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DADE'S INVARIANT CONJECTURE FOR GENERAL LINEAR AND UNITARY GROUPS IN NON-DEFINING CHARACTERISTICS

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Abstract. This paper is part of a program to study the conjecture of E. C. Dade on counting characters in blocks for several finite groups.

The invariant conjecture of Dade is proved for general linear and unitary groups when the characteristic of the modular representation is distinct from the defining characteristic of the groups.

Introduction

Let G be a finite group, r a prime and B an r-block. In [7] Dade introduced several conjectures on counting the numbers of ordinary irreducible characters in B with a fixed defect and announced that his final conjecture can be proved by verifying it for all non-abelian finite simple groups. The final conjecture has been verified for 15 sporadic simple groups, the simple Tits group, $L_2(q)$ and $Sz(2^{2m+1})$, and for ${}^2G_2(3^{2m+1})$, $G_2(q)$ (with $q \neq 3, 4$) and ${}^3D_4(q)$ in non-defining characteristics. The ordinary conjecture has been verified by Olsson, Uno and the author for S_n, ${}^2F_4(2^{2m+1})$ in non-defining characteristics, and general linear groups in the defining characteristic. In this paper, we prove the invariant conjecture for a general linear or unitary group G in non-defining characteristics. The reductions used in this paper can also be applied to other classical groups.

In Section 1, we fix some notation, state the invariant conjecture and prove three lemmas. In Section 2, we reduce the family of radical chains to a simple G-invariant subfamily, central radical chains $\mathcal{CR}(G)$. Given an r-block B, in Section 3, we first reduce the family $\mathcal{CR}(G)$ to its subfamily consisting of chains whose fixed-point subspace of each non-trivial subgroup has the same dimension as that of a defect group of B, then using the perfect isometries given by Broué, we reduce the final calculation for an r-block to that of the principal block. In the last section, we prove the invariant conjecture for the principal block using results of Fong, Srinivasan [10] and Olsson [12].

1. Dade's conjecture and lemmas

Let R be an r-subgroup of a finite group G. Then R is *radical* if $O_r(N(R)) = R$, where $O_r(N(R))$ is the largest normal r-subgroup of the normalizer $N(R) = N_G(R)$. Let $\mathrm{Blk}(G)$ be the set of r-blocks B of G, $D(B)$ a defect group of B and $\mathrm{Irr}(G)$ the

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set of all irreducible ordinary characters of G. If $H \leq G$, then denote $\mathrm{Blk}(H|B) = \{b \in \mathrm{Blk}(H) : b^G = B\}$ (in the sense of Brauer).

Given an r-subgroup chain

$$C : P_0 < P_1 < \cdots < P_w \tag{1.1}$$

of G, define the *length* $|C| = w$, $C_k : P_0 < P_1 < \cdots < P_k$, $C(C) = C_G(P_w)$, and

$$N(C) = N_G(C) = N(P_0) \cap N_G(P_1) \cap \cdots \cap N_G(P_w).$$

The chain C is said to be *radical* if it satisfies the following two conditions:

(a) $P_0 = O_r(G)$ and (b) $P_k = O_r(N(C_k))$ for $1 \leq k \leq w$.

Denote by $\mathcal{R} = \mathcal{R}(G)$ the set of all radical r-chains of G.

Suppose $1 \to G \to E \to \overline{E} \to 1$ is an exact sequence, so that E is an extension of G by $\overline{E}$. Then E acts on $\mathcal{R}$ by conjugation. Given $C \in \mathcal{R}(G)$ and $\xi \in \mathrm{Irr}(N_G(C))$, let $N_E(C)$ and $N_E(C,\xi)$ be the stabilizers of C and the pair (C,ξ) in E, respectively, so that

$$N_E(C,\xi)/N_G(C) \simeq N_{\overline{E}}(C,\xi) = N_E(C,\xi)G/G.$$

For $B \in \mathrm{Blk}(G)$, an integer $d \geq 0$ and $U \leq \overline{E}$, let $\mathrm{k}(N(C),B,d,U)$ be the number of characters in the set

$$\mathrm{Irr}(N(C),B,d,U) = \{\xi \in \mathrm{Irr}(N(C),B,d) : N_{\overline{E}}(C,\xi) = U\}.$$

Dade in [7] gives the following conjecture.

Dade's Invariant Conjecture. *If $O_r(G) = 1$ and B is an r-block of G with defect $\mathrm{d}(B) > 0$, then for any integer $d \geq 0$,*

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|}\mathrm{k}(N(C),B,d,U) = 0. \tag{1.2}$$

Let $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ be the automorphism and outer automorphism groups of G, respectively. Then we may suppose $\overline{E} \leq \mathrm{Out}(G)$.

Let m be a positive integer and $a(m)$ the integer such that $r^{a(m)}$ is the exact power of r dividing m. Set $a(G) = a(|G|)$ when G is a finite group and $a(\chi) = a(\chi(1))$ when χ is a character of G.

Let $H = H_1 \times H_2$ be a direct product of two finite groups H_1 and H_2, and let π_i be the natural projection from H onto H_i for $i = 1,2$. If $C \in \mathcal{R}(H)$ is given by (1.1), then each $\pi_i(P_t)$ is a radical subgroup of $\bigcap_{\ell=0}^{t-1} N_{H_i}(\pi_i(P_\ell))$. In general, $O_r(H_i) \leq \pi_i(P_1) \leq \pi_i(P_2) \leq \ldots \leq \pi_i(P_w)$ is not a radical chain of H_i. Let $\mathcal{Y} = \{\pi_i(P_\ell) : 0 \leq \ell \leq w\}$ and suppose $|\mathcal{Y}| = u$. Relabel the subgroups in $\mathcal{Y}$

$$\mathcal{Y} = \{Q_0, Q_1, \ldots, Q_u\}$$

such that $Q_i < Q_{i+1}$ for $0 \leq i \leq u-1$. Thus $Q_0 < Q_1 < \ldots < Q_u$ is a radical chain of H_i, which is denoted by $\pi_i(C)$.

(1A). *Let H_1 and H_2 be two finite groups, n a nonnegative integer, $B_i \in \mathrm{Blk}(H_i)$ and $\mathcal{R}(B_i)$ the subfamily of $\mathcal{R}(H_i)$ consisting of chains C such that $\mathrm{Blk}(N_{H_i}(C)|B_i) \neq \emptyset$.*

(a) *Suppose $\mathcal{X}_i \subseteq \mathrm{Irr}(H_i)$ for $i = 1,2$, and ψ is a defect preserving bijection between $\mathcal{X}_1$ and $\mathcal{X}_2$. In addition, let $K_i = H_i \wr \mathbf{S}(n)$, and let $\mathcal{X}_i \wr \mathbf{S}(n)$ be the subset of $\mathrm{Irr}(K_i)$ consisting of all characters covering a character of $(\mathcal{X}_i)^n$, where $\mathbf{S}(n)$ is the symmetric group on n letters. Then ψ can be extended to a defect preserving bijection Ψ between $\mathcal{X}_1 \wr \mathbf{S}(n)$ and $\mathcal{X}_2 \wr \mathbf{S}(n)$. Moreover, if τ is an automorphism* *of K_i for $i = 1, 2$ such that $(H_i)^\tau = H_i$, $\psi(\eta)^\tau = \psi(\eta^\tau)$ and $y^\tau = y$ for each $\eta \in \mathcal{X}_1$ and $y \in \mathbf{S}(n)$. Then τ is compatible with Ψ, that is, $\Psi(\chi)^\tau = \Psi(\chi^\tau)$ for $\chi \in \mathcal{X}_1 \wr \mathbf{S}(n)$.*

(b) *Let $H = H_1 \times H_2$ and $B = B_1 \times B_2 \in \mathrm{Blk}(H)$. In addition, let $\mathcal{L}(B_i)$ be an H_i-invariant subset of $\mathcal{R}(B_i)$, and $\mathcal{L}(B)$ a subset of $\mathcal{R}(B)$ consisting of all chains C such that $\pi_i(C) \in \mathcal{L}(B_i)$ for $i = 1, 2$. Suppose U is a subgroup of* $\mathrm{Out}(H)$ *generated by $\tau_1, \ldots, \tau_t$ and suppose $\tau_j = \pi_1(\tau_j) \times \pi_2(\tau_j)$, where each $\pi_i(\tau_j)$ is an automorphism of H_i. Let $U_i = \langle \pi_i(\tau_1), \ldots, \pi_i(\tau_t)\rangle$ for $i = 1, 2$. Then*

$$\sum_{C \in \mathcal{L}(B)/H} (-1)^{|C|} \mathrm{k}(N_H(C), B, d, U)$$

$$= \sum_{d_1+d_2=d} \left(\prod_{i=1}^{2} \Big(\sum_{C \in \mathcal{L}(B_i)/H_i} (-1)^{|C|} \mathrm{k}(N_{H_i}(C), B_i, d_i, U_i) \Big) \right).$$

Proof. (a) Let $(H_i)^n$ be the base subgroup of $K_i = H_i \wr \mathbf{S}(n)$, and $\mathcal{X}_1 = \{\xi_1, \ldots, \xi_k\}$. Then $\mathcal{X}_2 = \{\psi(\xi_1), \ldots, \psi(\xi_k)\}$. If $\chi \in \mathcal{X}_1 \wr \mathbf{S}(n)$, then the restriction $\chi|_{(H_1)^n}$ of χ to $(H_1)^n$ is a character in $(\mathcal{X}_1)^n$. If $\xi \in (\mathcal{X}_1)^n$ and m_i is the multiplicity of ξ_i in ξ, then $\mathbf{m} = (m_1, m_2, \ldots, m_k)$ is called the *type* of ξ and ξ has an extension $\tilde{\xi}$ to the stabilizer $(H_1)^n\mathbf{S}(\mathbf{m})$ of ξ in $H_1 \wr \mathbf{S}(n)$, where $\mathbf{S}(\mathbf{m})$ is the Young subgroup of $\mathbf{S}(n)$ of type $\mathbf{m}$. If $\zeta \in \mathrm{Irr}(\mathbf{S}(\mathbf{m}))$, then $\chi = \mathrm{Ind}_{(H_1)^n\mathbf{S}(\mathbf{m})}^{H_1 \wr \mathbf{S}(n)}(\tilde{\xi}\zeta) \in \mathrm{Irr}(K_1)$, and each character of $\mathcal{X}_1 \wr \mathbf{S}(n)$ is of this form. Define

$$\Psi(\chi) = \mathrm{Ind}_{(H_2)^n\mathbf{S}(\mathbf{m})}^{H_2 \wr \mathbf{S}(n)}(\psi(\tilde{\xi})\zeta),$$

where $\psi(\tilde{\xi})$ is an extension of the character $\psi(\xi) \in (\mathcal{X}_2)^n$ whose multiplicity of $\psi(\xi_i)$ is m_i. Then $\Psi(\chi) \in \mathcal{X}_2 \wr \mathbf{S}(n)$ and Ψ is a bijection. Moreover, $\mathrm{d}(\Psi(\chi)) = a((H_2)^n\mathbf{S}(\mathbf{m})) - a(\psi(\xi)) - a(\zeta)$ and $\mathrm{d}(\chi) = a((H_1)^n\mathbf{S}(\mathbf{m})) - a(\xi) - a(\zeta)$. Thus $\mathrm{d}(\chi) = \mathrm{d}(\Psi(\chi))$ if and only if

$$na(H_2) - a(\psi(\xi)) = na(H_1) - a(\xi).$$

But $na(H_2) - a(\psi(\xi)) = \sum_{j=1}^k m_j \mathrm{d}(\psi(\xi_j))$ and $na(H_1) - a(\xi) = \sum_{j=1}^k m_j \mathrm{d}(\xi_j)$, so Ψ is defect preserving. Since $\psi(\eta)^\tau = \psi(\eta^\tau)$ for $\eta \in \mathcal{X}_1$ and $y^\tau = y$ for $y \in \mathbf{S}(n)$, it follows that $\Psi(\chi)^\tau = \Psi(\chi^\tau)$.

(b) Given $C(0) \in \mathcal{L}(B)$ with length $s \geq 0$, let $X = X_1 \times X_2$ be the final subgroup of $C(0)$, where $X_i \leq H_i$. If $s \geq 1$, then X is a radical subgroup of $N_H(C(0)_{s-1}) = N_1 \times N_2$ with $X \neq O_r(N_H(C(0)_{s-1}))$, where $X_1 \leq N_1 \leq H_1$ and $X_2 \leq N_2 \leq H_2$. If $s = 0$, then $X_i = O_r(H_i)$ and we set $N_i = H_i$ for $i = 1, 2$. Let $\mathcal{L}(C(0))$ be the subset of $\mathcal{L}(B)$ consisting of all chains C such that $C_s = C(0)$. If $C \in \mathcal{L}(C(0))$ is given by (1.1), then $P_t = R_1(t) \times R_2(t)$ with $X_1 \leq R_1(t) \leq N_1$ and $X_2 \leq R_2(t) \leq N_2$ for $s \leq t \leq w$. Fix a radical subgroup Y_1 of $N_{N_1}(X_1)$ with $X_1 \neq Y_1$ and fix an integer $u \geq s+1$. Let $\mathcal{M}(C(0), Y_1, u)$ be the subset of $\mathcal{L}(C(0))$ consisting of all chains C given above such that $R_1(u) = Y_1$ and $R_1(t) = X_1$ for $s \leq t \leq u-1$ up to conjugacy in $N_H(C(0))$, so that we may suppose $P_u = Y_1 \times R_2(u)$ and $X_2 \leq R_2(u-1) \leq R_2(u)$. Let $\mathcal{M}^0(C(0), Y_1, u)$ and $\mathcal{M}^+(C(0), Y_1, u)$ be the subsets of $\mathcal{M}(C(0), Y_1, u)$ consisting of chains C such that $R_2(u-1) = R_2(u)$ and $R_2(u-1) \neq R_2(u)$, respectively. If $C \in \mathcal{M}^0(C(0), Y_1, u)$ is given by (1.1) with $u \geq s+2$, then

$$\varphi(C) : O_r(H) < P_1 < \ldots < P_{u-2} < P_u < \ldots < P_w$$

is a chain of $\mathcal{R}(B)$ and

$$N_{N_H(C(0))}(C) = N_{N_H(C(0))}(\varphi(C)). \tag{1.3}$$

Since $\pi_i(C) = \pi_i(\varphi(C))$, it follows that $\varphi(C) \in \mathcal{M}^+(C(0), Y_1, u-1)$. Conversely, if C is a chain of $\mathcal{M}^+(C(0), Y_1, u)$ given by (1.1) with $u \geq s+1$, then

$$\varphi(C) : O_r(H) < P_1 < \ldots < P_{u-1} < P < P_u < \ldots < P_w$$

is a chain of $\mathcal{M}^0(C(0), Y_1, u+1)$ and (1.3) still holds, where $P = X_1 \times R_2(u)$. Let

$$\mathcal{S}(C(0), Y_1) = \Bigg(\bigcup_{u \geq s+2} \mathcal{M}^0(C(0), Y_1, u)\Bigg) \cup \Bigg(\bigcup_{u \geq s+1} \mathcal{M}^+(C(0), Y_1, u)\Bigg). \tag{1.4}$$

Then $\varphi(C)$ is a map from $\mathcal{S} = \mathcal{S}(C(0), Y_1, u)$ to itself such that $\varphi(\varphi(C)) = C$ and $|\varphi(C)| = |C| \pm 1$.

Suppose $\chi \in \mathrm{Irr}(N_H(C), B, d)$ and $\chi^\tau = \chi$ for some $\tau \in U$. Then $\chi = \chi_1 \times \chi_2$ and $\tau = \pi_1(\tau) \times \pi_2(\tau)$, where $\chi_i \in \mathrm{Irr}(H_i)$ and $\pi_i(\tau) \in U_i$ for $i = 1, 2$. Thus $\chi_i^{\pi_i(\tau)} = \chi_i$ and $\chi \in \mathrm{Irr}(N_{H_1}(\pi_1(C)), B_1, d_1, U_1) \times \mathrm{Irr}(N_{H_2}(\pi_2(C)), B_2, d_2, U_2)$, where $d_i = \mathrm{d}(\chi_i)$ for $i = 1, 2$. Conversely, if $\chi_i \in \mathrm{Irr}(N_{H_i}(\pi_i(C)), B_i, d_i, U_i)$ with $d_1 + d_2 = d$ and if $\chi = \chi_1 \times \chi_2$, then $\chi \in \mathrm{Irr}(N_H(C), B, d, U)$, so that

$$\begin{aligned} &\mathrm{Irr}(N_H(C), B, d, U) \\ &= \bigcup_{d_1+d_2=d} \mathrm{Irr}(N_{H_1}(\pi_1(C)), B_1, d_1, U_1) \times \mathrm{Irr}(N_{H_2}(\pi_2(C)), B_2, d_2, U_2). \end{aligned} \tag{1.5}$$

Since $\pi_i(C) = \pi_i(\varphi(C))$ for $i = 1, 2$, $\mathrm{k}(N_H(C), B, d, U) = \mathrm{k}(N_H(\varphi(C)), B, d, U)$ and

$$\sum_{C \in \mathcal{S}/N_H(C(0))} (-1)^{|C|} \mathrm{k}(N_H(C), B, d, U) = 0$$

for all $d \geq 0$.

Let $C \in \mathcal{L}(C(0)) \backslash (\bigcup_{Y_1} \mathcal{S}(C(0), Y_1))$ be a chain given by (1.1), where Y_1 runs over all radical subgroups of $N_{N_1}(X_1)$ with $Y_1 \neq X_1$. Then either $P_t = X_1 \times R_2(t)$ for all $s \leq t \leq w$ or $P_{s+1} = Z_1 \times X_2$ for some radical subgroup Z_1 of $N_{N_1}(X_1)$ with $Z_1 \neq X_1$. Let

$$\mathcal{L}^*(B) = \mathcal{R}(B) \Bigg\backslash \Bigg(\bigcup_{C(0), Y_1} \mathcal{S}(C(0), Y_1)\Bigg),$$

where $C(0)$ runs over $\mathcal{R}(B)$ and Y_1 runs over all radical subgroups of $N_{N_1}(X_1)$ with $Y_1 \neq X_1$.

Suppose $C \in \mathcal{L}^*(B)$ is given by (1.1) with $P_t = R_1(t) \times R_2(t)$. If $R_1(1) = O_r(H_1)$, then $R_1(w) = O_r(H_1)$, and we set $u = u(C) = 0$. If $R_1(1) \neq O_r(H_1)$, then $R_2(1) = O_r(H_2)$, and there exists the largest integer $u = u(C) \geq 1$ such that $R_1(u-1) < R_1(u) = R_1(u+1)$. Thus $R_2(u) = O_r(H_2)$ and $R_2(t) \neq R_2(t-1)$ for all $u+1 \leq t \leq w$, so that $R_1(t) \neq R_1(t-1)$ for all $1 \leq t \leq u$ and $R_1(t) = R_1(t-1)$ for all $u+1 \leq t \leq w$. Define $\varphi_1(C) : O_r(H_1) < R_1(1) < \ldots < R_1(u)$ and $\varphi_2(C) : O_r(H_2) < R_2(u+1) < \ldots < R_2(w)$. Then $\varphi_i(C) = \pi_i(C) \in \mathcal{L}(B_i)$.

Let $C(0) \in \mathcal{L}^*(B)$ such that $u(C(0)) = |C(0)|$, and let $\mathcal{L}^*(C(0)) = \mathcal{L}^*(B) \cap \mathcal{L}(C(0))$. Then $N_H(C(0)) = N_1 \times H_2$ and $N_H(C) = N_1 \times N_{H_2}(\varphi_2(C))$, where $N_1 = N_{H_1}(\varphi_1(C(0)))$ and $C \in \mathcal{L}^*(C(0))$. Moreover, φ_2 induces a bijection between

$\mathcal{L}^*(C(0))$ and $\mathcal{L}(B_2)$, and $|\varphi_2(C)| = |C| - |C(0)|$. A similar proof to that of (1.5) shows that

$$\mathrm{k}(N_H(C), B, d, U) = \sum_{s+t=d} \mathrm{k}(N_1, B_1, s, U_1)\mathrm{k}(N_{H_2}(\varphi_2(C)), B_2, t, U_2)$$

and moreover,

$$\sum_{C\in\mathcal{L}^*(C(0))/N_H(C(0))} (-1)^{|C|}\mathrm{k}(N_H(C), B, d, U) = \sum_{t\geq 0}\sum_{s+t=d} (-1)^{|C(0)|}\mathrm{k}(N_1, B_1, s, U_1) \times \sum_{C\in\mathcal{L}(B_2)/H_2} (-1)^{|C|}\mathrm{k}(N_{H_2}(C), B_2, t, U_2).$$

Let $\mathcal{L}^+(B) = \mathcal{L}^*(B)\backslash(\bigcup_{C(0)}\mathcal{L}^*(C(0)))$, where $C(0)$ runs over all chains of $\mathcal{L}(B)$ such that $u(C(0)) = |C(0)|$. Then φ_1 induces a bijection from $\mathcal{L}^+(B)$ to $\mathcal{L}(B_1)$ such that $|\varphi_1(C)| = |C|$ and $N_H(C) = N_{H_1}(\varphi_1(C)) \times H_2$. Thus (b) follows by $\mathcal{L}^*(B)/H = \bigcup_{C\in\mathcal{L}^+(B)/H}\mathcal{L}^*(B)/N_H(C)$. □

Hypothesis (1B). *For a finite group X, suppose an X-invariant subfamily $\mathcal{L}(X)$ of $\mathcal{R}(X)$ has an X-invariant decomposition*

(1.6) $$\mathcal{L}(X) = \mathcal{L}_1(X) \cup \mathcal{L}_2(X) \qquad (\textit{disjoint})$$

satisfying the following condition:

There exists a bijection φ between the X-invariant subfamilies $\mathcal{L}_1(X)$ and $\mathcal{L}_2(X)$ such that for each $C \in \mathcal{L}_1(X)$ given by (1.1), $\varphi(C)$ is given by

(1.7) $$\varphi(C) : O_r(X) < P_1 < \ldots < P_v < Q < P_{v+1} < \ldots < P_w$$

for an integer $v = v(C)$ and a radical subgroup $Q = Q(C)$ of $N(C_v)$.

Set $v(\varphi(C)) = v(C) = v$. Let n be a positive integer, $H = X \wr \mathbf{S}(n)$, and let $Y = X_1 \times X_2 \times \ldots \times X_n$ be the base subgroup of H, where X_i is a copy of X. In addition, let π_i be the natural projection of Y onto X_i for $1 \leq i \leq n$ and $\mathcal{L}(H)$ the subfamily of $\mathcal{R}(Y)$ consisting of chains C such that $\pi_i(C) \in \mathcal{L}(X_i) = \mathcal{L}(X)$ for all i. Then $\mathcal{L}(H)$ is H-invariant.

Let $\mathcal{M}(H)/H$ be the subset of chains $C \in \mathcal{L}(H)$ given by (1.1) such that each $P_i = Q_i^n$ for some subgroup Q_i of X, and let $\mathcal{M}(H)$ be the subfamily of $\mathcal{L}(H)$ consisting of H-orbits of $\mathcal{M}(H)/H$. For $C \in \mathcal{M}(H)/H$, $\pi_i(C) = \pi_j(C)$ for all i, j, $|C| = |\pi_i(C)|$ and

$$N_H(C) = N_X(\pi(C)) \wr \mathbf{S}(n),$$

where $\pi(C) = \pi_1(C)$, viewed as a chain of X. In particular, π is a bijection between $\mathcal{M}(H)/H$ and $\mathcal{L}(X)/X$.

If $\mathbf{m} = (n_1, \ldots, n_t)$ is a sequence of positive integers such that $|\mathbf{m}| = n_1 + n_2 + \ldots + n_t = n$, then $X \wr \mathbf{S}(\mathbf{m}) = H_1 \times H_2 \times \ldots \times H_t$ is a subgroup of H, where each $H_i = X \wr \mathbf{S}(n_i)$. Let π_{n_i} be the projection from $X \wr \mathbf{S}(\mathbf{m})$ onto H_i for all i.

(1C). *In the notation above, suppose $C \in \mathcal{L}(H)$. Then there is a sequence $\mathbf{m} = \mathbf{m}(C) = (n_1, \ldots, n_t)$ of positive integers such that $\pi_{n_i}(C) \in \mathcal{M}(H_i)$, $|\mathbf{m}| = n$ and*

(1.8) $$N_H(C) = N_X(C(1)) \wr \mathbf{S}(n_1) \times N_X(C(2)) \wr \mathbf{S}(n_2) \times \ldots \times N_X(C(t)) \wr \mathbf{S}(n_t),$$

where $C(i) = \pi(\pi_{n_i}(C)) \in \mathcal{L}(X)$. *Moreover,* $\mathcal{L}(H)$ *satisfies Hypothesis* (1B) *and if* $C \in \mathcal{L}_1(H)$ *with* $N_H(C)$ *given by* (1.8), *then there is an integer* $s = s(C) \geq 1$ *such that*

$$\begin{aligned} N_H(\Phi(C)) = N_X(C(1)) \wr \mathbf{S}(n_1) \times \ldots \times N_X(C(s-1)) \wr \mathbf{S}(n_{s-1}) \\ \times N_X(\varphi(C(s))) \wr \mathbf{S}(n_s) \times N_X(C(s+1)) \wr \mathbf{S}(n_{s+1}) \\ \times \ldots \times N_X(C(t)) \wr \mathbf{S}(n_t), \end{aligned} \tag{1.9}$$

where $\Phi(C) \in \mathcal{L}_2(H)$. *In particular,* $C(s) \in \mathcal{L}_1(X)$.

Proof. Suppose R is a radical subgroup of the base subgroup $N_X(\pi(C))^n$ of $N_H(C)$, where C is a chain of $\mathcal{M}(H)/H$. Then

$$R = R_1^{n_1} \times R_2^{n_2} \times \ldots \times R_t^{n_t} \tag{1.10}$$

for some sequence $\mathbf{m} = (n_1, \ldots, n_t)$ with $|\mathbf{m}| = n$, where $R_i \neq R_j$ for $i \neq j$ viewed as subgroups of X. Thus

$$N_{N_H(C)}(R) = N_{N_X(\pi(C))}(R_1) \wr \mathbf{S}(n_1) \times \ldots \times N_{N_X(\pi(C))}(R_t) \wr \mathbf{S}(n_t) \tag{1.11}$$

and $N_{N_X(\pi(C))}(R_i) \wr \mathbf{S}(n_i) = N_{N_{H_i}(\pi_{n_i}(C))}(R_i^{n_i})$ for all $i \geq 1$.

If C is given by (1.1) and $R \neq O_r(N_H(C))$ such that $C' : O_r(H) < P_1 < \ldots < P_w < R$ is a chain of $\mathcal{L}(H)$, then $\pi_{n_i}(C') \in \mathcal{M}(H_i)/H_i$ and

$$N_{N_X(\pi(C))}(R_i) \wr \mathbf{S}(n_i) = N_X(C'(i)) \wr \mathbf{S}(n_i) = N_{H_i}(\pi_{n_i}(C'))$$

for all i, where $C'(i) = \pi(\pi_{n_i}(C')) \in \mathcal{L}(X)$. It follows by induction on n that for each $C \in \mathcal{L}(H)$, $N_H(C)$ is of the form (1.8) for some sequence $\mathbf{m} = \mathbf{m}(C)$.

Fix a sequence $\mathbf{m} = (n_1, \ldots, n_t)$, let $\mathcal{L}(\mathbf{m})$ be the subfamily of $\mathcal{L}(H)$ consisting of chains C such that $\mathbf{m}(C) = \mathbf{m}$, so that $\mathcal{L}(H)$ is a disjoint union of $\mathcal{L}(\mathbf{m})$'s, $N_H(C)$ satisfies (1.8) for each $C \in \mathcal{L}(\mathbf{m})$, $\mathcal{L}(\mathbf{m})$ is $X \wr \mathbf{S}(\mathbf{m})$-invariant and $N_H(C) = N_{X \wr \mathbf{S}(\mathbf{m})}(C)$.

Suppose $C \in \mathcal{L}(\mathbf{m})$ is given by (1.1), $C(i) = \pi(\pi_{n_i}(C))$ for all i, and for some fixed ℓ,

$$C(\ell) : O_r(X) < Q_1 < \ldots < Q_v < \ldots < Q_u, \tag{1.12}$$

where $v = v(C(\ell))$. Then each P_j has a decomposition

$$P_j = W_{j,1}^{n_1} \times W_{j,2}^{n_2} \times \ldots \times W_{j,t}^{n_t}, \tag{1.13}$$

where $W_{j,\ell} \leq \{O_r(X), Q_1, \ldots, Q_u\}$ for all j. Since $C(\ell) = \pi(\pi_{n_\ell}(C))$, there is an integer $m_\ell = m_\ell(C)$ such that $W_{m_\ell,\ell} = Q_v$ and $W_{m_\ell+1,\ell} \neq Q_v$, so that $W_{m_\ell+1,\ell} = Q_{v+1}$. If $Q = Q(C(\ell))$, then by definition, $Q = Q_{v+1}$ or $Q < Q_{v+1}$ according to whether $C(\ell) \in \mathcal{L}_2(X)$ or $C(\ell) \in \mathcal{L}_1(X)$. Thus

$$Q(n_\ell) = W_{m_\ell,1}^{n_1} \times \ldots \times W_{m_\ell,\ell-1}^{n_{\ell-1}} \times Q^{n_\ell} \times W_{m_\ell,\ell+1}^{n_{\ell+1}} \times \ldots \times W_{m_\ell,t}^{n_t} \leq P_{m_\ell+1} \tag{1.14}$$

and define

$$\Phi_{n_\ell}(C) : P_0 < \ldots < P_{m_\ell} < Q(n_\ell) \leq P_{m_\ell+1} < \ldots < P_w. \tag{1.15}$$

It follows that $\Phi_{n_\ell}(C) \in \mathcal{L}(\mathbf{m})$ and $N_H(\Phi_{n_\ell}(C))$ has the form (1.9) with $s = \ell$ when $\Phi_{n_\ell}(C) \neq C$.

Let $\mathcal{L}_1(\mathbf{m})$ be a subfamily of $\mathcal{L}(\mathbf{m})$ consisting of chains $C \in \mathcal{L}(\mathbf{m})$ such that

$$C = \Phi_{n_1}(C) = \ldots = \Phi_{n_{s-1}}(C), \quad \Phi_{n_s}(C) \neq C,$$

and define $s(C) = s$ for $C \in \mathcal{L}_1(\mathbf{m})$. In addition, let $\mathcal{L}_2(\mathbf{m})$ be the subfamily of $\mathcal{L}(\mathbf{m})$ consisting of the chains $C \in \mathcal{L}(\mathbf{m})$ such that $C = \Phi_{n_s}(C')$ for some $C' \in \mathcal{L}_1(\mathbf{m})$ with $s = s(C')$. Then $\mathcal{L}_1(\mathbf{m})$ and $\mathcal{L}_2(\mathbf{m})$ are $X \wr \mathbf{S}(\mathbf{m})$-invariant.

Suppose $C \in \mathcal{L}(\mathbf{m})\backslash(\mathcal{L}_1(\mathbf{m}) \cup \mathcal{L}_2(\mathbf{m})) \neq \emptyset$ and suppose $N_H(C)$ is given by (1.8). If $C(1)$ is given by (1.12) with $v = v(C(1))$. By (1.15), $\Phi_{n_1}(C) \in \mathcal{L}(\mathbf{m})$, and since $C \notin \mathcal{L}_1(\mathbf{m})$, it follows that $Q(n_1) = P_{m_1+1}$, where $Q(n_1)$ is given by (1.14). Let

$$\alpha(C) : P_0 < \ldots < P_{m_1} < P_{m_1+2} < \ldots < P_w, \tag{1.16}$$

so that $\alpha(C) \in \mathcal{L}(H)$. If $\alpha(C) \in \mathcal{L}(\mathbf{m})$, then $\Phi_{n_1}(\alpha(C)) = C$ and so $C \in \mathcal{L}_2(\mathbf{m})$, which is impossible. Thus $\mathbf{m}(\alpha(C)) \neq \mathbf{m}(C)$ and $N_H(C)$ is a proper subgroup of $N_H(\alpha(C))$. Since $\pi_{n_i}(\alpha(C)) \in \mathcal{M}(H_i)/H_i$ for all i and since $N_H(\alpha(C))$ is of the form (1.8), it follows that there is some n_i, say n_2, such that

$$\pi_{n_1+n_2}(\alpha(C)) \in \mathcal{M}(X \wr \mathbf{S}(n_1+n_2))/X \wr \mathbf{S}(n_1+n_2). \tag{1.17}$$

In particular, $\pi_{n_1}(\alpha(C)) = \pi_{n_2}(\alpha(C))$. If $P_j^+ = \prod_{k=3}^t W_{j,k}^{n_k}$ for all $j \geq 0$, then $P_{m_1} = Q_v^{n_1} \times Q_v^{n_2} \times P_{m_1}^+$, $P_{m_1+1} = Q^{n_1} \times Q_v^{n_2} \times P_{m_1}^+$ and $P_{m_1+2} = Z^{n_1} \times Z^{n_2} \times P_{m_1+2}^+$ for some r-subgroup Z of X. In particular, $Q_v < Q \leq Z$,

$$Q(n_2) = Q^{n_1} \times Q^{n_2} \times P_{m_1}^+$$

and $\Phi_{n_2}(C)$ is the chain

$$\Phi_{n_2}(C) : P_0 < \ldots < P_{m_1} < P_{m_1+1} < Q(n_2) \leq P_{m_1+2} < \ldots < P_w.$$

If $\Phi_{n_2}(C) = C$, then $Z = Q$, $P_{m_1}^+ = P_{m_1+2}^+$, $m_2 = m_1 + 1$, and $C \in \mathcal{L}_2(\mathbf{m})$. If $\Phi_{n_2}(C) \neq C$, then $C \in \mathcal{L}_1(\mathbf{m})$. This is impossible, so that $\mathcal{L}(\mathbf{m}) = \mathcal{L}_1(\mathbf{m}) \cup \mathcal{L}_2(\mathbf{m})$ and (1C) follows. □

Suppose b is a block of X with positive defect. For $C \in \mathcal{R}(X)$, let

$$\mathrm{Irr}(N_X(C), b) = \bigcup_{b_1 \in \mathrm{Blk}(N_X(C)|b)} \mathrm{Irr}(b_1)$$

and if $\mathcal{X} \subseteq \mathrm{Irr}(N_X(C))$, then denote $\mathrm{Irr}(\mathcal{X}, b) = \mathcal{X} \cap \mathrm{Irr}(N_X(C), b)$.

Hypothesis (1D). *Suppose an X-invariant subfamily $\mathcal{Q}(X)$ of $\mathcal{R}(X)$ has an X-invariant decomposition*

$$\mathcal{Q}(X) = \mathcal{Q}_1(X) \cup \mathcal{Q}_2(X) \cup \mathcal{S}(X) \qquad (disjoint) \tag{1.18}$$

satisfying the following two conditions:

(*a*) *The family $\mathcal{Q}_1(X) \cup \mathcal{Q}_2(X)$ satisfies Hypothesis* (1B) *and for each $C \in \mathcal{Q}_1(X)$, there is a defect preserving bijection ψ between* $\mathrm{Irr}(N_X(C), b)$ *and* $\mathrm{Irr}(N_X(\varphi(C)), b)$.

(*b*) *For each $C \in \mathcal{S}(X)$ given by* (1.1), *there are some maps φ such that $\varphi(C) \in \mathcal{S}(X)$ is given by* (1.7) *with $Q \leq P_{v+1}$, and*

$$\mathrm{Irr}(N_H(C), b) = \Omega_C(0) \bigcup_{\varphi} \Omega_C(\varphi) \quad (disjoint),$$

where φ runs over the maps with $C \neq \varphi(C)$. Moreover, there is a defect preserving bijection ψ between $\Omega_C(\varphi)$ and $\Omega_{\varphi(C)}(0)$, and if $|C| \neq 0$, then $\varphi(C') = C$ for a unique chain $C' \in \mathcal{S}(X)$ and map φ.

For $\tau \in \mathrm{Aut}(X)$ the family $\mathcal{Q}(X)$ is called τ*-invariant* if τ stabilizes $C \in \mathcal{Q}_1(X) \cup \mathcal{S}(X)$ if and only if τ stabilizes $\varphi(C)$, and each defect preserving bijection ψ is compatible with τ.

Let $b(Y) = b^n$ be the block of the base subgroup $Y = X^n$ of $H = X \wr \mathbf{S}(n)$. Since $C_H(D(b(Y))) \leq Y$, $b(Y)^H$ is a regular block B of H. Denote by $\mathcal{Q}(H)$ the subfamily of $\mathcal{R}(Y)$ consisting of the chains C such that $\pi_i(C) \in \mathcal{Q}(X)$ for each $i \geq 1$, where π_i is the natural projection of Y onto its i-th component $X_i = X$.

(1E). *In the notation above, suppose $\mathcal{Q}(X)$ satisfies Hypothesis* (1D) *for b.*

(*a*) *Then $\mathcal{Q}(H)$ also satisfies Hypothesis* (1D) *for the block $B = b(Y)^H$ and if $C_0(X) \in \mathcal{S}(X)$, then $\Omega_{C_0(H)}(0) = \Omega_{C_0(X)}(0) \wr \mathbf{S}(n)$, where $C_0(K)$ is the radical chain of a finite group K with length* 0*. Moreover, if $\tau \in \mathrm{Aut}(H)$ such that $X_i^\tau = X_i$ and $y^\tau = y$ for all $y \in \mathbf{S}(n)$ and if $\mathcal{Q}(X)$ is compatible with τ, then $\mathcal{Q}(H)$ is compatible with τ.*

(*b*) *Given $i = 1, 2$, suppose each M_i is a finite group and $\mathcal{Q}(M_i)$ is an M_i-invariant subfamily of $\mathcal{R}(M_i)$ satisfies Hypothesis* (1D) *for $B_i \in \mathrm{Blk}(M_i)$. Let $M = M_1 \times M_2$ and let $\mathcal{Q}(M)$ be the subfamily of $\mathcal{R}(M)$ consisting of chains C such that $\pi_i(C) \in \mathcal{Q}(M_i)$. Then $\mathcal{Q}(M)$ satisfies Hypothesis* (1D) *for the block $B_1 \times B_2$ and if each $C_0(M_i) \in \mathcal{S}(M_i)$, then $C_0(M) \in \mathcal{S}(M)$ and $\Omega_{C_0(M)}(0) = \Omega_{C_0(M_1)}(0) \times \Omega_{C_0(M_2)}(0)$. In addition, if $\tau \in \mathrm{Aut}(M)$ stabilizes each M_i and each $\mathcal{Q}(M_i)$ is τ-invariant, then $\mathcal{Q}(M)$ is τ-invariant.*

Proof. (a) Suppose $C \in \mathcal{Q}(H)$ and $b(C) \in \mathrm{Blk}(N_H(C))$. Then $N_H(C)$ is of the form (1.8) for some sequence $\mathbf{m}$, and $K = \prod_{i=1}^t N_X(C(i))^{n_i}$ is a normal subgroup of $N_H(C)$. We claim that $b(C) \in \mathrm{Blk}(N_H(C)|B)$ if and only if $b(C)$ covers a block $b_K \in \mathrm{Blk}(K|b(Y))$. By induction, we may suppose $t = 1$, so that $n = n_1$. Thus $N_H(C) = N_X(C(1)) \wr \mathbf{S}(n)$, K is the base subgroup of $N_H(C)$ and $b(C)$ is regular. If $(b_K)^Y = b(Y)$ and $b(C)$ covers b_K, then $(b_K)^{N_H(C)} = b(C)$, $(b_K)^H = B$ and $b(C)^H = B$. Conversely, if $b(C)^H = B$ and $b(C)$ covers b_K, then $((b_K)^Y)^H = B = b(Y)^H$, so that B covers both $(b_K)^Y$ and $b(Y)$, and hence $(b_K)^Y$ is conjugate to $b(Y)$ in H. But $b(Y) = b^n$ is H-invariant, so $(b_K)^Y = b(Y)$ and the claim holds. In particular,

$$\mathrm{Irr}(N_H(C), B) = \prod_{i=1}^{t} \mathrm{Irr}(N_X(C(i)), b) \wr \mathbf{S}(n_i). \tag{1.19}$$

Let $\mathcal{Q}_0(H)$ be the subfamily of $\mathcal{Q}(H)$ consisting of chains C such that in the decomposition of $N_H(C)$ given by (1.8), $C(i) \in \mathcal{Q}_1(X) \cup \mathcal{Q}_2(X)$ for some i. Then $\mathcal{Q}_0(H)$ is H-invariant and the same proof as that of (1C) with some obvious modifications shows that $\mathcal{Q}_0(H) = \mathcal{Q}_1(H) \cup \mathcal{Q}_2(H)$ satisfies Hypothesis (1B).

Suppose $C \in \mathcal{Q}_1(H)$ with $N_H(C)$ given by (1.8) and $\Phi(C) \in \mathcal{Q}_2(H)$ with $N_H(\Phi(C))$ given by (1.9). Then $C(s) \in \mathcal{Q}_1(X)$ and by Hypothesis (1D), there is a defect preserving bijection ψ between $\mathrm{Irr}(N_X(C(s)), b)$ and $\mathrm{Irr}(N_X(\varphi(C(s))), b)$. By (1A) (a) and (1.19), ψ can be extended to a defect preserving bijection Ψ between $\mathrm{Irr}(N_H(C), B)$ and $\mathrm{Irr}(N_H(\Phi(C)), B)$. Thus $\mathcal{Q}_0(H)$ satisfies Hypothesis (1D) for B, and moreover, if $\mathcal{Q}(X)$ is τ-invariant, then so is $\mathcal{Q}_0(H)$. We may suppose $\mathcal{Q}_1(X) = \mathcal{Q}_2(X) = \emptyset$, so that $\mathcal{Q}(H) = \mathcal{S}(H)$.

Suppose $C \in \mathcal{Q}(H)$ with $N_H(C)$ given by (1.8). Let $\Sigma_C(i) = \mathrm{Irr}(N_X(C(i)), b) \wr \mathbf{S}(n_i)$, $\Sigma_C(i)^0 = \Omega_{C(i)}(0) \wr \mathbf{S}(n_i)$ and $\Sigma_C(i)^+ = \Sigma_C(i) \backslash \Sigma_C(i)^0$. If $C = C_0(H)$, then

$n = n_1$ and we define $\Omega_C(0) = \Omega_{C_0(X)}(0) \wr \mathbf{S}(n)$. Suppose

$$\Omega_C(0) = \prod_{i \in I_0} \Sigma_C(i)^0 \times \prod_{j \in I_1} \Sigma_C(j), \tag{1.20}$$

where $I_0 \cup I_1$ is a partition of $\{1, 2, \dots, t\}$. By reordering, we may suppose $I_0 = \{1, 2, \dots, k\}$ and $I_1 = \{k+1, \dots, t\}$. In addition, if $\varphi(C(i)) = C(i)$ for all maps φ, then by definition, $\Omega_{C(i)}(0) = \mathrm{Irr}(N_X(C(i)), b)$ and so $\Sigma_C(i)^0 = \Sigma_C(i)$. Thus we may suppose for each $i \in I_0$, there is some map φ such that $\varphi(C(i)) \neq C(i)$, and suppose $I_0 \neq \emptyset$. Let $\Sigma_C^+ = \prod_{i=k+1}^t \Sigma_C(i)$ and $\Omega_C(+) = \mathrm{Irr}(N_H(C), B) \backslash \Omega_C(0)$. Then $\Omega_C(+)$ is a disjoint union

$$\Omega_C(+) = \bigcup_{\ell=1}^{k} (\prod_{i=1}^{\ell-1} \Sigma_C(i) \times \Sigma_C(\ell)^+ \times \prod_{i=\ell+1}^{k} \Sigma_C(i)^0 \times \Sigma_C^+). \tag{1.21}$$

For simplicity of notation, we suppose each $\mathrm{Irr}(N_X(C(\ell)), b) = \Omega_{C(\ell)}(0) \cup \Omega_{C(\ell)}(\varphi_\ell)$. Since $\Omega_{C(\ell)}(0) \cap \Omega_{C(\ell)}(\varphi_\ell) = \emptyset$, it follows that there is a defect preserving bijection between $\Sigma_C(\ell)^+$ and the disjoint union

$$\bigcup_{1 \le n_{\ell,2} \le n_\ell} \ \bigcup_{n_{\ell,1} + n_{\ell,2} = n_\ell} \Omega_{C(\ell)}(0) \wr \mathbf{S}(n_{\ell,1}) \times \Omega_{C(\ell)}(\varphi_\ell) \wr \mathbf{S}(n_{\ell,2}),$$

and we may identify these two sets. Fix nonnegative integers $n_{\ell,1}, n_{\ell,2}$ such that $n_{\ell,1} + n_{\ell,2} = n_\ell$ and $n_{\ell,2} \ge 1$. Define a chain $\Phi(C)$ as (1.15) such that

$$N_H(\Phi(C)) = \prod_{i \neq \ell} N_X(C(i)) \wr \mathbf{S}(n_i) \times N_X(C(\ell)) \wr \mathbf{S}(n_{\ell,1}) \times N_X(\varphi_\ell(C(\ell))) \wr \mathbf{S}(n_{\ell,2}).$$

This is possible since $\varphi_\ell(C(\ell)) \neq C(\ell)$. Thus $|\Phi(C)| = |C| + 1$. Let

$$\begin{aligned} \Omega_{\Phi(C)}(0) = \prod_{i=1}^{\ell-1} \Sigma_C(i) \times \Omega_{C(\ell)}(0) \wr \mathbf{S}(n_{\ell,1}) \\ \times \Omega_{\varphi_\ell(C(\ell))}(0) \wr \mathbf{S}(n_{\ell,2}) \times \prod_{i=\ell+1}^{k} \Sigma_C(i)^0 \times \Sigma_C^+ \end{aligned}$$

and $\Omega_C(\Phi) = \prod_{i=1}^{\ell-1} \Sigma_C(i) \times \Omega_{C(\ell)}(0) \wr \mathbf{S}(n_{\ell,1}) \times \Omega_{C(\ell)}(\varphi_\ell) \wr \mathbf{S}(n_{\ell,2}) \times \prod_{i=\ell+1}^{k} \Sigma_C(i)^0 \times \Sigma_C^+$. Then

$$\Omega_C(+) = \Omega_C(0) \bigcup_{\Phi} \Omega_C(\Phi)$$

and by (1A) (a), the bijection ψ between $\Omega_{C(\ell)}(\varphi_\ell)$ and $\Omega_{\varphi_\ell(C(\ell))}(0)$ can be extended to a defect preserving bijection Ψ between $\Omega_C(\Phi)$ and $\Omega_{\Phi(C)}(0)$. Since $\Omega_{\Phi(C)}(0)$ is also of the form (1.20), it follows by induction on $|C|$ that for each chain $C' \in \mathcal{S}(H)$, $\Omega_{C'}(0)$ is of the form (1.20) and $\mathrm{Irr}(N_H(C'), B)$ has a required decomposition.

If $I_0 = \emptyset$, then $\Omega_C(0) = \mathrm{Irr}(N_H(C), B)$ and $\Phi(C') = C$ for some $C' \in \mathcal{S}(H)$. In addition, if $\mathcal{Q}(X)$ is τ-invariant, then so is $\mathcal{Q}(H)$. This completes the proof.

(b) The proof of part (b) follows easily by that of (1A) (b) and part (a) above. □

2. Central radical chains

Let $q = p^f$ be a power of a prime p distinct from the odd prime r, $\epsilon = +$ or $-$, and let e be the multiplicative order of ϵq modulo r and $a = a(q^e - \epsilon)$. In addition, let $\mathrm{GL}^\epsilon(n,q) = \mathrm{GL}(n,q)$ or $\mathrm{U}(n,q)$ according to whether $\epsilon = +$ or $-$. The radical subgroups of G are classified by [2] and [3]. We shall follow the notation of [2].

Given integers $\alpha \geq 0$ and $\gamma \geq 0$, let Z_α be the cyclic group of order $r^{a+\alpha}$, E_γ an extraspecial group of order $r^{2\gamma+1}$ and $Z_\alpha E_\gamma$ the central product of Z_α and E_γ over $\Omega_1(Z_\alpha) = Z(E_\gamma)$, where $Z(H)$ denotes the center of a finite group H. Then $Z_\alpha E_\gamma$ can be embedded as a subgroup of $\mathrm{GL}^\epsilon(er^{\alpha+\gamma}, q)$, and its image $R_{\alpha,\gamma}$ is determined uniquely by $Z_\alpha E_\gamma$ up to conjugacy in $\mathrm{GL}^\epsilon(er^{\alpha+\gamma}, q)$.

Given an integer $m \geq 1$, the image $R_{m,\alpha,\gamma}$ of $R_{\alpha,\gamma}$ under the m-fold diagonal mapping in $\mathrm{GL}^\epsilon(mer^{\alpha+\gamma}, q)$ given by

$$g \mapsto \begin{pmatrix} g & & & \\ & g & & \\ & & \ddots & \\ & & & g \end{pmatrix},$$

is also determined up to conjugacy. The center $Z(R_{m,\alpha,\gamma})$ is cyclic of order $r^{a+\alpha}$, so that $\Omega_a(Z(R_{m,\alpha,\gamma})) = \langle z \rangle$, where z is an element of order r^a in $\mathrm{GL}^\epsilon(mer^{\alpha+\gamma}, q)$. Moreover,

$$C_{\mathrm{GL}^\epsilon(mer^{\alpha+\gamma},q)}(z) \simeq \mathrm{GL}^\epsilon(mr^{\alpha+\gamma}, q^e),$$

so that z is a primary element of $\mathrm{GL}^\epsilon(mer^{\alpha+\gamma}, q)$, where a semisimple element is primary if it has a unique elementary divisor.

For each nonnegative integer c, let A_c denote the elementary abelian group of order r^c represented by its regular permutation representation. For any sequence $\mathbf{c} = (c_1, c_2, \ldots, c_\ell)$ of nonnegative integers, let $A_\mathbf{c}$ be the wreath product $A_{c_1} \wr A_{c_2} \wr \ldots \wr A_{c_\ell}$, $|\mathbf{c}| = c_1 + c_2 + \ldots + c_\ell$, and let

$$R_{m,\alpha,\gamma,\mathbf{c}} = R_{m,\alpha,\gamma} \wr A_\mathbf{c}$$

be the wreath product in $\mathrm{GL}^\epsilon(d,q)$, where $d = mer^{\alpha+\gamma+|\mathbf{c}|}$. Then $R_{m,\alpha,\gamma,\mathbf{c}}$ is uniquely determined up to conjugacy in $\mathrm{GL}^\epsilon(d,q)$, which is called a *basic* subgroup of $\mathrm{GL}^\epsilon(d,q)$.

Let $A(R_{m,\alpha,\gamma,\mathbf{c}})$ be the intersection of all maximal normal abelian subgroups of a basic subgroup $R_{m,\alpha,\gamma,\mathbf{c}}$ and $Q = \Omega_a(A(R_{m,\alpha,\gamma,\mathbf{c}}))$. As shown in the proof of [2, (4.1)] $A(R_{m,\alpha,\gamma,\mathbf{c}}) = Z(R_{m,\alpha,\gamma})^{r^{|\mathbf{c}|}}$, so that $Q = \langle z \rangle^{r^{|\mathbf{c}|}}$, where $z \in \mathrm{GL}^\epsilon(mer^{\alpha+\gamma}, q)$ is a primary element of order r^a. Thus Q is a characteristic subgroup of $R_{m,\alpha,\gamma,\mathbf{c}}$ and will be called the *primary subgroup* of $R_{m,\alpha,\gamma,\mathbf{c}}$.

(2A). *Let $G = \mathrm{GL}^\epsilon(n,q)$, and let V be the underlying space of G and R a radical subgroup of G. Then there exists a corresponding decomposition*

$$V = V_0 \perp V_1 \perp \cdots \perp V_t, \quad R = R_0 \times R_1 \times \cdots \times R_t \tag{2.1}$$

such that $R_0 = \langle 1_{V_0} \rangle$ and R_i is a basic subgroup of $\mathrm{GL}^\epsilon(V_i)$ for $i \geq 1$, where $\mathrm{GL}^+(V_i) = \mathrm{GL}(V_i)$ or $\mathrm{GL}^-(V_i) = \mathrm{U}(V_i)$ according to whether V_i is a linear or unitary space, and $V_i \perp V_j = V_i \oplus V_j$ when V is linear. Moreover, the extraspecial components of R_i have exponent r for $i \geq 1$.

Proof. The proof is given by [2, (4A)] and [3, (2B)]. □

Given $H \le G = \mathrm{GL}^\epsilon(n,q) = \mathrm{GL}^\epsilon(V)$, let $C_V(H)$ and $[V,H]$ be the subspaces of V generated by the vectors of V fixed and moved by H, respectively.

Let $A(R)$ be the intersections of all the maximal normal abelian subgroups of an r-subgroup $R \le G$, $P(R) = \Omega_a(A(R))$ and $M_G(R) = O_r(N_G(P(R)))$. Then $A(R)$ and $P(R)$ are characteristic subgroups of R, and

$$N_G(R) \le N_G(P(R)) \le N_G(M_G(R)).$$

If R has a decomposition (2.1), then $P(R)$ will be called the *primary subgroup* of R and a *primary* subgroup of G is a primary subgroup of a subgroup R with decomposition (2.1). If Q is a primary subgroup of G, then there exists a corresponding decomposition

$$\begin{aligned} V &= M_0 \perp M_1 \perp \cdots \perp M_s, \\ Q &= X_0 \times X_1 \times \cdots \times X_s, \end{aligned} \tag{2.2}$$

where $X_0 = \langle 1_{M_0}\rangle$, $X_i = \langle z_i \rangle$ for $i \ge 1$ such that $[M_i, \langle z_i\rangle] = M_i$ and z_i is a primary element of order r^a in $\mathrm{GL}^\epsilon(M_i)$. In particular, $X_i = R_{u_i,0,0}$, where u_i is an integer such that $u_i e = \dim M_i$ for $i \ge 1$. So Q has a decomposition (2.1) and

$$C_G(Q) = \mathrm{GL}^\epsilon(M_0) \times \prod_{i=1}^{s} \mathrm{GL}^\epsilon(u_i, q^e). \tag{2.3}$$

Let $M(u) = \bigoplus_i M_i$ and $X(u) = \prod_i X_i$, where i runs over the indices such that $X_i = R_{u,0,0}$. Then

$$N_G(Q) = \mathrm{GL}^\epsilon(M_0) \times \prod_{u \ge 1} N_{\mathrm{GL}^\epsilon(M(u))}(X(u)),$$

$$N_{\mathrm{GL}^\epsilon(M(u))}(X(u)) = N_{\mathrm{GL}^\epsilon(ue,q)}(R_{u,0,0}) \wr \mathbf{S}(t_u), \tag{2.4}$$

where t_u is the number of basic components $R_{u,0,0}$ in $X(u)$. Moreover,

$$N_{\mathrm{GL}^\epsilon(ue,q)}(R_{u,0,0}) \simeq \langle \mathrm{GL}^\epsilon(u,q^e), \tau_u \rangle, \tag{2.5}$$

where $\tau_u \in \mathrm{GL}^\epsilon(ue,q)$ has order e acting as a field automorphism on $\mathrm{GL}^\epsilon(u,q^e)$.

In the rest of this paper we suppose $\gcd(r, q-\epsilon) = 1$, so that $O_r(G) = \langle 1_V \rangle$. In particular, r is odd and $e \ge 2$.

(2B). *Let $\mathrm{GL}^\epsilon(u,q^e) \wr \mathbf{S}(v) \le \mathrm{GL}^\epsilon(uve,q)$ and $W = O_r(\mathrm{GL}^\epsilon(u,q^e) \wr \mathbf{S}(v))$. If $(q,\epsilon,r) \ne (2,+,3)$, then $W = (R_{u,0,0})^v$. If $(q,\epsilon,r) = (2,+,3)$, then $W = (R_{u,0,0})^v$ or $\mathbb{Z}_3 \wr \mathbb{Z}_3$ according to whether $(u,v) \ne (1,3)$ or $(u,v) = (1,3)$. In particular, $P(W) = (R_{u,0,0})^v$ and $N_{\mathrm{GL}^\epsilon(uv,q^e)}(W) = N_{\mathrm{GL}^\epsilon(uv,q^e)}((R_{u,0,0})^v)$.*

Proof. For a finite group K, $O_r(K \wr \mathbf{S}(v)) = O_r(K)^v$ except when $r = 3$, $O_3(K) = K$ and $v = 3$, in which case, $O_3(K \wr \mathbf{S}(3)) = K \wr \mathbb{Z}_3$. Thus (2B) follows when $r \ge 5$.

Suppose $r = 3$ and $O_3(\mathrm{GL}^\epsilon(u,q^e)) = \mathrm{GL}^\epsilon(u,q^e)$. Then $u = 1$, $3^a = p^{ef} - \epsilon$, $p^{ef} = 3^a + \epsilon$ is even and $p = 2$. If $\epsilon = +$, then $ef = 2e_1$ is even, $3^a = (2^{e_1}-1)(2^{e_1}+1)$ and so $e_1 = 1$. Thus $e = 2$, $f = 1$ and $q = 2$. If $\epsilon = -$ and $a = 2a_1 + 1$ is odd, then $ef = 2e_1 + 1$ is odd and

$$(3^{a_1}-1)(3^{a_1}+1) = 3^{2a_1} - 1 = 2^{ef} - 2 \cdot 3^{2a_1} = 2(2^{2e_1} - 3^{2a_1}),$$

which is impossible, since $2|(3^{a_1}-1)$ and $2|(3^{a_1}+1)$ when $a_1 \ge 1$. Similarly, if $a = 2a_1$, then $(3^{a_1}-1)(3^{a_1}+1) = 2^{ef}$, and since 4 is not a common factor of $3^{a_1}-1$

and $3^{a_1}+1$, either $3^{a_1}-1=2$ or $3^{a_1}+1=2$, which is impossible since $e \geq 2$ and 3 is not a divisor of $q+1$. This implies (2B). $\square$

We say that a radical chain

$$C:\ 1 < P_1 < \cdots < P_w \tag{2.6}$$

of $G = \mathrm{GL}^\epsilon(n,q) = \mathrm{GL}^\epsilon(V)$ is *central radical* if $P_k = M_{N(C_{k-1})}(P_k)$ for $1 \leq k \leq w$. By (2.4) and (2.5), P_1 is a primary subgroup of G and if $(q,\epsilon,r) \neq (2,+,3)$, then by (2B), $P_k = P(P_k)$ is a primary subgroup of $C_G(C_{k-1})$ for $k \geq 1$. A central radical chain C has the following properties:

(a) There exists a decomposition

$$V = U_0 \perp U_1 \perp \cdots \perp U_v \tag{2.7}$$

such that $P_j = Q_{j,0} \times Q_{j,1} \times \cdots \times Q_{j,v}$ for all $j \geq 1$, where either $Q_{j,i}$ is a primary subgroup of $\mathrm{GL}^\epsilon(U_i)$ or $Q_{j,i}$ is a direct product of a primary subgroup and some $(\mathbb{Z}_3 \wr \mathbb{Z}_3)^\beta$ for all $j \geq 1$ and $i \geq 0$. In the latter case, $(q,\epsilon,r) = (2,+,3)$. Moreover, for each k with $1 \leq k \leq v$, there exists j such that $Q_{j,k} \neq 1$.

(b) $Q_{1,1} \neq \langle 1_{U_1} \rangle$.

(c) $Q_{w,0} = \langle 1_{U_0} \rangle$ and $[\,U_i, Q_{w,i}\,] = U_i$ for all $i \geq 1$.

(d) If $Q_{j,i} = \langle 1_{U_i} \rangle$, then $Q_{k,\ell} = \langle 1_{U_\ell} \rangle$ for all $k \leq j$ and $\ell \geq i$.

(e) If $Q_{j,i} \neq \langle 1_{U_i} \rangle$, then $[\,U_i, Q_{j,i}\,] = U_i$.

(f) Given $i \geq 1$, if j is the smallest index such that $Q_{j,i} \neq \langle 1_{U_i} \rangle$, then $Q_{j,k} = \langle 1_{U_k} \rangle$ for all $k \geq i+1$.

We shall call $v = z(C)$ the *size* of C. Denote by $\mathcal{CR} = \mathcal{CR}(G)$ or $\mathcal{MR} = \mathcal{MR}(G)$ the set of all central radical chains of G according to whether $(q,\epsilon,r) \neq (2,+,3)$ or $(q,\epsilon,r) = (2,+,3)$. Then $\mathcal{CR}$ and $\mathcal{MR}$ are G-invariant subfamilies of $\mathcal{R}$.

Suppose C is central radical with decomposition (2.7). In order to compute $N_E(C)$ for an extension E of G, we always suppose $\sigma = \sigma_0 \times \sigma_1 \times \ldots \times \sigma_v$ is a field automorphism of G such that each σ_i is a field automorphism of $\mathrm{GL}^\epsilon(U_i)$ of order f or $2f$ according to whether $\epsilon = +$ or $-$. In addition, suppose $\iota = \iota_0 \times \iota_1 \times \ldots \times \iota_v$ such that each ι_i is the inverse-transpose of $\mathrm{GL}^\epsilon(U_i)$, so that $[\,\sigma_i, \iota_i\,] = 1$ for all i and ι is a field automorphism of order 2 when $\epsilon = -$. We may always suppose $E \leq G \rtimes \langle \sigma, \iota \rangle$ and $\mathrm{Out}(G) = \langle \sigma, \iota \rangle$.

For $1 \leq i \leq t$, let I_i be a finite subset of positive integers, $\mathbf{n}_i = \{n_{i,j} : j \in I_i\}$ a set of positive integers, $\mathbf{n} = \{\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_t\}$, $\mathbf{X}_i = \{X_{i,j} : j \in I_i\}$ and $\mathbf{X} = \{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_t\}$, where $X_{i,j}$ is a subgroup of $\mathbf{S}(n_{i,j})$. In addition, let K be a subgroup of $\mathrm{GL}^\epsilon(m,q)$. Then $K \wr X_{1,j}$ for any $j \in I_1$ is a subgroup of $\mathrm{GL}^\epsilon(mn_{1,j},q)$. Let $K \wr \mathbf{X}_1 = \prod_{j \in I_1} K \wr X_{1,j}$ and

$$K \wr \mathbf{X} = K \wr \mathbf{X}_1 \wr \mathbf{X}_2 \wr \ldots \wr \mathbf{X}_t$$

be the subgroups of $\mathrm{GL}^\epsilon(m|\mathbf{n}_1|,q)$ and $\mathrm{GL}^\epsilon(m|\mathbf{n}|,q)$, respectively, where $|\mathbf{n}_i| = \sum_{j \in I_i} n_{i,j}$ and $|\mathbf{n}| = \sum_{i=1}^t |\mathbf{n}_i|$. If $X_{i,j} = \mathbf{S}(n_{i,j})$ for all i,j, then we set

$$K \wr \mathbf{S}(\mathbf{n}) = K \wr \mathbf{X}. \tag{2.8}$$

Given $1 \leq i \leq t$ and $j \in I_i$, let $\mathbf{c}_{i,j}$ be a sequence of nonnegative integers, $n_{i,j} = r^{|\mathbf{c}_{i,j}|}$ and $X_{i,j}$ a basic subgroup $A_{\mathbf{c}_{i,j}}$ of $\mathbf{S}(r^{|\mathbf{c}_{i,j}|})$. We set

$$K \wr A_{\mathbf{z}} = K \wr \mathbf{X}, \tag{2.9}$$

where $\mathbf{z} = \{\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_t\}$ such that $\mathbf{z}_i = \{\mathbf{c}_{i,j} : j \in I_i\}$. In particular, $R_{m,\alpha,\gamma,\mathbf{c}} = R_{m,\alpha,\gamma} \wr A_{\mathbf{z}}$ with $\mathbf{z} = \{\mathbf{c}\}$, and if K is an r-subgroup, then so is $K \wr A_{\mathbf{z}}$.

Suppose the chain C given by (2.6) is central radical with decomposition (2.7). Then for $1 \le j \le w$

$$P_j = P_{j,0} \times Q_{j,1} \times \cdots \times Q_{j,i-1} \times Q_{j,i},$$

where $P_{j,0}$ is the trivial subgroup of $\mathrm{GL}^\epsilon(C_V(P_j))$ and $Q_{j,k} \ne 1$ for $1 \le k \le i$. By the property (f), $i \le j$ and by (d), $C_V(P_j) = U_0 + \sum_{k=i+1}^{v} U_k$. Thus $N(C_j) = \mathrm{GL}^\epsilon(C_V(P_j)) \times N_{j,1} \times \cdots \times N_{j,i}$, where $N_{j,k} = \bigcap_{\ell=1}^{j} N_{\mathrm{GL}^\epsilon(U_k)}(Q_{\ell,k})$ for $1 \le k \le i$. In particular,

$$N(C) = \mathrm{GL}^\epsilon(U_0) \times N_1 \times N_2 \times \cdots \times N_v, \tag{2.10}$$

where $N_k = \bigcap_{\ell=1}^{w} N_{\mathrm{GL}^\epsilon(U_i)}(Q_{\ell,k})$ for all $k \ge 1$. Moreover, if $w = 1$, then $v = w = 1$ and $N_1 = N_{\mathrm{GL}^\epsilon(U_1)}(Q_{1,1})$ is given by (2.4). If $w \ge 2$, then by (2.4) (with $e = 1$),

$$N_k = \prod_{i \in I_0'} \left\langle \tau_i, \left(\prod_{j \in I_0} \mathrm{GL}^\epsilon(m_j, q^e) \wr \mathbf{S}(\mathbf{w}_j) \right) \right\rangle \wr \mathbf{S}(i), \tag{2.11}$$

where $|\tau_i| = e$, I_0' and I_0 are sets of some positive integers, and $\mathrm{GL}^\epsilon(m_j, q^e) \wr \mathbf{S}(\mathbf{w}_j)$ is defined by (2.8) with $K = \mathrm{GL}^\epsilon(m_j, q^e)$ and $\mathbf{w}_j = \mathbf{n}$.

An r-subgroup D is called a *quasi-radical* subgroup of $G = \mathrm{GL}^\epsilon(n,q)$ if $D = \prod_{i=0}^{t} D_i$ such that D_0 is the trivial subgroup of $\mathrm{GL}^\epsilon(C_V(D))$ and $D_i = R_i \wr A_{\mathbf{z}_i}$ for $i \ge 1$, where R_i is a radical subgroup of $\mathrm{GL}^\epsilon(m_i, q^e)$ and $R_i \wr A_{\mathbf{z}_i}$ is defined by (2.9). By [1, Theorem 2] and induction,

$$A(D) = D_0 \times \prod_{i=1}^{t} (A(R_i))^{|\mathbf{z}_i|}$$

and $P(D)$ is a primary subgroup of $\mathrm{GL}^\epsilon(n,q)$, called the *primary* subgroup of D. Since $P(M_G(D)) = P(D)$ by (2B), it follows that

$$N_G(D) \le N_G(P(D)) = N_G(M_G(D)). \tag{2.12}$$

(2C). *Let $G = \mathrm{GL}^\epsilon(n,q) = \mathrm{GL}^\epsilon(V)$, and let C be the central radical chain* (2.6) *with $w \ge 1$ and decomposition* (2.7).

(a) *Fix $1 \le i \le v$. Let $N_i = \bigcap_{\ell=1}^{w} N_{\mathrm{GL}^\epsilon(U_i)}(Q_{\ell,i})$, and let W be a radical subgroup of N_i. Then W is quasi-radical, $S = M_{N_i}(W)$ is radical in N_i and $S \le W$. In particular, $Q_{w,i} \trianglelefteq S$ and $N_{N_i}(W) \le N_{N_i}(S)$. Moreover, if $S = Q_{w,i}$, then W is radical in $N_i' = \bigcap_{\ell=1}^{w-1} N_{\mathrm{GL}^\epsilon(U_i)}(Q_{\ell,i})$ and $N_{N_i}(W) = N_{N_i'}(W)$, where $N_1' = \mathrm{GL}^\epsilon(U_1)$ when $w = 1$.*

(b) *If R is a radical subgroup of $N(C)$, then R is quasi-radical. If $D = M_{N(C)}(R)$, then D is radical in $N(C)$, $P_w \trianglelefteq D \le R$, and $N_{N(C)}(R) \le N_{N(C)}(D)$. In addition, if $P_w \ne D$, then*

$$C' : \ P_0 < P_1 < \cdots < P_w < D$$

is a central radical chain of G, R is radical in $N(C')$, and $N_{N(C')}(R) = N_{N(C)}(R)$. If $P_w = D$, then R is radical in $N(C_{w-1})$ and $N_{N(C_{w-1})}(R) = N_{N(C)}(R)$.

Proof. (a). Let $H = N_i$ decompose as (2.11). By [14, 2.2, 2.3 and 2.5] and induction, we may suppose $H = \langle \tau, \mathrm{GL}^\epsilon(m,q^e) \wr \mathbf{S}(\mathbf{z}) \rangle$. Since $\gcd(e,r) = 1$ and $\mathrm{GL}^\epsilon(m,q^e) \wr \mathbf{S}(\mathbf{z}) \trianglelefteq H$ and since $W \le \mathrm{GL}^\epsilon(m,q^e) \wr \mathbf{S}(\mathbf{z})$, it follows by [14, 2.1] that we may suppose $H = \mathrm{GL}^\epsilon(m,q^e) \wr \mathbf{S}(\mathbf{z})$. By [14, 2.3 and 2.5] and induction, we may suppose $H = \mathrm{GL}^\epsilon(m,q^e)$. Thus W is quasi-radical. By (2.12), $O_r(N_{N_i}(S)) = S$

and S is radical in N_i, so that $Q_{w,i} \le S$. Since $N_{N_i}(W) \le N_{N_i}(S)$, it follows that W is radical in $N_{N_i}(S)$, and so $S = O_r(N_{N_i}(S)) \le W$.

If $S = Q_{w,i}$, then $N_i = N_{N'_i}(S)$ and by (2.12),

$$N_{N'_i}(W) = N_{\mathrm{GL}^\epsilon(U_i)}(W) \cap N'_i \le N_{\mathrm{GL}^\epsilon(U_i)}(S) \cap N'_i = N_i,$$

so that $N_{N'_i}(W) = N_{N_i}(W)$ and W is radical in N'_i.

(b). Suppose R is a radical subgroup of $N(C)$. By (2.10) and [14, Lemma 2.2], $R = R(0) \times \prod_{i=1}^v R(i)$, where $R(0)$ and $R(i)$ are radical subgroups of $\mathrm{GL}^\epsilon(U_0)$ and N_i, respectively for all $i \ge 1$. By (2A) and part (a) above, $R(0)$ and $R(i)$ are quasi-radical, so is R. Let $D(0) = M_{\mathrm{GL}^\epsilon(U_0)}(R(0))$ and $D(i) = M_{N_i}(R(i))$. Then $D = D(0) \times \prod_{i=1}^v D(i) = M_{N(C)}(R)$ and $D(0)$ and $D(i)$ are radical in $\mathrm{GL}^\epsilon(U_0)$ and N_i, respectively. So D is radical in $N(C)$, and the chain C' defined in (b) is central radical. Since $N_{N(C)}(R) \le N_{N(C)}(D)$, it follows that $N_{N(C)}(R) = N_{N_{N(C)}(D)}(R) = N_{N(C')}(R)$.

Finally, if $D = P_w$, then $R(0) = D(0) = \langle 1_{U_0} \rangle$ and $D(i) = Q_{w,i}$. By part (a), each $R(i)$ is radical in $N'_i = \bigcap_{\ell=1}^{w-1} N_{\mathrm{GL}^\epsilon(U_i)}(Q_{\ell,i})$ for $i \ge 1$, so that R is radical in $N(C_{w-1})$, $N_{N(C_{w-1})}(R) \le N_{N(C_{w-1})}(D) = N(C)$ and $N_{N(C_{w-1})}(R) = N_{N(C)}(R)$. This proves (2C). □

Remark. Suppose $(q, \epsilon, r) = (2, +, 3)$ and $C \in \mathcal{MR}(G)$ given by (2.6). Let

$$P(C) : 1 < P(P_1) < P(P_2) < \ldots < P(P_w)$$

and $\mathcal{CR} = \mathcal{CR}(G) = \{P(C) : C \in \mathcal{MR}(G)\}$. Then each subgroup of a chain in $\mathcal{CR}$ is a primary subgroup of G, the map $C \mapsto P(C)$ is a bijection between $\mathcal{MR}(G)$ and $\mathcal{CR}$ and by (2.12), $N_G(C) = N_G(P(C))$. Moreover, if E is an extension of G, then

$$N_E(C) = N_E(P(C)).$$

It follows that we can identify C with $P(C)$ and view $\mathcal{CR}$ as a subfamily of $\mathcal{R}$.

(2D). *Let* $G = \mathrm{GL}^\epsilon(n, q) = \mathrm{GL}^\epsilon(V)$, $B \in \mathrm{Blk}(G)$ *with defect* $\mathrm{d}(B) \ge 1$, $d \ge 0$ *an integer and* $U \le \mathrm{Out}(G)$. *Then*

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \mathrm{k}(N(C), B, d, U) = \sum_{C \in \mathcal{CR}/G} (-1)^{|C|} \mathrm{k}(N(C), B, d, U).$$

Proof. Let $\mathcal{M}$ be the complementary G-invariant subfamily $\mathcal{R} \backslash \mathcal{CR}$ of $\mathcal{CR}$ in $\mathcal{R}$. It suffices to show that

$$\sum_{C \in \mathcal{M}/G} (-1)^{|C|} \mathrm{k}(N(C), B, d, U) = 0.$$

Suppose $C \in \mathcal{M}$ is given by (1.1). Let $m = m(C) \ge 0$ be the largest integer such that $C_m \in \mathcal{CR}$ and $C_{m+1} \notin \mathcal{CR}$, so that $0 \le m \le w - 1$. Since P_{m+1} is radical in $N(C_m)$, it follows that P_{m+1} is quasi-radical, $D = M_{N(C_m)}(P_{m+1})$ is radical in $N(C_m)$, and $P_m \trianglelefteq D$. Define a map φ from $\mathcal{M}$ to itself such that

$$\varphi(C) : \begin{cases} 1 < P_1 < \ldots < P_m < D < P_{m+1} < \ldots < P_w & \text{if } P_m \ne D, \\ 1 < P_1 < \ldots < P_{m-1} < P_{m+1} < \ldots < P_w & \text{if } P_m = D. \end{cases}$$

By (2C), $N(C) = N(\varphi(C))$, $\varphi(\varphi(C)) = C$ and $|\varphi(C)| = |C| \pm 1$. Since $N_E(C) = N_E(\varphi(C))$ for an extension E of G, (2D) follows. □

3. More reductions

Let $\mathcal{F} = \mathcal{F}_q$ be the set of polynomials serving as elementary divisors for all semisimple elements of $G = \mathrm{GL}^\epsilon(n,q)$. If $G = \mathrm{GL}(n,q)$, then $\mathcal{F}$ consists of all monic irreducible polynomials over the field $\mathbb{F}_q$ of q elements with non-zero roots. Suppose $G = \mathrm{U}(n,q)$ and let $\Delta(T) = T^m + a_{m-1}T^{m-1} + \ldots + a_1T + a_0$ be a monic polynomial of $\mathbb{F}_{q^2}[T]$ with $a_0 \neq 0$, and $\tilde{\Delta}(T) = (a_0^{-1})^q T^m \Delta^q(T^{-1})$. Then

$$\mathcal{F}_1 = \{\Delta : \Delta \text{ is monic, irreducible}, \Delta \neq T, \Delta = \tilde{\Delta}\},$$
$$\mathcal{F}_2 = \{\Delta\tilde{\Delta} : \Delta \text{ is monic, irreducible}, \Delta \neq T, \Delta \neq \tilde{\Delta}\}$$

and $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$.

A semisimple element $s \in G$ has a primary decomposition $V = \sum_{\Gamma\in\mathcal{F}} V_\Gamma$, $s = \prod_{\Gamma\in\mathcal{F}} s_\Gamma$, where each $s_\Gamma \in \mathrm{GL}^\epsilon(V_\Gamma)$ has a unique elementary divisor $\Gamma \in \mathcal{F}$ of multiplicity $m_\Gamma(s)$. Let d_Γ be the degree of $\Gamma \in \mathcal{F}$, $\delta_\Gamma = d_\Gamma$ (respectively, $\epsilon_\Gamma = +$) except when $\epsilon = -$ and $\Gamma \in \mathcal{F}_2$ (respectively, $\Gamma \in \mathcal{F}_1$), in which case $\delta_\Gamma = \frac{1}{2}d_\Gamma$ (respectively, $\epsilon_\Gamma = -$), and let e_Γ be the multiplicative order of $\epsilon_\Gamma q^{\delta_\Gamma}$ modulo r. The Brauer pairs (R,b) of G have been labelled by Broué [4, (3.1)]. This labelling is by ordered triples (R,s,κ), where s is a semisimple r'-element of the dual group G^* of G, and $\kappa = \prod_{\Gamma\in\mathcal{F}} \kappa_\Gamma$ is a product of partitions κ_Γ such that each κ_Γ is an e_Γ-core. This labelling extends the labelling [10, (5D)], by Fong and Srinivasan for block $B \in \mathrm{Blk}(G)$ by ordered pair (s,κ). Since $G \simeq G^*$, we may identify G^* with G, so that $s \in G$.

Given $C \in \mathcal{CR}$ with a final subgroup P_w, let $C_V(C)$ be the fixed-point subspace $C_V(P_w)$ of P_w in V. Fix an integer $m \geq 1$, suppose V decomposes as (2.7) with $\dim U_1 = me$, and let

$$V(m) = U_1, \qquad V(m)' = U_0 \perp U_2 \perp \cdots \perp U_v,$$

so that $V = V(m) \perp V(m)'$. Set $G(m) = \mathrm{GL}^\epsilon(V(m))$ and $G(m)' = \mathrm{GL}^\epsilon(V(m)')$.

Let X be a primary radical subgroup of $G(m)$ with $[\,V(m),X\,] = V(m)$, and $\mathcal{S}(X)$ the subset of $\mathcal{CR}(G)$ consisting of all chains C such that its first non-trivial subgroup is $P_X = X \times \langle 1_{V(m)'}\rangle$. Then

$$N(C_1) = N(P_X) = N_X \times G(m)',$$

where $N_X = N_{G(m)}(X)$.

(3A). *In the notation above, suppose $B \in \mathrm{Blk}(G)$ with defect group $D(B) \neq 1$ such that $m \neq \dim[\,V, D(B)\,]$. Then*

$$\sum_{C\in\mathcal{S}(X)/N(P_X)} (-1)^{|C|}\mathrm{k}(N(C),B,d,U) = 0 \tag{3.1}$$

for all integers $d \geq 0$ and $U \leq \mathrm{Out}(G)$. If $m = \dim[\,V, D(B)\,]$, then we may suppose $C_V(C) = C_V(D(B)) = V(m)'$ for $C \in \mathcal{S}(X)$.

Proof. Let $\mathcal{CR}(N_X \times G(m)')$ and $\mathcal{CR}(N_X)$ be the subfamilies of $\mathcal{R}(N_X \times G(m)')$ and $\mathcal{R}(N_X)$, respectively consisting of chains C given by (1.1) with $P(P_i) = P_i$ for all $i \geq 1$. Here we have a similar identification to that in the remark after (2C) when $(q,\epsilon,r) = (2,+,3)$. If $C \in \mathcal{S}(X)$, then $C : 1 < P_X < P_2 < \ldots < P_w$ and $\varphi(C) : P_X < P_2 < \ldots < P_w$ is a chain of $\mathcal{CR}(N_X \times G(m)')$, $|C| = |\varphi(C)|+1$, $N(C) = N_{N_X\times G(m)'}(\varphi(C))$ and φ is a bijection between $\mathcal{S}(X)$ and $\mathcal{CR}(N_X \times G(m)')$.

Let π_m and π' be the projections of $N(P_X)$ onto N_X and $G(m)'$, respectively. Suppose $C \in \mathcal{CR}(N_X \times G(m)')$ is given by (1.1). Then $\pi_m(C) \in \mathcal{CR}(N_X)$, $\pi'(C) \in$

$\mathcal{CR}(G(m)')$ and $N(C) = N_{N_X}(\pi_m(C)) \times N_{G(m)'}(\pi'(C))$, where $\pi_m(C)$ and $\pi'(C)$ are defined as (1A). Conversely, if $C \in \mathcal{R}(N_X \times G(m)')$ such that $\pi_m(C) \in \mathcal{CR}(N_X)$ and $\pi'(C) \in \mathcal{CR}(G(m)')$, then by definition, $C \in \mathcal{CR}(N_X \times G(m)')$.

Given $C \in \mathcal{S}(X)$, let $C_X = \pi_m(\varphi(C))$ and $C' = \pi'(\varphi(C))$, so $C(C) = C_{N_X}(C_X) \times C_{G(m)'}(C')$ and

$$N(C) = N_{N_X \times G(m)'}(\varphi(C)) = N_{N_X}(C_X) \times N_{G(m)'}(C').$$

Let (s, κ) be the label of B. If $b(C) \in \mathrm{Blk}(N(C)|B)$, then $b(C) = b(C)_X \times b(C)'$, where $b(C)_X$ and $b(C)'$ are blocks of $N_{N_X}(C_X)$ and $N_{G(m)'}(C')$, respectively. Let $b_X \in \mathrm{Blk}(C_{N_X}(C_X))$ and $b' \in \mathrm{Blk}(C_{G(m)'}(C'))$ such that $b(C)_X = b_X^{N_{N_X}(C_X)}$ and $b(C)' = {b'}^{N_{G(m)'}(C')}$. If $b = b_X \times b'$, then $b^G = B$, so that (P_w, b) is a Brauer B-pair. Let $(P_w, s_m \times s'_m, \kappa_m \times \kappa'_m)$ be the label of the pair (P_w, b), where $s_m \in C_{N_X}(C_X)$ and $s'_m \in C_{G(m)'}(C')$. By the Broué-Puig theorem, [4, (3.2)], s and $s_m \times s'_m$ are conjugate in $\mathrm{GL}^\epsilon(V)$. Since $X \leq O_r(C_{N_X}(C_X)) \leq D(b_X)$, it follows that $\kappa_m = -$ is the empty set and $\kappa = \kappa'_m$. Let B_m and B'_m be the blocks of $G(m)$ and $G(m)'$ labelled by $(s_m, -)$ and (s'_m, κ), respectively. Then $b_X^{G(m)} = b(C)_X^{G(m)} = B_m$ and ${b'}^{G(m)'} = {b(C)'}^{G(m)'} = B'_m$. The blocks B_m and B'_m depend only on the decomposition $s_m \times s'_m$ of s in $G(m) \times G(m)'$ and not on the choice of $C \in \mathcal{S}(X)$.

Let $\sigma' = \sigma_0 \times \sigma_2 \times \sigma_3 \times \ldots \times \sigma_v$ and $\iota' = \iota_0 \times \iota_2 \times \iota_3 \times \ldots \times \iota_v$, so that $\sigma = \sigma_1 \times \sigma'$ and $\iota = \iota_1 \times \iota'$. Suppose $U = \langle \gamma, \rho \rangle \leq \mathrm{Out}(G) = \langle \sigma, \iota \rangle$, where $\gamma = \sigma^\ell$ and $\rho = \iota^i$ for some integers ℓ and i. Let $\gamma_X = \sigma_1^\ell$, $\gamma' = {\sigma'}^\ell$, $\rho_X = \iota_1^i$, $\rho' = {\iota'}^i$, $U_X = \langle \gamma_X, \rho_X \rangle$ and $U' = \langle \gamma', \rho' \rangle$. Since σ_1 and ι_1 are field and the inverse-transpose maps of $\mathrm{GL}^\epsilon(V(m))$, respectively, we may suppose they stabilize X, so that $\sigma_1, \iota_1 \in \mathrm{Aut}(N_X)$. It follows by (1A) (b) that

$$\begin{aligned}
&\sum_{C \in \mathcal{S}(X)/N(P_X)} (-1)^{|C|} \mathrm{k}(N(C), B, d, U) \\
(3.2) \qquad &= - \sum_{u+t=d} \sum_{C_X} (-1)^{|C_X|} \mathrm{k}(N_{N_X}(C_X), B_m, u, U_X) \\
&\quad \times \sum_{C'} (-1)^{|C'|} \mathrm{k}(N_{G(m)'}(C'), B'_m, t, U'),
\end{aligned}$$

where C_X and C' run over $\mathcal{CR}(N_X)/N_X$ and $\mathcal{CR}(G(m)')/G(m)'$, respectively.

Since $D(B_m)$ and $D(B'_m)$ can be viewed as Sylow r-subgroups of $C_{G(m)}(s_m)$ and $C_{G(m)'}(s'_m)$, respectively, we may suppose

$$P_X \leq D(B_m) \times D(B'_m) \leq D(B).$$

In addition, we may suppose $Z(D(B)) \leq Z(D(b_X)) \times Z(D(b'))$ and $D(b') \leq D(B'_m)$. Thus B'_m has defect 0 if and only if $m = \dim[\,V, D(B)\,]$.

If $m \neq \dim[\,V, D(B)\,]$, then $m < \dim[\,V, D(B)\,]$ and $\dim V(m)' < \dim V$. If $\dim V(m)' \leq e$, then $\mathrm{GL}^\epsilon(V(m)')$ has a cyclic Sylow r-subgroup and Dade's invariant conjecture for B'_m follows by [8, Theorem 7.11]. By induction on $\dim V$,

$$\sum_{C \in \mathcal{CR}(G(m)')/G(m)'} (-1)^{|C|} \mathrm{k}(N_{G(m)'}(C), B'_m, t, U') = 0$$

for all integers $t \geq 0$. Thus (3.1) follows by (3.2).

If $m = \dim[V, D(B)]$, then B'_m has defect 0, $D(B'_m) = 1$ and $V(m)' = C_V(P_X) = C_V(D(B))$. We may suppose each non-trivial subgroup of a chain $C \in \mathcal{S}(X)$ is a subgroup of $D(B)$, so $C_V(C) = C_V(D(B)) = V(m)'$. This proves (3A). □

Let $\mathcal{CR}(B) = \{C \in \mathcal{CR} : \mathrm{Blk}(N(C)|B) \neq \emptyset\}$, and let

$$\mathcal{CR}^*(B) = \mathcal{CR}(B) \setminus \left(\bigcup_X \mathcal{S}(X) \right)$$

and $\mathcal{CR}^*(G) = \bigcup_B \mathcal{CR}^*(B)$, where $P_X = X \times \langle 1_{C_V(P_X)} \rangle$ runs over primary subgroups of G such that $\dim C_V(P_X) \neq \dim C_V(D(B))$, and B runs over all blocks of G with positive defect. If $C \in \mathcal{CR}^*(B)$, then by (3A), we may suppose $C_V(C) = C_V(D(B))$.

Let χ be an irreducible unipotent character of $G = \mathrm{GL}^{\epsilon}(V) = \mathrm{GL}^{\epsilon}(n,q)$. If $\epsilon = +$, then χ is rational and ι induces the dual map on $\mathrm{Irr}(G)$, so that ι stabilizes χ. Similarly, the field automorphism σ also stabilizes χ. If $\epsilon = -$, then χ is a linear combination of some $R_T^G(1)$ with rational coefficients, where each T is a σ-invariant maximal torus of G. So σ also stabilizes χ. In both cases, $\chi^\tau = \chi$ for $\tau \in G \rtimes \langle \sigma, \iota \rangle$.

Suppose $\chi_{s,\mu}$ is an irreducible character of G labelled by (s, χ_μ), where χ_μ is an irreducible unipotent character of $C_G(s)$. If $\tau \in \langle \sigma, \iota \rangle$, then we may suppose τ stabilizes χ_μ, $\chi^\tau_{s,\mu}$ is labelled by (s^τ, χ_μ) and so

$$\chi^\tau_{s,\mu} = \chi_{s^\tau,\mu}. \tag{3.3}$$

In addition, $\mathrm{d}(\chi_{s,\mu}) = a(G) - a(\chi_{s,\mu})$ and $a(\chi_{s,\mu}) = a(|G\colon C_G(s)|) + a(\chi_\mu)$. Thus

$$\mathrm{d}(\chi_{s,\mu}) = a(C_G(s)) - a(\chi_\mu) = \mathrm{d}(\chi_\mu). \tag{3.4}$$

Let B be a block of $G = \mathrm{GL}^{\epsilon}(V)$ labelled by (s, κ), D a defect group of B, $V_0 = C_V(D)$ and $V_+ = [V, D]$. Then $D = D_0 \times D_+$ and we may suppose $s \in C_G(D) = \mathrm{GL}^{\epsilon}(V_0) \times C_+$, where $D_0 = \langle 1_{V_0} \rangle$, $D_+ \leq G_+ = \mathrm{GL}^{\epsilon}(V_+)$ and $C_+ = C_{\mathrm{GL}^{\epsilon}(V_+)}(D_+)$. Thus $s = s_0 \times s_+$ and we may suppose $\sigma = \sigma_0 \times \sigma_+$, $\iota = \iota_0 \times \iota_+$ such that $s^\sigma = s_0^{\sigma_0} \times s_+^{\sigma_+}$ and $s^\iota = s_0^{\iota_0} \times s_+^{\iota_+}$, where $s_0 \in \mathrm{GL}^{\epsilon}(V_0)$ and $s_+ \in C_+$. Suppose $U \leq \mathrm{Out}(G) = \langle \sigma, \iota \rangle$, so that $U = \langle x, \rho \rangle$, where $x = \sigma^\ell$ and $\rho = \iota^i$ for some integers $\ell, i \geq 0$. Let $x_0 = \sigma_0^\ell$, $\rho_0 = \iota_0^i$, $x_+ = \sigma_+^\ell$, $\rho_+ = \iota_+^i$, $U_0 = \langle x_0, \rho_0 \rangle$ and $U_+ = \langle x_+, \rho_+ \rangle$. Then $U_0 \leq \mathrm{Out}(\mathrm{GL}^{\epsilon}(V_0)) = \langle \sigma_0, \iota_0 \rangle$ and $U_+ \leq \mathrm{Out}(G_+) = \langle \sigma_+, \iota_+ \rangle$.

(3B). *In the notation above, let $B(0)$ and $B(+)$ be blocks of $G_0 = \mathrm{GL}^{\epsilon}(V_0)$ and G_+ labelled by (s_0, κ) and $(s_+, -)$, respectively. Then there exists a defect preserving bijection between $\mathrm{Irr}(B)$ and $\mathrm{Irr}(B(+))$. Moreover, there exists a bijection φ between $\mathcal{CR}^*(B)$ and $\mathcal{CR}^*(B(+))$ such that $N(C) = G_0 \times N_{G_+}(\varphi(C))$ for $C \in \mathcal{CR}^*(B)$ with $|C| \geq 1$. In addition,*

$$\mathrm{k}(N(C), B, d, U) = \mathrm{k}(N_{G_+}(\varphi(C)), B(+), d, U_+) \tag{3.5}$$

for any integers $d \geq 0$ and $U \leq \mathrm{Out}(G)$.

Proof. Let $\chi_{sy,\mu}$ be an irreducible character of $\mathrm{Irr}(B)$ labelled by (sy, χ_μ), where y is an r-element of $C_G(s)$. Then $y = 1_{V_0} \times y_+$ for some element $y_+ \in C_{G_+}(s_+)$, and $sy = \prod_\Gamma (sy)_\Gamma$ and $\mu = \prod_\Gamma \mu_\Gamma$, where μ_Γ is a partition of $m_\Gamma(sy)$. Suppose κ_Γ and $Q(\mu_\Gamma)$ are the e_Γ-core and e_Γ-quotient of μ_Γ, respectively. Then $\kappa = \prod_\Gamma \kappa_\Gamma$ and μ_Γ is uniquely determined by κ_Γ and $Q(\mu_\Gamma)$. Let $(\mu_\Gamma)_+$ be a partition of $m_\Gamma(s_+y_+)$ with $Q(\mu_\Gamma)$ as its e_Γ-quotient and $-$ as its e_Γ-core, and let $\mu_+ = \prod_\Gamma (\mu_\Gamma)_+$. Then the character $\xi_{s_+y_+,\mu_+}$ of G_+ labelled by (s_+y_+, χ_{μ_+}) is a character of $\mathrm{Irr}(B(+))$ and each character of $\mathrm{Irr}(B(+))$ is of this form. Define $\Psi(\xi_{s_+y_+,\mu_+}) = \chi_{sy,\mu}$. By [10, Theorem (7A)], Ψ is a bijection between $\mathrm{Irr}(B(+))$ and $\mathrm{Irr}(B)$. In addition, by (3.4) and the hook-length formula [10, (1.15)], Ψ is defect preserving. If $\tau = \tau_0 \times \tau_+ \in \mathrm{Out}(G)$ for some $\tau_0 \in \mathrm{Out}(\mathrm{GL}^\epsilon(V_0))$ and $\tau_+ \in \mathrm{Out}(G_+)$, then $\chi^\tau_{sy,\mu} = \chi_{(sy)^\tau,\mu}$ and $\xi^{\tau_+}_{s_+y_+,\mu_+} = \xi_{(s_+y_+)^{\tau_+},\mu_+}$. It follows that

$$\Psi(\xi)^\tau = \Psi(\xi^{\tau_+})$$

for any $\tau \in \langle\sigma, \iota\rangle$, so that $\mathrm{k}(G, B, d, U) = \mathrm{k}(G_+, B(+), d, U_+)$.

Suppose $C \in \mathcal{CR}^*(B)$ is given by (1.1) with $|C| \geq 1$. We may suppose $P_t = \langle 1_{V_0}\rangle \times P_+(t)$ for all $t \geq 1$, where $P_+(t) \leq G_+$. Define $\varphi(C) : 1 < P_+(1) < P_+(2) < \ldots < P_+(w)$. Then $\varphi(C) \in \mathcal{CR}^*(B(+))$ and $N(C) = G_0 \times N_{G_+}(\varphi(C))$. Since $\mathrm{Irr}(B(0))$ has exactly one character of defect 0, it follows that $\mathrm{k}(G_0, B(0), t) = 1$ or 0 according to whether $t = 0$ or $t \neq 0$. This and a similar proof to that of (1.5) imply (3.5). □

Let B be a block of $G = \mathrm{GL}^\epsilon(n, q) = \mathrm{GL}^\epsilon(V)$ with defect group D. By (3B), we may suppose $[V, D] = V$, so that $n = ue$ and B is labelled by $(s, -)$. Let $V = \sum_\Gamma V_\Gamma$, $s = \prod_\Gamma s_\Gamma$ be the primary decomposition of s and $L = C_G(s) = \prod_\Gamma L_\Gamma$, where $L_\Gamma = \mathrm{GL}^{\epsilon_\Gamma}(m_\Gamma(s), q^{\delta_\Gamma}) = C_{\mathrm{GL}^\epsilon(V_\Gamma)}(s_\Gamma)$. We may choose the decomposition such that σ and ι stabilizes each V_Γ. Thus $\sigma = \prod_\Gamma \sigma_\Gamma$ and $\iota = \prod_\Gamma \iota_\Gamma$, where σ_Γ and ι_Γ induce fields and the inverse-transpose maps on $\mathrm{GL}^\epsilon(V_\Gamma)$, respectively. Suppose U is a subgroup of $\mathrm{Out}(G) = \langle\sigma, \iota\rangle$, so that $U = \langle x, \rho\rangle$ with $x = \sigma^\ell$ and $\rho = \iota^i$ for some integers $\ell, i \geq 0$. Let $x_\Gamma = \sigma^\ell_\Gamma$, $\rho_\Gamma = \iota^i_\Gamma$ and $U_\Gamma = \langle x_\Gamma, \rho_\Gamma\rangle$. Then $U_\Gamma \leq \mathrm{Out}(\mathrm{GL}^\epsilon(V_\Gamma)) = \langle\sigma_\Gamma, \iota_\Gamma\rangle$.

Let π_Γ be the natural projection onto V_Γ and $C \in \mathcal{CR}^*(B)$. We may suppose D is a Sylow r-subgroup of L and each subgroup of C is a subgroup of D. Thus C is an r-subgroup chain of L. Let $\mathcal{Y}_\Gamma = \{\pi_\Gamma(P_\ell) : 0 \leq \ell \leq w\}$ and relabel the subgroups of $\mathcal{Y}_\Gamma$ such that $\mathcal{Y}_\Gamma = \{1 = W_0 < W_1 < \ldots < W_{w'}\}$. Define $\pi_\Gamma(C) : 1 = W_0 < W_1 < \ldots < W_{w'}$, so that $\pi_\Gamma(C)$ is an r-subgroup chain of L_Γ.

Choose a representative set $\mathcal{CR}^*(B)/G$ for the G-orbits in $\mathcal{CR}^*(B)$ such that each chain $C \in \mathcal{CR}^*(B)/G$ is also an r-subgroup chain of D. Let B_L be the block of L labelled by $(s, -)$. Then $B_L = \prod_\Gamma B_\Gamma$ for some $B_\Gamma \in \mathrm{Blk}(L_\Gamma)$ labelled by $(s_\Gamma, -)$.

(3C). *In the notation above, suppose $\mathcal{L}(B_\Gamma)$ is the set of r-subgroup chains of L_Γ consisting of all L_Γ-conjugates of $\pi_\Gamma(C)$ for all $C \in \mathcal{CR}^*(B)/G$. Then*

$$\sum_{C\in\mathcal{CR}^*(B)/G} (-1)^{|C|}\mathrm{k}(N(C), B, d, U)$$
$$= \sum_{u_\Gamma}\Bigg(\prod_{\Gamma'}\Big(\sum_{C\in\mathcal{L}(B_{\Gamma'})/L_{\Gamma'}} (-1)^{|C|}\mathrm{k}(N_{L_{\Gamma'}}(C), B_{\Gamma'}, u_\Gamma, U_{\Gamma'})\Big)\Bigg),$$

where both Γ and Γ' run over elementary divisors of s, and u_Γ runs over nonnegative integers such that $\sum_\Gamma u_\Gamma = d$.

Proof. Suppose $C \in \mathcal{CR}^*(B)/G$ has a non-trivial final subgroup Q. Then $Q \leq D \leq L$, Q is a primary subgroup of G and $[V, Q] = V$. In addition, $C_G(Q)$ and $N(Q)$ are given by (2.3) and (2.4), respectively, with $M_0 = C_V(Q) = 0$. Thus $C_G(Q) = C(C)$ and $N(C) \leq N(Q)$. Let $H = C_G(Q)$ and $K = C_H(s)$, so that $K = C_L(Q)$.

Suppose $B_H \in \mathrm{Blk}(H|B)$ and $B_K \in \mathrm{Blk}(K|B)$. Then both blocks are labelled by $(s, -)$, and R^H_K is a perfect isometry between $\mathrm{Irr}(B_K)$ and $\mathrm{Irr}(B_H)$. Let $\psi_{sy,\mu} \in \mathrm{Irr}(B_K)$, $\varphi_{sy,\mu'} = R^H_K(\psi_{sy,\mu}) \in \mathrm{Irr}(B_H)$, and let $N_L(\psi_{sy,\mu})$ and $N_G(\varphi_{sy,\mu'})$ be the stabilizers of $\psi_{sy,\mu}$ and $\varphi_{sy,\mu'}$ in $N_L(C)$ and $N(C)$, respectively, where $y \in D$. If $z \in N_L(C)$ stabilizes $\psi_{sy,\mu}$, then $s^z = s$, $K^z = K$, $(R_K^H)^z = R_K^H$, and $(\varphi_{sy,\mu'})^z = \varphi_{sy,\mu'}$. Conversely, if $z \in N(C)$ stabilizes $\varphi_{sy,\mu'}$, then $(sy)^z$ is H-conjugate to sy, so that $s^{zh} = s$ for some $h \in H$. Thus $zh \in N_L(C)$, $(R_K^H)^{zh} = R_K^H$ and $(\psi_{sy,\mu})^{zh} = \psi_{sy,\mu}$. It follows that

$$N_G(\varphi_{sy,\mu'})/H \simeq N_L(\psi_{sy,\mu})/K.$$

By equalities (2.3) and (2.4), $\varphi_{sy,\mu'}$ and $\psi_{sy,\mu}$ have extensions to $N_{N(Q)}(\varphi_{sy,\mu'})$ and $N_{N_L(Q)}(\psi_{sy,\mu})$, and so they have extensions to $N_G(\varphi_{sy,\mu'})$ and $N_L(\psi_{sy,\mu})$, respectively. Since σ is a field automorphism and ι is the inverse-transpose map, we may suppose $s^\tau \in \langle s \rangle$ and $Q^\tau = Q$ for $\tau = \sigma$ or ι, so $K^\tau = K$ and $(R_K^H)^\tau = R_K^H$. Moreover, since we only consider the action of τ on $\mathrm{Irr}(B_H)$ and $\mathrm{Irr}(B_K)$, it follows by (2.4) and (2.5) that we may suppose τ commutes with each element of $N(Q)/H$ and $N_L(Q)/K$.

Since a defect group of B_H is conjugate to a Sylow r-subgroup of $C_H(s) = K$, it follows that B_H and B_K have the same defect. Now R_K^H, $\mathrm{Ind}_{N_G(\varphi_{sy,\mu'})}^{N(C)}$ and $\mathrm{Ind}_{N_L(\psi_{sy,\mu})}^{N_L(C)}$ are defect preserving. By Clifford theory, there is a defect preserving bijection Ψ between $\mathrm{Irr}(N(C),B)$ and $\mathrm{Irr}(N_L(C),B_L)$ such that $\Psi(\chi^\tau) = \Psi(\chi)^\tau$ for each $\chi \in \mathrm{Irr}(N(C),B)$ and $\tau \in \mathrm{Out}(G)$, so

$$\mathrm{k}(N(C),B,d,U) = \mathrm{k}(N_L(C),B_L,d,U). \tag{3.6}$$

Since L is a regular subgroup of G, R_L^G is a perfect isometry and $(R_L^G)^\tau = R_L^G$ for $\tau \in \mathrm{Out}(G)$, so that (3.6) still holds when $|C| = 0$.

Suppose $C \in \mathcal{CR}^*(B)/G$ is given by (2.6). For $1 \le i \le w$, let $(P_i)_\Gamma = \pi_\Gamma(P_i)$ be the subgroup of L_Γ. Then $P_i \le \prod_\Gamma (P_i)_\Gamma \le \prod_\Gamma (P_{i+1})_\Gamma$ and $N_L(P_i) \le \prod_\Gamma N_{L_\Gamma}((P_i)_\Gamma)$.

Let $\mathcal{M}$ be the subfamily of $\mathcal{CR}^*(B)$ consisting of all G-conjugates of chains $C \in \mathcal{CR}^*(B)/G$ such that $P_i \ne \prod_\Gamma (P_i)_\Gamma$ for some i. Given $C \in \mathcal{M}$, let $v = v(C)$ be the largest integer such that $P_v \ne \prod_\Gamma (P_v)_\Gamma$. Let

$$C : 1 < P_1 < \ldots < P_v < P_{v+1} < \ldots < P_w \tag{3.7}$$

be a chain of $\mathcal{M}$ with $v = v(C)$, so that $P_{v+1} = \prod_\Gamma (P_{v+1})_\Gamma$ and $\prod_\Gamma (P_v)_\Gamma \le P_{v+1}$. Let $\mathcal{M}^0$ and $\mathcal{M}^+$ be the subfamilies of $\mathcal{M}$ consisting of chains C such that $\prod_\Gamma (P_v)_\Gamma = P_{v+1}$ and $\prod_\Gamma (P_v)_\Gamma < P_{v+1}$, respectively. If $C \in \mathcal{M}^0$ is given by (3.7), then define

$$g(C) : 1 < P_1 < \ldots < P_v < P_{v+2} < \ldots < P_w,$$

so that $g(C) \in \mathcal{M}^+$ and $N_L(C) = N_L(g(C))$. If $C \in \mathcal{M}^+$ is given by (3.7), then define

$$h(C) : 1 < P_1 < \ldots < P_v < \prod_\Gamma (P_v)_\Gamma < P_{v+1} < \ldots < P_w,$$

so that $h(C) \in \mathcal{M}^0$ and $N_L(C) = N_L(h(C))$. In addition, since $\tau = \prod_\Gamma \tau_\Gamma$ for each $\tau \in \langle \sigma, \iota \rangle$, it follows that $N_{\langle \sigma,\iota \rangle}(C) = N_{\langle \sigma,\iota \rangle}(g(C))$ or $N_{\langle \sigma,\iota \rangle}(h(C))$ according to whether $C \in \mathcal{M}^0$ or $\mathcal{M}^+$. Since gh and hg are identities, it follows that

$$\sum_{C \in \mathcal{M}/L} (-1)^{|C|} \mathrm{k}(N_L(C),B,d,U) = 0$$

for all d and $U \leq \langle \sigma, \iota \rangle$. We may suppose $C \notin \mathcal{M}$, so that $P_i = \prod_\Gamma (P_i)_\Gamma$ for all i. Thus ($3C$) follows by (3.6) and the same proof as that of (1A) (b) with some obvious modifications. □

4. The proof of the invariant conjecture

Let $\mathcal{F}_q^k = \mathcal{F}_q^k(r)$ be the subset of $\mathcal{F} = \mathcal{F}_q$ consisting of all polynomials whose roots have multiplicative order r^k. In addition, let $\mathcal{F}_q(r,a) = \mathcal{F}_q^0 \cup \mathcal{F}_q^1 \cup \ldots \cup \mathcal{F}_q^a$ and $\mathcal{F}_q(r) = \bigcup_{k \geq 0} \mathcal{F}_q^k(r)$. Then $\mathcal{F}_q^0 = \{T-1\}$ and by [12, (1) and (2)], $|\mathcal{F}_q(r,a)| = 1 + (r^a - 1)/e$ and $|\mathcal{F}_q^{a+i}| = (r^a - r^{a-1})/e$ for $i \geq 1$ (note that the number e here is the e' in [11] and [12]). Moreover, by [11, (1.5)], $d_\Gamma = e$ or er^i according to whether $\Gamma \in \mathcal{F}_q(r,a) \backslash \{T-1\}$ or $\mathcal{F}_q^{a+i}$, and each elementary divisor of an r-element is an element of $\mathcal{F}_q(r)$. Following the notation of [12], we write

$$\mathcal{F}_q(r,a) = \{\Delta(0,j) : 0 \leq j \leq (r^a - 1)/e\}, \qquad \Delta(0,0) = T-1,$$

and for $i \geq 1$

$$\mathcal{F}_q^{a+i}(r) = \{\Delta(i,j) : 1 \leq j \leq (r^a - r^{a-1})/e\}.$$

A sequence $(w_i) = (w_0, w_1, w_2, \ldots, w_\ell)$ of nonnegative integers is called an *r-weight sequence* of u if

$$\sum_{i=0}^{\ell} w_i r^i = u.$$

Given such a sequence (w_i), let $\Omega_G((w_i))$ be the subset of $\mathrm{Irr}(B_0)$ consisting of irreducible characters $\chi_{y,\mu}$ of $G = \mathrm{GL}^\epsilon(ue, q)$ such that

$$\sum_{j=0}^{1+(r^a-1)/e} m_{0,j} = w_0, \qquad \sum_{j=1}^{(r^a - r^{a-1})/e} m_{i,j} = w_i$$

for $i \geq 1$ and $\mu = \prod_{\ell,j} \mu_{\ell,j}$, where $m_{\ell,j} = m_{\Delta(\ell,j)}(y)$ and $\mu_{\ell,j}$ is a partition of $m_{\ell,j}$ for all ℓ and j. As shown in the proof of [12, Proposition 6]

$$|\Omega_G((w_i))| = \mathrm{k}(e + (r^a - 1)/e, w_0) \prod_{i \geq 1} \mathrm{k}((r^a - r^{a-1})/e, w_i), \tag{4.1}$$

where $\mathrm{k}(s,t)$ is the number of s-tuples $(\mu_1, \ldots, \mu_s)$ of partitions such that $\sum_{j=1}^{s} |\mu_j| = t$. Moreover,

$$\mathrm{Irr}(B_0) = \bigcup_{(w_i)} \Omega_G((w_i)) \qquad \text{(disjoint)}, \tag{4.2}$$

where (w_i) runs over all r-weight sequences of u. An r-weight sequence (w_i) determines a unique partition $\lambda = (\lambda_0^{\beta_0}, \ldots, \lambda_\ell^{\beta_\ell})$ of u, called an *r-weight partition*, where $\lambda_i = r^i$ and $\beta_i = w_i$ for all $i \geq 0$.

(4A). *Let $K = \mathrm{GL}^\epsilon(r^j, q^e)$, $H = \mathrm{GL}^\epsilon(tr^j, q^e)$ and $K \wr \mathbf{S}(t) \leq H$, where $j \geq 0$ and $t \geq 1$. In addition, let $\mathcal{X} = \{\xi_\Delta : \Delta \in (\mathcal{F}_{q^e}^{a+j})^*\}$, where $(\mathcal{F}_{q^e}^{a+j})^* = \mathcal{F}_{q^e}^{a+j}$ or $\mathcal{F}_{q^e}(r,a)$ according to whether $j \geq 1$ or $j = 0$, and ξ_Δ is the irreducible character of the principal block $B_0(K)$ labelled by $(\Delta, 1)$. Thus $\mathcal{X} = \mathrm{Irr}(B_0(K))$ when $j = 0$. Then there is a defect preserving bijection Ψ between $\Omega_H((w_i))$ and $\mathcal{X} \wr \mathbf{S}(t)$, where $w_i = \delta_{ij} t$ and $\mathcal{X} \wr \mathbf{S}(t)$ is defined by (1A). Moreover, Ψ is compatible with $\tau \in \mathrm{Out}(H)$.*

Proof. If $\chi = \chi_{y,\mu} \in \Omega_H((w_i))$, then $\sum_{\Delta\in(\mathcal{F}_{q^e}^{a+j})^*} m_\Delta(y) = t$ and $\mu = \prod_\Delta \mu_\Delta$, where $\mu_\Delta \vdash m_\Delta(y)$. Let $\xi \in \mathrm{Irr}(K^n)$ such that $m_{\xi_\Delta}(\xi) = m_\Delta(y)$, where $m_{\xi_\Delta}(\xi)$ is the multiplicity of ξ_Δ in ξ. Then the stabilizer of ξ in $K \wr \mathbf{S}(t)$ is $K^t\mathbf{S}(\mathbf{m})$ and ξ has an extension $\tilde{\xi}$ to $K^t\mathbf{S}(\mathbf{m})$, where

$$\mathbf{m} = (m_{\Delta(j,1)}(y), \dots , m_{\Delta(j,r^a-r^{a-1})}(y))$$

or $(m_{\Delta(0,0)}(y), \dots , m_{\Delta(0,r^a)}(y))$ according to whether $j \ge 1$ or $j = 0$. Define

$$\Psi(\chi) = \mathrm{Ind}_{K^t\mathbf{S}(\mathbf{m})}^{K\wr\mathbf{S}(t)}(\tilde{\xi}\phi_\mu),$$

where $\phi_\mu = \prod_\Delta \phi_{\mu_\Delta}$ with ϕ_{μ_Δ} the irreducible character of $\mathbf{S}(m_\Delta(y))$ labelled by μ_Δ. Since $\mathrm{d}(\chi) = \mathrm{d}(\chi_\mu)$, it follows that

$$\mathrm{d}(\chi) = \sum_\Delta (m_\Delta(y)(a+j) + a(\mathbf{S}(m_\Delta(y))) - a(\chi_\mu) = t(a+j) + a(\mathbf{S}(\mathbf{m})) - a(\chi_\mu).$$

Similarly, $\mathrm{d}(\Psi(\chi)) = a(K^t\mathbf{S}(\mathbf{m})) - a(\xi\phi_\mu)$. But $a(\xi) = \sum_\Delta m_\Delta(y)a(\xi_\Delta)$ and

$$a(\xi_\Delta) = a(|\mathrm{GL}^\epsilon(r^j,q^e)\colon \mathrm{GL}^\epsilon(1,q^{er^j})|) = r^j a + a(\mathbf{S}(r^j)) - (a+j),$$

so by [10, (8.5)], $\mathrm{d}(\Psi(\chi)) = t(a+j) + a(\mathbf{S}(\mathbf{m})) - a(\phi_\mu) = \mathrm{d}(\chi)$. Thus Ψ is a defect preserving bijection between $\Omega_H((w_i))$ and $\mathcal{X} \wr \mathbf{S}(t)$.

If $\tau \in \mathrm{Out}(H)$, then $\chi_{y,\mu}^\tau = \chi_{y^\tau,\mu}$. The base group K^t is uniquely determined up to conjugacy in H, we may suppose $K^\tau = K$ and moreover, we may suppose $[\tau, x] = 1$ for $x \in \mathbf{S}(t)$, so that $\Psi(\chi)^\tau = \Psi(\chi^\tau)$. This proves (4A). □

Let $B \in \mathrm{Blk}(G)$ with defect group D. By (3B), we may suppose $[V, D] = V$ and $G = \mathrm{GL}^\epsilon(eu, q)$. Let $C \in \mathcal{CR}^*(B)$ be a chain given by (2.6) such that the Sylow subgroups of $C(C)$ are abelian. Then by (2.10) and (2.11), $N(C) = N_1 \times \dots \times N_v$ and each

$$N_k = \prod_{i\in I_0'} \left\langle \tau_i, \left(\prod_{j\in I_0} \mathrm{GL}^\epsilon(m_j,q^e) \wr \mathbf{S}(\mathbf{w}_j) \right) \right\rangle \wr \mathbf{S}(i),$$

so that $m_j \le r - 1$. Let Q be a Sylow r-subgroup of $C(C)$ and

$$C^* : 1 < P_1 < \dots < P_w \le Q. \tag{4.3}$$

Then $Q = (R_{1,0,0,0})^u$ and $N(C^*) = N_1^* \times \dots \times N_v^*$ such that for $k \ge 1$

$$N_k^* = \prod_{i\in I_0'} \left\langle \tau_i, \left(\prod_{j\in I_0} \mathrm{GL}^\epsilon(1,q^e) \wr \mathbf{S}(m_j) \wr \mathbf{S}(\mathbf{w}_j) \right) \right\rangle \wr \mathbf{S}(i).$$

(4B). *In the notation above, suppose $C(C)$ has an abelian Sylow r-subgroup Q and $B_0 = B_0(G)$. Then*

$$\mathrm{k}(N(C), B_0, d, U) = \mathrm{k}(N(C^*), B_0, d, U)$$

for all integers $d \ge 0$ and $U \le \mathrm{Out}(G)$. Moreover, Dade's invariant conjecture holds for the principal block with an abelian defect group.

Proof. We may suppose $C \ne C^*$. Let $K = \mathrm{GL}^\epsilon(m_j, q^e)$ and $K^* = \mathrm{GL}^\epsilon(1, q^e) \wr \mathbf{S}(m_j)$. By (4A), there is a defect preserving bijection ψ between $\mathrm{Irr}(B_0(K))$ and $\mathrm{Irr}(B_0(K^*))$. Let $W = K \wr \mathbf{S}(\mathbf{w}_j)$ and $W^* = K^* \wr \mathbf{S}(\mathbf{w}_j)$. By (1A) (a), there is a defect preserving bijection Ψ between $\mathrm{Irr}(B_0(W))$ and $\mathrm{Irr}(B_0(W^*))$. Since τ_i is a field automorphism of $\mathrm{GL}^\epsilon(U_i)$, it follows that we may suppose $[\tau_i, y] = 1$ for each permutation matrix $y \in \mathrm{GL}^\epsilon(U_i)$, so that $\Psi(\chi)^{\tau_i} = \Psi(\chi^{\tau_i})$. Thus Ψ

can be extended to a defect preserving bijection Ψ^* between $\mathrm{Irr}(B_0(N(C)))$ and $\mathrm{Irr}(B_0(N(C^*)))$. Similarly, if $\tau \in \mathrm{Out}(G)$, then we may suppose $[\tau, y] = 1$ for each permutation matrix $y \in G$ and $[\tau, \tau_i] = 1$ (see the remark after (2.7)). If τ stabilizes C^*, then $C^\tau = C$. If $C^\tau = C$, then $Q^{\tau h} = Q$ for some $h \in C(C)$. It follows that $\Psi^*(\chi)^\tau = \Psi^*(\chi^\tau)$ for any $\chi \in \mathrm{Irr}(B_0(N(C)))$.

If B_0 has an abelian defect group, then $C(C)$ has an abelian Sylow subgroup for each $C \in \mathcal{CR}^*$. Suppose $C \in \mathcal{CR}^*(B_0)$ is given by (2.6). Define $\varphi(C) = C_{w-1}$ or C^* according to whether $C = C^*$ or $C \neq C^*$. Then $\varphi(C) \in \mathcal{CR}^*(B_0)$, $\varphi(\varphi(C)) = C$ and $|\varphi(C)| = |C| \pm 1$. This implies that

$$\sum_{C \in \mathcal{CR}^*(B_0)/G} (-1)^{|C|} \mathrm{k}(N(C), B_0, d, U) = 0.$$

This proves (4B). □

Let $R_u = R_{u,0,0,0}$ be a basic subgroup of $G = \mathrm{GL}^\epsilon(eu, q)$, so that $H = C_G(R_u) = \mathrm{GL}^\epsilon(u, q^e)$. If $\lambda = (\lambda_1^{\beta_1}, \lambda_2^{\beta_2}, \ldots, \lambda_\ell^{\beta_\ell})$ is a partition of u with $\lambda_1 < \lambda_2 < \ldots < \lambda_\ell$, then

$$R_\lambda = \prod_{j=1}^{\ell} (R_{\lambda_j})^{\beta_j} \tag{4.4}$$

is a primary subgroup of G and H, $C(R_\lambda) = \prod_{j=1}^{\ell} \mathrm{GL}^\epsilon(\lambda_j, q^e)^{\beta_j}$ and

$$N_H(R_\lambda) = \prod_{j=1}^{\ell} \mathrm{GL}^\epsilon(\lambda_j, q^e) \wr \mathbf{S}(\beta_j).$$

Conversely, each primary subgroup of G and H with no nonzero fixed-point on the underlying space is determined by a partition of u.

For $K \leq G$, we denote by $\mathcal{CR}^*(K)$ the subfamily of $\mathcal{R}(C_K(P(O_r(K))))$ consisting of the chains C such that each subgroup of C is primary in G. Here we have the same identification as that of the remark of (2C) when $(q, \epsilon, r) = (2, +, 3)$. Let $\mathcal{CR}_H^*(\lambda)$ be the subfamily of $\mathcal{CR}^*(H)$ consisting of all chains C whose first subgroup is equal to R_λ. If $C \in \mathcal{CR}_H^*(\lambda)$ is given by (1.1) with $R_\lambda \neq O_r(H)$, then $g(C) : R_\lambda < P_2 < \ldots < P_w$ is a chain of $\mathcal{CR}^*(N_H(R_\lambda))$, $|g(C)| = |C| - 1$, $N_H(C) = N_{N_H(R_\lambda)}(g(C))$ and g is a bijection between $\mathcal{CR}_H^*(\lambda)$ and $\mathcal{CR}^*(N_H(R_\lambda))$. We can identify $\mathcal{CR}_H^*(\lambda)$ with $\mathcal{CR}^*(N_H(R_\lambda))$.

(4C). *Let $H = \mathrm{GL}^\epsilon(u, q^e)$ for $u \geq 1$. Then $\mathcal{Q}(H) = \mathcal{CR}^*(H)$ satisfies Hypothesis* (1D) *for the principal block $B_0 = B(H)$ and it is τ-invariant for each $\tau \in \mathrm{Out}(H)$. In addition, $C_0(H) \in \mathcal{S}(H)$ such that $\Omega_{C_0(H)}(0) = \emptyset$ or $\Omega_H((\delta_{it}))$ according to whether u is not a power of r or $u = r^t$.*

Proof. If $u = 1$, then $\mathcal{Q}(H) = \{C_0(H)\}$ and by definition, $\mathrm{Irr}(H, B_0) = \Omega_H((1, 0)) = \Omega_{C_0(H)}(0)$. Thus $\mathcal{Q}_1(H) = \mathcal{Q}_2(H) = \emptyset$ and $\mathcal{S}(H) = \mathcal{Q}(H)$.

If $1 < u < r$, then H has an abelian Sylow r-subgroup. Let $\mathcal{Q}_1(H) = \{C \in \mathcal{Q}(H) : C \neq C^*, |C| \neq 0\}$, $\mathcal{Q}_2(H) = \{C \in \mathcal{Q}(H) : C = C^*, |C| \neq 1,\}$ and $\mathcal{S}(H) = \{C_0(H), C_0(H)^*\}$, where C^* is defined by (4.3). Set $\Omega_{C_0(H)}(0) = \emptyset$, $\varphi(C_0(H)) = C_0(H)^*$, $\Omega_{C_0(H)}(\varphi) = \Omega_H((u, 0)) = \mathrm{Irr}(H, B_0)$ and $\Omega_{C_0(H)^*}(0) = \mathrm{Irr}(N_H(C_0(H)^*), B_0)$. It follows by (4A) and (4B) that $\mathcal{Q}(H) = \mathcal{Q}_1(H) \cup \mathcal{Q}_2(H) \cup \mathcal{S}(H)$ satisfies Hypothesis (1D) and is τ-invariant for $\tau \in \mathrm{Out}(H)$.

Let $\lambda = (\lambda_0^{\beta_0}, \lambda_1^{\beta_1}, \dots, \lambda_\ell^{\beta_\ell})$ be a partition of u such that $R_\lambda \neq O_r(H)$, and let $K_i = \mathrm{GL}^\epsilon(\lambda_i, q^e)$ and $W_i = \mathrm{GL}^\epsilon(\lambda_i, q^e) \wr \mathbf{S}(\beta_i)$ for $1 \le i \le \ell$. Thus $\lambda_i < u$ for all i, and by induction, $\mathcal{Q}(K_i) = \mathcal{CR}^*(K_i)$ satisfies Hypothesis (1D) for $B_0(K_i)$ and is τ-invariant for $\tau \in \mathrm{Out}(K_i)$. By (1E), $\mathcal{Q}(W_i) = \mathcal{CR}^*(W_i)$ and $\mathcal{Q}(N_H(R_\lambda)) = \mathcal{CR}^*_H(\lambda)$ both satisfy Hypothesis (1D) for $B_0(W_i)$ and $B_0(N_H(R_\lambda))$ and they are τ-invariant for $\tau \in \mathrm{Out}(W_i)$ and $\tau \in \mathrm{Out}(N_H(R_\lambda))$, respectively.

Suppose λ_i is not a power of r for some i. Then $\Omega_{C_0(K_i)}(0) = \emptyset$, $\Omega_{C_0(W_i)}(0) = \Omega_{C_0(K_i)}(0) \wr \mathbf{S}(\beta_i) = \emptyset$ and so $\Omega_{C_0(N_H(R_\lambda))}(0) = \prod_{j=0}^{\ell} \Omega_{C_0(W_j)}(0) = \emptyset$. In particular, by (1A) (b),

$$\sum_{C \in \mathcal{CR}^*_H(\lambda)/N_H(R_\lambda)} (-1)^{|C|} \mathrm{k}(N_H(C), B_0, d, U) = 0.$$

Suppose each λ_i is a power of r, and we may suppose $\lambda_i = r^i$ for all $i \ge 0$. Thus $(\beta_0, \beta_1, \dots, \beta_\ell)$ is an r-weight sequence, $\Omega_{C_0(K_i)}(0) = \Omega_{K_i}((\delta_{ji}))$ and

$$\Omega_{C_0(N_H(R_\lambda))}(0) = \prod_{i=0}^{\ell} \Omega_{K_i}((\delta_{ji})) \wr \mathbf{S}(\beta_i).$$

It follows by (4A) that there is a defect preserving bijection between $\Omega_{C_0(N_H(R_\lambda))}(0)$ and $\Omega_H((\beta_0, \beta_1, \dots, \beta_\ell))$, which is compatible with $\tau \in \mathrm{Out}(H)$.

Let $\varphi_\lambda(C_0(H))$ be the chain $O_r(H) < R_\lambda$, $\Omega_{C_0(H)}(0) = \emptyset$ or $\Omega_H((\delta_{jt}))$ according to whether u is not a power of r or $u = r^t$, and $\Omega_{C_0(H)}(\varphi_\lambda) = \Omega_H((\beta_0, \beta_1, \dots, \beta_\ell))$. Then

$$\mathrm{Irr}(H, B_0) = \Omega_{C_0(H)}(0) \bigcup_\lambda \Omega_{C_0(H)}(\varphi_\lambda) \quad (disjoint),$$

where λ runs over the r-weight partitions of u. Now

$$\mathcal{CR}^*(H) = \bigcup_\mu \mathcal{CR}^*_H(\mu) \quad (disjoint),$$

where μ runs over the partitions of u such that μ is not the partition (u). Thus $\mathcal{Q}(H) = \mathcal{CR}^*(H)$ satisfies Hypothesis (1D) for $B_0(H)$ and $\Omega_{C_0(H)}(0)$ given above, and $\mathcal{Q}(H)$ is τ-invariant for $\tau \in \mathrm{Out}(H)$. This completes the proof. □

(4D). *Let $G = \mathrm{GL}^\epsilon(n,q) = \mathrm{GL}^\epsilon(V)$ such that $O_r(G) = 1$.*

(a) *If $n = me$ for some integer m, then*

$$\sum_{C \in \mathcal{CR}^*(G)/G} (-1)^{|C|} \mathrm{k}(N_G(C), B_0, d, U) = 0$$

for any integer $d \ge 0$ and any $U \le \mathrm{Out}(G)$.

(b) *Dade's invariant conjecture holds for each r-block B of G with defect $\mathrm{d}(B) \ge 1$.*

Proof. (a) Suppose $G = \mathrm{GL}^\epsilon(me,q)$ and $C \in \mathcal{CR}^*_G(\lambda)$, where $\lambda = (\lambda_0^{\beta_0}, \lambda_1^{\beta_1}, \dots, \lambda_u^{\beta_u})$ is a partition of m. If λ_i is not a power of r and $K = \mathrm{GL}^\epsilon(\lambda_i, q^e)$, then by (4C), $\mathcal{CR}^*(K)$ satisfies Hypothesis (1D) and is τ-invariant for each $\tau \in \mathrm{Out}(K)$, so that

$$\sum_{C \in \mathcal{CR}^*(K)/K} (-1)^{|C|} \mathrm{k}(N_K(C), B_0, d, U_K) = 0,$$

where $U_K \leq \mathrm{Out}(K)$. If $L = \langle \tau_i, \mathrm{GL}^\epsilon(\lambda_i, q^e)\rangle$, then

$$\sum_{C\in\mathcal{CR}^*(L)/L} (-1)^{|C|}\mathrm{k}(N_L(C), B_0, d, U_L) = 0,$$

for $U_L \leq \mathrm{Out}(\mathrm{GL}^\epsilon(e\lambda_i, q))$, since $\mathcal{CR}^*(L) = \mathcal{CR}^*(K)$ and $\mathcal{CR}^*(K)$ is τ-invariant for each $\tau \in \mathrm{Out}(\mathrm{GL}^\epsilon(e\lambda_i, q))$. Thus by (1E),

$$\sum_{C\in\mathcal{CR}^*(W)/W} (-1)^{|C|}\mathrm{k}(N_W(C), B_0, d, U_W) = 0,$$

where $W = \langle \tau_i, \mathrm{GL}^\epsilon(\lambda_i, q^e)\rangle \wr S(\beta_i)$, $U_W \leq \mathrm{Out}(\mathrm{GL}^\epsilon(e\beta_i\lambda_i, q))$. By (1A) (b),

$$\sum_{C\in\mathcal{CR}_G^*(\lambda)/N_G(R_\lambda)} (-1)^{|C|}\mathrm{k}(N_{N_G(R_\lambda)}(C), B_0, d, U) = 0.$$

We may suppose $\lambda_i = r^i$ for each i, so that $\beta = (\beta_0, \beta_1, \dots, \beta_u)$ is an r-weight sequence. Thus

$$|\Omega_G((\beta_0, \beta_1, \dots, \beta_u))| = \mathrm{k}(e + (r^a - 1)/e, \beta_0)\prod_{i=1}^{u}\mathrm{k}((r^a - r^{a-1})/e, \beta_i)$$

and $\mathrm{Irr}(B_0(G)) = \bigcup_\beta \Omega_G(\beta)$, where β runs over all r-weight sequences. Let $\Omega^2_{K_i} = \mathrm{Irr}(B_0(K_i))\backslash\Omega_{K_i}((w^i_\ell))$, where $w^i_\ell = \delta_{\ell i}$ and $(w^i_\ell) = (w^i_0, w^i_1, \dots)$. We may suppose $[\,y, \tau_i\,] = 1$ for $y \in \mathrm{Out}(\mathrm{GL}^\epsilon(er^i, q))$. Then by (4C),

$$\sum_{C\in\mathcal{CR}^*(L)/L, |C|\neq 0} (-1)^{|C|}\mathrm{k}(N_L(C), B_0, d, U_L) + \mathrm{k}(\langle\tau_i, \Omega^2_{K_i}\rangle, d, U_L) = 0,$$

where $L = \langle\tau_i, K_i\rangle$, $U_L \leq \mathrm{Out}(\mathrm{GL}^\epsilon(er^i, q))$ and $\langle\tau_i, \Omega^2_{K_i}\rangle$ denotes the characters of $\mathrm{Irr}(L)$ covering $\Omega^2_{K_i}$. By (1A) and (1E),

$$\sum_{C\in\mathcal{CR}_G^*(\lambda)/N_G(R_\lambda), |C|\neq 1} (-1)^{|C|}\mathrm{k}(N_{N_G(R_\lambda)}(C), B_0, d, U)$$
$$= \mathrm{k}(\prod_{i=0}^{u}(\langle\tau_i, \Omega^2_{K_i}\rangle \wr \mathbf{S}(\beta_i)), d, U).$$

Therefore

$$\sum_{C\in\mathcal{CR}_G^*(\lambda)/N_G(R_\lambda)} (-1)^{|C|}\mathrm{k}(N_{N_G(R_\lambda)}(C), B_0, d, U)$$
$$= -\mathrm{k}(\prod_{i=1}^{u}(\langle\tau_i, \Omega_{K_i}((w^i_\ell))\rangle \wr \mathbf{S}(\beta_i)), d, U).$$

Now $|\Omega_{K_i}((w^i_\ell))| = r^a - r^{a-1}$ or r^a according to whether $i \geq 1$ or $i = 0$, and each character of $\Omega_{K_i}((w^i_\ell))$ is labelled by $(\Delta, 1)$ for some $\Delta \in (\mathcal{F}^{a+i}_{q^e})^*$, where $(\mathcal{F}^{a+i}_{q^e})^* = \mathcal{F}^{a+i}_{q^e}$ or $\mathcal{F}_{q^e}(r, a)$ according to whether $i \geq 1$ or $i = 0$. Since τ_i induces a field automorphism of order e on $K_i = \mathrm{GL}^\epsilon(r^i, q^e)$, it follows that

$$|\langle\tau_i, \Omega_{K_i}((w^i_\ell))\rangle| = (r^a - r^{a-1})/e \quad \text{or} \quad e + (r^a - 1)/e$$

according to whether $i \geq 1$ or $i = 0$. If a character $\chi \in \langle\tau_i, \Omega_{K_i}((w^i_\ell))\rangle$ is not an extension of the trivial character, then χ is labelled by $(\Delta', 1)$ for some $\Delta' \in (\mathcal{F}^{a+i}_q)^*$, where $(\mathcal{F}^{a+i}_q)^*$ is defined similarly as before. It follows from (1A) (a) that there exists a defect preserving bijection, compatible with elements of $\mathrm{Out}(G)$, between

$$\prod_{i=0}^{u}\langle \tau_i, \Omega_{K_i}((w_\ell^i))\rangle \wr \mathbf{S}(\beta_i)$$

and $\Omega_G((\beta_0,\beta_1,\ldots,\beta_u))$. Thus

$$\sum_{C\in\mathcal{CR}^*_G(\lambda)/N_G(R_\lambda)} (-1)^{|C|}\mathrm{k}(N_{N_G(R_\lambda)}(C), B_0, d, U) \\ + \mathrm{k}(\Omega_G((\beta_0,\beta_1,\ldots,\beta_u)), d, U) = 0$$

for all r-weight partitions $\lambda = (\lambda_0^{\beta_0},\ldots,\lambda_u^{\beta_u})$ of m. This proves part (a).

(b) If $B = B_0 = B_0(G)$ is the principal block, then Dade's invariant conjecture for B follows by (3B) and part (a) above.

Suppose $B \neq B_0$ and by (3B), we may suppose B is labelled by $(s,-)$. By (3C) and (1A), we may suppose $L = C_G(s) = L_\Gamma = \mathrm{GL}^{\epsilon_\Gamma}(m_\Gamma(s), q^{\delta_\Gamma})$, so that $s = s_\Gamma$. Let $R = O_r(Z(L_\Gamma))$ and let $C : 1 < P_1 < \ldots < P_w$ be a chain of $\mathcal{L}(B_\Gamma)$ (see (3C)).

Suppose $R = 1$, so that r and $q^{\delta_\Gamma} - \epsilon_\Gamma$ are coprime. Let $Q_i = \Omega_{a+\alpha_\Gamma}(O_r(C_{L_\Gamma}(P_i)))$ for all $i \geq 1$, where $\alpha_\Gamma = a(\delta_\Gamma)$. Then

$$\varphi(C) : 1 < Q_1 < \ldots < Q_w$$

is a chain of $\mathcal{CR}^*(L_\Gamma)$ and $N_{L_\Gamma}(C) = N_{L_\Gamma}(\varphi(C))$. Since $P_i = \Omega_a(Q_i)$, it follows that $N_{\mathrm{Out}(L_\Gamma)}(C,\xi) = N_{\mathrm{Out}(L_\Gamma)}(\varphi(C),\xi)$ for any $\xi \in \mathrm{Irr}(N_{L_\Gamma}(C))$, and $\mathcal{CR}^*(B_\Gamma) = \{\varphi(C) : C \in \mathcal{L}(B_\Gamma)\}$. But $s \in Z(L_\Gamma)$, so

$$\mathrm{k}(N_{L_\Gamma}(C), B_\Gamma, d, U) = \mathrm{k}(N_{L_\Gamma}(C), B_0(L_\Gamma), d, U)$$

for all integers $d \geq 0$ and $U \leq \mathrm{Out}(L_\Gamma)$. Thus Dade's invariant conjecture for B_Γ follows by part (a).

Suppose $R \neq 1$, so that $\Omega_a(R) \leq P_1$. Define

$$\varphi(C) : \begin{cases} 1 < \Omega_a(R) < P_1 < \ldots < P_w & \text{if } \Omega_a(R) \neq P_1, \\ 1 < P_2 < \ldots < P_w & \text{if } \Omega_a(R) = P_1. \end{cases}$$

Then $\varphi : \mathcal{L}(B_\Gamma) \to \mathcal{L}(B_\Gamma)$, $N_{L_\Gamma}(C) = N_{L_\Gamma}(\varphi(C))$, and $N_{\mathrm{Out}(L_\Gamma)}(C,\xi) = N_{\mathrm{Out}(L_\Gamma)}(\varphi(C),\xi)$ for any $\xi \in \mathrm{Irr}(N_{L_\Gamma}(C))$, $|\varphi(C)| = |C| \pm 1$ and $\varphi(\varphi(C)) = C$. It follows that

$$\sum_{C\in\mathcal{L}(B_\Gamma)/L_\Gamma} (-1)^{|C|}\mathrm{k}(N_{L_\Gamma}(C), B_\Gamma, d, U) = 0$$

for all integers $d \geq 0$ and $U \leq \mathrm{Out}(L_\Gamma)$. This completes the proof. □

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References

[1] J. L. Alperin, *Large abelian subgroups of p-groups*, Trans. Amer. Math. Soc. **117** (1965), 10-20. MR **30:**180

[2] J. L. Alperin and P. Fong, *Weights for symmetric and general linear groups*, J. Algebra **131** (1990), 2-22. MR **91h:**20014

[3] Jianbei An, *Weights for classical groups*, Trans. Amer. Math. Soc. **342** (1994), 1-42. MR **94e:**20015

[4] M. Broué, *Les ℓ-blocs des groupes* $\mathrm{GL}(n,q)$ *et* $\mathrm{U}(n,q^2)$ *et leurs structures locales*, Séminaire Bourbaki Astérisque **133-134** (1986), 159-188. MR **87e:**20021

[5] M. Broué, *Isométries parfaites, types de blocs, catégories dérivées.* Astérisque **181-182** (1990), 61-92. MR **91i:**20006

[6] E. Dade, *Counting characters in blocks*, I, Invent. Math. **109** (1992), 187-210. MR **93g:**20021

[7] E. Dade, *Counting characters in blocks,* II.9. *in* Representation Theory of Finite Groups (R. Solomon, editor), Ohio State University Math. Res. Inst. Publ. 6, de Gruyter, Berlin 1997. MR **99b:**20016

[8] E. Dade, *Counting characters in blocks with cyclic defect groups*, I, J. Algebra **186** (1996), 934-969. MR **98b:**20013

[9] W. Feit, The representation theory of finite groups, North Holland, 1982. MR **83g:**20001

[10] P. Fong and B. Srinivasan, *The blocks of finite general linear and unitary groups*, Invent. Math. **69** (1982), 109-153. MR **3k:**20013

[11] G. O. Michler and J. B. Olsson, *Character correspondences in finite general linear, unitary and symmetric groups*, Math. Z. **184** (1983), 203-233. MR **85e:**20014

[12] J. B. Olsson, *On the number of characters in blocks of finite general linear, unitary and symmetric groups*, Math. Z. **186** (1984), 41-47. MR **85d:**20008

[13] J. B. Olsson and K. Uno, *Dade's conjecture for general linear groups in the defining characteristic*, Proc. London Math. Soc. **72** (1996), 359-384. MR **97b:**20010

[14] J. B. Olsson and K. Uno, *Dade's conjecture for symmetric groups*, J. Algebra **176** (1995), 534-560. MR **96h:**20025

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