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TAME AND WILD COORDINATES OF $K[z][x,y]$

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Abstract. Let K be a field of characteristic zero. We characterize coordinates and tame coordinates in $K[z][x,y]$, i.e. the images of x respectively under all automorphisms and under the tame automorphisms of $K[z][x,y]$. We also construct a new large class of wild automorphisms of $K[z][x,y]$ which maps x to a concrete family of nice looking polynomials. We show that a subclass of this class is stably tame, i.e. becomes tame when we extend its automorphisms to automorphisms of $K[z][x,y,t]$.

Introduction

The well known theorem of Jung-Van der Kulk [10], [12] gives that the automorphisms of the polynomial algebra in two variables $K[x,y]$ over a field K are tame, i.e. they can be decomposed as products of affine and triangular automorphisms. The combinatorial description of $\mathrm{Aut}K[x,y]$ as the amalgamated free product of the subgroups of affine and upper triangular automorphisms is also well known; see the book of Cohn [5]. Several algorithms have been discovered which determine whether a homomorphism of $K[x,y]$ is an automorphism and, if it is, decompose it as a product of linear and triangular automorphisms; see [4], [5], [9]. Recent results of Shpilrain and the second author [17] over a field of characteristic 0 give that one can also determine whether a polynomial $p(x,y)\in K[x,y]$ is a coordinate, i.e. is the image of x under some automorphism $P=(p,q)$ of $K[x,y]$ and to find a concrete P with this property.

On the other hand, very little is known about the automorphisms of the polynomial algebra $R[x,y]$ where R is some commutative algebra. The first result in this direction was the famous example of Nagata [15] of an automorphism of the $K[z]$-algebra $K[z][x,y]$ which is not tame. Then Wright [20] described the structure of the group of tame automorphisms of $R[x,y]$ over any principal ideal domain as the amalgamated free product in the same way as over a field. Wright showed as well that the group of all automorphisms of $R[x,y]$ is also an amalgamated free product of the affine group and one more group of automorphisms which, when R is not a field, properly contains the group of upper triangular automorphisms.

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The purpose of this paper is to commence the systematic study of the $K[z]$-automorphisms and coordinates of $K[z][x,y]$ when K is a field of characteristic 0. Applying the results of [20] for $K[z][x,y]$ and of [17] for $K[x,y]$, we characterize the tame coordinates of $K[z][x,y]$. To specify, a polynomial $p(x,y) \in K[z][x,y]$ is a tame coordinate if and only if 1 can be obtained from the partial derivatives $p_x = \partial p/\partial x$ and $p_y = \partial p/\partial y$ using the Euclidean algorithm. Hence the problem whether a polynomial $p(x,y) \in K[z][x,y]$ is a tame coordinate can be solved using division of polynomials only. In particular, our algorithm produces a concrete tame automorphism of $K[z][x,y]$ sending x to $p(x,y)$. We also use recent results of Daigle and Freudenburg [6] on locally nilpotent derivarions and characterize the coordinates of $K[z][x,y]$, i.e. the images of x under the automorphisms of $K[z][x,y]$. Namely, we shall see that $p(x,y) \in K[z][x,y]$ is a coordinate in $K[z][x,y]$, if and only if $p(x,y)$ is a coordinate as an element of $K(z)[x,y]$ and has a unimodule gradient, which means that $K[z][x,y]$ is generated as an ideal by the partial derivatives p_x and p_y. As in the tame case, our approach allows us to solve effectively the problem whether a polynomial is a coordinate, but this time we have to apply Gröbner bases techniques instead of the Euclidean algorithm.

The Nagata automorphism was conjectured to be wild considered as an automorphism of the polynomial algebra $K[x,y,z]$ in three variables; see [15]. On the other hand, Martha Smith [19] discovered another important property of the Nagata automorphism. She showed that it is stably tame and becomes tame if we extend it to an automorphism of $K[z][x,y,t]$ fixing t. Recently Alev [1], the authors of this paper and Gutierrez [8], and Le Bruyn [13] found some new evidence that the Nagata automorphism should be wild considered as an automorphism of $K[x,y,z]$. Naturally it is important to have new examples of automorphisms of polynomial algebras in more than two variables which may serve as candidates of wild automorphisms.

Considered as an automorphism of $K(z)[x,y]$, the Nagata automorphism is a conjugate of an elementary automorphism by another elementary automorphism. Hence, the Nagata automorphism is a product of three elementary automorphisms of $K(z)[x,y]$. We study automorphisms of $K[z][x,y]$ which have a similar presentation. In particular, we consider polynomials of the form $p(x,y) = x + q(a(x) + by)$, where $q(w) \in K[z][w]$, $a(x) \in K[z][x]$, $b \in K[z]$, and study the problem when these polynomials are coordinate in $K[z][x,y]$. We establish that if $q(w)$ is divisible by all irreducible factors of $b(z)$, then p is coordinate and is the image of x under an automorphism which is a product of three elementary automorphisms of $K(z)[x,y]$. In the special case when $q(w)$ is divisible by $b(z)$ itself, we show that p is really a "Nagata like" coordinate and is the image of an automorphism of $K[z][x,y]$ which is a conjugate of an elementary automorphism by another elementary automorphism of $K(z)[x,y]$. We also show that the automorphisms with $q(w)$ divisible by $b(z)$ are stably tame. In this way, our results give a new large family of automorphisms of $K[z][x,y]$ which may be considered as candidates of wild automorphisms of $K[x,y,z]$.

1. Preliminaries

In this paper we fix a field K of characteristic 0 and, if not explicitly stated, consider commutative unitary K-algebras only. If R is an algebra, we denote by $R[x,y]$ the polynomial algebra in two variables x,y over R. Sometimes we use the notation p_x and p_y, respectively, for the partial derivatives $\partial p/\partial x$ and $\partial p/\partial y$ of $p \in R[x,y]$. In what follows we assume that R is one of the algebras K, $K[z]$ or $K(z)$. We denote the endomorphisms of $R[x,y]$ as $F = (f_1(x,y), f_2(x,y))$ assuming that f_1 and f_2 are respectively the images of x and y (and that F is an R-algebra endomorphism). If F is an automorphism, then its inverse is denoted by F^{-1}. We accept similar notation for endomorphisms of polynomial algebras in more than two variables. The Jacobian matrix of $F = (f_1, f_2) \in \mathrm{End}R[x,y]$ is

$$J_F = J_F(x,y) = \begin{pmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{pmatrix}.$$

The composition $F \circ G$ of two endomorphisms $F = (f_1, f_2)$ and $G = (g_1, g_2)$ is

$$F \circ G = F(G) = (f_1(g_1, g_2), f_2(g_1, g_2)).$$

The chain rule gives that

$$J_{F\circ G}(x,y) = J_F(g_1, g_2) J_G(x,y).$$

In particular, if F is an automorphism, then its Jacobian matrix is invertible over R.

Definition 1.1. An automorphism of $R[x,y]$ is called *tame* if it belongs to the subgroup of $\mathrm{Aut}R[x,y]$ generated by the affine automorphisms and the triangular automorphisms, the latter being defined as

$$F = (\alpha x + f(y), \beta y),$$

where $\alpha, \beta \in R^*$ and $f \in R[y]$ is a polynomial which does not depend on x. The automorphisms which are not tame are called *wild*.

Remark 1.2. The notion of tame and wild automorphism depends on the ground algebra R. For example, Nagata [15] constructed the following automorphism of $K[x,y,z]$ defined by

$$N = (x - 2(y^2 + zx)y - (y^2 + zx)^2 z, y + (y^2 + zx)z, z),$$

which is wild considered as an automorphism of $K[z][x,y]$. It is still unknown whether N is tame as an automorphism of $K[x,y,z]$ and the famous conjecture of Nagata states that N is wild.

Definition 1.3. Any automorphic image $p = p(x,y) \in R[x,y]$ of x is called a *coordinate polynomial* or simply a *coordinate*. If $P = (p,q)$ for some tame automorphism P of $R[x,y]$, then p is a *tame coordinate*. If all automorphisms P which send x to p are wild, then the coordinate p is also called *wild*.

Remark 1.4. If p is a tame coordinate of $R[x,y]$, then all automorphisms Q of $R[x,y]$ with the property $Q = (p,q)$ are tame. Indeed, if $P = (p, q_0)$ for some tame automorphism P of $R[x,y]$ and $Q = (p,q) \in \mathrm{Aut}R[x,y]$, then the automorphism $S = Q \circ P^{-1}$ fixes x. It is easy to see that $S = (x, \alpha y + g(x))$ for some $\alpha \in R^*$ and $g(x) \in R[x]$, i.e. S is tame. Since P is also tame, this implies that Q is tame.

Let $GE_2(R[x,y])$ be the subgroup of $GL_2(R[x,y])$ generated by elementary and diagonal matrices. By the Jung-Van der Kulk theorem and the chain rule, the Jacobian matrix of every automorphism of $K[x,y]$ belongs to $GE_2(K[x,y])$. On the other hand, the weak Jacobian theorem of Wright [20] states that if the Jacobian matrix of an endomorphism F of $K[x,y]$ belongs to $GE_2(K[x,y])$, then F is an automorphism. This result was generalized to coordinate polynomials by Shpilrain and the second author [17]. They showed that $p(x, y) \in K[x, y]$ is a coordinate polynomial if and only if the vector $(\partial p/\partial x, \partial p/\partial y)$ is the first row of some matrix in $GE_2(K[x, y])$. An equivalent form of this statement is that $(\partial p/\partial x, \partial p/\partial y)$ can be brought to $(1, 0)$ by the Euclidean algorithm.

The results of Wright [20] and Shpilrain and Yu [17] are based on the description of Wright [20] of $GE_2(K[x_1, \dots, x_n])$ as an amalgamated free product. Recall the definition of the amalgamated free product (see e.g. [14] or [20]). If A, B and C are three abstract groups such that $\phi : B \to A$ and $\psi : B \to C$ are two embeddings, then the amalgamated free product $G = A *_B C$ is generated by $A \cup C$ and the defining relations of G are the defining relations of A and C together with the defining relations $\phi(b) = \psi(b)$, $b \in B$. If we consider B as a subgroup of A and C and assume that $A \cap C = B$, then the elements of $G = A *_B C$ can be presented as

$$g = ba_1^{\varepsilon_1} c_1 a_2 c_2 \dots a_k c_k^{\varepsilon_2},$$

where $b \in B$, $a_i \in A$, $c_i \in C$, and $\varepsilon_1, \varepsilon_2$ are equal to 1 or 0, depending on whether or not a_1 and c_k participate in the expression of g. It is well known that g is different from 1, provided that $a_1, \dots, a_k$ and $c_1, \dots, c_k$ do not belong to B. The description of $GE_2(K[x_1, \dots, x_n])$ given by Wright [20] is the following.

Theorem 1.5 (Wright [20])**.** *The group $GE_2(K[x_1, \dots, x_n])$ is isomorphic to the amalgamated free product of the subgroup $GL_2(K)$ and the subgroup $B_2(K[x_1, \dots, x_n])$ of all lower triangular matrices with polynomial entries.*

We assume that we can perform concrete calculations with the elements of K. We also introduce an arbitrary ordering on the monomials of $K[x, y, z]$ which allows induction and is preserved under multiplication. For example, we may consider the usual lexicographic ordering or the *deg-lex ordering*, comparing the monomials of $K[x, y, z]$ first by total degree and then lexicographically with $x > y > z$. (See e.g. [2] for different orderings of $K[x_1, \dots, x_n]$.) We say that a matrix

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in GE_2(K[x, y, z])$$

can be decomposed into a product of diagonal and elementary matrices using the Euclidean algorithm if, in each step of bringing a to its diagonal form by elementary transformations, we multiply it from the right by an elementary matrix which decreases the leading terms of the first row of a. For example, if the leading term of a_{11} is equal to the leading term of ba_{12} for some monomial $b \in K[x, y, z]$, then we are allowed to replace a by

$$ua = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}.$$

In particular, if f and g are two relatively prime polynomials in $K[x, y, z]$ and we reach 1 applying the usual Euclidean algorithm to f and g, then the corresponding operations can be written in a matrix form as

$$\begin{pmatrix} a_{11} & a_{12} \end{pmatrix} e_1 \dots e_k = \alpha \begin{pmatrix} 1 & 0 \end{pmatrix}, \ \alpha \in K^*,$$

where $e_1, \dots, e_k$ are (alternatively lower and upper) triangular matrices

$$e_i = \begin{pmatrix} 1 & h_i \\ 0 & 1 \end{pmatrix}, \ e_{i+1} = \begin{pmatrix} 1 & 0 \\ h_{i+1} & 1 \end{pmatrix}.$$

We shall make use of a result of the thesis of Park [16]. Since the journal version of [16] has not been published yet, a proof for the case of polynomials in two variables can be found in [17]. A careful study of the proof in [17] shows that it works for any number of variables and any ordering.

Proposition 1.6. *Every matrix in $GE_2(K[x, y, z])$ can be decomposed into a product of diagonal and elementary matrices using only the Euclidean algorithm.*

Remark 1.7. If (f, g) is a pair of polynomials brought to $(1, 0)$ by the Euclidean algorithm in k steps

$$\begin{pmatrix} a_{11} & a_{12} \end{pmatrix} e_1 \dots e_k = \alpha \begin{pmatrix} 1 & 0 \end{pmatrix}, \; \alpha \in K^*,$$

then the matrix $e_1 \dots e_k$ belongs to $GE_2(K[x, y, z])$ and one presentation as an element of $GL_2(K) *_{B_2(K)} B_2(K[x, y, z])$ can be obtained replacing the upper triangular matrices $(e_{11} + e_{22}) + h_i e_{12}$ by the products $(e_{12} + e_{21})((e_{11} + e_{22}) + h_i e_{21})(e_{12} + e_{21})$ if h_i is not a constant.

2. Characterization of Tame and Wild Coordinates in $K[z][x, y]$

In this section we characterize tame coordinates in $K[z][x, y]$. Then we give algorithms which recognize whether a polynomial $p(x, y) \in K[z][x, y]$ is a tame coordinate and, if it is, find concrete tame automorphisms sending x to $p(x, y)$. Finally, we characterize all coordinates in $K[z][x, y]$ and give a procedure which determines whether a polynomial is a coordinate of $K[z][x, y]$. In particular, we are able to find effectively a lot of wild automorphisms of $K[z][x, y]$, giving in this way new candidates for wild automorphisms of $K[x, y, z]$, all of them fixing z as in the example suggested by Nagata.

To prove our main results on tame automorphisms, we modify the considerations in the proof of the weak Jacobian theorem in [20] and the description of the coordinate polynomials in $K[x, y]$ in [17].

Lemma 2.1 (Compare with the proof of the weak Jacobian theorem, [20])**.** *Let ξ: $K[x, y, z] \to \mathbb{Z}^3$ be the degree function on $K[x, y, z]$ induced by the lexicographic ordering $x > y > z$, i.e. $\xi(f) = (d_1, d_2, d_3)$ if the leading term of $f \neq 0$ is $\alpha x^{d_1} y^{d_2} z^{d_3}$, $\alpha \in K^*$. Let $a_1, \dots, a_k$ be matrices in $GL_2(K)$ which do not belong to the lower triangular group $B_2(K)$ and let $c_i = (e_{11} + e_{22}) + f_i e_{21}$, $i = 1, \dots, k$, where $f_1, \dots, f_k$ are polynomials of positive degree. Then the row-matrices*

$$(u_0 \; v_0) = (1 \; 0), \; (u_i \; v_i) = (u_{i-1} \; v_{i-1}) a_i c_i,$$

$i = 1, \dots, k$, satisfy the equation

$$\xi(u_i) = \xi(v_i) + \xi(f_i).$$

Proof. We apply induction on i. If $a_1 = \alpha_{11} e_{11} + \alpha_{12} e_{12} + \alpha_{21} e_{21} + \alpha_{22} e_{22}$, $\alpha_{12} \neq 0$, then concrete calculation shows that

$$u_1 = \alpha_{11} + \alpha_{12} f_1, \; v_1 = \alpha_{12},$$

and $\xi(u_1) = \xi(f_1)$, $\xi(v_1) = \xi(\alpha_{12}) = (0, 0, 0)$. Similarly, if $a_i = \beta_{11} e_{11} + \beta_{12} e_{12} + \beta_{21} e_{21} + \beta_{22} e_{22}$, $\beta_{12} \neq 0$, then

$$u_i = (\beta_{12} f_i + \beta_{11}) u_{i-1} + (\beta_{22} f_i + \beta_{21}) v_{i-1}, \; v_i = \beta_{12} u_{i-1} + \beta_{22} v_{i-1},$$

and $\xi(u_i) = \xi(u_{i-1}) + \xi(f_i)$, $\xi(v_i) = \xi(u_{i-1})$, which completes the proof by induction.

The following theorem is one of the main results of this section. The proof uses some ideas of the proof of [17, Theorem 1.1].

Theorem 2.2. *The following statements for $p(x,y) \in K[z][x,y]$ are equivalent:*
(i) *The polynomial $p(x,y)$ is a tame coordinate in $K[z][x,y]$;*
(ii) *There exists a matrix $g \in GE_2(K[x,y,z])$ such that*

$$(p_x\ p_y)g = (1\ 0),$$

i.e. $(p_x\ p_y)$ can be brought to $(1\ 0)$ by elementary transformations;
(iii) *Applying the Euclidean algorithm to p_x and p_y, the result is equal to 1.*

Proof. The equivalence of (ii) and (iii) follows immediately from Proposition 1.6. The implication (i) $\Longrightarrow$ (ii) is a consequence of the chain rule and the fact that $K[z]$ is a Euclidean domain (hence $GE_2(K[z]) = GL_2(K[z])$ and the Jacobian matrix of any affine automorphism of $K[z][x,y]$ belongs to $GE_2(K[x,y,z])$). Therefore, it is sufficient to show that (ii) implies (i). Let

$$(p_x\ p_y)g = (1\ 0)$$

for some matrix $g = g(x,y) \in GE_2(K[x,y,z])$ (depending on x,y and on the "constant" z) and let

$$g = a_1c_1a_2c_2\dots a_kc_k^{\varepsilon},$$

where $a_1,\dots,a_k \in GL_2(K)$ and $a_2,\dots,a_k$ do not belong to the lower triangular group $B_2(K)$, $c_i = (e_{11}+e_{22}) + f_ie_{21}$ for some nonconstant polynomials $f_i \in K[x,y,z]$, $i = 1,\dots,k$, and $\varepsilon = 0,1$. Since

$$\begin{pmatrix} u & v \end{pmatrix}\begin{pmatrix} 1 & 0 \\ f_k & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

implies that $(u\ v) = (1\ 0)$, without loss of generality we may assume that $\varepsilon = 0$. Let

$$a_1 = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix},\ g(x,y) = a_1g_1(x,y),$$

$$q(x,y) = p(\alpha_{11}x + \alpha_{12}y, \alpha_{21}x + \alpha_{22}y).$$

Replacing x and y respectively by $\alpha_{11}x + \alpha_{12}y$ and $\alpha_{21}x + \alpha_{22}y$ in $(p_x\ p_y)g(x,y) = (1\ 0)$, easy calculations show that $(p_x\ p_y)a_1$ goes to $(q_x\ q_y)$, i.e. to the gradient of $q(x,y)$ and we obtain a new equation

$$(q_x\ q_y)g'(x,y) = (1\ 0),\ g'(x,y) = g_1(\alpha_{11}x + \alpha_{12}y, \alpha_{21}x + \alpha_{22}y)$$

where the decomposition $g' = c_1a_2\dots c_{k-1}a_k$ starts with c_1. Let ξ be the degree function on $K[x,y,z]$ induced by the lexicographic ordering $x > y > z$. Since

$$(q_x\ q_y) = (1\ 0)a_k^{-1}c_{k-1}^{-1}\dots a_2^{-1}c_1^{-1},$$

and $c_1^{-1} = (e_{11}+e_{22}) - f_1e_{21}$, Lemma 2.1 gives that $\xi(q_x) = \xi(q_y) + \xi(f_1)$. First, we assume that f_1 essentially depends on x. Let, for example, $\xi(q_y) = (d_1,d_2,d_3)$, $\xi(f_1) = (e_1,e_2,e_3)$ and $e_1 > 0$. Then $\xi(q_x) = (d_1+e_1, d_2+e_2, d_3+e_3)$. Since $q_{xy} = q_{yx}$, we obtain that $\xi(q_{xy}) = (d_1+e_1-1, d_2+e_2, d_3+e_3)$ and $\xi(q_{yx}) = (d_1, d_2-1, d_3)$, if $d_2 > 0$, or $\xi(q_{yx}) \le (d_1-1, d_2', d_3')$ for some d_2', d_3', if $d_2 = 0$. Since this is impossible, we derive that f_1 does not depend on x. Let $f(y) \in K[z][y]$ be a polynomial such that $f_y = f_1$. Replacing x by $x + f(y)$ in the equation $(q_x\ q_y)g_1 = (1\ 0)$, we obtain that $(q_x\ q_y)c_1$ goes to the gradient of $p_1(x,y) = q(x+f(y),y)$. In this way we obtain the equation $((p_1)_x\ (p_1)_y)g_2 = (1\ 0)$, where $g_2 = a_2c_2\dots c_{k-1}a_k$. By inductive arguments we may assume that $p_1(x,y)$ is a tame coordinate. Since both

$$A_1 = (\alpha_{11}x + \alpha_{12}y, \alpha_{21}x + \alpha_{22}y), \quad F_1 = (x + f(y), y)$$

are tame automorphisms, we obtain that $p(x,y)$ is also a tame coordinate.

Remark 2.3. The theorem of Wright concerning the structure of the group $GE_2(K[z][x,y])$ holds for any Euclidean domain R instead of $K[z]$. Nevertheless, the proof of Theorem 2.2 does not hold for Euclidean domains R of characteristic 0 which do not contain the field $\mathbb{Q}$. The reason is that in the inductive step of the proof we have used that the integral of the polynomial $f(y) \in K[z][y]$ also belongs to $K[z][y]$. For instance, Theorem 2.2 is no longer true even for $R = \mathbb{Z}$. A simple example is the following. If we replace in the Nagata automorphism

$$N = (x - 2(y^2 + zx)y - (y^2 + zx)^2 z, y + (y^2 + zx)z) \in \mathrm{Aut}K[z][x,y],$$

z by 2, we obtain the automorphism

$$N_0 = (p,q) = (x - 2(y^2 + 2x)y - 2(y^2 + 2x)^2, y + 2(y^2 + 2x)) \in \mathrm{Aut}\mathbb{Z}[x,y],$$

and $p_x = 1 - 4(y + 2(y^2 + 2x))$, $p_y = -4y(y + 2(y^2 + 2x)) - 2(y^2 + 2x)$,

$$p_y - yp_x = -y - 2(y^2 + 2x),\ p_x - 4(p_y - yp_x) = 1.$$

Similarly, $q_x = 4$, $q_y = 1 + 4y$, $q_y - yq_x = 1$. Hence both p and q satisfy the hypothesis of Theorem 2.2 (iii). On the other hand, we shall see that N_0 is a wild automorphism of $\mathbb{Z}[x,y]$. Following the proof of Wright [20, p. 247] for existence of wild automorphisms of $R[x,y]$ where R is a principal ideal domain which is not a field, our considerations are the following. Let

$$I = \{A_i = (\alpha_i x + \beta_i y, \alpha_i' x + \beta_i' y)\}$$

be a set of nontrivial left coset representatives of

$$GL_2(\mathbb{Q}) = \{F = (\alpha x + \beta y, \alpha' x + \beta' y) \mid \alpha, \beta, \alpha', \beta' \in K,\ J_F \text{ invertible}\}$$

considered as a subgroup of $\mathrm{Aut}\mathbb{Q}[x,y]$ modulo $B_2(\mathbb{Q}) = \{(\alpha x + \beta y, \alpha' x) \mid \alpha, \alpha' \in \mathbb{Q}^*, \beta \in \mathbb{Q}\}$. (Comparing with [20], we change left and right for cosets and lower and upper for triangular automorphisms because of the difference in the definitions of $F \circ G$ and J_F.) As in [20], we can choose I as a subset of $GL_2(\mathbb{Z})$. Also let

$$J = \{C = (x + y^2 f(y), y) \mid f \in \mathbb{Q}\}$$

be the set of triangular automorphisms of $\mathbb{Q}[x,y]$ corresponding to polynomials without linear components. Then the set W of all automorphisms

$$P = C_1 \circ A_1 \circ C_2 \circ \dots \circ A_{k-1} \circ C_k \circ B,$$

such that $A_i \in I$, $C_i \in J$, $B \in B_2(\mathbb{Z})$ and $C_1 \circ A_1 \circ \dots \circ A_{l-1} \circ C_l$ is not an automorphism of $\mathbb{Z}[x,y]$ for $l < k$, is a subgroup of $\mathrm{Aut}\mathbb{Z}[x,y]$. The subgroup $\mathrm{Aut}^0\mathbb{Z}[x,y]$ of all preserving the augmentation automorphisms of $\mathbb{Z}[x,y]$ has the presentation

$$\mathrm{Aut}^0\mathbb{Z}[x,y] \cong GL_2(\mathbb{Z}) *_{B_2(\mathbb{Z})} W.$$

In particular, if $P = C_1 \circ A_1 \circ \dots \circ A_{k-1} \circ C_k \in W$ and $k > 1$, then P is a wild automorphism of $\mathbb{Z}[x,y]$. In our case, we choose I in such a way that it contains

$A = (x, y + 4x)$, we define $C = (x + y^2/2, y)$ and a direct calculation shows that $N_0 = C^{-1} \circ A \circ C$. Hence N_0 is a wild automorphism of $\mathbb{Z}[x, y]$.

Corollary 2.4. *There is an algorithm which determines whether a polynomial $p(x, y) \in K[z][x, y]$ is a tame coordinate and, if the answer is affirmative, produces a concrete tame automorphism of $K[z][x, y]$ which sends x to $p(x, y)$.*

Proof. Given a polynomial $p \in K[z][x, y]$, we want to decide whether it is a tame coordinate. The algorithm can be derived from the proof of Theorem 2.2. We fix some ordering in $K[x, y, z]$ (e.g. the lexicographic or the deg-lex ordering satisfying $x > y > z$).

Step 1. Take the partial derivatives $q_1 = p_x$, $q_2 = p_y$.

Step 2. If the leading monomial (l.m.) of q_1 is not divisible by the leading monomial of q_2 (or vise versa), then p is not a tame coordinate. If $\text{l.m.}(q_1) = h \cdot \text{l.m.}(q_2)$ (or $\text{l.m.}(q_2) = h \cdot \text{l.m.}(q_1)$) for some monomial $h \in K[z][x, y]$, then go to

Step 3. Set $q_1' = q_1 - hq_2$ and $q_2' = q_2$ (or $q_1' = q_1$ and $q_2' = q_2 - hq_1$). If, say, $\text{l.m.}(q_1') = \alpha \in K^*$, then p is a tame coordinate in $K[z][x, y]$. If $\text{l.m.}(q_1') = 0$, then p is a tame coordinate if and only if $\text{l.m.}(q_2) = \alpha \in K^*$. If $\text{l.m.}(q_1') \notin K$, then repeat Step 2 upon replacing q_1 with q_1' and q_2 with q_2'.

The algorithm for constructing some tame automorphism of $K[z][x, y]$ which sends x to p is also contained in the proof of Theorem 2.2. For simplicity we assume that we always know that $p = p(x, y)$ is a tame coordinate.

Step 1. Define an automorphism $T = (t_1(x, y), t_2(x, y)) = (x, y)$ of $K[z][x, y]$.

Step 2. Take $Q = (q_1, q_2) = (p_x, p_y)$.

Step 3. If one of the polynomials q_1 or q_2 is equal to a nonzero constant α, then $p = \alpha x + f(y)$ (or $p = \alpha y + f(x)$). Replace T by $T = (t_1(\alpha x + f(y), y), t_2(\alpha x + f(y)))$ (or by $T = (t_1(x, \alpha y + f(x)), t_2(x, \alpha y + f(x))))$. The automorphism T is the inverse of an automorphism of $K[z][x, y]$ which sends x to the initially given $p(x, y)$.

If neither q_1 nor q_2 is a constant, then let the result of the division of q_1 with q_2 be $q_1 = hq_2 + r$ (or $q_2 = hq_1 + r$). Then $h = h(y) \in K[z][y]$ does not depend on x (or $h = h(x) \in K[z][x]$ does not depend on y). Choose $f(y) \in K[z][y]$ such that $f_y = h(y)$ (or $f(x) \in K[z][x]$ such that $f_x = h(x)$). Replace $p(x, y)$ and T respectively by $p(x + f(y), y)$ and $T = (t_1(\alpha x + f(y), y), t_2(\alpha x + f(y), y))$ (or respectively by $p(x, y + f(x))$ and $T = (t_1(x, \alpha y + f(x)), t_2(x, \alpha y + f(x))))$. Then go to Step 2.

If we want to determine all automorphisms sending x to $p(x, y)$, we can use Remark 1.4.

Corollary 2.5. *There is an algorithm which determines whether a polynomial $p(x, y)$ in the free associative algebra $K[z]\langle x, y\rangle$ is a tame coordinate and, if the answer is affirmative, produces a concrete tame automorphism sending x to $p(x, y)$.*

Proof. By the proof of Theorem 1.2 in [18], for every field R and every coordinate $q_0(x, y) \in R[x, y]$ there exists a unique coordinate $q_1(x, y) \in R\langle x, y\rangle$ which maps onto $q_0(x, y)$ under the natural homomorphism $R\langle x, y\rangle \to R[x, y]$. Since every tame automorphism of $K[z][x, y]$ is induced by a tame automorphism of $K[z]\langle x, y\rangle$, this implies that if $p_0(x, y) \in K[z][x, y]$ is a tame coordinate, then it is an image of a unique tame coordinate $p_1(x, y) \in K[z]\langle x, y\rangle$. Our algorithm for recognizing the tame coordinates in $K[z]\langle x, y\rangle$ is the following. If $p(x, y) \in K[z]\langle x, y\rangle$, take its image p_0 under the natural homomorphism $K[z]\langle x, y\rangle \to K[z][x, y]$. Apply the algorithm in Corollary 2.4 to p_0. If p_0 is not a tame coordinate in $K[z][x, y]$, then p

is not a tame coordinate in $K[z]\langle x,y\rangle$. If p_0 is a tame coordinate in $K[z][x,y]$, then find a product of elementary automorphisms $F=(p_0,q_0)=F_1\circ F_2\circ\ldots\circ F_n$ which sends x to p_0. Take the preimage $G=(p_1,q_1)$ of F under the natural isomorphism $\mathrm{Aut}(K(z)\langle x,y\rangle)\to\mathrm{Aut}(K(z)[x,y])$. Then p is a tame coordinate if and only if $p_1=p$.

It is also interesting to characterize effectively all coordinates (in particular, the nontame ones) in $K[z][x,y]$. The following theorem is based on recent results of Daigle and Freudenburg [6]. Recall that a derivation δ of $K[X]=K[x_1,\ldots,x_n]$ is called *locally nilpotent* if for any $u\in K[X]$ there exists a positive n such that $\delta^n(u)=0$. Following [6, Definition 2.2], for any polynomial $p=p(x,y)\in K[z][x,y]$ we define a mapping $\Delta_p : K[z][x,y]\longrightarrow K[z][x,y]$ by

$$\Delta_p=-p_y\frac{\partial}{\partial x}+p_x\frac{\partial}{\partial y},$$

i.e. $\Delta_p(f)$ is equal to the determinant of the Jacobian matrix of the endomorphism $F=(p,f)$ of $K[z][x,y]$, $f\in K[z][x,y]$. It is easy to see that Δ_p is locally nilpotent if p is a coordinate of $K(z)[x,y]$.

Theorem 2.6. *The polynomial $p=p(x,y)\in K[z][x,y]$ is a coordinate if and only if p is a coordinate in $K(z)[x,y]$ with unimodular gradient in $K[z][x,y]$.*

Proof. Clearly, if $p=p(x,y)\in K[z][x,y]$ is a coordinate in $K[z][x,y]$, then it is also a coordinate in $K(z)[x,y]$ and for every automorphism $F=(p,q)$ of $K[z][x,y]$ sending x to p, the determinant of the Jacobian matrix of F is a nonzero constant in K. Hence p is with unimodular gradient and this gives the "easy" part of the theorem.

Now, let $p\in K[z][x,y]$ be a coordinate in $K(z)[x,y]$ and with unimodular gradient in $K[z][x,y]$. Restricted to the case of $K[z][x,y]$, Proposition 2.3 of [6] states that every $p\in K[z][x,y]$ which is a coordinate in $K(z)[x,y]$ and with unimodular gradient in $K[z][x,y]$ has the property that the derivation Δ_p defined above is locally nilpotent and its kernel $\mathrm{Ker}\Delta_p$ is equal to $K[z][p]$. Since

$$\Delta_p(x)=-p_y,\ \Delta_p(y)=p_x,$$

we obtain that the ideals of $K[z][x,y]$ generated respectively by the image of Δ_p and by p_x,p_y coincide and, by the unimodularity of the gradient, are equal to $K[z][x,y]$. By the implication (3) $\Rightarrow$ (2) of [6, Theorem 2.5], every locally nilpotent derivation Δ of $K[z][x,y]$ such that the ideal of $K[z][x,y]$ generated by the image of Δ coincides with $K[z][x,y]$ satisfies also the condition $K[z][x,y]=(\mathrm{Ker}\Delta)[q]$ for some $q\in K[z][x,y]$. Hence

$$K[z][x,y]=(\mathrm{Ker}\Delta)[q]=(K[z][p])[q]=K[z][p,q].$$

Hence the endomorphism $F=(p,q)$ of $K[z][x,y]$ is an automorphism and p is a coordinate in $K[z][x,y]$.

Example 2.7. Let every irreducible factor of $b(z)\in K[z]$ be a divisor of $c(z)\in K[z]$ and let $a(x,z)\in K[x,z]$ be such that $b(z)$ does not divide some coefficient $a_i(z)$, $i>0$, of $a_x=\sum_{i=0}^n a_i(z)x^i$. Then for any $q(u,z)\in K[u,z]$ with $\deg_u q>0$, the polynomial $p(x,y,z)=x+q(a(x,z)+b(z)y,z)c(z)$ is a wild coordinate in $K[z][x,y]$.

Proof. Applying Theorem 2.2, we calculate

$$p_x = 1 + c(z)a_x q_u(a(x,z) + b(z)y, z),$$
$$p_y = b(z)c(z)q_u(a(x,z) + b(z)y, z).$$

It is easy to see that the first steps of the Euclidean algorithm for p_x and p_y in $K[x,y,z]$ are the same as for a_x and $b(z)$ in $K[x,z]$ and in this way we cannot reach 1 because some summand $a_i(z)x^i$ of a_x is not divisible by $b(z)$. Now, we apply Theorem 2.6. Since every irreducible factor of $b(z)$ divides $c(z)$, we obtain that no solution of the equation $p_y = 0$ in the algebraic closure of K is a solution of the equation $p_x = 0$. By the Hilbert Nullstellensatz this implies that the ideal of $K[x,y,z] = K[z][x,y]$ generated by p_x and p_y is equal to $K[z][x,y]$. On the other hand, $p_x - a_x p_y/b = 1$ in $K(z)[x,y]$ and hence (see [17] or our Theorem 2.2), we obtain that p is a coordinate in $K(z)[x,y]$.

Corollary 2.8. *There is an algorithm which determines whether a polynomial $p(x,y) \in K[z][x,y]$ is a coordinate.*

Proof. Given a polynomial $p(x,y) \in K[z][x,y]$, we apply the following procedure.

Step 1. We choose some ordering on $K[x,y,z]$ and calculate the reduced Gröbner basis of the ideal of $K[x,y,z]$ generated by p_x and p_y (see e.g. [2] for the necessary background on Gröbner bases). This ideal coincides with $K[x,y,z]$ if and only if its reduced Gröbner basis consists of one nonzero constant only. Hence, if the obtained reduced Gröbner basis consists of a nonzero constant, then we continue with Step 2. Otherwise, we conclude that p is not a coordinate of $K[z][x,y]$.

Step 2. Working on $K(z)$ instead on $K[z]$, we apply the algorithm of Corollary 2.4 and determine whether p is a coordinate in $K(z)[x,y]$. If the answer is negative, then p cannot be a coordinate in $K[z][x,y]$. Otherwise, p is a coordinate of $K[z][x,y]$.

3. New class of wild automorphisms

For $R = K, K(z), K[z]$, let $\mathrm{Aut}^0 R[x,y]$ be the group of preserving the augmentation automorphisms, i.e. $F = (f_1, f_2) \in \mathrm{Aut}^0 R[x,y]$ means that the polynomials f_1 and f_2 have no constant terms. Every automorphism G of $R[x,y]$ can be decomposed as $G = T(F)$, where $G = (x+a, y+b)$ for some $a, b \in R$ and $F = (f_1, f_2) \in \mathrm{Aut}^0 R[x,y]$. Hence, it is sufficient to study $\mathrm{Aut}^0 R[x,y]$ instead of $\mathrm{Aut} R[x,y]$. By the results of Wright [20], the group $T^0(R[x,y])$ of tame automorphisms of $R[x,y]$ preserving the augmentation has a similar description as a free amalgamated product as $\mathrm{Aut} K[x,y]$. In particular, $T^0(R[x,y])$ is generated by the elementary automorphisms

$$F = (x, y + f(x)), G = (x + g(y), y),\ f(x) \in R[x], g(y) \in R[y], f(0) = g(0) = 0,$$

and the diagonal automorphisms $D_{\alpha,\beta} = (\alpha x, \beta y)$, $\alpha, \beta \in R^*$. Since $D_{\alpha,\beta} = D_{\alpha\beta,1} \circ D_{\beta^{-1},\beta}$ and $D_{\beta^{-1},\beta}$ is a product of elementary linear automorphisms (because $SL_2(R) = E_2(R)$), we obtain that every automorphism P in $T^0(R[x,y])$ has the form $P = H_1 \circ H_2 \circ \ldots \circ H_k \circ D_{\alpha,1}$, where each H_i is an elementary automorphism.

Every augmentation preserving automorphism P of $K[z][x,y]$ can be considered as an automorphism in $\mathrm{Aut}^0 K(z)[x,y]$ and has a presentation $P = H_1 \circ H_2 \circ \ldots \circ H_k \circ D_{\alpha,1}$, where each H_i is an elementary automorphism of $K(z)[x,y]$ and $0 \neq \alpha \in K(z)$. The linear components of P and $H_1 \circ H_2 \circ \ldots \circ H_k$ are in $GL_2(K[z])$ and $SL_2(K(z))$, respectively. This easily implies that $\alpha \in K^*$ and both $D_{\alpha,1}$ and $H_1 \circ H_2 \circ \ldots \circ H_k$ are automorphisms of $K[z][x,y]$. This observation shows that, studying the automorphisms of $K[z][x,y]$, it is sufficient to consider the automorphisms which are products of elementary automorphisms of $K(z)[x,y]$. We state the following naturally arising problem.

Problem 3.1. Let T_k be the subgroup of $\mathrm{Aut}^0 K[z][x,y]$ generated by those automorphisms of $\mathrm{Aut}^0 K[z][x,y]$ which can be presented as a product of not more than k elementary (and preserving the augmentation) automorphisms of $K(z)[x,y]$. Is it true that the sequence of subgroups

$$T_1 \subseteq T_2 \subseteq T_3 \subseteq \dots$$

satisfies the ascending chain condition?

Obviously, every automorphism in $T^0(K[z][x,y])$ is a product of some $D_{\alpha,1}$ and an automorphism in T_1. More generally, every automorphism of $K[z][x,y]$ is a product of a $D_{\alpha,1}$ and an automorphism in T_k. Therefore, the minimal k with this property can serve as a measure "how wild" is the automorphism.

Remark 3.2. It is easy to see that $T_1 = T_2$. Indeed, if $P = F \circ G \in T_2$, where

$$F = (x, y + f(x)),\ G = (x + g(y), y),\ f(x) \in K(z)[x], g(y) \in K(z)[y],$$

then obviously

$$P = (x + g(y), y + f(x + g(y))) \in \mathrm{Aut}K[z][x,y],$$

which means that $g(y) \in K[z][y]$. Hence $F = P \circ G^{-1} \in \mathrm{Aut}K[z][x,y]$ and again we obtain that $f(x) \in K[z][x]$. Therefore, both F and G are tame automorphisms of $K[z][x,y]$ and $P \in T_1$, i.e. $T_1 = T_2$.

Example 3.3. By [15], the automorphism of Nagata

$$N = (x - 2(y^2 + zx)y - (y^2 + zx)^2 z, y + (y^2 + zx)z)$$

considered as an automorphism of $K(z)[x,y]$ has the presentation $N = G^{-1} \circ F \circ G$, where

$$F = (x, y + z^2 x),\ G = (x + \frac{y^2}{z}, y).$$

Since $N \in T_3$ and the Nagata automorphism is wild considered as an automorphism of $K[z][x,y]$, by Remark 3.2 we obtain that T_2 is a proper subgroup of T_3.

Now we start to study the automorphisms in T_3 of the form $P = F^{-1} \circ G \circ F$ (where $F = (x, y + f(x))$ and $G = (x + g(y), y)$ are elementary automorphisms of $K(z)[x,y]$. (Up to the order of the variables these automorphisms are "Nagata like".) Since

$$P = (x + g(y + f(x)), y + f(x) - f(x + g(y + f(x)))),$$

we shall also describe the corresponding coordinate polynomials $x + g(y + f(x))$.

Lemma 3.4. *Let $f(x) \in K(z)[x]$, $g(y) \in K(z)[y]$ be such that $f(0) = g(0) = 0$ and $g(y + f(x)) \in K[z][x,y]$. Then $f(x)$ and $g(y)$ have the presentations*

$$f(x) = \frac{a(x)}{b(z)},\ g(y) = d(b(z)y),$$

where $a(x) \in K[z][x]$, $d(y) \in K[z][y]$, $a(0) = d(0) = 0$, and $b(z) \in K[z]$.

Proof. We rewrite $f(x)$ in the form $f(x) = a(x)/b(z)$, where $a = a(x) \in K[z][x]$ (and $a(0) = 0$), $b(z) \in K[z]$ and $a(x) = a(x,z)$ and $b(z)$ are relatively prime in $K[x,z]$. In the formula

$$g(y + f(x)) = g\left(y + \frac{a(x)}{b(z)}\right) = \sum_{k=0}^{n} \frac{g^{(k)}(y)a^k(x)}{k!b^k(z)} \in K[z][x,y],$$

where $g^{(k)} = g^{(k)}(y)$ is the k-th derivative of $g(y)$ (with respect to y), we replace x by 0 and obtain

$$g(y) = g(y + f(0)) \in K[z][y].$$

Hence

$$\sum_{k=1}^{n} \frac{g^{(k)}(y)a^k(x)}{k!b^k(z)} \in K[z][x,y],$$

$$a\left(\frac{g'b^{n-1}}{1!} + \frac{g''ab^{n-2}}{2!} + \dots + \frac{g^{(n-1)}a^{n-2}b}{(n-1)!} + \frac{g^{(n)}a^{n-1}}{n!}\right) \equiv 0 \pmod{b^n},$$

and since a and b are relatively prime, we obtain that

$$\frac{g'b^{n-1}}{1!} + \frac{g''ab^{n-2}}{2!} + \dots + \frac{g^{(n-1)}a^{n-2}b}{(n-1)!} + \frac{g^{(n)}a^{n-1}}{n!} \equiv 0 \pmod{b^n}.$$

Replacing x by 0, we obtain that $g'b^{n-1} \equiv 0 \pmod{b^n}$, hence $g' \equiv 0 \pmod{b}$ and the coefficient of y^k in $g(y)$ is divisible by $b(z)$ for $k \geq 1$. Let $g'(y) = b(z)g_1(y)$ for some $g_1(y) \in K[z][y]$. Now, our congruence has the form

$$a\left(\frac{g_1'b^{n-2}}{2!} + \frac{g_1''ab^{n-3}}{3!} + \dots + \frac{g_1^{(n-2)}a^{n-3}b}{(n-1)!} + \frac{g_1^{(n-1)}a^{n-2}}{n!}\right) \equiv 0 \pmod{b^{n-1}}.$$

Again, the coefficient of y^k in $g_1(y)$ is divisible by $b(z)$ for $k \geq 1$ and hence the coefficient of y^{k+1} in $g(y)$ is divisible by $b^2(z)$ for any $k \geq 1$. Continuing in this way, we obtain that the coefficient of y^k in $g(y)$ is divisible by $b^k(z)$ for all k, i.e. $g(y) = d(b(z)y)$ for some $d(y) \in K[z][y]$.

Lemma 3.5. *Let $a(x) \in K[z][x]$, $d(y) \in K[z][y]$, $b(z) \in K[z]$, $a(0) = d(0) = 0$, and let $a(x) = a(x,z)$ and $b(z)$ be relatively prime in $K[x,z]$. If the polynomial $a(x + d(a(x) + b(z)y)) - a(x)$ is divisible by $b(z)$, then the polynomial $d(y)$ is also divisible by $b(z)$.*

Proof. Let $b(z) = \prod_{i=1}^{s} b_i^{p_i}(z)$, where $b_1(z), \dots, b_s(z)$ are the different irreducible factors of $b(z)$. It is sufficient to show that $d(y)$ is divisible by all polynomials $b_i^{p_i} = b_i^{p_i}(z)$. Since $b_i^{p_i}$ divides the polynomial $a(x + d(a(x) + b(z)y)) - a(x)$, it divides also $a(x + d(a(x))) - a(x)$. We want to show that the polynomial $d(y)$ is also divisible by $b_i^{p_i}$, assuming that $a(x)$ is relatively prime with b_i. Working modulo $b_i^{p_i}$, we obtain that

$$0 \equiv a(x + d(a(x))) - a(x) \equiv \sum_{k=1}^{n} \frac{a^{(k)}(x)}{k!} d^k(a(x)) \pmod{b_i^{p_i}}.$$

Let $a(x) \equiv x^q a_1(x) \pmod{b_i}$, $a_1(x) \in K[z][x]$, and let q be the maximal integer with this property, i.e. $a_1(0) \not\equiv 0 \pmod{b_i}$. Since $a(0) = 0$, we have that $q \geq 1$. Let us first assume that $d(y)$ is not divisible by b_i. Let r be the maximal integer with $d(y) \equiv y^r d_1(x) \pmod{b_i}$, $d_1(y) \in K[z][y]$, $d_1(0) \not\equiv 0 \pmod{b_i}$. Again $r \geq 1$. For

$k \leq q$ the derivative $a^{(k)}(x)$ has the form $a^{(k)}(x) \equiv x^{q-k}c_k(x) \pmod{b_i}$, $c_k(x) \in K[z][x]$, $c_k(0) \not\equiv 0 \pmod{b_i}$. Hence

$$a^{(k)}(x)d^k(a(x)) \equiv x^{q-k}c_k(x)((x^q a_1(x))^r d_1(x^q a_1(x)))^k \equiv x^{q-k+kqr}e_k(x) \pmod{b_i},$$

where $e_k(x) \in K[z][x]$ and $e_k(0) \not\equiv 0 \pmod{b_i}$. For $k > q$ we have that

$$a^{(k)}(x)d^k(a(x)) \equiv a^{(k)}((x^q a_1(x))^r d_1(x^q a_1(x)))^k \pmod{b_i},$$

and $a^{(k)}(x)d^k(a(x))$ is divisible by x^{kqr} modulo b_i. If $qr > 1$, then the degrees t_k of x^{t_k} dividing $a^{(k)}(x)d^k(a(x))$ modulo b_i satisfy the inequalities

$$t_1 = q - 1 + qr < t_2 = q - 2 + 2qr < \ldots < t_q = q^2 r < (q+1)qr \leq t_{q+1}, \ldots, t_n.$$

Hence $a(x + d(a(x))) - a(x) \equiv x^{q-1+qr}a_0(x) \pmod{b_i}$, $a_0(x) \in K[z][x]$ and $a_0(0) \equiv e_1(0) \not\equiv 0 \pmod{b_i}$. But this contradicts the assumption that $a(x+d(a(x)))-a(x) \equiv 0 \pmod{b_i}$. The case $qr = 1$, i.e. $q = r = 1$, is similar. Again $a'(x)d(a(x)) \equiv xc_1(x) \pmod{b_i}$ with $c_1 \in K[z][x]$, $c_1(0) \not\equiv 0 \pmod{b_1}$ and $a^{(k)}(x)d^k(a(x))$ is divisible by x^2 modulo b_i for $k > 1$. Again $a(x+d(a(x)))-a(x) \equiv xa_0(x) \pmod{b_i}$, $a_0(x) \in K[z][x]$ and $a_0(0) \equiv e_1(0) \not\equiv 0 \pmod{b_i}$ which is a contradiction. Hence in all cases we have that b_i divides $d(y)$. Let $d(y) = b_i^t d_0(y)$, where $t \geq 1$ and $d_0(y) \in K[z][y]$ is relatively prime with b_i. Then the congruence

$$\sum_{k=1}^{n} \frac{a^{(k)}(x)}{k!} d^k(a(x)) \equiv 0 \pmod{b_i^{p_i}}$$

has the form

$$\sum_{k=1}^{n} \frac{a^{(k)}(x)}{k!} d_0^k(a(x)) b_i^{kt} \equiv 0 \pmod{b_i^{p_i}}$$

and, if $t < p_i$, we obtain that $a'(x)d_0(a(x)) \equiv 0 \pmod{b_i}$. Since $a(x)$ is relatively prime with b_i and $a(0) = 0$, we obtain that $a'(x)$ is also relatively prime with b_i and, hence, b_i divides $d_0(a(x))$. But, as above, we see that this is impossible. Therefore $t \geq p_i$ and $b_i^{p_i}$ divides $d(x)$. Hence b also divides $d(y)$.

The derivation δ is called *triangular* if $\delta(x_i) \in K[x_{i+1}, \ldots, x_n]$, $i = 1, \ldots, n$. Clearly, every triangular derivation is locally nilpotent. For every locally nilpotent derivation δ and for any element w in the kernel of δ, the mapping of $K[X]$

$$\exp(w\delta) = \sum_{n \geq 0} \frac{w^n \delta^n}{n!}$$

is a well defined automorphism of $K[X]$. Finally, the automorphism $F = (f_1, \ldots, f_n)$ of $K[x_1, \ldots, x_n]$ is *stably tame* if its extension $(f_1, \ldots, f_n, x_{n+1}, \ldots, x_{n+m})$ to an automorphism of $K[x_1, \ldots, x_{n+m}]$ for some $m > 0$ is a tame automorphism of $K[x_1, \ldots, x_{n+m}]$. Similarly, one can define stably tame automorphisms of $R[x_1, \ldots, x_n]$ for any commutative algebra R.

The next result describes the tame and the wild automorphisms of the form $P = F^{-1} \circ G \circ F \in T_3$ (where $F = (x, y + f(x))$ and $G = (x + g(y), y)$ are elementary automorphisms of $K(z)[x,y]$) and shows that all automorphisms of this form are stably tame, i.e. they are really "Nagata like". Also, it turns out that these automorphisms are as those considered in Example 2.7.

Theorem 3.6. *Let $0 \neq f(x) \in K(z)[x]$, $0 \neq g(y) \in K(z)[y]$, where $f(0) = g(0) = 0$ and $f(x) = a(x)/b(z)$, $a(x) \in K[z][x]$, $b(z) \in K[z]$, the polynomials a and b being relatively prime in $K[x, z]$. Define the automorphisms F and G of $K(z)[x, y]$ by*

$$F = (x, y + f(x)),\ G = (x + g(y), y).$$

(i) *The composition $P = F^{-1} \circ G \circ F$ is an automorphism of $K[z][x, y]$ if and only if $g(y) = b(z)q(b(z)y)$ for some $q(y) \in K[z][y]$. In this case P is equal to the automorphism $\exp(w\delta)$, where δ is the triangular derivation of $K[z][x, y]$ defined by*

$$\delta(x) = b(z),\ \delta(y) = -a'(x)$$

and $w = q(a(x) + b(z)y)$ is in the kernel of δ.

(ii) *Let $P = F^{-1} \circ G \circ F$ be an automorphism of $K[z][x, y]$. Then P is wild (as an automorphism of $K[z][x, y]$) if and only if the coefficient of x^k in $f(x)$ is not in $K[z]$ for some $k > 1$.*

(iii) *If $P = F^{-1} \circ G \circ F \in \mathrm{Aut}K[z][x, y]$, then P is stably tame and becomes tame as an automorphism of $K[z][x, y, t]$ (acting identically on t).*

Proof. (i) Direct computations show that

$$P = (p_1, p_2) = (x + g(y + f(x)), y + f(x) - f(x + g(y + f(x)))).$$

If P is an automorphism of $K[z][x, y]$, then $p_1, p_2 \in K[z][x, y]$. Applying Lemma 3.4 to p_1, we obtain that $g(y) = d(b(z)y)$ for some $d(y) \in K[z][y]$ and $b(z) \in K[z]$. Now Lemma 3.5 gives that $d(y) = b(z)q(y)$, $q(y) \in K[z][y]$.

Now, let $g(y) = b(z)q(b(z)y)$ for some $q(y) \in K[z][y]$. We define the derivation δ of $K[z][x, y]$ by $\delta(x) = b(z)$, $\delta(y) = -a'(x)$. Clearly, δ is triangular. It is easy to verify that $w = q(a(x) + b(z)y)$ is in the kernel of δ. Hence $\exp(w\delta)$ is an automorphism of $K[z][x, y]$ and we have to show that P and $\exp(w\delta)$ are equal on the generators x and y of $K[z][x, y]$. Again, direct calculations show that

$$\exp(w\delta)(x) = x + w\delta x = x + bq(a(x) + by) = p_1,$$

$$\exp(w\delta)(y) = y + \sum_{k\geq 1} \frac{w^k\delta^k y}{k!} = y - \frac{1}{b}\sum_{k\geq 1} \frac{a^{(k)}(x)b^kq^k(a(x) + by)}{k!}$$

$$= y + \frac{1}{b}\,(a(x) - a(x + bq(a(x) + by)) = p_2.$$

(ii) Since $P \in \mathrm{Aut}K[z][x, y]$, by (i) we obtain that $g(y) = b(z)q(b(z)y)$ for some $q(y) \in K[z][y]$ and

$$P = \left(x + bq(a(x) + by), y + \frac{1}{b}(a(x) - a(x + bq(a(x) + by)))\right).$$

Hence $x + bq(a(x) + by)$ is a coordinate in $K[z][x, y]$. If the coefficient of x^k in $f(x)$ is not in $K[z]$ for some $k > 1$, this means that the coefficient of x^{k-1} of $a_x(x, z)$ is not divisible by $b(z)$ and Example 2.7 shows that P is a wild automorphism of $K[z][x, y]$. Now, let the coefficients of x^k in $f(x)$ be in $K[z]$ for all $k > 1$, i.e. $f(x) = c(z)/b(z)x + x^2f_0(x, z)$, where $c(z)$ and $b(z)$ are relatively prime in $K[z]$ and $f_0(x, z) \in K[x, z]$. If $Q = (x, y - x^2f_0(x, z))$, then the automorphism P is tame if and only $P_1 = Q^{-1} \circ P \circ Q = F_1^{-1} \circ G \circ F_1$ is tame, where $F_1 = (x, y + c(z)x/b(z))$. In this case P_1 is equal to $\exp(w\delta)$, where $w = q(c(z)x + b(z)y)$ and the derivation

δ satisfies $\delta(x) = b(z)$, $\delta(y) = -c(z)$. Since $b(z)$ and $c(z)$ are relatively prime in $K[z]$, there exist $b'(z), c'(z) \in K[z]$ such that $b'c - c'b = 1$, i.e. the endomorphism

$$S = (x_1, y_1) = (cx + by, c'x + b'y)$$

of $K[z][x, y]$ is a linear (and hence tame) automorphism. Now we calculate P_1 with respect to the generators x_1, y_1:

$$\delta(x_1) = 0,\ \delta(y_1) = c'b - b'c = -1,\ w = q(x_1),$$

$$P_1(S) = P_1(x_1, y_1) = (\exp(w\delta)(x_1), \exp(w\delta)(y_1)) = (x_1, y_1 - q(x_1)).$$

Since P_1 is elementary with respect to the tame system of generators x_1, y_1, we obtain that P_1 and hence P are tame.

(iii) The statement follows immediately from the proof of the result of Martha Smith [19] that for any triangular derivation δ of $K[X] = K[x_1, \dots, x_n]$ and any $w \in \mathrm{Ker}\delta$, the automorphism $\exp(w\delta)$ is stably tame. If we extend our δ to a derivation of $K[z][x, y, t]$ by $\delta(t) = 0$, we obtain that $w = q(a(x) + by)$ is still in the kernel of δ,

$$\exp(w\delta)(x, y, t) = (p_1, p_2, t) = Q^{-1} \circ E \circ Q \circ E^{-1},$$

where $Q = (x, y, t + w)$, $E = \exp(t\delta)$, and Q and E are tame automorphisms of $K[z][x, y, t]$.

Remark 3.7. Wright [20], in the proof of Corollary of Theorem 5, p. 247, gave a family of wild automorphisms of $R[x, y]$, where R is a principal ideal domain. For $R = K[z]$ his automorphisms are given by

$$P = (x + b^3(y + \frac{x^2}{b})^2, y + \frac{x^2}{b} - \frac{1}{b}(x + b^3(y + \frac{x^2}{b})^2)^2),$$

where $b = b(z) \in K[z]$ is not a constant. All these automorphisms can be obtained from Theorem 3.6 for

$$f(x) = \frac{x^2}{b},\ g(y) = b(by)^2.$$

Theorem 3.6 (iii) has the following generalization which is obtained by multiple application of the idea of Martha Smith [19] and gives new stably tame automorphisms. Notice that for $A = (y, x)$ and the triangular automorphism $F = (x, y + f(x))$, we have that $A^{-1} = A$ and the conjugate $A \circ F \circ A$ is equal to the automorphism $(x + f(y), y)$ which is triangular with respect to another ordering of the variables.

Theorem 3.8. *Let $a_i \in K[x, z]$, $b_i \in K[z]$, $f_i = a_i/b_i$, $i = 1, \dots, k$, $q \in K[y, z]$ be any polynomials and let $A = (y, x)$, $F_i = (x, y + f_i(x))$, $i = 1, \dots, k$, be automorphisms of $K(z)[x, y]$. Then there exist sufficiently large positive integers $m_1, \dots, m_k$ such that for $G = (x + b_1^{m_1} \dots b_k^{m_k} q(y), y)$ the mapping*

$$P = F_k^{-1} \circ A \circ F_{k-1}^{-1} \circ A \circ \dots \circ F_1^{-1} \circ G \circ F_1 \circ A \circ \dots \circ F_{k-1} \circ A \circ F_k$$

is a stably tame automorphism of $K[z][x, y]$.

Proof. Let $q(t) = \sum_{j=0}^{n} c_j(z)y^j$ for some polynomials $c_j \in K[z]$. Let r_i, s_i, $i = 1,\dots,k$, be positive integers and let $m_i = r_i + (n+1)s_i$. Clearly,

$$b_1^{m_1}q(y) = b_1^{s_1}\sum_{j=0}^{n}(c_j b_1^{r_1+(n-j)s_1})(b_1^{s_1}y)^j = b_1^{s_1}q_1(b_1^{s_1}y),$$

where $q_1(y) = \sum_{j=0}^{n} c_j b_1^{r_1+(n-j)s_1}y^j \in K[z][y]$. Similarly, we obtain that

$$b_1^{m_1}\dots b_k^{m_k}q(y) = b_1^{s_1}\dots b_k^{s_k}q_k(b_1^{s_1}\dots b_k^{s_k}y)$$

for some $q_k \in K[z][y]$. As in the proof of Theorem 3.6 (iii), for $k = 1$ we fix $s_1 = 1$ and obtain that

$$F_1^{-1}\circ G\circ F_1 = \exp(w\delta),$$

where $\delta = b_2^{s_2}\dots b_k^{s_k}\delta_0$, $\delta_0(x) = -b_1$, $\delta_0(y) = (a_1)_x$,

$$w = b_2^{s_2}\dots b_k^{s_k}q_k(b_2^{s_2}\dots b_k^{s_k}(a(x)+b_1y)) \in \mathrm{Ker}\delta,$$

and, extending its action on $K[z][x,y,t_1]$, the automorphism $F_1^{-1}\circ G\circ F_1$ is expressed in terms of the triangular automorphisms

$$\exp(b_2^{s_2}\dots b_k^{s_k}t_1\delta_0),\ (x,y,z,t_1 + b_2^{s_2}\dots b_k^{s_k}q_k(b_2^{s_2}\dots b_k^{s_k}(a(x)+b_1y)),$$

and their inverses. Continuing in this way, we obtain that the automorphism $A\circ F_{k-1}^{-1}\circ\dots\circ G\circ\dots\circ F_{k-1}\circ A$ becomes tame in $K[z][X]$ for $X = \{x_1,\dots x_{k+1}\} = \{x,y,t_1,\dots,t_{k-1}\}$ and is a composition of automorphisms of the type

$$G_i = (x_1 + b_k^{s_k}u_i(x_2,\dots,x_{k+1}), x_2,\dots,x_{k+1})$$

for some polynomials $u_i \in K[z][x_2,\dots,x_{k+1}]$ and up to a permutation of the variables. Hence we have to show that $F_k^{-1}\circ G_i\circ F_k$ is an automorphism of $K[z][x_1,\dots,x_{k+1}]$ which becomes stably tame in $K[z][x_1,\dots,x_{k+2}]$, where s_k is big enough and F_k is of one the following types (again up to a permutation of the variables):

$$F_k = (x_1 + a_k(x_2)/b_k, x_2,\dots,x_{k+2}),$$

$$F_k = (x_1, x_2 + a_k(x_3)/b_k, x_3,\dots,x_{k+2}),$$

$$F_k = (x_1, x_2 + a_k(x_1)/b_k, x_3,\dots,x_{k+2}).$$

Direct calculations show that in the first two cases the composition $F_k^{-1}\circ G_i\circ F_k$ is triangular for s_k sufficiently large, hence tame. In the third case we may consider $F_k^{-1}\circ G_i\circ F_k$ as an automorphism of $K(x_3,\dots,x_{k+1})[z][x_1,x_2]$ which keeps invariant $K[x_3,\dots,x_{k+1},z][x_1,x_2]$. As for $k=1$ we obtain that $F_k^{-1}\circ G_i\circ F_k$ is a composition of tame automorphisms of $K[x_3,\dots,x_{k+1},z][x_1,x_2,x_{k+2}]$ and this completes the proof.

Theorem 3.6 shows that the polynomials

$$p(x,y) = x + bq(a(x)+by),\ a(x)\in K[z][x],\ b(z)\in K[z],\ q(u)\in K[z][u],$$

are images of x under automorphisms of $K[z][x,y]$ of a very special form. Now we are interested in the more general polynomials as those considered in Example 2.7. We study the polynomials of the form

$$p(x,y) = x + d(a(x)+by),$$

where $a(x) \in K[z][x]$, $b(z) \in K[z]$, $d(u) \in K[z][u]$ and $a(0) = d(0) = 0$. If $e(z) \in K[z]$ is a divisor of $b(z)$ and $a(x)$, we may replace $a(x)$ with $a_1(x) = a(x)/e(z)$, $b(z)$ with $b_1(z) = b(z)/e(z)$, $d(u)$ with $d_1(u) = d(e(z)u)$ and without loss of generality we may assume that $a(x)$ and $b(z)$ are relatively prime in $K[x, z]$. Besides, if $p(x, y) \in K[z][x, y]$ is a coordinate polynomial, then it has a unimodular gradient with respect to x and y. In our case this means that

$$p_x = 1 + a_x(x)d_u(a(x) + by),\ p_y = bd_u(a(x) + by)$$

generate $K[z][x, y]$ as an ideal. By the Hilbert Nullstellensatz this is equivalent with the statement that $p_x = p_x(x, y, z)$ and $p_y = p_y(x, y, z)$ have no common zeros (x_0, y_0, z_0) in the algebraic closure of K. Since $a(x, z)$ and $b(z)$ are relatively prime and $a(0, z) = 0$, we obtain that $a_x(x)$ is also relatively prime with $b(z)$. If z_0 is a zero of $b(z)$ and $d(u) = d(u, z)$ does not vanish for $z = z_0$, we derive that $p_x(x, y, z_0) = 1 + a_x(x, z_0)d_u(a(x, z_0))$ is a nonzero polynomial of x. The only case when p_x and p_y have no common zeros is when this polynomial is a constant, i.e. $d(u)$ has the form $d(u) = e(z)u + b_0(z)u^2d_1(u)$, where $e(z) \in K[z]$ is relatively prime with $b(z)$ and $b_0(z)$ is the product of all irreducible components of $b(z)$. Hence, with this exception only, we may assume that the polynomial $d(u)$ is divisible by $b_0(z)$, i.e. is of the form considered in Example 2.7. Finding an automorphism which sends x to $p(x, y)$, we also bring some light on the automorphisms of $K[z][x, y]$ which are products of three elementary automorphisms of $K(z)[x, y]$.

Lemma 3.9. *Let $P = (p_1, p_2)$ be an automorphism of $K(z)[x, y]$ such that $p_1, p_2 \in K[z][x, y]$. Then P is an automorphism of $K[z][x, y]$ if its Jacobian matrix is invertible in $GL_2(K[z][x, y])$.*

Proof. We make use of the Keller theorem (see [11] or [3]): If P is an endomorphism of $K[X]$ which induces an automorphism of $K(X)$ and the Jacobian matrix J_P of P is invertible in $GL_n(K[X])$, then P is an automorphism of $K[X]$. In our case, since P fixes z, the determinant of the Jacobian matrix of P considered as an endomorphism of $K[x, y, z]$ coincides with the determinant of the Jacobian matrix considering P as an endomorphism of $K[z][x, y]$. The proof is completed by the Keller theorem because P induces an automorphism of $K(x, y, z)$.

Theorem 3.10. *Let $0 \neq a(x) := a(x, z) \in K[z][x]$, $0 \neq q(u) := q(u, z) \in K[z][u]$, $a(0) = q(0) = 0$, $0 \neq b(z) \in K[z]$ and let $a(x, z), b(z)$ be relatively prime in $K[x, z]$. Let $b_0(z)$ be the product of all irreducible factors of $b(z)$. Then the polynomial*

$$p(x, y) = x + b_0q(a(x) + by) := x + b_0(z)q(a(x, z) + b(z)y, z)$$

is an image of x under an automorphism of $K[z][x, y]$ which is a product of three elementary automorphisms of $K(z)[x, y]$. Moreover, $p(x, y)$ is a wild coordinate if $b(z)$ does not divide the coefficient of x^i in $a_x(x)$ for some $i > 0$.

Proof. We shall find a polynomial $c(v) \in K[z][v]$, $c(0) = 0$, such that $P = F_2 \circ G \circ F_1 = (p, r)$ is an automorphism of $K[z][x, y]$ for some $r \in K[z][x, y]$, where

$$F_1 = \left(x, y + \frac{a(x)}{b}\right),\ F_2 = \left(x, y + \frac{c(x)}{b}\right),\ G = (x + b_0q(by), y)$$

are automorphisms of $K(z)[x, y]$. Direct calculations show that

$$P = (p(x, y), r(x, y)) = \left(x + b_0q(a(x) + by), y + \frac{1}{b}\left(a(x) + c(x + b_0q(a(x) + by))\right)\right)$$

and the Jacobian matrix of P is equal to

$$J_P = \begin{pmatrix} 1+b_0q_ua_x & b_0q_ub \\ \frac{1}{b}(a_x+c_v(1+b_0q_ua_x)) & 1+c_vb_0q_u \end{pmatrix},$$

where $a_x = a_x(x)$, $q_u = q_u(a(x)+by)$, $c_v = c_v(x+b_0q(a(x)+by))$. It is easy to see that the determinant of J_P is equal to 1. By Lemma 3.9, the proof of the theorem will be completed if we show that $r(x,y) \in K[z][x,y]$ for some $c(v) \in K[z][v]$. Clearly, $r(x,y)$ is in $K[z][x,y]$ if and only if

$$a(x) + c(x + b_0(z)q(a(x) + b(z)y)) \equiv 0 \pmod{b(z)}.$$

This is equivalent to $a(x) + c(x + b_0(z)q(a(x))) \equiv 0 \pmod{b(z)}$. Since $b(z)$ is a divisor of $b_0^n(z)$ for some n, it is sufficient to show that the congruence

$$a(x) + c(x + b_0(z)q(a(x))) \equiv 0 \pmod{b_0^n(z)}$$

has a solution $c(v) \in K[z][v]$ for every $n > 0$. Clearly, the congruence has a solution $c_1(v) \equiv -a(v) \pmod{b_0(z)}$. Let us assume that we have found a solution $c_m(v)$ modulo $b_0^m(z)$. Then writing $c(v)$ in the form $c(v) = c_m(v) + b_0^m(z)d(v)$, we see that the congruence

$$a(x) + c_m(x + b_0(z)q(a(x))) + b_0^m(z)d(x + b_0(z)q(a(x))) \equiv 0 \pmod{b_0^{m+1}(z)}$$

has a solution $d(v) \in K[z][v]$, namely

$$d(v) \equiv -\frac{1}{b_0^m(z)}(a(v) + c_m(v + b_0(z)q(a(v)))) \pmod{b_0(z)}.$$

Remark 3.11. The proof of Theorem 3.10 gives a description of the generators of the group T_3 defined in Problem 3.1. It is generated by all elementary automorphisms of $K[z][x,y]$ and the automorphisms P defined by

$$P = (p,r) = \left(x + q(a(x) + b(z)y), y + \frac{a(x)}{b(z)} + \frac{1}{d(z)}c(x + q(a(x) + b(z)y))\right),$$

where $a(x), b(z), c(v), d(z), q(u)$ are nonzero polynomials, $a(x) \in K[z][x]$, $b(z), d(z) \in K[z]$, $c(v) \in K[z][v]$, $q(u) \in K[z][u]$, $a(0) = c(0) = q(0) = 0$, such that $r \in K[z][x,y]$. The careful study of the proof of Theorem 3.10 gives that it is sufficient to choose in the set of generators of T_3 only one P for each coordinate polynomial r, i.e. fixing r we fix also $c(v)$ and $d(z)$.

Up till now, all known stably tame automorphisms of $K[X]$ can be obtained by the method of Martha Smith [19] and are exponential automorphisms of locally nilpotent derivations; see [7]. The automorphisms of Theorem 3.6 and 3.8 are also of this kind. We do not know whether the automorphisms involved in Theorem 3.10 can be obtained as compositions (considered as automorphisms of $K[z][x,y]$) of stably tame exponential automorphisms of locally nilpotent derivations.

Problem 3.12. (i) Are the automorphisms in the group $T_3 \subset \mathrm{Aut}K[z][x,y]$ stably tame?

(ii) Are all automorphisms of $K[z][x,y]$ stably tame?

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