{1}{\epsilon}\right)^{p+2} \mu(|u_j| > \epsilon) \\
 &\leq \epsilon^\gamma \int_{|u_j| \geq \frac{1}{\epsilon}} |u_j|^{p+2+\gamma} + \epsilon^p \int_{|u_j| \leq \epsilon} |u_j|^2 + C_\epsilon \mu(|u_j| > \epsilon)
 \end{aligned}$$

where $0 < \gamma < 4 - p$. Since $p + \gamma < 4$, we have

$$\int_{|u_j| \geq \frac{1}{\epsilon}} |u_j|^{2+p+\gamma} \leq \int_{\mathbb{R}^2} |u_j|^{2+p+\gamma} \leq C(|u_j|_2^2 + |v_j|_2^2 + |\partial_x u_j|_2^2)^{\frac{2+p+\gamma}{2}} \leq C_1$$

and

$$\int_{|u_j| \geq \frac{1}{\epsilon}} |u_j|^2 \leq \int_{\mathbb{R}^2} |u_j|^2 \leq C_2$$

where C, C_1 and C_2 are some constants. Choosing ϵ sufficiently small, we obtain

$$\mu(|u_j| > \epsilon) \geq \frac{\alpha - \epsilon^\gamma C_1 - \epsilon^p C_2}{C_\epsilon} = \delta > 0.$$

It follows from Lemma 2.9 that $\mu(B \cap (|u_0| > \frac{\epsilon}{2})) > \delta_0$ for the unit ball B because $u_j \rightarrow u_0$ a.e. in \mathbb{R}^2 . This implies that $u_0 \neq 0$ a.e. in \mathbb{R}^2 .

Step 3. $m = L_c^1(u_0)$.

Proof of Step 3. By Lemma 2.7, we deduce that

$$(2.12) \quad Q(u_j) - Q(u_j - u_0) - Q(u_0) \rightarrow 0,$$

and

$$(2.13) \quad L_c^1(u_j) - L_c^1(u_j - u_0) - L_c^1(u_0) \rightarrow 0.$$

Now suppose that $Q(u_0) > 0$. Then from the fact that $Q(u_j) \leq 0$ we obtain that $Q(u_j - u_0) \leq 0$ as $j \rightarrow \infty$. By the definition of m , it follows that $L_c^1(u_j - u_0) \geq m$. Since $L_c^1(u_j) \rightarrow m$, it follows from (2.13) that $L_c^1(u_0) \leq 0$. That is,

$$(2.14) \quad \frac{c}{2} \int_{\mathbb{R}^2} u_0^2 + \frac{3p-4}{6p} \int_{\mathbb{R}^2} (v_0^2 + (\partial_x u_0)^2) \leq 0.$$

This contradicts $u_0 \neq 0$, a.e. Consequently, $Q(u_0) \leq 0$. Therefore,

$$m \leq L_c^1(u_0) \leq \liminf_{j \rightarrow \infty} L_c^1(u_j) = m$$

and $m = L_c^1(u_0)$.

Step 4. $Q(u_0) = 0$.

Proof of Step 4. Suppose that $Q(u_0) < 0$. We try to get a contradiction. Toward this end, we choose a small $\lambda > 0$, and find that

$$Q(\lambda u_0) = \lambda^2 \int_{\mathbb{R}^2} (v_0^2 + (\partial_x u_0)^2) - \frac{3p}{2(p+1)(p+2)} \lambda^{p+2} \int_{\mathbb{R}^2} u_0^{p+2} > 0.$$

It follows from the continuity that there exists $\lambda_0 \in (0, 1)$ such that $Q(\lambda_0 u_0) = 0$. Therefore, by the definition of m , it yields a contradiction in the following:

$$\begin{aligned} m &\leq L_c^1(\lambda_0 u_0) = \lambda_0^2 \left(\frac{c}{2} \int_{\mathbb{R}^2} u_0^2 + \frac{3p-4}{6p} \int_{\mathbb{R}^2} (v_0^2 + (\partial_x u_0)^2) \right) \\ &< \frac{c}{2} \int_{\mathbb{R}^2} u_0^2 + \frac{3p-4}{6p} \int_{\mathbb{R}^2} (v_0^2 + (\partial_x u_0)^2) = L_c^1(u_0) = m. \end{aligned}$$

Hence it proves that $Q(u_0) = 0$.

Step 5. $u_0 = \varphi_c \in G_c$.

Proof of Step 5. In fact, it follows from Step 3 and Step 4 that

$$(2.15) \quad m = \sigma = \inf\{L_c(u) \mid u \neq 0, Q(u) = 0\} = L_c(u_0).$$

Hence, there exists a constant $\lambda \in \mathbb{R}$ such that

$$L'_c(u_0) + \lambda Q'(u_0) = 0.$$

It remains to show that $\lambda = 0$ so that $L'_c(u_0) = 0$. By using $Q(u_0) = 0$, it yields that

$$\begin{aligned} 0 &= \langle L'(u_0) + \lambda Q'(u_0), u_0 \rangle \\ &= \int_{\mathbb{R}^2} (cu_0^2 + (\partial_x u_0)^2 + v_0^2) - \frac{1}{p+1} \int_{\mathbb{R}^2} u_0^{p+2} \\ &\quad + \lambda \left(2 \int_{\mathbb{R}^2} ((\partial_x u_0)^2 + v_0^2) - \frac{3p}{2(p+1)} \int_{\mathbb{R}^2} u_0^{p+2} \right) \\ &= \int_{\mathbb{R}^2} (cu_0^2 + (2\lambda + 1)(v_0^2 + (\partial_x u_0)^2)) - \frac{2(p+2)}{3p} \int_{\mathbb{R}^2} (v_0^2 + (\partial_x u_0)^2) \\ &\quad - \lambda(p+2) \int_{\mathbb{R}^2} (v_0^2 + (\partial_x u_0)^2) \\ &= \int_{\mathbb{R}^2} cu_0^2 + \left(\frac{p-4}{3p} - \lambda p \right) \int_{\mathbb{R}^2} (v_0^2 + (\partial_x u_0)^2). \end{aligned}$$

It follows from $c > 0$ that $\frac{p-4}{3p} - \lambda p < 0$. On the other hand, let $w^\eta(x, y) = \eta^{\frac{3}{2}} u_0(\eta x, \eta^2 y)$. It is easy to see that

$$\begin{aligned} 0 &= \left\langle (L'_c + \lambda Q')(u_0), \frac{\partial w^\eta}{\partial \eta} \Big|_{\eta=1} \right\rangle = \partial_\eta (L_c + \lambda Q)(w^\eta) \Big|_{\eta=1} \\ &= Q(u_0) + \lambda \left(2 \int_{\mathbb{R}^2} (v_0^2 + (\partial_x u_0)^2) - \frac{9p^2}{4(p+1)(p+2)} \int_{\mathbb{R}^2} u_0^{p+2} \right). \end{aligned}$$

If $\lambda \neq 0$, then

$$2 \int_{\mathbb{R}^2} (v_0^2 + (\partial_x u_0)^2) - \frac{9p^2}{4(p+1)(p+2)} \int_{\mathbb{R}^2} u_0^{p+2} = 0.$$

Since $Q(u_0) = 0$, it follows that

$$0 = 2 \int_{\mathbb{R}^2} (v_0^2 + (\partial_x u_0)^2) - \frac{3p}{2} \int_{\mathbb{R}^2} (v_0^2 + (\partial_x u_0)^2) = \frac{4-3p}{2} \int_{\mathbb{R}^2} v_0^2 + (\partial_x u_0)^2 < 0$$

because $p > \frac{4}{3}$. Hence $\lambda = 0$ and $L'(u_0) = 0$. On the other hand, suppose $u \in Y$ satisfies $L'_c(u) = 0$. We have

$$Q(u) = \partial_\eta L_c(u^\eta)|_{\eta=1} = \left\langle L'_c(u), \frac{du^\eta}{d\eta} \Big|_{\eta=1} \right\rangle = 0$$

where $u^\eta = \eta^{\frac{3}{2}}u(\eta x, \eta^2 y)$. It follows from (2.15) that $L_c(u_0) \leq L_c(u)$. This implies that $u_0 = \varphi_c \in G_c$. The proof of Lemma 2.11 is complete.

To prove Theorem 1.1, we also need the following two lemmas.

Lemma 2.12. *Let $1 \leq p < 4$ and $\varphi_c \in G_c$. Then*

$$(2.16) \quad L_c(\varphi_c) = \inf \left\{ L_c(u) \mid u \in Y, u \neq 0, \int_{\mathbb{R}^2} u_x^2 = \int_{\mathbb{R}^2} (\partial_x \varphi_c)^2 \right\}.$$

Proof. Since $K = L_c(u) - \frac{1}{2} \int_{\mathbb{R}^2} u_x^2$, it follows from Lemma 2.10 that

$$\begin{aligned} & \inf \left\{ L_c(u) \mid u \in Y, u \neq 0, \int_{\mathbb{R}^2} u_x^2 = \int_{\mathbb{R}^2} (\partial_x \varphi_c)^2 \right\} \\ &= \inf \left\{ K(u) + \frac{1}{2} \int_{\mathbb{R}^2} (\partial_x \varphi_c)^2 \mid u \in Y, u \neq 0, \int_{\mathbb{R}^2} u_x^2 = \int_{\mathbb{R}^2} (\partial_x \varphi_c)^2 \right\} \\ &= \inf \left\{ K(u) \mid u \in Y, u \neq 0, \int_{\mathbb{R}^2} u_x^2 = \int_{\mathbb{R}^2} (\partial_x \varphi_c)^2 \right\} + \frac{1}{2} \int_{\mathbb{R}^2} (\partial_x \varphi_c)^2 \\ &= \frac{1}{2} \int_{\mathbb{R}^2} (\partial_x \varphi_c)^2 = L_c(\varphi_c). \end{aligned}$$

Lemma 2.13. *Let $1 \leq p < 4$ and $\varphi_c \in G_c$. Then*

$$(2.17) \quad L_c(\varphi_c) = \inf \left\{ L_c(u) \mid u \in Y, u \neq 0, \int_{\mathbb{R}^2} u^{p+2} = \int_{\mathbb{R}^2} \varphi_c^{p+2} \right\}.$$

Proof.

$$\begin{aligned} L_c(\varphi_c) &= \inf \left\{ L_c(u) \mid u \in Y, u \neq 0, \int_{\mathbb{R}^2} u^{p+2} = \int_{\mathbb{R}^2} \varphi_c^{p+2} \right\} \\ &= \frac{1}{2} \inf \left\{ \int_{\mathbb{R}^2} (cu^2 + v^2 + u_x^2) \mid u \in Y, u \neq 0, \int_{\mathbb{R}^2} u^{p+2} = \int_{\mathbb{R}^2} \varphi_c^{p+2} \right\} \\ &\quad - \frac{1}{(p+1)(p+2)} \int_{\mathbb{R}^2} \varphi_c^{p+2} = \frac{1}{2} J_c(\varphi_c) - \frac{1}{(p+1)(p+2)} \int_{\mathbb{R}^2} \varphi_c^{p+2} = L_c(\varphi_c). \end{aligned}$$

3. INVARIANT SETS

In this section we construct some invariants for the flow of the KP equation (1.1). Using those invariant properties of the solution and a virial identity which has been showed in [TuFa], we are able to show the blow-up result in Section 4. Toward this end, we begin to define the invariants in the following:

$$K_1^c = \{ u \in Y \mid u \neq 0, L_c(u) < L_c(\varphi_c), Q(u) \geq 0 \},$$

$$K_2^c = \{ u \in Y \mid u \neq 0, L_c(u) < L_c(\varphi_c), Q(u) < 0 \},$$

$$\begin{aligned}
 R_1^c &= \left\{ u \in Y \mid u \neq 0, L_c(u) < L_c(\varphi_c), \int_{\mathbb{R}^2} u_x^2 \leq \int_{\mathbb{R}^2} (\partial_x \varphi_c)^2 \right\}, \\
 R_2^c &= \left\{ u \in Y \mid u \neq 0, L_c(u) < L_c(\varphi_c), \int_{\mathbb{R}^2} u_x^2 > \int_{\mathbb{R}^2} (\partial_x \varphi_c)^2 \right\}, \\
 J_1^c &= \left\{ u \in Y \mid u \neq 0, L_c(u) < L_c(\varphi_c), \int_{\mathbb{R}^2} u^{p+2} \geq \int_{\mathbb{R}^2} \varphi_c^{p+2} \right\}, \\
 J_2^c &= \left\{ u \in Y \mid u \neq 0, L_c(u) < L_c(\varphi_c), \int_{\mathbb{R}^2} u^{p+2} < \int_{\mathbb{R}^2} \varphi_c^{p+2} \right\}.
 \end{aligned}$$

The following lemma is crucial to obtain the blow-up result.

Lemma 3.1 (Invariant sets). *Suppose $1 \leq p < 4$ and $c > 0$. Let u_0 be the initial data such that the corresponding solution $u(t)$ of KP equation (1.1) is in $C([0, T]; Y)$ for some $T > 0$ and satisfies $E(u(t)) = E(u_0)$ and $V(u(t)) = V(u_0)$ for $0 \leq t < T$. Then*

- a) $\frac{4}{3} < p < 4$, $u_0 \in K_i^c$ implies that $u(t) \in K_i^c, \forall 0 \leq t < T$,
- b) $u_0 \in R_i^c$ implies that $u(t) \in R_i^c, \forall 0 \leq t < T$, and
- c) $u_0 \in J_i^c$ implies that $u(t) \in J_i^c, \forall 0 \leq t < T$ where $i = 1, 2$. Moreover, if $u_0 \in K_2^c$, then $Q(u(t)) < -\frac{3p}{2}(L_c(\varphi_c) - L_c(u_0))$ for $0 \leq t < T$.

Proof. Here we only consider the invariance of K_2^c , since for K_1^c the proof is similar and the proof of invariance of R_1^c and J_i^c are also similar due to Lemmas 2.12 and 2.13.

Let $u_0 \in K_2^c$. Since $E(u(t)) = E(u_0)$ and $V(u(t)) = V(u_0)$, we have

$$(3.1) \quad L_c(u(t)) = E(u(t)) + cV(u(t)) = L_c(u_0) < L_c(\varphi_c).$$

Suppose $u(t_0) \notin K_2^c$ for some $t_0 \in (0, T)$; that is, $Q(u(t_0)) \geq 0$. By $Q(u(0)) = Q(u_0) < 0$ and the continuity of $Q(u(t))$ with respect to t , there exists $t_1 \in (0, t_0]$ such that $Q(u(t_1)) = 0$. Therefore, applying Lemma 2.11 yields a contradiction

$$L_c(\varphi_c) > L_c(u(t_1)) \geq \inf \{ L_c(u) \mid u \neq 0, Q(u) = 0 \} = L_c(\varphi_c).$$

This implies that $u(t) \in K_2^c$ for $0 \leq t < T$. To prove the final inequality, we use the definition of m and the fact

$$m = \inf \{ L_c(u) \mid u \in Y, u \neq 0, Q(u) = 0 \} = L_c(\varphi_c)$$

which is proved in Lemma 2.11.

Suppose $u_0 \in K_2^c$. Then we have $u(t) \in K_2^c$, i.e. $L_c(u(t)) < L_c(\varphi_c)$ and $Q(u(t)) < 0$ for $t \geq 0$. Since

$$Q(\lambda u) = \lambda^2 \int_{\mathbb{R}^2} (v^2 + u_x^2) - \frac{3p}{2(p+1)(p+2)} \lambda^{p+2} \int_{\mathbb{R}^2} u^{p+2} > 0$$

for some sufficiently small $\lambda > 0$, there exists $\lambda_0 \in (0, 1)$, such that $Q(\lambda_0 u) = 0$ and

$$\begin{aligned}
 L_c(\varphi_c) &\leq L_c(\lambda_0 u) = \lambda_0^2 \left(\frac{c}{2} \int_{\mathbb{R}^2} u^2 + \frac{3p-4}{6p} \int_{\mathbb{R}^2} (v^2 + u_x^2) \right) \\
 &< \frac{c}{2} \int_{\mathbb{R}^2} u^2 + \frac{3p-4}{6p} \int_{\mathbb{R}^2} (v^2 + u_x^2) = L_c^1(u(t)).
 \end{aligned}$$

Therefore, $Q(u(t)) < -\frac{3p}{2}(L_c(\varphi_c) - L_c(u_0))$.

Remark. If $u_0 \in K_2^c \cap J_2^c$, then it follows from the conservation laws $E(u)$ and $V(u)$ that $\|u(t)\|_Y \leq C(\varphi_c, \|u_0\|_Y)$. Consequently, the solution $u(t)$ of the KP equation is bounded globally for $t \geq 0$; blow-up cannot occur in finite time in $Y(\mathbb{R}^2)$. However, we do have a blow-up solution which is only due to the *transverse* dispersion; we will state it in the next section.

4. FINITE BLOW-UP TIME

In Section 1, we mention that the solution of (1.1) blows up in finite time if the initial data u_0 satisfies $E(u_0) < 0$ and $p \geq 4$ [TuFa, Sa1]. Using the invariant sets, we are able to extend this blow-up result to allow the energy E to be even positive and also $\frac{4}{3} \leq p < 4$. We start to prove the case of $p > \frac{4}{3}$ and consider the critical case $p = \frac{4}{3}$ later.

Theorem 4.1 (Improved blow-up). *Let $\frac{4}{3} < p < 4$ with $p = \frac{n_1}{n_2}$, where n_1 is an even integer and n_2 an odd integer. Assume*

- i) *Let $u_0 \in Y$ and $yu_0 \in L^2$, and*
- ii) *$u_0 \in K_2^c \cap R_2^c \cap J_2^c$.*

Let u be the solution of the KP equation (1.1) in $C([0, T]; Y)$ with $u(0) = u_0$, conserved energy E and momentum V . Then there exists a blow-up time $T_0 < \infty$ such that

$$(4.1) \quad \lim_{t \rightarrow T_0^-} |\partial_y u(t)|_2 = \infty.$$

Proof. If $u(t)$ remains in Y , we define $I(t) = \int_{\mathbb{R}^2} y^2 u^2(x, y, t) dx dy$, where u is the solution of the KP equation (1.1). Using the virial identity in [TuFa, WaAbSe, Sa1] yields

$$(4.2) \quad \begin{aligned} \frac{d^2 I(t)}{dt^2} &= 4pE(u(t)) + 2(4 - p) \int_{\mathbb{R}^2} v^2 - 2p \int_{\mathbb{R}^2} u_x^2 \\ &= 8 \left(Q(u(t)) - \int_{\mathbb{R}^2} u_x^2 + \frac{p}{(p + 1)(p + 2)} \int_{\mathbb{R}^2} u^{p+2} \right). \end{aligned}$$

It follows from Lemma 3.1 that

$$(4.3) \quad \begin{aligned} \frac{d^2 I(t)}{dt^2} &< 8 \left(-\frac{3p}{2} (L_c(\varphi_c) - L_c(u_0)) - \int_{\mathbb{R}^2} (\partial_x \varphi_c)^2 \right. \\ &\quad \left. + \frac{p}{(p + 1)(p + 2)} \int_{\mathbb{R}^2} \varphi_c^{p+2} \right) \\ &= -12p\epsilon_0 \end{aligned}$$

where $\epsilon_0 = L_c(\varphi_c) - L_c(u_0)$. In the last step of the above proof, we used the fact

$$(4.4) \quad \int_{\mathbb{R}^2} (\partial_x \varphi_c)^2 = \frac{p}{(p + 1)(p + 2)} \int_{\mathbb{R}^2} \varphi_c^{p+2}$$

which can be obtained from $\frac{\partial}{\partial \lambda} L_c(\varphi_c^\lambda)|_{\lambda=1} = 0$ where $\varphi_c^\lambda = \lambda \varphi_c(\lambda x, \lambda y)$. Therefore, it follows that $\lim_{t \rightarrow T_0} I(t) = 0$ for some $T_0 < \infty$, And the blow-up result can be deduced from the conserved momentum $V(u)$ and the classical inequality

$$|u|_2^2 \leq 2|yu|_2 |\partial_y u|_2.$$

The proof of Theorem 4.1 is complete.

Remark 1. If $u_0 \in R_2^c \cap J_2^c$ and $p \geq 2$, then $E(u_0) > 0$. This can be verified as follows.

Since

$$\int_{\mathbb{R}^2} (\partial_x u_0)^2 > \int_{\mathbb{R}^2} (\partial_x \varphi_c)^2, \quad \int_{\mathbb{R}^2} u_0^{p+2} < \int_{\mathbb{R}^2} \varphi_c^{p+2},$$

and

$$\int_{\mathbb{R}^2} (\partial_x \varphi_c)^2 = \frac{p}{(p+1)(p+2)} \int_{\mathbb{R}^2} \varphi_c^{p+2},$$

we have

$$\begin{aligned} E(u_0) &= \frac{1}{2} \int_{\mathbb{R}^2} (\partial_x u_0)^2 + \frac{1}{2} \int_{\mathbb{R}^2} v_0^2 - \frac{1}{(p+1)(p+2)} \int_{\mathbb{R}^2} u_0^{p+2} \\ &> \frac{1}{2} \int_{\mathbb{R}^2} (\partial_x \varphi_c)^2 + \frac{1}{2} \int_{\mathbb{R}^2} v_0^2 - \frac{1}{(p+1)(p+2)} \int_{\mathbb{R}^2} \varphi_c^{p+2} \\ &= \int_{\mathbb{R}^2} v_0^2 + \frac{p-2}{2p} \int_{\mathbb{R}^2} (\partial_x \varphi_c)^2 \geq \frac{1}{2} \int_{\mathbb{R}^2} v_0^2 > 0. \end{aligned}$$

Remark 2. Since $E(\varphi_c) = \frac{3p-4}{6p} (|\partial_x \varphi_c|_2^2 + |\partial_x^{-1} \partial_y \varphi_c|_2^2)$ and $E(\varphi_c) > 0$ when $p > \frac{4}{3}$, we are able to choose $u_0 = \varphi_c^\lambda$ which is sufficiently close to φ_c such that $E(u_0) > 0$.

Remark 3. A similar blow-up result can be obtained for a generalized Kadomtsev-Petviashvili (KP) equation in the three-dimensional case [Liu]

$$(KP-3D) \quad \begin{cases} u_t + u_{xxx} + u^p u_x = v_y + w_z \\ v_x = u_y \\ w_x = u_z \end{cases} \quad (x, y) \in \mathbb{R}^3, \quad t \geq 0.$$

The solution of KP-3D blows up in finite time for all $p \geq 1$.

Next, we consider the critical case $p = \frac{4}{3}$.

Theorem 4.2. Let $p = \frac{4}{3}$. Assume

- i) $u_0 \in Y, \partial_y u_0 \in L^2$ and $yu_0 \in L^2$,
- ii) $u_0 \in R_2^c \cap J_2^c$, and
- iii) $E(u_0) < 0$.

Let u be the solution of the KP equation (1.1) in $C([0, T]; Y)$ with $u(0) = u_0$ and satisfy $E(u(t)) = E(u_0)$ and $V(u(t)) = V(u_0)$. Then there exists a blow-up time $T_0 < \infty$ such that

$$\lim_{t \rightarrow T_0} |\partial_y u(t)|_2 = \infty.$$

In order to show Theorem 4.2, we need the following basic lemma.

Lemma 4.3. Let $p = \frac{4}{3}$. Then $E(\varphi_c) = 0$ for $\varphi_c \in G_c$, and

$$2 \int_{\mathbb{R}^2} (\partial_x \varphi_c)^2 = \frac{4-p}{(p+1)(p+2)} \int_{\mathbb{R}^2} \varphi_c^{p+2}.$$

Proof. Define $\phi_c^\lambda = \lambda \varphi_c(\lambda x, \lambda y)$. It can be easily verified that

$$0 = \frac{d}{d\lambda} L_c(\varphi_c^\lambda)|_{\lambda=1} = \int_{\mathbb{R}^2} (\partial_x \varphi_c)^2 - \frac{p}{(p+1)(p+2)} \int_{\mathbb{R}^2} \varphi_c^{p+2};$$

that is,

$$2 \int_{\mathbb{R}^2} (\partial_x \varphi_c)^2 = \frac{2p}{(p+1)(p+2)} \int_{\mathbb{R}^2} \varphi_c^{p+2} = \frac{4-p}{(p+1)(p+2)} \int_{\mathbb{R}^2} \varphi_c^{p+2}$$

because $p = \frac{4}{3}$. To prove $E(\varphi_c) = 0$, it follows from the fact $Q(\varphi_c) = 0$ that

$$\begin{aligned} E(\varphi_c) + cV(\varphi_c) &= \frac{c}{2} \int_{\mathbb{R}^2} + \frac{1}{2} \int_{\mathbb{R}^2} ((\partial_x^{-1} \partial_y \varphi_c)^2 + (\partial_x \varphi_c)^2) - \frac{1}{(p+1)(p+2)} \int_{\mathbb{R}^2} \varphi_c^{p+2} \\ &= \frac{c}{2} \int_{\mathbb{R}^2} + \frac{3p-4}{6p} \int_{\mathbb{R}^2} ((\partial_x^{-1} \partial_y \varphi_c)^2 + (\partial_x \varphi_c)^2) = \frac{c}{2} \int_{\mathbb{R}^2} = cV(\varphi_c). \end{aligned}$$

This implies that $E(\varphi_c) = 0$.

Proof of Theorem 4.2. By Lemma 3.1, the solution $u(t) \in R_2^c$ and $u(t) \in J_2^c$ for $0 \leq t < T$; that is, $|u(t)|_{p+2}^{p+2} < |\varphi_c|_{p+2}^{p+2}$ and $|\partial_x u(t)|_2^2 > |\partial_x \varphi_c|_2^2$ where $\varphi_c \in G_c$. It follows from the viral identity and Lemma 4.3 that

$$\begin{aligned} (4.5) \quad I''(t) &= 16E(u(t)) - 8 \int_{\mathbb{R}^2} u_x^2 + \frac{4(4-p)}{(p+1)(p+2)} \int_{\mathbb{R}^2} u^{p+2} \\ &< 16E(u_0) - 8|\partial_x \varphi_c|_2^2 + \frac{4(4-p)}{(p+1)(p+2)} \int_{\mathbb{R}^2} \varphi_c^{p+2} = 16E(u_0). \end{aligned}$$

Since $E(u_0) < 0$, the inequality (4.4) implies $\lim_{t \rightarrow T_0} I(t) = 0$ for a finite time $T_0 < \infty$.

Therefore,

$$\lim_{t \rightarrow T_0} |\partial_y u(t)|_2 = \infty.$$

This completes the proof of Theorem 4.2.

5. STRONG INSTABILITY OF SOLITARY WAVES

In the proof of Theorem 4.1, we find that one assumption for the blow-up $u_0 \in R_2^c \cap J_2^c$ can be relaxed to $u_0 \in N_c$, for $2 < p < 4$, where

$$N_c = \{ u \in Y \mid u \neq 0, L_c(u) < L_c(\varphi_c), P(u) > 0 \}$$

and

$$P(u) = \int_{\mathbb{R}^2} u_x^2 - \frac{p}{(p+1)(p+2)} \int_{\mathbb{R}^2} u^{p+2}.$$

A strong instability of solitary-wave solutions φ_c . can be obtained in the following.

Theorem 5.1. *Let $2 < p < 4$ with $p = \frac{n_1}{n_2}$, where n_1 is an even integer and n_2 an odd integer. Suppose $\varphi_c \in Y$ is the solitary-wave solution of the KP equation (1.1) with $c > 0$. For any $\delta > 0$, there is an initial data $u_0 \in Y$ with $\|u_0 - \varphi_c\|_Y < \delta$, such that the solution u of (1.1) with $u(0) = u_0$ blows up in finite time. More precisely,*

$$\lim_{t \rightarrow T^-} |\partial_y u(t)|_2 = \infty.$$

To prove Theorem 5.1, we need the following lemmas.

Lemma 5.2. *Let $\tau = \inf\{ L_c(u) \mid u \neq 0, P(u) = 0 \}$. Suppose $2 < p < 4$. Then $\varphi_c \in G_c$ if and only if $P(\varphi_c) = 0$ and $\tau = L_c(\varphi_c)$.*

Since the proof is similar to that of Lemma 2.11, we only give a sketch of the proof.

Sketch of the proof. $P(\varphi_c) = 0$ is given by the equation (4.4). Let

$$L_c^2(u) = L_c(u) - \frac{1}{p}P(u).$$

Then we have

$$L_c^2(u) = \frac{c}{2} \int_{\mathbb{R}^2} u^2 + \frac{1}{2} \int_{\mathbb{R}^2} v^2 + \frac{p-2}{2p} \int_{\mathbb{R}^2} u_x^2 > 0$$

when $2 < p$. It can be shown that $\tau = m_1$ where

$$m_1 = \inf\{L_c^2(u) \mid u \in Y, u \neq 0, P(u) \leq 0\}.$$

In fact, suppose $P(u) < 0$. There exists a small number $\lambda \in (0, 1)$ such that $P(\lambda_0 u) = 0$ and

$$\tau \leq L_c(\lambda_0 u) = \lambda_0^2 \left(\frac{c}{2} \int_{\mathbb{R}^2} u^2 + \frac{1}{2} \int_{\mathbb{R}^2} v^2 + \frac{p-2}{2p} \int_{\mathbb{R}^2} u_x^2 \right) < L_c^2(u).$$

This implies that $\tau = m_1$. On the other hand, since $L_c^2(u) > 0$, there exists a minimizing sequence of $\{u_j\}, u_j \neq 0, \forall j$ such that $P(u_j) \leq 0, L_c^2(u_j) \rightarrow m_1$, and $u_j \rightarrow u_0$ weakly in $V(\mathbb{R}^2)$ as well as in L^{2+p} . In next step, we need to show $\inf_j |u_j|_{p+2}^{p+2} > 0$, so that $u_0 \neq 0$, a.e. In fact, if $|u_j|_{p+2}^{p+2} \rightarrow 0$, then by $P(u_j) \leq 0$, we deduce that $|\partial_x u_j|_2 \rightarrow 0$. It follows from the anisotropic Sobolev embedding that

$$|\partial_x u_j|_2^2 \leq |u_j|_2^{\frac{4-p}{2}} |v_j|_2^{\frac{p}{2}} |\partial_x u_j|_2^p \leq C |\partial_x u_j|_2^p.$$

Because $p > 2$, it yields a contradiction $1 \leq C |\partial_x u_j|_2^{p-2}$. To show the rest of the proof, we use the weak convergence lemma (Lemma 2.7) and follow the proofs in Step 3, Step 4 and Step 5 of Lemma 2.11.

By the proof of Lemma 3.1, we are also able to obtain the following lemma.

Lemma 5.3. *N_c is invariant for the solution of the KP equation (1.1). That is, $u_0 \in N_c$ implies that the corresponding solution $u(t) \in N_c$ for $t \geq 0$.*

Now by using Lemmas 5.2 and 5.3, it is easy to obtain the following lemma which is crucial to prove the instability of solitary waves (Theorem 5.1).

Lemma 5.4. *Let $2 < p < 4$. Assume*

- i) $u_0 \in Y$ and $yu_0 \in L^2$, and
- ii) $u_0 \in K_2^c \cap N_c$.

Let u be the solution of the KP equation (1.1) in $C([0, T]; Y)$ with $u(0) = u_0, E(u(t)) = E(u_0)$ and $V(u(t)) = V(u_0)$. Then there exists a blow-up time $T_0 < \infty$ such that

$$(5.1) \quad \lim_{t \rightarrow T_0} |\partial_y u(t)|_2 = \infty.$$

Proof. It follows from the proof of Theorem 4.1 that

$$\frac{d^2 I}{dt^2} = 8(Q(u(t)) - P(u(t))).$$

By Lemma 5.3, we obtain that $P(u(t)) > 0$, for $t \geq 0$. Hence

$$(5.2) \quad \frac{d^2 I}{dt^2} < 8Q(u(t)) < -12p\epsilon_0$$

and a blow-up time T_0 can easily be deduced. Using this lemma we are able to prove Theorem 5.1.

Proof of Theorem 5.1. We define $u_0(x, y) = \chi_\omega(x, y) = \sigma^a(\omega)\varphi_\omega(\sigma^b x, \sigma^e y)$, where $\sigma^{2a-b-e}(\omega) = \frac{|\varphi_c|_2^2}{|\varphi_\omega|_2^2}$, $b = \frac{ap}{2} + \eta$, $e = ap - \epsilon - \eta$ with $\epsilon, \eta > 0$ and $a < 0$. It is easy to see that $|\chi_\omega|_2^2 = |\varphi_c|_2^2$, $\chi_c = \varphi_c$, and $\sigma(c) = 1$. On the other hand,

$$\sigma'(\omega)|_{\omega=c} = \frac{-d''(c)}{d'(c)(2a - b - e)} = \frac{-d''(c)}{\frac{1}{2}|\varphi_c|_2^2(\frac{4-3p}{2}a - \epsilon)} > 0$$

for small ϵ . By the smooth of χ_ω with respect to ω , it is found that χ_ω is sufficiently close to φ_c as $\omega \rightarrow c$. Let $u(t)$ be the solution of the KP equation (1.1) with $u(0) = u_0$. By the smooth of φ_ω [BoSa2], the solution $u \in C([0, T]; Y)$ has the conserved energy E and the momentum V . By Lemma 5.3, it remains to show $\chi_\omega \in K_2^c \cap N_c$ as $\omega \rightarrow c$. More precisely, $Q(\chi_\omega) < 0$, $P(\chi_\omega) > 0$, and $L_c(\chi_\omega) < L_c(\varphi_c)$ for some $\omega > c$. By a simple computation, we have

$$Q(\chi_\omega) = \left(\frac{1}{2}\sigma^{2a+e-3b}(\omega) + \sigma^{2a+b-e}(\omega) - \frac{3}{2}\sigma^{a(p+2)-e-b}(\omega) \right) |\varphi_\omega|_2^2.$$

At $\omega = c$, we have $Q(\chi_\omega)|_{\omega=c} = 0$ and the differentiation of $Q(\chi_\omega)$ with respect to ω at $\omega = c$ yields

$$\begin{aligned} \frac{dQ(\chi_\omega)}{d\omega}|_{\omega=c} &= \left(\frac{1}{2}(2a + e - 3b) + (2a + b - e) - \frac{3}{2}(a(p + 2) - e - b) \right) \sigma'(c)|\partial_x \varphi_c|_2^2 \\ &= (e + b - \frac{3}{2}ap)\sigma'(c)|\partial_x \varphi_c|_2^2 = -\epsilon\sigma'(c)|\partial_x \varphi_c|_2^2 < 0. \end{aligned}$$

This implies that $Q(\chi_\omega) < Q(\chi_c) = Q(\varphi_c) = 0$ for $\omega > c$. Next, we have

$$P(\chi_\omega) = \sigma^{1+2b}(1 - \sigma^{ap-2b})|\partial_x \varphi_\omega|_2^2$$

and at $\omega = c$

$$\frac{dP}{d\omega}|_{\omega=c} = (2b - ap)\sigma'(c)|\partial_x \varphi_c|_2^2 = 2\eta\sigma'(c)|\partial_x \varphi_c|_2^2 > 0.$$

This implies that $P(\chi_\omega) > P(\chi_c) = P(\varphi_c) = 0$ for $\omega > c$.

Finally, we need to show $L_c(\chi_\omega) < L_c(\varphi_c)$ for $\omega > c$. Let $L_\omega(u) = E(u) + \omega V(u)$. A simple computation yields

$$\begin{aligned} L_\omega &= \frac{\omega}{2}\sigma^{2a-b-e}|\varphi_\omega|_2^2 + \frac{1}{2}\sigma^{2a-3b+e}|\partial_x^{-1}\partial_y \varphi_\omega|_2^2 \\ &\quad + \frac{1}{2}\sigma^{2a+b-e}|\partial_x \varphi_\omega|_2^2 - \frac{1}{(p+1)(p+2)}\sigma^{a(p+2)-b-e}|\varphi_\omega|_{p+2}^{p+2} \\ &= \left(\frac{4-p}{4p}\sigma^{2a-b-e} + \frac{1}{4}\sigma^{2a-3b+e} + \frac{1}{2}\sigma^{2a+b-e} - \frac{1}{p}\sigma^{a(p+2)-b-e} \right) |\partial_x \varphi_\omega|_2^2. \end{aligned}$$

In the above computation, we use the fact that

$$|\partial_x^{-1}\partial_y \varphi_\omega|_2^2 = \frac{1}{2}|\partial_x \varphi_\omega|_2^2, \quad \frac{\omega}{2}|\varphi_\omega|_2^2 = \frac{p-4}{4p}|\partial_x \varphi_\omega|_2^2,$$

$$\frac{1}{(p+1)(p+2)}|\varphi_\omega|_{p+2}^{p+2} = \frac{1}{p}|\partial_x \varphi_\omega|_2^2, \quad \text{and} \quad d(\omega) = \frac{1}{2}|\partial_x \varphi_\omega|_2^2.$$

Differentiation of $L_\omega(\chi_\omega)$ with respect to ω at $\omega = c$ yields the formulae

$$\frac{dL_\omega(\chi_\omega)}{d\omega}|_{\omega=c} = d'(c) = \frac{1}{2}|\varphi_c|_2^2$$

and

$$\begin{aligned}
\frac{d^2 L_\omega(\chi_\omega)}{d\omega^2} \Big|_{\omega=c} &= d''(c) + \left(\frac{1}{2}(2a+b-e)^2 + \frac{4-p}{4p}(2a-b-e)^2 \right. \\
&\quad \left. + \frac{1}{4}(2a-3b+e)^2 - \frac{1}{p}(a(p+2)-b-e)^2 \right) 2d(c)(\sigma'(c))^2 \\
&= d''(c) + \left(\frac{1}{2}(2a+b-e)^2 + \frac{4-p}{4p}(2a-b-e)^2 \right. \\
&\quad \left. + \frac{1}{4}(2a-3b+e)^2 - \frac{1}{p}(a(p+2)-b-e)^2 \right) \frac{2}{(2a-b-e)^2} \frac{4-3p}{4-p} d''(c) \\
&= \frac{2(4-3p)}{(2a-b-e)^2(4-p)} g(\epsilon, \eta) d''(c),
\end{aligned}$$

where

$$\begin{aligned}
g(\epsilon, \eta) &= \frac{1}{2} \left(\frac{4-p}{2} a + \epsilon + 2\eta \right)^2 - \frac{(4-p)^2}{4p(3p-4)} \left(\frac{4-3p}{2} a \epsilon \right)^2 \\
&\quad + \frac{1}{4} \left(\frac{4-3p}{2} a - 4\epsilon - \eta \right)^2 - \frac{1}{p} \left(\frac{4-p}{2} a + \epsilon \right)^2.
\end{aligned}$$

Now, we claim $\frac{d^2 L_\omega(\chi_\omega)}{d\omega^2} \Big|_{\omega=c} < 0$. Since $d''(c) < 0$ and $4-3p < 0$, it suffices to show that $g(\epsilon, \eta) < 0$ for small ϵ and η . In fact, $g(0, 0) = 0$,

$$\frac{\partial g}{\partial \eta}(0, \eta) = 2 \left(\frac{4-p}{2} a + 2\eta \right) - \frac{1}{2} \left(\frac{4-3p}{2} a - \eta \right),$$

and $\frac{\partial g}{\partial \eta}(0, 0) = \frac{1}{4}(12-p)a < 0$. This implies that $g(0, \eta) < g(0, 0) = 0$ for small $\eta > 0$ and therefore $g(\epsilon, \eta) < 0$ for small $\epsilon > 0$ and $\eta > 0$. It follows from $\frac{d^2 L_\omega(\chi_\omega)}{d\omega^2} \Big|_{\omega=c} < 0$ that

$$\begin{aligned}
L_c(\chi_\omega) &= L_\omega(\chi_\omega) + V(\omega_\omega)(c-\omega) = L_\omega(\chi_\omega) + V(\varphi_c)(c-\omega) \\
&< L_c(\varphi_c) + \frac{d}{d\omega} L_\omega(\chi_\omega) \Big|_{\omega=c} (\omega-c) + V(\varphi_c)(c-\omega) \\
&= L_c(\varphi_c) + V(\varphi_c)(\omega-c) + V(\varphi_c)(c-\omega) = L_c(\varphi_c)
\end{aligned}$$

for ω sufficiently near c . Therefore the result of strong instability of solitary waves can be obtained by Lemma 5.4.

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