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LOCAL DERIVATIONS ON C^*-ALGEBRAS ARE DERIVATIONS

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ABSTRACT. Kadison has shown that local derivations from a von Neumann algebra into any dual bimodule are derivations. In this paper we extend this result to local derivations from any C^*-algebra $\mathfrak{A}$ into any Banach $\mathfrak{A}$-bimodule $\mathfrak{X}$. Most of the work is involved with establishing this result when $\mathfrak{A}$ is a commutative C^*-algebra with one self-adjoint generator. A known result of the author about Jordan derivations then completes the argument. We show that these results do not extend to the algebra $C^1[0,1]$ of continuously differentiable functions on $[0,1]$. We also give an automatic continuity result, that is, we show that local derivations on C^*-algebras are continuous even if not assumed a priori to be so.

1. INTRODUCTION

A continuous operator T from a Banach algebra $\mathfrak{A}$ into a Banach $\mathfrak{A}$-bimodule $\mathfrak{X}$ is a *local derivation* if for each a in $\mathfrak{A}$ there is a continuous derivation D_a from $\mathfrak{A}$ into $\mathfrak{X}$ with $D_a(a) = T(a)$. This concept was introduced by Kadison [5] who showed that if $\mathfrak{A}$ is a von Neumann algebra and $\mathfrak{X}$ is a dual bimodule, then all local derivations are in fact derivations. In this paper we show (Theorem 5.3) that this result holds for all C^*-algebras $\mathfrak{A}$ and all Banach $\mathfrak{A}$-bimodules $\mathfrak{X}$. Using [4, Theorem 6.3], it is enough to show the result when $\mathfrak{A}$ is a C^*-algebra generated by a single self-adjoint element. It turns out that the general result follows fairly immediately once the case $\mathfrak{X} = (\mathfrak{A}\hat{\otimes}\mathfrak{A})^*$ has been settled. Accordingly, most of the work in the paper is involved in establishing the case $\mathfrak{A} = C_0(\mathbb{R})$, $\mathfrak{X} = (C_0(\mathbb{R})\hat{\otimes}C_0(\mathbb{R}))^*$ (Proposition 5.1). To get this result we need to consider "local multipliers" (defined by replacing "derivation" by "multiplier" in the definition above) and show that they are all multipliers (Proposition 4.1; see also Corollary 5.4). This result in turn depends on Proposition 3.1 which says that in certain circumstances every local operator, defined as an operator T such that the germ of $T(a)$ at each point of the structure space depends only on the germ of a there, is a multiplier. The main ingredient in this is Proposition 2.1, showing that the diagonal in $\mathbb{R}^2$ is a set of synthesis for $C_0(\mathbb{R})\hat{\otimes}C_0(\mathbb{R})$.

In Section 6 we consider the regular Banach algebra $C^1[0,1]$ of continuously differentiable functions in $[0,1]$. This has local derivations which are not derivations, but if we restrict attention to symmetric modules, all local derivations are derivations. In Section 7 we consider automatic continuity for local derivations. Our

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definition of local derivation requires them to be continuous but, for local derivations on C^*-algebras, this requirement is redundant since the other conditions for the local derivation are only satisfied by continuous maps.

2. The diagonal as a set of synthesis

Let $\mathfrak{A}$ be a commutative regular semisimple Banach algebra with maximal ideal space Φ. Let F be a closed subset of Φ. Then F is a set of synthesis if every element a of $\mathfrak{A}$ for which $\hat{a}$ is zero on F can be approximated in norm by elements b such that $\hat{b}$ has compact support disjoint from F. If $\mathfrak{A}$ has a bounded approximate identity $\{e_\alpha\}$ where $\hat{e}_\alpha$ has compact support for each α, then the previous sentence remains true with "compact" omitted.

We will be concerned with algebras $\mathfrak{A}\hat{\otimes}\mathfrak{A}$ where $\mathfrak{A}$ is as in the previous paragraph. It is easy to see that if $\mathfrak{A}$ is unital, then the maximal ideal space of $\mathfrak{A}\hat{\otimes}\mathfrak{A}$ is $\Phi\times\Phi$ and the Gelfand transform is $(a\otimes b)^\wedge(\varphi,\psi)=\hat{a}(\varphi)\hat{b}(\psi)$. A nonzero multiplicative linear functional φ on an ideal J in an algebra A can be extended to A by defining $\varphi(a)=\varphi(aj)/\varphi(j)$ $(a\in A,\ j\in J,\ \varphi(j)\neq 0)$ and taking $\mathfrak{A}\hat{\otimes}\mathfrak{A}$ is an ideal in $\mathfrak{A}^1\hat{\otimes}\mathfrak{A}^1$, where $\mathfrak{A}^1$ is the algebra obtained by adjoining an identity to $\mathfrak{A}$, the result in the previous sentence holds for nonunital algebras also.

If, in addition, $\mathfrak{A}$ satisfies the approximation property so that linear functionals on $\mathfrak{A}\hat{\otimes}\mathfrak{A}$ of the form $a\otimes b\to f(a)g(b)$ $(a,b\in\mathfrak{A}, f,g\in\mathfrak{A}^*)$ are total for $\mathfrak{A}\hat{\otimes}\mathfrak{A}$ (that is, the intersection of their kernels is $\{0\}$), then because Φ is total for $\mathfrak{A}$ by semisimplicity, we see that $\Phi\times\Phi$ is total for $\mathfrak{A}\hat{\otimes}\mathfrak{A}$ so that $\mathfrak{A}\hat{\otimes}\mathfrak{A}$ is semisimple.

We denote the diagonal of $\Phi\times\Phi$ by $\Delta(\Phi)$, that is,

$$\Delta(\Phi)=\{(\varphi,\varphi)\colon \varphi\in\Phi\}$$

and put

$$J_\Delta(\mathfrak{A})=\{t\colon t\in\mathfrak{A}\hat{\otimes}\mathfrak{A},\ \hat{t}=0 \text{ on } \Delta(\Phi)\},$$

$$J^0_\Delta(\mathfrak{A})=\{t\colon t\in\mathfrak{A}\hat{\otimes}\mathfrak{A},\ \hat{t} \text{ has support disjoint from } \Delta(\Phi)\}.$$

The product map on $\mathfrak{A}$ extends to an operator π of norm 1 from $\mathfrak{A}\hat{\otimes}\mathfrak{A}$ to $\mathfrak{A}$. It satisfies

$$\pi(t)^\wedge(\varphi)=\hat{t}(\varphi,\varphi),\quad (t\in\mathfrak{A}\hat{\otimes}\mathfrak{A},\ \varphi\in\Phi).$$

Thus $J_\Delta(\mathfrak{A})=\ker\pi$.

We shall show

Proposition 2.1. *The diagonal is a set of synthesis for* $C_0(\mathbb{R})\hat{\otimes}C_0(\mathbb{R})=V_0(\mathbb{R})$.

To do this we will first show the corresponding result for $A(\mathbb{R})$, the subalgebra of functions which are Fourier transforms of functions in $L^1(\mathbb{R})$. Note that both $A(\mathbb{R})$ and $C_0(\mathbb{R})$ have bounded approximate identities consisting of functions with compact support.

Curtis and Loy [1, Theorem 3.10] have shown that $\mathfrak{A}$ is amenable if and only if $\ker\pi$ has a bounded approximate identity. It is easy to see that the diagonal is a set of synthesis, that is, $J^0_\Delta(\mathfrak{A})$ is dense in $J_\Delta(\mathfrak{A})$, if the approximate identity can be chosen from $J^0_\Delta(\mathfrak{A})$.

Proof of Proposition 2.1. Put $M=\{a\colon a\in A(\mathbb{R}), a(0)=0\}$.

Then M has a bounded approximate identity and, either by direct calculation or by using the fact that one-point sets in locally compact abelian groups are sets of synthesis, we see that it has a bounded approximate identity $\{e_\alpha\}$ consisting of elements which are zero in a neighbourhood of 0. If $t = \sum a_i \otimes b_i \in A(\mathbb{R})\hat{\otimes}A(\mathbb{R})$ has $t(0,\psi) = 0$ for all ψ in $\mathbb{R}$, then taking a bounded approximate identity $\{f_\beta\}$ in $A(\mathbb{R})$, we have

$$t = \sum (a_i - a_i(0)f_\beta) \otimes b_i + f_\beta \otimes \sum a_i(0)b_i \in M\hat{\otimes}A(\mathbb{R})$$

because $(f_\beta \otimes \sum a_i(0)b_i)(\phi,\psi) = f_\beta(\varphi)t(0,\psi) = 0$. Thus $\{e_\alpha \otimes f_\beta\}$ is a bounded approximate identity for the set of such elements t and each $e_\alpha \otimes f_\beta$ is zero in a neighbourhood of $\{0\} \times \mathbb{R}$. The algebra $A(\mathbb{R})\hat{\otimes}A(\mathbb{R})$ is, via the Fourier transform, isomorphic with $L^1(\mathbb{R})\hat{\otimes}L^1(\mathbb{R}) = L^1(\mathbb{R}\times\mathbb{R})$. The map $(x,y) \mapsto (x, x+y)$ is an isomorphism of the group $\mathbb{R}\times\mathbb{R}$ and induces an automorphism ω of $L^1(\mathbb{R}\times\mathbb{R}) \simeq A(\mathbb{R})\hat{\otimes}A(\mathbb{R})$ which, in terms of this latter space, is given by

$$(\omega t)(\varphi,\psi) = t(\varphi+\psi,\psi).$$

Since this maps $\{0\}\times\mathbb{R}$ to the diagonal, $\{\omega(e_\alpha \otimes f_\beta)\}$ is a bounded approximate identity for $J_\Delta(A(\mathbb{R}))$ with $\omega(e_\alpha \otimes f_\beta) \in J^0_\Delta(A(\mathbb{R}))$.

As $A(\mathbb{R})$ is norm dense in $C_0(\mathbb{R})$, $\{\omega(e_\alpha \otimes f_\beta)\}$ is a bounded approximate identity for $J_\Delta(C_0(\mathbb{R}))$ from $J^0_\Delta(C(\mathbb{R}))$ if $J_\Delta(A(\mathbb{R}))$ is dense in $J_\Delta(C(\mathbb{R}))$. If $\varepsilon > 0$ and $v \in J_\Delta(C(\mathbb{R}))$, then there is $t \in A(\mathbb{R})\hat{\otimes}A(\mathbb{R})$ with $\|t - v\|_V < \varepsilon$. Then $\pi(t) \in A(\mathbb{R})$ and $\|\pi(t)\|_\infty = \|\pi(t) - \pi(v)\|_\infty < \varepsilon$. Define s_n and e_n in $A(\mathbb{R})$ by induction satisfying

$$\begin{gathered} s_0 = \pi(t), \quad \|e_n\| \le 1, \\ \|s_n - s_n e_n\|_A < 2^{-n}\varepsilon, \\ s_{n+1} = s_n - s_n e_n \quad (\text{so } \|s_n\|_A < 2^{-n}\varepsilon,\ n > 0). \end{gathered}$$

This is possible because $A(\mathbb{R})$ has a bounded approximate identity of norm 1. Then $t' = \sum s_n \otimes e_n \in A(\mathbb{R})\hat{\otimes}A(\mathbb{R})$ and $\pi(t') = \sum s_n e_n = \sum s_n - s_{n+1} = s_0 = \pi(t)$. Thus $t - t' \in J_\Delta(A(\mathbb{R}))$ and $\|t - t' - v\|_V < 3\varepsilon$ because $\|t'\|_V < 2\varepsilon$. □

3. Multipliers and local operators

Let $\mathfrak{A}$ be a commutative regular semisimple Banach algebra and let $\mathfrak{X}$ be a left Banach $\mathfrak{A}$-module. For $x \in \mathfrak{X}$, the *annihilator* $A(x)$ of x is

$$A(x) = \{a \colon a \in \mathfrak{A},\ ax = 0\}.$$

It is clearly a closed ideal in $\mathfrak{A}$ and its hull is called the *support* of x, denoted by $\operatorname{supp} x$. A continuous operator T from $\mathfrak{A}$ to $\mathfrak{X}$ is called a *local operator* if

$$\operatorname{supp} Ta \subseteq \operatorname{supp} a$$

for all a in $\mathfrak{A}$. In this we are treating $\mathfrak{A}$ as a left module over itself and the support of an element a is just the closure of the set of points at which $\hat{a}$ is nonzero.

A (right) *multiplier* from $\mathfrak{A}$ to $\mathfrak{X}$ is bounded operator T from $\mathfrak{A}$ to $\mathfrak{X}$ with $T(ab) = aT(b)$ $(a, b \in \mathfrak{X})$. Of course, if $\mathfrak{A}$ is unital with unit e, then $T(a) = aT(e)$ $(a \in \mathfrak{A})$ so multipliers are given by multiplication by a fixed element of $\mathfrak{X}$. We could define local operators and multipliers from one $\mathfrak{A}$ module to another but we will not need to use these. It must be admitted that our use of "local" above conflicts with its use in "local multiplier" and "local derivation" below, however, to remove this conflict we would need to replace well established terminology by unfamiliar terminology and we have decided to accept the conflict rather than do this.

It is easy to see that if T is a multiplier and $ab = 0$, then $aT(b) = T(ab) = 0$ so that $A(a) \subseteq A(Ta)$ giving $\operatorname{supp} Ta \subseteq \operatorname{supp} a$; that is T is a local operator. We shall show that the converse holds for some algebras.

Proposition 3.1. *Let $\mathfrak{X}$ be an essential left Banach $C_0(\mathbb{R})$-module and let T be a continuous local operator from $C_0(\mathbb{R})$ to $\mathfrak{X}$. Then T is a multiplier.*

Proof. Consider first the case $\mathfrak{X} = M(\mathbb{R})$ where the action of $C_0(\mathbb{R})$ is the usual product of a function and a measure. Then $\tau(a \otimes b) = (T(a), b)$ $(a, b \in C_0(\mathbb{R}))$ defines a linear functional τ on $V_0(\mathbb{R})$. Note that the support of a measure μ in $M(\mathbb{R})$ defined above is the same as its support as usually defined. The local property shows that if a and b have disjoint supports, then so do $T(a)$ and b and hence $\tau(a \otimes b) = 0$. Let J be the closed linear span in $V_0(\mathbb{R})$ of

$$\{a \otimes b\colon a, b \in C_0(\mathbb{R}),\ \operatorname{supp} a \cap \operatorname{supp} b = \varnothing\}.$$

Then J is a closed ideal and it is easy to see its hull is the diagonal. Thus $J = J_\Delta(C_0(\mathbb{R}))$ and τ is zero on $J_\Delta(C_0(\mathbb{R})) = \ker \pi$. As π is surjective, there is $\mu \in M(\mathbb{R})$ with $\pi^*\mu = \tau$ and so

$$(T(a), b) = \int a(t)b(t)d\mu(t) \quad (a, b \in C_0(\mathbb{R})).$$

It follows immediately that $(T(ac), b) = (T(a), cb) = (cT(a), b)$ so T is a multiplier.

We now consider the general case. Let $F \in \mathfrak{X}^*$ and consider the map K_F of $\mathfrak{X}$ into $M(\mathbb{R})$ given by

$$((K_F x), a) = F(ax), \qquad a \in C_0(\mathbb{R}), x \in \mathfrak{X}.$$

It is easy to check that this is a $C_0(\mathbb{R})$ module map and so $\operatorname{supp} K_F x \subseteq \operatorname{supp} x$ for all x in $\mathfrak{X}$. Thus $K_F T$ is a local operator with range in $M(\mathbb{R})$ and so is a multiplier. Hence for all a, b in $C_0(\mathbb{R})$, $K_F(aT(b) - T(ab)) = aK_F T(b) - K_F T(ab) = 0$. However, if $x \in \mathfrak{X}$ with $K_F x = 0$ for all $F \in \mathfrak{X}^*$, then

$$F(ax) = 0 \quad (a \in C_0(\mathbb{R}), F \in \mathfrak{X}^*)$$

so that $ax = 0$ and hence $x = 0$ because $\mathfrak{X}$ is essential. Thus $aT(b) = T(ab)$ for all a, b in $C_0(\mathbb{R})$ and T is a multiplier. $\square$

Note that the hypothesis that $\mathfrak{X}$ is essential is needed because any map from $C_0(\mathbb{R})$ into $\{x\colon ax = 0,\ a \in \mathfrak{A}\}$ is a local operator which is a multiplier if and only if it is zero. Note also that for $\mathfrak{A} = C^1[0,1]$, $\mathfrak{X} = C[0,1]$, differentiation is a local operator which is not a multiplier.

4. Local multipliers

Let $\mathfrak{A}$ be a commutative semisimple regular Banach algebra and let $\mathfrak{X}$ be a left Banach $\mathfrak{A}$-module. A *local multiplier* from $\mathfrak{A}$ to $\mathfrak{X}$ is a continuous operator T such that for each $a \in \mathfrak{A}$ there is a (right) multiplier T_a from $\mathfrak{A}$ to $\mathfrak{X}$ with $T(a) = T_a(a)$. We have $bT(a) = bT_a(a) = T_a(ba) = 0$ if $ba = 0$ so $\operatorname{supp} T(a) \subseteq \operatorname{supp} a$. Thus a local multiplier is a local operator.

Proposition 4.1. *Let I be an interval* (*open or closed*) *in $\mathbb{R}$ and let $\mathfrak{X}$ be a left Banach $C_0(I)$-module. If T is a local multiplier from $C_0(I)$ to $\mathfrak{X}$, then T is a multiplier.*

Proof. Note that if I is compact, then $C_0(I)$ is just $C(I)$. Consider first the case $I = \mathbb{R}$. If $b, c \in C_0(\mathbb{R})$, $a = bc$ and T_a is a multiplier with $T_a(a) = T(a)$, then $T(a) = T_a(bc) = bT_a(c)$ so $T(a)$ lies in the essential submodule of $\mathfrak{X}$ whenever a is a product and hence for all a in $C_0(\mathbb{R})$. Thus by Proposition 3.1, T is a multiplier. The general result follows because $C_0(I)$ is a quotient of $C_0(\mathbb{R})$ so we can lift local multipliers on $C_0(I)$ to local multipliers on $C_0(\mathbb{R})$. □

5. Local derivations

"Local derivations" is defined in Section 1. If $\mathfrak{A}$ is an amenable Banach algebra and $\mathfrak{X}$ is a dual $\mathfrak{A}$ module, then there is $x_a \in \mathfrak{X}$ with $D_a(b) = bx_a - x_a b$ for all b in $\mathfrak{A}$. Thus in this situation a continuous linear operator T is a local derivation if and only if for each element a of $\mathfrak{A}$ there is x_a in $\mathfrak{X}$ with $T(a) = ax_a - x_a a$. This section is mainly devoted to proving

Proposition 5.1. *If T is a local derivation from $C_0(\mathbb{R})$ to $V_0(\mathbb{R})^*$, then it is a derivation.*

The module structure on $V_0(\mathbb{R})^*$ is the dual of the action of $C_0(\mathbb{R})$ on $C_0(\mathbb{R})\hat{\otimes} C_0(\mathbb{R})$ given by $a(b \otimes c) = ab \otimes c$, $(b \otimes c)a = b \otimes ca$ $(a, b, c \in C_0(\mathbb{R}))$. For any subset E of $\mathbb{R}$ we put, as usual

$$k(E) = \{a \colon a \in C_0(\mathbb{R}),\ a = 0 \text{ on } E\}$$

and for two subintervals I_1 and I_2 we put $V_0(I_1, I_2)$ for the closed linear span in $V_0(\mathbb{R})$ of elements $a_1 \otimes a_2$ where $a_i \in k(\mathbb{R}\backslash I_i)$.

Lemma 5.2. *Let I_1 and I_2 be compact intervals in $\mathbb{R}$ and let $\theta \in V_0(\mathbb{R})^*$.*
(i) *If $a \in k(I_1)$, then $\theta a \in V_0(I_1, I_2)^\perp$.*
(ii) *If $a \in k(I_2)$, then $a\theta \in V_0(I_1, I_2)^\perp$.*
(iii) *If $a \in C_0(\mathbb{R})$ and $a = 1$ on I_1, then $\theta - \theta a \in V_0(I_1, I_2)^\perp$.*
(iv) *If $a \in C_0(\mathbb{R})$ and $a = 1$ on I_2, then $\theta - a\theta \in V_0(I_1, I_2)^\perp$.*

Proof. Let $c_i \in k(\mathbb{R}\backslash I_i)$. For (i) we have $(\theta a, c_1 \otimes c_2) = (\theta, ac_1 \otimes c_2) = 0$. For (iii) we have $(\theta - \theta a, c_1 \otimes c_2) = (\theta, (c_1 - ac_1) \otimes c_2) = 0$. The other two statements are proved similarly. □

Proof of Proposition 5.1. Let I_1 and I_2 be disjoint compact intervals. We need to extend Proposition 4.1 to the algebras $k(I_1)$ and $k(I_2)$. If the intervals are nonempty the algebras are isomorphic with $C_0(\mathbb{R}) \oplus C_0(\mathbb{R})$. Any local multiplier T on the direct sum restricts to a local multiplier on each summand and so is a multiplier there by Proposition 4.1. If $a = a_1 \oplus a_2$ and $b = b_1 \oplus b_2$ are elements of $C_0(\mathbb{R}) \oplus C_0(\mathbb{R})$, then

$$\begin{aligned} T(ab) &= T(a_1b_1 \oplus a_2b_2) = T(a_1b_1) + T(a_2b_2) \\ &= a_1T(b_1) + a_2T(b_2) = (a_1 \oplus a_2)T(b_1 \oplus b_2) = aT(b) \end{aligned}$$

because by writing $b_2 = b_2'b_2''$ we have $a_1T(b_2) = a_1T(b_2'b_2'') = a_1b_2'T(b_2'') = 0$ and similarly $b_2T(a_1) = 0$. The quotient map $q \colon V_0(\mathbb{R})^* \to V_0(\mathbb{R})^*/V_0(I_1, I_2)^\perp$ is a module map. It follows that qT is a local derivation. Denote the restriction of qT to $k(I_1)$ by S For $a \in k(I_1)$ there is $\psi_a \in V_0(\mathbb{R})^*$ with $T(a) = a\psi_a - \psi_a a$ so that $S(a) = aq\psi_a$ because $q(\psi_a a) = 0$ by Lemma 5.2. Thus S is a local left multiplier and hence a left multiplier so that there is $\varphi(I_1, I_2)$ in $V_0(I_1, I_2)^*$ with

$$qT(a) = a\varphi(I_1, I_2) \quad (a \in k(I_1)). \tag{1}$$

A similar calculation shows that there is $\psi(I_1, I_2)$ in $V_0(I_1, I_2)^*$ with

$$qT(b) = \psi(I_1, I_2)b \quad (b \in k(I_2)). \tag{2}$$

We will now prove that as I_1 and I_2 are disjoint, then

$$\varphi(I_1, I_2) = -\psi(I_1, I_2).$$

Let $a \in C_0(\mathbb{R})$ with $a = 0$ on I_1 and $a = 1$ on I_2. Then $qT(a) = a\varphi(I_1, I_2) = \varphi(I_1, I_2)$ by Lemma 5.2. Similarly, arguing with $b - a$ in place of a where $b \in C_0(\mathbb{R})$ has $b = 1$ on $I = I_1 \cup I_2$ we get $qT(b - a) = \psi(I_1, I_2)$. However, as T is a local derivation, there is $\theta \in V(I)^*$ with $T(b) = b\theta - \theta b$ where $b\theta - \theta b = (b\theta - \theta) - (\theta b - \theta) \in V_0(I_1, I_2)^\perp$ by Lemma 5.2. Thus $qTb = 0$ and so $qT(a) = -\psi(I_1, I_2) = \varphi(I_1, I_2)$ as required.

It follows that

$$qT(a) = a\varphi(I_1, I_2) - \varphi(I_1, I_2)a$$

for all a in $\mathfrak{A}$ because by (1) and the lemma this holds for $a \in k(I_1)$; by (2), the lemma, and $\varphi = -\psi$ it holds for $a \in k(I_2)$ and $C_0(\mathbb{R}) = k(I_1) + k(I_2)$. Thus qT is a derivation into $V_0(I_1, I_2)^*$.

Consider δT given by $\delta T(a, b) = aT(b) - T(ab) + T(a)b$. It is of course a 2 cocycle from $C_0(\mathbb{R})$ with values in $V_0(\mathbb{R})^*$. However, because qT is a derivation δT maps into $V_0(I_1, I_2)^\perp$ and since this holds for all choices of I_1 and I_2, δT maps into $(\operatorname{Span}\{V_0(I_1, I_2) \colon I_1 \cap I_2 = \varnothing\})^\perp$. In the proof of Proposition 3.1 we showed that this span is dense in $J_\Delta(C_0(\mathbb{R}))$ so that δT maps into $(J_\Delta C_0(\mathbb{R}))^\perp$ which we have seen is isomorphic as a $C_0(\mathbb{R})$ bimodule with $M(\mathbb{R})$. By [2, Proposition 8.2] every 2 cocycle from $C_0(\mathbb{R})$ to $M(\mathbb{R})$ is a coboundary so there is a map S from $C_0(\mathbb{R})$ to $M(\mathbb{R})$ with $\delta S = \delta T$. We have $S = T + S - T$ where T is a local derivation, $S - T$ is a derivation and hence a local derivation so S is a local derivation into $V_0(\mathbb{R})^*$ with values in $\operatorname{Im} \pi^*$. (It is not immediately obvious that this is the same as saying that S is a local derivation into $\operatorname{Im} \pi^*$.) Because $C_0(\mathbb{R})$ is amenable, all derivations from $C_0(\mathbb{R})$ into $V_0(\mathbb{R})^*$ are inner so if $a \in C_0(\mathbb{R})$, there is $F_a \in V_0(\mathbb{R})^*$ with $S(a) = aF_a - F_a a$. But $S(a) = \pi^*\mu$ for some $\mu \in M(\mathbb{R})$ so for all $v \in V_0(\mathbb{R})$

$$\int \pi(v)\, d\mu = -(F_a, av - va).$$

If $v \in \operatorname{Ker} \pi$, this gives $(F_a, av - va) = 0$. However, $\operatorname{Ker} \pi$ has a bounded approximate identity $\{e_n\}$, so if $v \in V_0(\mathbb{R})$, then $ve_n \in \ker \pi$ so $(F_a, ave_n - ve_n a) = 0$. But $ave_n - ve_n a = (av - va)e_n \to av - va$ because $av - va \in \ker \pi$, so $(F_a, av - va) = 0$. Thus $Sa = 0$ so $S = 0$ and $T = -(S - T)$ is a derivation. □

Theorem 5.3. *Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{X}$ a Banach $\mathfrak{A}$ bimodule. Then every local derivation from $\mathfrak{A}$ to $\mathfrak{X}$ is a derivation.*

Proof. The result is proved by extending Proposition 5.1 in a number of steps. First we show it holds with $V_0(\mathbb{R})^*$ replaced by any $C_0(\mathbb{R})$ bimodule $\mathfrak{X}$ which is essential on both left and right. Let $F \in \mathfrak{X}^*$ and define $L_F \colon \mathfrak{X} \to V_0(\mathbb{R})^*$ by

$$L_F(x)(a \otimes b) = F(bxa) \quad (a, b \in C_0(\mathbb{R}), x \in \mathfrak{X}).$$

It is straightforward to check that L_F is a bimodule map and hence if T is a local derivation into $\mathfrak{X}$, then $L_F T$ is a local derivation into $V_0(\mathbb{R})^*$. Thus

$$L_F(aT(b) - T(ab) + T(a)b) = 0 \quad (a, b \in C_0(\mathbb{R}), F \in \mathfrak{X}^*). \tag{3}$$

However, if $L_F(x) = 0$ for all F in $\mathfrak{X}^*$, then $F(axb) = 0$ $(F \in \mathfrak{X}^*, a, b \in C_0(\mathbb{R}))$ so that $axb = 0$ $(a, b \in C_0(\mathbb{R}))$; hence taking a and b as elements of a bounded approximate identity we get $x = 0$. Thus (3) implies that T is a derivation.

Next we extend the result by replacing $C_0(\mathbb{R})$ by $C(K)$ where K is any compact subset of $\mathbb{R}$. The restriction map R maps $C_0(\mathbb{R})$ onto $C(K)$. If $\mathfrak{X}$ is an essential $C(K)$ module, then defining $ax = (Ra)x$, $xa = x(Ra)$, $\mathfrak{X}$ becomes an essential $C_0(\mathbb{R})$ module. If T is a local derivation on $C(K)$, then TR is a local derivation on $C_0(\mathbb{R})$. Thus by what we have proved, TR is a derivation so $T(RaRb) = TR(ab) = aTR(b) + TR(a)b = RaT(Rb) + T(Ra)Rb$ for all a, b in $C_0(\mathbb{R})$ so T is a derivation on $C(K)$.

Now suppose that M is a maximal ideal in $C(K)$ and $\mathfrak{X}$ is an M bimodule (not assumed to be essential). Then $C(K)$ is the algebra obtained by adjoining an identity to M and $\mathfrak{X}$ becomes a unital $C(K)$ module if we put $\mathbf{1}x = x = x\mathbf{1}$ for all $x \in \mathfrak{X}$. If T is a local derivation from M into $\mathfrak{X}$, then defining $T(1) = 0$ extends T to a local derivation from $C(K)$ to $\mathfrak{X}$. As T thus extended is a derivation, the original T was a derivation.

If T is a local derivation from $C(K)$ into a $C(K)$ bimodule $\mathfrak{X}$ (not assumed to be essential), then we can take $\lambda \in \mathbb{R} \backslash K$ and consider $C(K)$ as a maximal ideal in $C(K \cup \{\lambda\})$. Thus by what we have already proved, T is a derivation.

We now take the case of a general C^*-algebra $\mathfrak{A}$ and let T be a local derivation $\mathfrak{A} \to \mathfrak{X}$ and $a \in \mathfrak{A}$ with $a = a^*$. The closed subalgebra $\mathfrak{A}(a)$ of $\mathfrak{A}$ generated by a is isomorphic either to an algebra $C(K)$ (if it is unital) or to a maximal ideal in such an algebra. The restriction of T to $\mathfrak{A}(a)$ is a local derivation and hence a derivation. We thus have $T(a^2) = aT(a) + T(a)a$ for all self-adjoint elements a of $\mathfrak{A}$. If a and b are two self-adjoint elements, then applying this to the polarisation identity $(a+b)^2 - a^2 - b^2 = ab + ba$ yields

$$T(ab + ba) = aT(b) + T(b)a + T(a)b + bT(a).$$

Since this is linear in a and b, it extends to linear combinations of self-adjoint elements, that is to all a, b in $\mathfrak{A}$. Thus T is a Jordan derivation and hence a derivation [3, Theorem 6.3]. □

Corollary 5.4. *Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{X}$ a Banach right $\mathfrak{A}$ module. Let T be a local left multiplier from $\mathfrak{A}$ to $\mathfrak{X}$. Then T is a multiplier from $\mathfrak{A}$ to $\mathfrak{X}$.*

Proof. By saying that T is a local left multiplier we mean that for each a in $\mathfrak{A}$ there is $T_a \colon \mathfrak{A} \to \mathfrak{X}$ with $T_a(a) = T(a)$ and $T_a(bc) = T_a(b)c$. We make $\mathfrak{X}$ an $\mathfrak{A}$ bimodule by putting $ax = 0$ $(a \in \mathfrak{A},\ x \in \mathfrak{X})$. Then a map from $\mathfrak{A}$ to $\mathfrak{X}$ is a left multiplier if and only if it is a derivation and consequently the same holds for local multipliers and derivations. Thus T is a local derivation, hence a derivation and hence a left multiplier. □

6. The algebra $C^1[0,1]$

As usual, $C^1[0,1]$ is the algebra of complex valued functions on $[0,1]$ with continuous first derivative. Multiplication is pointwise and the norm is $\|a\| = \|a\|_\infty + \|a'\|_\infty$ where $\|a\|_\infty$ is the sup-norm.

Example. There are local multipliers from $C^1[0,1]$ into $C^1[0,1]^*$ which are not multipliers. There are local derivations from $C^1[0,1]$ into an essential $C^1[0,1]$ bimodule which are not derivations.

Denote the functional $b \mapsto b(0)$ in $C^1[0,1]$ by δ_0. Consider the map $T(a) = a'(0)\delta_0$ $(a \in C^1[0,1])$. To show that T is a local multiplier it is enough to show that for each $a \in C^1[0,1]$ there is an element F_a of $C^1[0,1]^*$ with

$$(Ta)(b) = a'(0)b(0) = F_a(ba) \ (b \in C^1[0,1]),$$

If $a(0) \neq 0$, then we can take $F_a = (a'(0)/a(0))\delta_0$. If $a(0) = 0$, then we can take F_a as the functional $c \mapsto c'(0)$ so that $F_a(ba) = (ab)'(0) = a'(0)b(0)$. Obviously T is not a multiplier because $T(ba) = (ab)'(0)\delta_0$ and $bT(a) = a'(0)b(0)\delta_0$.

Let M be the ideal in $C^1[0,1]$ of functions which are zero at 1. Then $T_0 = T/M$ is a local multiplier from M to $C^1[0,1]^*$. If we consider $C^1[0,1]^*$ as an M bimodule X with trivial action on the right and the usual multiplication on the left, then T_0 is a local derivation from M to $C^1[0,1]^*$ which is not a derivation. Adjoining a unit to M in the usual way given an algebra M^1 isomorphic with $C^1[0,1]$ and X is a unital M^1 module if we put $\mathbf{1}x = x = x\mathbf{1}$ $(x \in X)$. The local derivation T_0 extends to a local derivation D from M^1 to X by defining $D(\mathbf{1}) = 0$ and D is not a derivation because T_0 is not.

Theorem 6.1. *Let X be a symmetric $C^1[0,1]$ module and let T be a local derivation from $C^1[0,1]$ into X. Then T is a derivation.*

Proof. Consider first the case $X = C^1[0,1]^*$. If D is a derivation from $C^1[0,1]$ into X and $j \in C^1[0,1]$ is the function $x \mapsto x$, then for any polynomial P, $D(P) = P'D(j)$. For $a \in C[0,1]$ let $(\mathcal{T}a)(t) = \int_0^t a(s)ds$. Then $\mathcal{T}a \in C^1[0,1]$ and $a \mapsto D(\mathcal{T}a)(\mathbf{1})$ is a continuous linear functional on $C[0,1]$. Thus there is $\mu \in M[0,1]$ with $D(\mathcal{T}a)(\mathbf{1}) = \int a d\mu$ for all a in $C^1[0,1]$. Thus if P and Q are polynomials, then

$$\begin{aligned} D(P)(Q) &= (QD(P))(\mathbf{1}) = (P'QD(j))(\mathbf{1}) \\ &= D(\mathcal{T}P'Q)(\mathbf{1}) = \int P'Q\, d\mu. \end{aligned}$$

Since the polynomials are dense in $C^1[0,1]$, this shows that

$$D(a)(b) = \int a'b\, d\mu \qquad (a, b \in C^1[0,1]).$$

The inclusion $\iota\colon C^1[0,1] \to C[0,1]$ has adjoint ι^* which is an injection of $M[0,1]$ into a subspace M of $C^1[0,1]^*$ which is in fact a submodule, though it is not closed. We have seen that $D(a) = \iota(a'\mu)$ so D maps into M. Thus if T is a local derivation from $C^1[0,1]$ into $C^1[0,1]^*$, then T maps into M. Under the norm $\| \|_0$ for which ι^* is an isometry from $M[0,1]$ to M, M is a Banach space so the closed graph theorem shows that T is a continuous map from $C[0,1]$ into $(M, \| \|_0)$. Moreover, because T is a local derivation, we have seen that for each a in $C[0,1]$ there is a measure μ_a in $M[0,1]$ with $T(a) = \iota^*(a'\mu_a)$. If we denote the map from M to $M[0,1]$ inverse to ι^* by k and define $S(c) = kT(\mathcal{T}c)(c \in C[0,1])$, then $S(c) = k\iota^*(c\mu_{Sc}) = c\mu_{\mathcal{T}c}$ so S is a local multiplier. By the remarks at the beginning of Section 4, S is a multiplier so there is $\mu \in M[0,1]$ with $S(c) = c\mu$ for all c in $C[0,1]$. Taking $a = Sc$ this gives $T(a) = \iota^*S(a') = a'\iota^*\mu$ for all $a \in C^1[0,1]$ with $a(0) = 0$. As T is a local derivation so $T(\mathbf{1}) = 0$, the equation $T(a) = a'\iota^*\mu$ holds for all a in $C^1[0,1]$ showing that T is a derivation.

The argument in Proposition 3.1 using K_F now shows that if $\mathfrak{X}$ is a unital symmetric $C^1[0,1]$ module, then every local derivation from $C^1[0,1]$ into $\mathfrak{X}$ is a derivation. If $\mathfrak{X}$ is not unital and D is a derivation from $C^1[0,1]$ into $\mathfrak{X}$, then

$D(\mathbf{1}) = 0$ so $D(a) = D(\mathbf{1}a) = \mathbf{1}D(a)$ so D maps into the essential part, $\mathfrak{X}_1 = \{x\colon \mathbf{1}x = x\}$ of $\mathfrak{X}$. Thus any local derivation also maps into $\mathfrak{X}_1$ and so we can use the result for the essential case. □

7. Automatic continuity

There are a number of situations in which all multipliers or derivations are continuous; see [6]. In this section we consider the same question for local derivations from C^*-algebras, that is, we show that the hypothesis that T be continuous in the definition of local derivation is redundant (the condition that D_a is continuous is redundant by the result quoted above). Our approach parallels that of Sections 3, 4 and 5; that is, we consider local operators then local multipliers and finally local derivations. The results for general modules follow from the case of $\mathfrak{A}^*$ and $(\mathfrak{A}\hat{\otimes}\mathfrak{A})^*$ as in the earlier sections and the result for general C^*-algebras follows from the commutative case by [1].

Proposition 7.1. *Let Ω be a locally compact topological space and let $\mathfrak{A} = C_0(\Omega)$ be a commutative C^*-algebra. Let $\mathfrak{X}$ be a left Banach $\mathfrak{A}$ module and let T be a local operator from $\mathfrak{A}$ to $\mathfrak{X}$. Then T has a finite number of discontinuity values.*

Proof. Our notation follows that of [7, Theorem 2.3]. For any open set G in Ω put $X(G) = k(G), Y(G) = \{x\colon x \in \mathfrak{X},\ \operatorname{supp} x \subseteq \Omega\backslash G\}$. The present result follows directly from the result referred to. □

It is easy to find discontinuous local operators. For example if $\mathfrak{X} = M(\Omega)$ and ω is a nonisolated point of Ω, then the map $T(a) = F(a)\delta_\omega$ is a discontinuous local operator where δ_ω is unit mass at ω and F is a discontinuous linear functional with $F(a) = 0$ for all a which are zero in a neighbourhood of ω.

Proposition 7.2. *Let Ω be a locally compact topological space, let $\mathfrak{A} = C_0(\Omega)$ and let T be a local multiplier, not assumed a priori to be continuous, from $\mathfrak{A}$ into a left Banach $\mathfrak{A}$-module $\mathfrak{X}$. Then T is continuous.*

Proof. We consider the case $\mathfrak{X} = \mathfrak{A}^* = M(\Omega)$ first. The proof (at the beginning of Section 4) that a local multiplier is a local operator applies even if T is not assumed to be continuous. Thus, by Proposition 7.1, T has only a finite number of discontinuity points; call them $\omega_1, \dots, \omega_n$. Let $\mu \in \mathfrak{S}$, the separating space of T [7, p. 7]. If $\omega \in \Omega$ is not a discontinuity value, then there is an open neighbourhood N of ω for which $Q(N)T$ is continuous. Thus if $a_n \to 0$ in $\mathfrak{A}$ and $T(a_n) \to \mu$ in $\mathfrak{X}$, then $Q(N)\mu = 0$ and $|\mu|(N) = 0$. Hence μ is supported on the discontinuity values. Let χ_j be the characteristic function of the one-point set $\{\omega_j\}$ and put $\chi = \mathbf{1} - \sum \chi_j$. Then $\chi\mu = 0$ for all $\mu \in \mathfrak{S}$ so χT is continuous. For each j, $T_j = \chi_j T$ is a local multiplier with values in $\mathbb{C}\delta_j$ where δ_j is unit mass at ω_j. If $a \in \mathfrak{A}$, there is $\mu_a \in M(\Omega)$ with $T(a) = a\mu_a$. Thus $T_j(a) = \chi_j a\mu_a \in \mathbb{C}\delta_j$ so $T_j(a) = f_j(a)\delta_j$ for some linear functional f_j. However, $\chi_j a = a(\omega_j)\chi_j$ so $f_j = 0$ if $a(\omega_j) = 0$. Thus f_j and hence T_j is continuous. It follows that $T = \chi T + \sum T_j$ is continuous.

Using the maps K_F from Proposition 3.1 and the closed graph theorem we can extend what we have proved to any essential left Banach $\mathfrak{A}$ module $\mathfrak{X}$. It applies to nonessential modules too because the comments at the beginning of Section 4 show that T maps into the essential submodule of $\mathfrak{X}$. □

Proposition 7.3. *Let Ω be a locally compact space, let $\mathfrak{A} = C_0(\Omega)$ and let T be a local derivation, not assumed a priori to be continuous, from $\mathfrak{A}$ into a Banach $\mathfrak{A}$ bimodule $\mathfrak{X}$. Then T is continuous.*

First we need a lemma.

Lemma 7.4. *Let F be a finite set in $\Omega \times \Omega$. The F is a set of synthesis for $\mathfrak{A}\hat{\otimes}\mathfrak{A}$.*

Proof. Let $\omega, \omega' \in \Omega$. There is a bounded net $\{u_\alpha\}$ in $\mathfrak{A}$ with $u_\alpha = 1$ in a neighbourhood of ω and $\|u_\alpha a\| \to 0$ for all a in $\mathfrak{A}$ with $a(\omega) = 0$. There is a similar net $\{v_\beta\}$ corresponding to ω'. Then $\{u_\alpha \otimes v_\beta\}$ is a bounded net with $u_\alpha \otimes v_\beta = 1$ in a neighbourhood of (ω, ω') and $(u_\alpha \otimes v_\beta)t \to 0$ if $t = a \otimes b$ with either $a(0) = 0$ or $b(\omega') = 0$ and hence, if t is in the closed linear span of such tensors. If $t = \sum a_j \otimes b_j$ with $\sum \|a_j\|\,\|b_j\| < \infty$ and $\sum a_j(\omega)b_j(\omega') = 0$, then choosing $e_0 \in \mathfrak{A}$ with $e_0(\omega') = 1$ we have

$$t = \sum a_j \otimes (b_j - b_j(\omega')e_0) + \sum b_j(\omega')a_j \otimes e_0$$

where $(b_j - b_j(\omega')e_0)(\omega') = 0$ and $(\sum b_j(\omega')a_j)(\omega) = \sum a_j(\omega)b_j(\omega') = 0$ so $\|(u_\alpha \otimes v_\beta)t\| \to 0$. Choosing a net $\{u_\alpha \otimes u_\beta\}$ for each point $p_1, \ldots, p_n$ in F, call it $\{u^j_\alpha \otimes v^j_\beta\}$, we have

$$t = \lim_{\alpha,\beta} t - \sum t(u^j_\alpha \otimes v^j_\beta)$$

which expresses t as a limit of functions zero in the neighbourhood of each point of E. $\square$

Proof of Proposition 7.3. As in the proof of Theorem 5.3, we can adjoin a unit to $\mathfrak{A}$ and extend T by defining $T\mathbf{1} = 0$. Thus we need consider only the case of compact spaces Ω and essential $\mathfrak{A}$ modules $\mathfrak{X}$. Consider first the case $\mathfrak{X} = (\mathfrak{A}\hat{\otimes}\mathfrak{A})^*$. For any set S we denote $S \times S$ by S^2. For any subset E of Ω or Ω^2, $k(E)$ is the kernel of E in $\mathfrak{A}$ or $\mathfrak{A}\hat{\otimes}\mathfrak{A}$ and E' is the complement of E in Ω or $\Omega \times \Omega$ as appropriate. If $E \subseteq \Omega^2$, then $q(E)$ is the quotient map $\mathfrak{X} \to \mathfrak{X}/k(E)^\perp = k(E)^*$, that is, the operation of restricting elements of $\mathfrak{X}$ to $k(E)$. For an open set G in Ω let T_G be the restriction of $q(G \times \Omega)T$ to $k(\Omega\backslash G)$. For $a \in k(\Omega\backslash G)$ and $v \in k(G \times \Omega)$ we have $av = 0$ so if $F \in \mathfrak{X}$, then $Fa \in k(G \times \Omega)^\perp$ and $q(G \times \Omega)(Fa) = 0$. Thus $k(\Omega\backslash G)$ acts trivially on the right of $\mathfrak{X}/k(G \times \Omega)^\perp$ and T_G is a local multiplier from $k(\Omega\backslash G)$ to $\mathfrak{X}/k(G \times \Omega)^\perp$. Hence T_G is continuous by Proposition 7.2.

In the same way the restriction of $q(G \times \Omega)T$ to $k(\Omega\backslash G)$ is continuous. We want to show that the restriction of $q(G^2)T$ to $k(\Omega\backslash G)$ is continuous. By the closed graph theorem this follows from what we have proved if we can show that $k(G \times \Omega)^\perp \cap k(\Omega \times G)^\perp = k(G^2)^\perp$ and this is equivalent to $k(G^2) = [k(G \times \Omega) + k(\Omega \times G)]^-$. Since each side of this equation is a closed ideal in $\mathfrak{A}\hat{\otimes}\mathfrak{A}$ with hull $\overline{G}^2$, this follows if $\overline{G}^2$ is a set of synthesis.

For any finite open cover $C = \{C_1, \ldots, C_n\}$ of Ω, take a partition $\{\rho_j : j = 1, \ldots, n\}$ of the identity subordinate to C, choose $x_j \in C_j$ and for $a \in \mathfrak{A}$ put $U_C a = \sum_j a(x_j)\rho_j$ (where we take the undefined term $a(x_j)\rho_j$ to be zero if $C_j = \varnothing$. If $\varepsilon > 0$ and C has the property that $|a(x) - a(x')| < \varepsilon$ if $x, x' \in C_i$ for some i, then $(U_C a)(x)$ is a convex combination of $a(x_j)$ with $|a(x_j) - a(x)| < \varepsilon$ so $\|a - U_C a\| < \varepsilon$. Thus directing the C by refinement we have $\|U_C\| \le 1$ and $U_C a \to a$. Every cover C has a refinement such that if $C_i \cap G = \varnothing$, then $\overline{C}_i \cap \overline{G} = \varnothing$ and we restrict attention to such covers. Moreover, for C_i with $G \cap C_i \neq \varnothing$ we select $x_i \in G \cap C_i$. Put $G' = \bigcup\{C_i : C_i \in C, C_i \cap G \neq \varnothing\}$, $G'' = \bigcup\{C_i : C_i \in C, C_i \cap G = \varnothing\}$.

If $t = \sum a_i \otimes b_i \in k(G \times G)$, then

$$\sum_i (U_C a_i)(x)(U_C b_i)(y) = \sum_{jk} \left(\sum_i a_i(x_j) b_i(y_k) \right) \rho_j(x) \rho_k(y).$$

Thus if $x, y \in \Omega \backslash G''$, so that the x_j and y_k appearing with nonzero values of $\rho_j(x)\rho_k(y)$ are in G, the right side of the equation is zero. As $\Omega \backslash G''$ is a neighbourhood of $\overline{G}$ and $(U_C \otimes U_C)t \to t$, the result follows.

We define a point ω of Ω to be a *discontinuity value* if the restriction of T to $k(\Omega \backslash N)$ is discontinuous for all open neighbourhoods N of ω (note that this is not the same definition as on p. 15 of [7]). We will show that T has only finitely many discontinuity values. If not, we can choose $\{U_n\}$ and $\{V_n\}$ as in [7, Theorem 2.3]. Moreover, replacing U_n by a subset we can assume that $\overline{U}_n \cap \overline{V}_n = \varnothing$. Put $W_n = \bigcup_{j \neq n} U_j$. We have $\overline{U}_n \cap \overline{W}_n = \varnothing$. On $k(\Omega \backslash U_n)$, T is discontinuous and $q(U_n \times U_n)T$ is continuous so $T_n = q(\Omega^2 \backslash (\Omega \backslash \overline{W}_n)^2)T$ is discontinuous by the closed graph theorem because $k(U_n^2) + k(\Omega^2 \backslash (\Omega \backslash \overline{W}_n)^2) = \mathfrak{A} \hat{\otimes} \mathfrak{A}$, so $k(U_n^2)^\perp \cap k(\Omega^2 \backslash (\Omega \backslash \overline{W}_n)^2)^\perp = \{0\}$. Let S_n denote the restriction of $q(W_n^2)T$ to $k(\Omega \backslash W_n)$. We have seen that S_n is continuous. Choose $a_i \in k(\Omega \backslash U_i)$ with $\|a_i\| \leq 2^{-i}$ and $\|T_i a_i\| \geq 2^i + \|S_i\|$ $(i = 1, 2, \dots)$ and put $a = \sum a_i$, $b_i = a - a_i$. Then $\|a\| \leq 1$ and $\|b_i\| \leq 1$. Also $b_i \in k(\Omega \backslash W_i)$ so $\|S_i\| \geq \|q(W_i^2)Tb_i\| \geq \|q(\Omega^2 \backslash (\Omega \backslash \overline{W}_i)^2)Tb_i\|$ because $k(W_i^2)^\perp \subseteq k(\Omega^2 \backslash (\Omega \backslash \overline{W}_i)^2)^\perp$. Thus

$$\begin{aligned} \|Ta\| &\geq \|q(\Omega^2 \backslash (\Omega \backslash \overline{W}_i)^2)Ta\| \\ &\geq \|T_i a_i\| - \|q(\Omega^2 \backslash (\Omega \backslash \overline{W}_i)^2)Tb_i\| \\ &\geq 2^i + \|S_i\| - \|S_i\| = 2^i. \end{aligned}$$

As $\|Ta\|$ is finite, this cannot hold for all i, so the set of discontinuity values is finite.

Let $x_1, x_2 \in \Omega$ with $x_1 \neq x_2$. Choose open neighbourhoods N_1, N_2 of x_1, x_2 with $\overline{N}_1 \cap \overline{N}_2 = \varnothing$. Consider $q((N_1 \times N_2)')T$. We have

$$(N_1 \times N_2)' = (N_1' \times \Omega) \cup (\Omega \times N_2') \supseteq N_1' \times N_2'$$

so $k(N_1' \times N_2')^\perp \subseteq (k(N_1 \times N_2)')^\perp$ and $q((N_1 \times N_2)')T$ is continuous on $k(N_1)$ because $q((N_1' \times N_2'))T$ is. Similarly, $q((N_1 \times N_2)')T$ is continuous on $k(N_2)$. Thus $q((N_1 \times N_2)')T$ is continuous on $\mathfrak{A}$ because $k(N_1) + k(N_2) = \mathfrak{A}$.

Let x be an element of Ω which is not a discontinuity value and let U, V be open neighbourhoods of x with T continuous on $k(U')$ and $\overline{V} \subseteq U$. Consider $q((V \times V)')T$. It is continuous on $k(U')$ because T is. We have $\overline{V}' \times \overline{V}' \subseteq (V \times V)'$ so $k(\overline{V}' \times \overline{V}')^\perp \subseteq k((V \times V)')^\perp$ and $q((V \times V)')T$ is continuous on $k(V)$ because $q((\overline{V}' \times \overline{V}'))T$ is. Because $k(U') + k(V) = \mathfrak{A}$, $q((V \times V)')T$ is continuous. Let $\mathfrak{S}$ be the separating set for T. The continuity of $q((N_1 \times N_2)')T$ above shows that $\mathfrak{S} \subseteq k((N_1 \times N_2)')^\perp$ and the continuity of $q(V' \times V')T$ shows that $\mathfrak{S} \subseteq k(V' \times V')^\perp$ so $\mathfrak{S} \subseteq J^\perp$ where J is the closed linear span of the $k((N_1 \times N_2)')$ and $k(V \times V')$ over all possible choices of N_1, N_2 and V. Clearly J is a closed ideal and its hull does not contain any points (x, x') with $x \neq x'$ or with $x = x'$ where x is not a discontinuity value. As finite sets are sets of synthesis, this shows that $J = kh(J)$

where

$$h(J) \subseteq \{(x, x)\colon x \text{ is a discontinuity value}\}.$$

In the proof of Lemma 7.4 we showed that $\mathbf{1}\otimes\mathbf{1}-\sum_j u_\alpha^j \otimes v_\beta^j = e_{\alpha\beta}$ is a bounded approximate identity for J. Let $\mathfrak{X}_0$ denote the essential submodule of $\mathfrak{X}$ considered as a J module. If D is a derivation from $\mathfrak{A}$ into $\mathfrak{X}$, then there is $F \in \mathfrak{X}$ with $D(a) = aF - Fa$ $(a \in \mathfrak{A})$ so $D(a) = (a \otimes 1 - 1 \otimes a)F \in \mathfrak{X}_0$ because $a \otimes 1 - 1 \otimes a \in J$. Thus the range of T lies in $\mathfrak{X}_0$ and hence $\mathfrak{S} \subseteq \mathfrak{X}_0$. If $s \in \mathfrak{X}_0 \cap J^\perp$, then $s = \lim_{\alpha\beta} e_{\alpha\beta}s$ so $s(a) = \lim_{\alpha\beta} s(e_{\alpha\beta}a) = 0$ because $e_{\alpha\beta}a \in J$. Thus $\mathfrak{X}_0 \cap J^\perp = 0$ so $\mathfrak{S} = \{0\}$ and T is continuous.

The result is extended to general essential $\mathfrak{A}$ modules by using the maps L_F. If $\mathfrak{S}$ is the separating space for T, then $L_F T$ is continuous by what we have shown so $L_F\mathfrak{S} = \{0\}$. Since this holds for all F in $\mathfrak{X}^*$ we see $\mathfrak{S} = \{0\}$ and so T is continuous. □

Theorem 7.5. *Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{X}$ a Banach $\mathfrak{A}$-bimodule. If T is a local derivation, not assumed a priori to be continuous, from $\mathfrak{A}$ into $\mathfrak{X}$, then T is continuous.*

Proof. By Proposition 7.4, the restriction of T to any closed commutative self-adjoint subalgebra is continuous so the result follows from [1]. □

It is reasonable to ask how far these results apply to other Banach algebras. We give two answers to this question. One is to show that they do not extend to $C^1[0,1]$ and the other is to state conditions on $\mathfrak{A}$ under which the proof of Proposition 7.3 applies.

Let $\mathfrak{A} = C^1[0,1]$ and let $\mathfrak{X}$ be the two-dimensional submodule of $\mathfrak{A}^*$ generated by the functionals δ and δ' where $\delta a = a(0)$, $\delta' a = a'(0)$ $(a \in \mathfrak{A})$. If $\xi = \lambda\delta + \mu\delta'$, then $a\xi = (\lambda a(0) + \mu a'(0))\delta + a(0)\mu\delta'$. Since $\mathfrak{A}$ is unital, all multipliers are continuous. The only condition imposed on T by saying that it is a local multiplier is that if $a(0) = 0$, then $T(a) \in \mathbb{C}\delta$. Thus any linear map from $\mathfrak{A}$ to $\mathbb{C}\delta$ which satisfies this is a local multiplier and so there are discontinuous local multipliers.

In this situation there are discontinuous derivations and hence discontinuous local derivations. It is enough to show this with $\mathfrak{A}$ replaced by M, the ideal of functions which are 0 at 0. Let σ be a linear functional on M and put $Ta = \sigma(a)\delta$ $(a \in M)$. Then T is a derivation if $\sigma(ab) = 0$ $(a, b \in M)$. The ideal in M of functions a with $a(t) = O(t^2)$ as $t \to 0$ has infinite codimension and contains all products ab. Thus these are discontinuous functionals σ which are zero on this ideal.

Proposition 7.3 can be generalised to

Theorem 7.6. *Let $\mathfrak{A}$ be a commutative regular Banach algebra which satisfies the approximation property. Suppose further that every maximal ideal M in $\mathfrak{A}^1$ has a bounded approximate identity whose elements have Gelfand transforms which are 0 in the neighbourhood of M. Then every local derivation from $\mathfrak{A}$ to any Banach $\mathfrak{A}$-bimodule is continuous.*

Here $\mathfrak{A}^1$ denotes the algebra obtained by adjoining an identity to $\mathfrak{A}$. We will give only a brief indication of how the proof of Proposition 7.3 can be adapted. The proof of Proposition 7.1 applies in any regular commutative Banach algebra. To extend Proposition 7.2 we use Proposition 7.1 to show that the set E of discontinuity values is finite. Thus E is a set of synthesis and $k(E)$ has a bounded approximate identity.

If $s \in \mathfrak{S}$, then $\operatorname{supp} s \subseteq E$ so if $j \in k(E)$, then $js = 0$. Denote the restriction of T to $k(E)$ by T_0 and its separating space by $\mathfrak{S}_0$. If $j \in k(E)$, then there are j_1 and j_2 in $k(E)$ with $j_1 j_2 = j$ and a multiplier S with $T_0(j) = S(j) = S(j_1 j_2) = j_1 S(j_2)$ so the range of T_0 lies in the $k(E)$ essential submodule of $\mathfrak{A}^*$. Thus $\mathfrak{S}_0 \subseteq \mathfrak{S} \cap k(E)\mathfrak{A}^* = \{0\}$ and T_0 is continuous. As $k(E)$ is of finite codimension in $\mathfrak{A}$ this shows that T is continuous.

The proof of Lemma 7.4 requires no essential changes. In the proof of Proposition 7.3 the main difficulty is that we have no reason to expect that $\overline{G}$ is a set of synthesis so $\overline{G}^2$ may not be. However, if G' is an open set with $G' \supseteq \overline{G}$, then by regularity $k(G' \times G') \subseteq k(G \times \Omega) + k(\Omega \times G)$ so the restriction of $q((G')^2)T$ to $k(\Omega \backslash F)$ is continuous by the same argument as before. Then we change the definition of S_n by replacing $q(W_n^2)T$ by $q((W_n^*)^2)T$ where $W_n^* \supseteq \overline{W}_n$ and $(W_n^*)^- \cap U_n^- = \varnothing$.

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