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TRACIALLY AF C^*-ALGEBRAS

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Abstract. Inspired by a paper of S. Popa and the classification theory of nuclear C^*-algebras, we introduce a class of C^*-algebras which we call tracially approximately finite dimensional (TAF). A TAF C^*-algebra is not an AF-algebra in general, but a "large" part of it can be approximated by finite dimensional subalgebras. We show that if a unital simple C^*-algebra is TAF then it is quasidiagonal, and has real rank zero, stable rank one and weakly unperforated K_0-group. All nuclear simple C^*-algebras of real rank zero, stable rank one, with weakly unperforated K_0-group classified so far by their K-theoretical data are TAF. We provide examples of nonnuclear simple TAF C^*-algebras. A sufficient condition for unital nuclear separable quasidiagonal C^*-algebras to be TAF is also given. The main results include a characterization of simple rational AF-algebras. We show that a separable nuclear simple TAF C^*-algebra A satisfying the Universal Coefficient Theorem and having $K_1(A) = 0$ and $K_0(A) = \mathbf{Q}$ is isomorphic to a simple AF-algebra with the same K-theory.

1. Introduction

In recent years there has been an explosion of results in the classification of (simple) nuclear C^*-algebras of real rank zero, starting with G. A. Elliott's paper [Ell2]. For example, all separable nuclear purely infinite simple C^*-algebras satisfying the Universal Coefficient Theorem ([KP], [K1] and [Ph]) are classified by their K-theory, and simple nuclear C^*-algebras in a large class of C^*-algebras of real rank zero and stable rank one with weakly unperforated K_0 are also classified by their K-theory data ([EG]; see also [G] and [D1]). The second class of simple C^*-algebras mentioned above is constructed from direct limits of so-called homogeneous C^*-algebras with slow dimension growth. Since it is not known whether there are any simple C^*-algebras of real rank zero which are neither purely infinite nor of stable rank one, the next goal in classification theory is to classify all separable nuclear simple C^*-algebras (satisfying the UCT) of real rank zero and stable rank one. To do this one needs to establish a classification result for C^*-algebras that are not assumed to be direct limits of some special form. However, recent developments suggest that one may further assume that simple C^*-algebras are quasidiagonal (see [BK1] and [Po]). It is certainly very important to have an abstract characterization of the second class of nuclear simple C^*-algebras mentioned above. A recent result of S. Popa [Po] says that unital separable simple quasidiagonal C^*-algebras have some very interesting local approximation properties. For

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example, he shows that a unital separable simple C^*-algebra A with sufficiently many projections (for example, A has real rank zero) is quasidiagonal if and only if it has the following property: for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exists a finite dimensional C^*-subalgebra $F \subset A$ with $1_F = p$ such that, for all $a \in \mathcal{F}$,

(1) $\|pa - ap\| < \varepsilon$, and

(2) $pap \in_\varepsilon F$.

We say that a unital C^*-algebra A is *tracially approximately finite dimensional* (TAF) if, in addition,

(3) $1 - p$ is arbitrarily "smaller" than p (see 2.1 for the formal definition).

Recall that a C^*-algebra A is said to be AF if, for any $\varepsilon > 0$ and any finite subset $\mathcal{F}$, there exists a finite dimensional C^*-subalgebra $B \subset A$ such that $\mathcal{F} \subset_\varepsilon B$. Roughly speaking, a C^*-algebra is TAF if a "large" part of it is AF. If we think AF-algebras are C^*-algebras which can be approximated by finite dimensional C^*-subalgebras in norm, then TAF-algebras can be viewed as C^*-algebras which can be approximated by finite dimensional C^*-subalgebras in "measure" (or rather in trace). It turns out that this notion is very useful to classification theory. In fact, all simple nuclear C^*-algebras classified in [EG] are TAF. In particular, many TAF C^*-algebras are not AF. Moreover, for any countable weakly unperforated graded ordered group $(G_0, (G_0)_+, G_1)$ with the Riesz decomposition property, there exists a separable nuclear simple TAF C^*-algebra A with $(K_0(A), K_0(A)_+, K_1(A)) = (G_0, (G_0)_+, G_1)$. On the other hand, we show in Section 3 that every simple TAF C^*-algebra is quasidiagonal and has real rank zero, stable rank one and weakly unperforated K_0. These results suggest that, perhaps, the class of separable nuclear simple TAF C^*-algebras is the right class to study (as the replacement for the class of separable nuclear simple C^*-algebras of real rank zero and stable rank one). It is certainly tempting to conjecture that every quasidiagonal simple C^*-algebra of real rank zero, stable rank one and with weakly unperforated K_0 is TAF. One of the main results of this paper is the following characterization of rational simple AF-algebras: A unital nuclear separable simple TAF C^*-algebra A satisfying the Universal Coefficient Theorem and with $K_1(A) = 0$ and $K_0(A) = \mathbf{Q}$ (the additive ordered group of rational numbers) is isomorphic to a unital simple AF-algebra with the same K-theory. We regard this as the first step in classifying nuclear simple C^*-algebras of real rank zero without assuming any special inductive limit structure. We would like to point out that the requirement that A satisfies the UCT is a natural condition, since we want to use K-theory as the invariant. One should also note that there are no known examples of nuclear C^*-algebras that do not satisfy the UCT, and all C^*-algebras in the so-called "bootstrap" class satisfy the UCT.

We also give a class of unital simple C^*-algebras which includes many non-nuclear simple TAF C^*-algebras. In particular, using examples given in [D2], we show that there are unital separable simple TAF C^*-algebras with unique normalized trace that are not nuclear. For separable simple nuclear quasidiagonal C^*-algebras, a sufficient condition for TAF is given.

The paper is organized as follows. In Section 2, we give the definition of tracially approximately finite dimensional C^*-algebras. We show that all simple C^*-algebras of real rank zero that are classified in [EG] are in fact TAF. Some elementary properties of TAF C^*-algebras are discussed. We show that, under a suitable assumption, ideals and quotients of TAF C^*-algebras are TAF. In Section 3, we discuss simple TAF C^*-algebras. We show that any unital simple TAF C^*-algebra is quasidiagonal and has real rank zero, stable rank one and weakly unperforated K_0-group. In Section 4, we use examples from [D2] to exhibit a class of unital TAF simple C^*-algebras which are constructed from general separable residually finite dimensional C^*-algebras. Some of them are not nuclear. From these C^*-algebras, we construct a unital separable simple quasidiagonal TAF C^*-algebra with unique normalized quasitrace which is not nuclear. In Section 5, we discuss strong NF-algebras, introduced in [BK1]. We show that every unital strong NF-algebra A of real rank zero has the local approximation property introduced by S. Popa mentioned earlier (i.e., (1) and (2) hold). We also give a sufficient condition for unital separable simple nuclear quasidiagonal C^*-algebras of real rank zero to be TAF. In Section 6, we give the main characterization theorem.

The following terminology and notation will be used in this paper.

(i) Let A be a C^*-algebra. We denote by $D(A)$ the Murray–von Neumann equivalence classes of projections in A.

(ii) Two projections in A are said to be equivalent if they are Murray–von Neumann equivalent. We write $p \preceq q$ if p is equivalent to a projection in qAq.

(iii) Let $p, q \in A$ be two projections, and let k be a positive integer. We write $k[p] \le [q]$ if qAq contains k mutually orthogonal projections each of which is equivalent to p.

(iv) Let $a \in A$ be a positive element. We denote by $Her(a)$ the hereditary C^*-subalgebra of A generated by a. We write $p \preceq a$ if p is equivalent to a projection in $Her(a)$.

(v) An element in A is said to be *full* if the ideal generated by the element is A itself. Every nonzero element in a simple C^*-algebra is full.

(vi) Let $\varepsilon > 0$, and let $\mathcal{F}$ and S be subsets of A. We write $x \in_\varepsilon S$, if there exists $y \in S$ such that $\|x - y\| < \varepsilon$, and write $\mathcal{F} \subset_\varepsilon S$, if $x \in_\varepsilon S$ for all $x \in \mathcal{F}$.

(vii) We denote by $QT(A)$ (and $T(A)$) the normalized quasi-trace (or trace) space of A.

(viii) All ideals in this paper are closed two-sided ideals.

2. Tracially AF-algebras

2.1. Definition. Let A be a unital C^*-algebra. We say that A is *tracially approximately finite dimensional* (TAF for brevity) if it satisfies the following: For any $\varepsilon > 0$, any integer $n > 0$, any finite subset $\mathcal{F}$ of A which contains a non-zero element x_1, and any full $a \in A_+$ there exists a finite dimensional C^*-subalgebra $F \subset A$ with $p = 1_F$ such that

(1) $\|[p, x]\| < \varepsilon$ for all $x \in \mathcal{F}$;

(2) $pxp \in_\varepsilon F$ for all $x \in \mathcal{F}$ and $\|px_1p\| \ge \|x_1\| - \varepsilon$;

(3) $n[1-p] \le [p]$ in $D(A)$ and $1 - p \preceq a$.

A non-unital C^*-algebra is said to be TAF if $\tilde{A}$ is TAF.

2.2. Proposition. *Every unital commutative TAF C^*-algebra is in fact an AF-algebra, or more precisely, $A \cong C(X)$ for some compact totally disconnected space.*

Proof. In a commutative C^*-algebra, if p and q are two mutually orthogonal projections with $q \preceq p$, then $q = 0$. So, in the definition of TAF, we see that $1 - p = 0$. This implies that A is an AF-algebra. □

2.3. *Remark.* A TAF C^*-algebra is not in general an AF-algebra, but a "large" part of it is approximately finite dimensional. Originally, traces were used to measure the "size" of the finite dimensional part. However we prefer not to assume that the C^*-algebras admit a trace. Moreover, in the absence of "fundamental comparability", traces may not even give a good measurement. In order to make sure that the finite dimensional part is "large", we use the somewhat complicated looking condition (3), which certainly causes some technical difficulties. However when A is simple, both conditions (2) and (3) can be much simplified (see 3.8 and 3.9).

When (quasi)traces are good measurements for projections we have the following proposition, which is the reason that we use the term "tracially AF".

2.4. Proposition. *Let A be a unital simple C^*-algebra of real rank zero, stable rank one and with weakly unperforated $K_0(A)$. Then A is TAF if and only if for any $\varepsilon > 0$ and $\sigma > 0$, and any finite subset $\mathcal{F}$ of A, there exists a finite dimensional C^*-subalgebra $F \subset A$ with $p = 1_F$ such that*

(1) $\|[p, x]\| < \varepsilon$ for all $x \in \mathcal{F}$,

(2') $pxp \in_\varepsilon F$ for all $x \in \mathcal{F}$, and

(3') $\tau(1 - p) < \sigma$ for all normalized quasi-traces on A.

Proof. We only need to prove the "if" part.

Let $\varepsilon > 0$, n, a finite subset $\mathcal{F}$, a nonzero $x_1 \in \mathcal{F}$ and a positive element a be given. We will show that there exists a finite dimensional C^*-subalgebra $F \subset A$ with $p = 1_F$ such that

(i) $\|[p, x]\| < \varepsilon$ for all $x \in \mathcal{F}$;

(ii) $pxp \in_\varepsilon F$ for all $x \in \mathcal{F}$, and $\|p|x_1|p\| \geq \||x_1|\| - \varepsilon$;

(iii) $n[1 - p] \leq [p]$ in $D(A)$, and $1 - p \preceq a$.

Note that, if $\|[p, x_1]\|$ is small, $\|[p, x_1^*]\|$ is also small. So $\|(px_1p)^*(px_1p)\|$ is close to $\|px_1^*x_1p\|$. So we see it suffices to prove the above. To save notation, we may assume that $x_1 \geq 0$.

For any $0 < \eta < \|x_1\|$, let $f(t) \geq 0$ be a continuous function on $[0, \|x_1\|]$ such that $f(t) = 1$ if $|t| \geq \|x_1\| - \eta/2$ and $f(t) = 0$ if $|t| < \|x_1\| - \eta$. Since A has real rank zero, we obtain a nonzero projection $e \in Her(f(x_1))$. Let E be the spectral projection of x_1 corresponding to the open subset $[\|x_1\| - \eta, 1]$ in A^{**}. We have

$$Ex_1E \geq (\|x_1\| - \eta)E.$$

Since $eE = Ee = e$, we have

$$ex_1e \geq (\|x_1\| - \eta)e.$$

Let $\mathcal{G} = \{e\} \cup \mathcal{F}$. Since A has real rank zero, there exists a nonzero projection $q \in Her(a)$. Let $d = \inf\{\tau(q) : \tau \in QT(A)\}\,(> 0)$. Choose $\sigma > 0$ so that $\sigma < min(1/(n + 1), d)$.

We obtain a finite dimensional C^*-subalgebra $F \subset A$ with $p = 1_F$ such that

(a) $\|[p, x]\| < \eta$ for all $x \in \mathcal{G}$;

(b) $pxp \in_\eta F$ for all $x \in \mathcal{G}$;

(c) $\tau(1 - p) < \sigma$ for all $\tau \in QT(A)$.

Thus we have (i). Since A has real rank zero, stable rank one and weakly unperforated $K_0(A)$, it follows from [BH] that (iii) holds.

For (ii), we note that

$$pex_1ep \geq (\|x_1\| - \eta)pep.$$

We also have $\|ep - pe\| < \eta$. With sufficiently small σ, we obtain $\|pep\| \geq 1 - \varepsilon/4$. Then, we have

$$\|px_1p\| \geq \|epx_1pe\| \geq \|pex_1ep\| - \eta \geq (\|x_1\| - \eta)\|pep\| - \eta$$

$$\geq (\|x_1\| - \eta)(1 - \varepsilon/4) - \eta = \|x_1\| - (\varepsilon/4\|x_1\| + \eta) + \varepsilon\eta/4.$$

So (ii) also follows with sufficiently small η. ◻

It is obvious that every AF-algebra is TAF. Popa ([Po]) shows that every unital simple quasidiagonal C^*-algebra of real rank zero satisfies conditions (1) and (2) (see also Section 5). Proposition 2.6 (below) shows that, for any countable weakly unperforated graded ordered group $(G_0, (G_0)_+, G_1)$ with the Riesz decomposition property, there exists a unital simple TAF C^*-algebra A with $(K_0(A), K_0(A)_+, K_1(A)) = (G_0, (G_0)_+, G_1)$. In particular, there are TAF C^*-algebras that are not AF (whenever K_0 has torsion, or K_1 is nontrivial). Non-nuclear TAF C^*-algebras are given in section 4. Other examples are given in section 5.

2.5. Let $A = \lim(A_n, \phi_n)$, where $A_n = \bigoplus_{j=1}^{m(n)} P_{n(j)} M_{n(j)}(C(X_{n(j)}))P_{n(j)}$, each $X_{n(j)}$ is a finite connected CW complex and $P_{n(j)}$ is a projection in $M_{n(j)}(C(X_{n(j)}))$. When A is simple, A is said to have *slow dimension growth* if

$$\lim_{n\to\infty} \max_{1\leq j\leq m(n)} \frac{\dim(X_{n(j)})}{\mathrm{rank}(P_{n(j)})} = 0.$$

We denote by $\mathcal{C}$ the class of those simple unital C^*-algebras of the above form (with slow dimension growth) with real rank zero. The classification theorem in [EG] combined with the reduction theorem in [G] and [D1] shows that the C^*-algebras in $\mathcal{C}$ are classified by their scaled ordered groups $(K_0(A), K_0(A)_+, [1_A], K_1(A))$. It also shows that, given any countable weakly unperforated graded ordered group $(G_0, (G_0)_+, G_1)$ with the Riesz decomposition property, there is a C^*-algebra $A \in \mathcal{C}$ such that $(K_0(A), K_0(A)_+, K_1(A)) = (G_0, (G_0)_+, G_1)$.

2.6. Proposition. *Let $A \in \mathcal{C}$. Then A is a (unital) simple TAF C^*-algebra.*

Proof. Let A be a unital simple C^*-algebra in $\mathcal{C}$. It follows from [G] (see also 3.2 in [D1]) that $A = \lim_{n\to\infty}(B_n, \phi_n)$, where each $B_n = \bigoplus P_{n(j)} M_{n(j)}(C(X_{n(j)})P_{n(j)}$ and $X_{n(j)}$ is a finite connected CW complex of dimension ≤ 3. Since A has real rank zero, for any $\delta > 0$, by 1.4.5 in [EG] (see also 2.5 in [Su]), each partial map of ϕ_n has spectral variation SPV less than δ (see 1.4.5 in [EG]). Now since $\dim(X_{n(j)}) \leq 3$, it follows from 2.21 in [EG] and 2.4 that A is TAF. ◻

We will show that when A is separable, a quotient of a TAF C^*-algebra is TAF and, under a mild condition, every ideal of a TAF C^*-algebra is TAF.

2.7. Proposition. *Let A be a non-unital C^*-algebra. Then A is TAF if and only if the following holds: For any $\varepsilon > 0$, any integer $n > 0$, any finite subset $\mathcal{F}$ of A which contains a non-zero element x_1, and any full $a \in \tilde{A}_+$ there exist two mutually orthogonal projections $p_1, p_2 \in A$ and a finite dimensional C^*-subalgebra $F \subset A$ with $p_1 = 1_F$ such that*

(1) $\|[p_i, x]\| < \varepsilon$ for all $x \in \mathcal{F}$, $i = 1, 2$;

(2) $p_1xp_1 \in_\varepsilon F$ for all $x \in \mathcal{F}$ and $\|p_1x_1p_1\| \geq \|x_1\| - \varepsilon$, and $\|(p_1+p_2)x - x\| < \varepsilon$;

(3) there are mutually orthogonal and mutually equivalent non-zero projections $q_1, q_2, ..., q_n \in (1 - p_2)A(1 - p_2)$ such that $p_2 \preceq q_i$ for each i and $p_2 \preceq a$.

Proof. First consider the "if" part. Let $\mathcal{G} = \{z_1, z_2, ..., z_m\}$ be a finite subset of $\tilde{A}$. We may write $z_i = \lambda_i + x_i$, where λ_i is a scalar multiple of the identity and $x_i \in A$. Set $\mathcal{F} = \{x_1, x_2, ..., x_m\}$. Suppose that p_1, p_2 and F satisfy (1), (2) and (3) in the proposition. Define $F_1 = \mathbf{C} \cdot (1 - (p_1 + p_2)) \oplus F$ and $p = 1 - p_2$. We see easily that, with $\mathcal{G}$ being the finite subset of $\tilde{A}$, the projection p and the finite dimensional C^*-subalgebra F_1 satisfy (1), (2) and (3) in 2.1. So $\tilde{A}$ is TAF.

To see the "only if" part, we assume that $\tilde{A}$ is TAF. Let $\mathcal{F} \subset A$, ε, n and a full element $a \in \tilde{A}_+$ be given. Let p, F satisfy (1), (2) and (3) in the definition of TAF (with $\varepsilon/8$ replacing ε). Let $\pi : \tilde{A} \to \mathbf{C}$ be the quotient map. Then either $\pi(1-p) = 0$ or $\pi(p) = 0$. But then (3) in 2.1 implies that $\pi(1-p) = 0$, or, equivalently, $1 - p \in A$. Write $F = M_{k_1} \oplus \cdots \oplus M_{k_s}$. Since $\pi(F) = \mathbf{C}$, and M_{k_i} is simple, we conclude that

$$F = \mathbf{C} \cdot d \oplus M_{k_2} \oplus \cdots \oplus M_{k_s},$$

where $1 - d \in A$ and $F_1 = M_{k_2} \oplus \cdots M_{k_s} \in A$. Suppose that $\mathcal{F} = \{x_1, x_2, ..., x_m\}$ ($\subset A$). Let $y_i \in F$ be such that

$$\|y_i - px_ip\| < \varepsilon/8, \quad i = 1, 2, ..., n.$$

Then $\|\pi(y_i)\| < \varepsilon/8$. So $y_i = \lambda_i \cdot d \oplus z_i$ with $z_i \in F_1$ and $|\lambda_i| < \varepsilon/4$. Therefore

$$\|z_i - px_ip\| < \varepsilon/2, \quad i = 1, 2, ..., n.$$

We also have

$$\|(1-d)px_ip - px_ip\| < \varepsilon/2, \quad i = 1, 2, ..., n.$$

Therefore

$$\begin{aligned} \|((1-p) + (p-d))x_i - x_i\| &= \|(p-d)px_i - px_i\| \\ &\le \|(p-d)px_ip - px_ip\| + \varepsilon/2 < \varepsilon. \end{aligned}$$

Now define $p_1 = (p - d)$ and $p_2 = (1 - p)$ with the finite dimensional C^*-subalgebra F_1. Then (1), (2) and (3) in the proposition follow. □

2.8. Corollary. *Let A be a separable non-unital C^*-algebra of TAF. Then A admits an approximate identity consisting of projections.*

Proof. This follows from (2) in 2.7. □

2.9. Proposition. *Let A be a TAF C^*-algebra, and let I be an ideal. If I has a positive full element, then A/I is TAF.*

(One should note that if I is σ-unital then I has a positive full element. So, in particular, every quotient of a separable TAF C^*-algebra is TAF.)

Proof. We may assume that A has a unit. From the definition, one sees that the only thing that we need to check is that for every positive full element $a \in A/I$, there is a positive full element $b \in A$ such that $\pi(b) = a$, where $\pi : A \to A/I$ is the quotient map. Take a positive full element c of I and a positive element $h \in A$ with $\pi(h) = a$. Set $b = c + h$. We claim that b is full. Let J be an ideal generated by b. Since $\pi(b) = a$ and a is full in A/I, $J + I = A$. Since $Her(b) \subset J$ and $c \le b$, it follows that $c \in Her(b)$. Therefore $c \in J$. Hence $I \subset J$. So $J = A$. This implies that b is full. □

The following two lemmas are well known.

2.10. Lemma. *For any $\varepsilon > 0$ there exists $\delta > 0$ which sastisfies the following: For any C^*-algebra A and $a \in A_+$ with $0 \le a \le 1$ and a projection $p \in A$ such that*

$$\|ap - p\| < \delta,$$

there exists a projection $q \in Her(a)$ with

$$\|p - q\| < \varepsilon.$$

2.11. Lemma (cf. 2.11 in [Ln2])**.** *Let $f \in C([-1,1])$. For any $\varepsilon > 0$ there exists $\delta > 0$ satisfying the following: For any C^*-algebra A and $a \in A_{s.a.}$ with $\|a\| \le 1$ and a projection $p \in A$ such that*

$$\|[p, a]\| < \delta,$$

we have

$$\|f(a)p - f(pap)\| < \varepsilon.$$

2.12. Lemma. *Let A be a C^*-algebra which satisfies the local approximation property of Popa, i.e., it satisfies conditions* (1) *and* (2) *in 2.1. Then every non-zero hereditary C^*-subalgebra of A contains a non-zero projection (i.e., A has property (SP)).*

Proof. Let B be a hereditary C^*-subalgebra of A. Fix $0 < r < 1$. Define $f_r \in C([0,1])$ as follows:

$$f_r(t) = \begin{cases} 1 & \text{if } r/2 \le t \le 1; \\ \text{linear} & \text{if } r/4 \le t < r/2; \\ 0 & \text{if } 0 \le t < r/4. \end{cases}$$

For any $\sigma > 0$, there exists $d > 0$ such that

$$|f_r(t) - f_r(x)| < \sigma$$

if $|x - t| < d$.

Let $\delta > 0$ with $\delta < d/2$ and let e be a positive element of B with $\|e\| = 1$. Let $\mathcal{F} = \{e, f_r(e)\}$. Since A satisfies conditions (1) and (2) in 2.1, there exists a finite dimensional C^*-subalgebra $F \subset A$ with $1_F = p$ such that

$$\|px - xp\| < \delta$$

for $x \in \mathcal{F}$, and there exists an element $a \in F$ such that

$$\|pep - a\| < \delta \quad \text{and} \quad \|pep\| > r.$$

It is easy to see that we may assume that $0 \le a \le 1$.

For any $\varepsilon > 0$, by Lemma 2.11 we choose δ so small that

$$\|f_r(e)p - f_r(pep)\| < \varepsilon/4.$$

Since $a \in F$, we may write

$$a = \sum_{i=1}^{n} \lambda_i p_i,$$

where $p_1, p_2, ..., p_n$ are mutually orthogonal projections. By choosing even smaller δ, without loss of generality, we may assume that $\|a\| \ge r$. So we may assume that

$\lambda_1 \geq r$. Thus there is a nonzero projection $e_0 \in F$ such that $f_r(a)e_0 = e_0$. We now estimate that

$$\|f_r(e)e_0 - e_0\| = \|f_r(e)pe_0 - e_0\| \leq \|(f_r(e)p - f_r(pep))e_0\|$$
$$+\|(f_r(pep) - f_r(a))e_0\| + \|f_r(a)e_0 - e_0\| < \varepsilon/2 + \varepsilon/2 + 0 = \varepsilon.$$

It follows from Lemma 2.10 that, if ε is sufficiently small, B contains a non-zero projection q with $\|q - e_0\| < 1$. □

2.13. Lemma. *Let A be a unital TAF C^*-algebra, and let B be a unital full hereditary C^*-subalgebra of A. Then B satisfies the following: For any $\varepsilon > 0$, any integer $n > 0$, any finite subset $\mathcal{F}$ of B which contains a non-zero element x_1, and any full element a in B_+, there exists a finite dimensional C^*-subalgebra $F \subset B$ with $p = 1_F$ such that*
(1) $\|[p, x]\| < \varepsilon$ for all $x \in \mathcal{F}$;
(2) $pxp \in_\varepsilon F$ for all $x \in \mathcal{F}$, and $\|px_1p\| \geq \|x_1\| - \varepsilon$;
(3') $1 - p \preceq a$.

Proof. Let $\delta > 0$ be positive, let $\mathcal{F}$ be a finite subset of B which contains a nonzero element x_1 and $e = 1_B$, let n be an integer, and let $a \in B_+$ be a full element in B. Note that a is a full element in A. Since A is TAF, there exists a finite dimensional C^*-subalgebra $F \subset A$ with $p = 1_F$ such that
(1) $\|[p, x]\| < \delta$ for all $x \in \mathcal{F}$;
(2) $pxp \in_\delta F$ for all $x \in \mathcal{F}$, and $\|px_1p\| \geq \|x_1\| - \delta$;
(3) $n[1 - p] \leq [p]$ in $D(A)$, and $1 - p \preceq a$.
So we have

$$\|pep - (pep)^2\| < \delta \quad \text{and} \quad \|epe - (epe)^2\| < \delta.$$

So for any $\eta > 0$, if δ is sufficiently small, we obtain two projections $e_1 \in F$ and $e_2 \in B$ such that

$$\|e_1 - pep\| < \eta, \|e_2 - epe\| < \eta,$$

$$\|e_1 - e_2\| < \eta \quad \text{and} \quad \|(e - e_2) - (1 - p)(e - e_2)(1 - p)\| < \eta.$$

With small η and δ, there is a unitary $U \in A$ such that

$$U^* e_1 U = e_2.$$

In particular, $F_1 = U^* e_1 F e_1 U$ is a finite dimensional C^*-subalgebra of B.

For any $\varepsilon > 0$, if δ and η are small enough, we have (note that $e = 1_B$, $\mathcal{F} \subset B$ and $1_{F_1} = e_2$)
(1′) $\|[x, e_2]\| < \varepsilon$ for all $x \in \mathcal{F}$,
(2′) $e_2 x e_2 \in_\varepsilon F_1$, $\|e_2 x_1 e_2\| > \|x_1\| - \varepsilon$.
Since we also have

$$\|(e - e_2) - [(1 - p)(e - e_2)(1 - p)]\| < \eta$$

(with sufficiently small δ and η), it follows that $e - e_1$ is (unitarily) equivalent to a subprojection of $1 - p$. Since $1 - p \preceq a$, we have (3') $e - e_2 \preceq a$. □

2.14. Lemma. *Let A be a unital C^*-algebra and let I be an ideal. Suppose that $a \in \tilde{I}_+$ is a full element of $\tilde{I}_+$. Then a is a full element in A (we identify the unit of $\tilde{I}$ with that of A).*

Proof. It is easy to see that $a = \lambda + b$ for some $b \in I$ and some $\lambda > 0$. Let J be the ideal generated by a in A. Since a is full in $\tilde{I}$, $\tilde{I} \subset J$. Thus $1 \in J$. This implies that a is full in A. ☐

2.15. Theorem. *Let A be a TAF C^*-algebra. Let I be an ideal of A which contains an approximate identity consisting of projections. Then I is TAF. Conversely, if I is TAF, I admits an approximate identity consisting of projections.*

Proof. The converse follows from 2.8. Note that, if I is unital, then 1_I is in the center. It is obvious from the definition that I is TAF. We may assume that A is unital. Fix $\varepsilon > 0$, an integer n, a full element $a \in \tilde{I}$ and a finite subset $\mathcal{F} \subset I$ containing a non-zero element x_1. Here we assume that the identity of $\tilde{I}$ is the same as that of A. In particular, by 2.14, a is full in A.

Without loss of generality, we may assume that there exists $e \in I$ such that $ex = x$ for all $x \in \mathcal{F}$. Note also, by 2.14, that a is full in A.

We will use the proof of 2.13 and its notation (with $B = eIe$). Also, we note that the proof of 2.13 actually proves that 2.13 holds if the condition that B is full is replaced by the condition that a is a full element in A (not necessarily in B). From this we have

(1) $\|[x, e_2]\| < \varepsilon$ for all $x \in \mathcal{F}$,

(2) $e_2 x e_2 \in_\varepsilon F_1$, $\|e_2 x_1 e_2\| > \|x_1\| - \varepsilon$.

We also have

(3') $e - e_2 \preceq a$.

Suppose that $p_1, p_2, ..., p_n$ are mutually orthogonal projections in pAp such that $[1-p] \le [p_i]$ $(i = 1, 2, ..., n)$.

Since $[e-e_2] \le [1-p]$ (as in the proof of 2.13), $[e-e_2] \le [p_i]$ in $D(A)$. Therefore there are $v_i \in A$ with

$$v_i v_i^* = e - e_2 \quad \text{and} \quad v_i^* v_i \le p_i, \quad i = 1, 2, ..., n.$$

Since I is an ideal, $v_i \in I$ and $v_i^* v_i \in I$. Let U be the unitary in the proof of 2.13. Note that $e_2 = U^* e_1 U$. We obtain

$$\|(e - e_2) U^* p U\| = \|(e - e_2) e_2 U^* p U\| = 0.$$

Thus $U^* v_i^* v_i U \in (1 - (e - e_2)) I (1 - (e - e_2))$. Therefore we obtain

(3) $n[e - e_2] \le [1 - (e - e_2)]$ in $D(I)$ and $e - e_2 \preceq a$.

So it follows from 2.7 that I is TAF. ☐

The reader probably realizes that the notion of TAF could be modified in several ways. A further discussion is probably needed. When C^*-algebras are not simple, it is difficult to compare the size of projections in general. Fortunately, we are mostly interested in simple C^*-algebras. In the next section we will consider only simple TAF C^*-algebras. TAF C^*-algebras behave as expected when they are simple.

3. Simple TAF C^*-algebras

In this section we will show that a unital simple TAF C^*-algebra is quasidiagonal and has real rank zero, stable rank one and a weakly unperforated K_0-group.

3.1. Lemma (cf. 1.8 in [Cu])**.** *Let A be a simple C^*-algebra with property (SP) (i.e., every nonzero hereditary C^*-subalgebra of A contains a nonzero projection), and let p and q be two nonzero projections in A. Then there exists a nonzero projection $e \in A$ with $e \le p$ and $[e] \le [q]$.*

3.2. Lemma. *Let A be a non-elementary simple C^*-algebra with property (SP). Then for any nonzero projection $p \in A$ and any integer $n > 0$, there are $n+1$ mutually orthogonal projections $q_1, q_2, ..., q_n, q_{n+1}$ such that $q_1 \neq 0$, $[q_1] = [q_i]$, $i = 1, 2, ..., n$, and $p = q_1 + q_2 + \cdots + q_{n+1}$.*

Proof. The simple C^*-algebras pAp is non-elementary. Therefore (by [AS]), there is a positive element $a \in pAp$ with $sp(a) = [0,1]$. In particular, there are $n+1$ mutually orthogonal positive elements $a_1, a_2, ..., a_{n+1} \in pAp$ such that $\|a_i\| = 1$, $i = 1, 2, ..., n+1$. Consider $B_i = Her(a_i)$. Then, there are non-zero projections $p_i \in B_i$. By applying Lemma 3.1 repeatedly, we obtain a non-zero projection $e \in p_1Ap_1$ such that

$$[e] \le [p_i] \text{ in } D(A).$$

The lemma then follows. □

Recall that a C^*-algebra A is said to have *cancellation of projections* if whenever $p \oplus e$ is equivalent to $q \oplus e$ in $M_m(A)$, where p, $q \in A$ are projections and $e \in M_m(A)$ is a projection, then p is equivalent to q.

3.3. Lemma. *Let A be a simple TAF C^*-algebra. Then A has cancellation of projections.*

Proof. Let p and q be two projections in A and let $u \in A$ be a partial isometry such that

$$u^*u = p \quad \text{and} \quad uu^* = q.$$

It suffices to show that $1-p$ is equivalent to $1-q$. To this end, we assume that $1 - p \neq 0$.

For any $0 < \delta < 1/2$, there exists a finite dimensional C^*-subalgebra $F \subset A$ with $1_F = P$ such that

(1) $\|[P, x]\| < \delta$ for $x \in \mathcal{F}$, where $\mathcal{F} = \{p, q, (1-p), (1-q), u, u^*\}$,

(2) $PxP \in_\delta F$ for all $x \in \mathcal{F}$ and $\|P(1-p)P\| \ge 1 - \delta$,

(3) $2[1-P] \le [P]$ in $D(A)$.

We will perturb the elements $PuP, Pu^*P, (1-P)u(1-P)$ and $(1-P)u^*(1-P)$. A standard perturbation argument (with sufficiently small δ) enables us to assume, without loss of generality, that $p = p_1 + p_2$, $q = q_1 + q_2$ such that $p_1, q_1 \in (1-P)A(1-P)$, and p_1 and q_1 are equivalent in $(1-P)A(1-P)$, $p_2, q_2 \in F$, and they are equivalent in F. Furthermore, by (2) above, we may assume that $P - p_2 \neq 0$.

Since F is finite dimensional, we conclude that $P - p_2$ is unitarily equivalent to $P - q_2$ in F. Let u_1 be a unitary in $F \subset PAP$ such that

$$u_1^*(P - p_2)u_1 = P - q_2.$$

Define $U_1 = (1-P) + u_1$. Then U_1 is a unitary of A. By replacing q by $U_1^*pU_1$, we may assume that $P - p_2 = P - q_2$. Denote this nonzero projection in A by d.

By 3.2, we can write $d = d_1 + d_2 + d_3$, where d_1, d_2, d_3 are mutually orthogonal projections such that d_1 and d_2 are nonzero equivalent projections and $[d_1] \le [d_3]$ in $D(A)$. Let

$$B = (1 - (d_2 + d_3))A(1 - (d_2 + d_3)).$$

Note that now we have p and $q \in B$. We apply Lemma 2.13. Since A is simple, every hereditary C^*-subalgebra is full. Therefore, by 2.13, every unital hereditary

C^*-subalgebra is TAF. By applying the same argument above to B (using 2.13), we may assume that

$$p = p_1' + p_2' \quad \text{and} \quad q = q_1' + q_2',$$

such that p_1' and q_1' are equivalent in QBQ for some projection $Q \in B$ with $[Q] \leq [d_1]$, and p_2' and q_2' are unitarily equivalent in a (unital) finite dimensional C^*-subalgebra $F_1 \subset (1_B - Q)B(1_B - Q)$. Therefore $(1_B - Q) - p_2'$ and $(1_B - Q) - q_2'$ are unitarily equivalent in F_1. Thus it suffices to show that $Q - p_1' + d_2$ is equivalent to $Q - q_1' + d_2$ in $(Q + d_2)A(Q + d_2)$. There exists a projection $d_2' \leq d_2$ such that $[Q] = [d_2']$ in $D(A)$. It suffices to show that $Q - p_1' + d_2'$ is equivalent to $Q - q_1' + d_2'$ in $(Q + d_2')A(Q + d_2')$. We have (in $(Q + d_2')B(Q + d_2') \cong M_2(QBQ)$) the matrix decomposition

$$p_1' = \begin{pmatrix} p_1' & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad q_1' = \begin{pmatrix} q_1' & 0 \\ 0 & 0 \end{pmatrix}.$$

Since p_1' is equivalent to q_1' in QBQ, it is well known that p_1' is unitarily equivalent to q_1' in $(Q + d_2')A(Q + d_2')$ (cf. 4.3 in [Bl]). Therefore $Q - q_1' + d_2$ is equivalent to $Q - p_1' + d_2$. □

3.4. Theorem. *Every unital simple TAF C^*-algebra is quasidiagonal and has real rank zero and stable rank one.*

Proof. Let A be a unital TAF C^*-algebra. We first show that A has real rank zero. Fix a self-adjoint element $x \in A$ and $\varepsilon > 0$. We will show that there is an invertible self-adjoint element $z \in A$ with $\|z - x\| < \varepsilon$.

Let $f \in C([-\|x\|, \|x\|])$ with $0 \leq f \leq 1$ and $f(t) = 1$ if $|t| < \varepsilon/128$, and $f(t) = 0$ if $|t| \geq \varepsilon/64$. If $f(x) = 0$, then x is invertible. So we assume that $f(x) \neq 0$. Let $B = Her(f(x))$. Then, by Lemma 2.12, there exists a non-zero projection $e \in B$. Since eAe is simple, by 3.2, we may write $e = e_1 + e_2$, where e_1 and e_2 are mutually orthogonal nonzero projections with $[e_1] \leq [e_2]$.

For any $\delta > 0$, there exist a projection $p \in A$ and a finite dimensional C^*-subalgebra $F \subset A$ with $1_F = p$ such that

$$\|[p, x]\| < \delta \quad \text{and} \quad \|[p, e_1]\| < \delta,$$

and there exist a self-adjoint element $y \in F$ and a positive element $b \in F$ such that

$$\|pxp - y\| < \delta \quad \text{and} \quad \|pe_1p - b\| < \delta$$

and $[1 - p] \leq [e_1]$. With sufficiently small $\delta\,(< 1/4)$, we obtain projections $p_1 \in (1 - p)A(1 - p)$ and $p_2 \in F$ such that

$$\|e_1 - (p_1 + p_2)\| < 4\delta.$$

With $\delta < 1/8$, we know that e_1 is unitarily equivalent to $p_1 + p_2$. Since $p_1 \leq 1 - p$ and $[1 - p] \leq [e_1]$, by Lemma 3.3, we know that $[1 - p - p_1] \leq [p_2]$. In particular, $p_2 \neq 0$. Note that

$$\|e_1 x\| < \varepsilon/64.$$

Therefore, with sufficiently small δ, we have

$$\|p_2 y\| < \varepsilon/32, \qquad \|ye_2\| < \varepsilon/32, \qquad \|p_1 x\| < \varepsilon/32.$$

So

$$\|y - (p - p_2)y(p - p_2)\| < \varepsilon/16$$

and

$$\|(1-p)x(1-p)-(1-p-p_1)x(1-p-p_1)\|<\varepsilon/16.$$

Since $(p-p_2)F(p-p_2)$ is finite dimensional, there exists an invertible self-adjoint element $z_2 \in (p-p_2)F(p-p_2)$ such that

$$\|(p-p_2)y(p-p_2)-z_2\|<\varepsilon/16 \quad \text{and} \quad \|pxp-z_2\|<\varepsilon/4$$

(if δ is sufficiently small). Let $p_2' \le p_2$ be a projection such that $[1-p-p_1]=[p_2']$. In $((1-p-p_1)+p_2')A((1-p-p_1)+p_2')$, set

$$z_1=(1-p-p_1)x(1-p-p_1)+(\varepsilon/16)v+(\varepsilon/16)v^*,$$

where $vv^*=(1-p-p_1)$ and $v^*v=p_2'$. (In the matrix representation, we have

$$z_1=\begin{pmatrix} x' & \varepsilon/16 \\ \varepsilon/16 & 0 \end{pmatrix},$$

where $x'=(1-p-p_1)x(1-p-p_1)$.) Clearly z_1 is an invertible self-adjoint element in $((1-p-p_1)+p_2')A((1-p-p_1)+p_2')$.

Put $z=z_1+(\varepsilon/16)p_1+(\varepsilon/16)(p_2-p_2')+z_2$. Then z is an invertible self-adjoint element in A. We estimate that

$$\begin{aligned}
\|x-z\| &\le \|x-(pxp+(1-p)x(1-p))\|+\|(pxp+(1-p)x(1-p))-z\| \\
&< \varepsilon/16+\|(pxp+(1-p)x(1-p))-(pxp+(1-p-p_1)x(1-p-p_1))\| \\
&\qquad +\|(pxp+(1-p-p_1)x(1-p-p_1))-z\| \\
&< \varepsilon/16+\varepsilon/16+\|pxp-z_2\|+\|(1-p-p_1)x(1-p-p_1))-z_1\| \\
&\qquad +\|(\varepsilon/16)p_1+(\varepsilon/16)(p_2-p_2')\| \\
&< \varepsilon/16+\varepsilon/16+\varepsilon/16+\varepsilon/4+\|(1-p-p_1)x(1-p-p_1)-z_1\|+\varepsilon/16 \\
&< 7\varepsilon/16+\varepsilon/16+\varepsilon/16<\varepsilon.
\end{aligned}$$

This implies that A has real rank zero. It follows from [Po] that A is quasidiagonal. Since $D(A)$ has cancellation, by [BH, III, 2.4], A has stable rank one. □

3.5. Lemma. *Let A be a unital simple non-elementary C^*-algebra of real rank zero and stable rank one. Suppose that p and q are two non-zero projections in A. If $n[q]<n[p]$ in $D(A)$ for some positive integer n, then there exists a non-zero projection $p' \le p$ with $p-p' \ne 0$ such that*

$$n[q]<n[p']$$

in $D(A)$.

Proof. Let $p_1=p, p_2, ..., p_n \in M_n(A)$ and $q_1=q, q_2, ..., q_n \in M_n(A)$ be mutually orthogonal projections such that $[p_i]=[p]$ and $[q_i]=[q]$. Suppose that $v \in M_n(A)$ is such that

$$vv^*=q_1+q_2+\cdots+q_n$$

and $d=p_1+p_2+\cdots+p_n-v^*v$ is a non-zero projection. Since A has real rank zero and is a simple non-elementary C^*-algebra, there are n mutually orthogonal non-zero projections $d_1, d_2, ..., d_n \in dAd$. Then, by repeatedly applying 3.1, we obtain equivalent non-zero projections $e_1, e_2, ..., e_n$ such that $e_i \le d_i$ and $e_i \preceq p_i$, $i=1,2,...,n$. Without loss of generality (to save notation), we may assume that $e_i \le p_i$ and $e_i \preceq d_i$. Since A has stable rank one, we conclude that

$$p_1+p_2+\cdots+p_n-d \preceq (p_1-e_1)+(p_2-e_2)+\cdots+(p_n-e_n).$$

Note that $[p_i - e_i] = [p_1 - e_1]$ in $D(A)$. Put $p' = p - e_1$. □

3.6. Theorem. *Let A be a unital simple TAF C^*-algebra. Then A has weakly unperforated $K_0(A)$. Furthermore, if p and q are two non-zero projections in A with $n[q] < n[p]$ in $D(A)$, then $[q] < [p]$ in $D(A)$.*

Proof. By Theorem 3.4, A has real rank zero and stable rank one. So Lemma 3.5 applies. So there exists a non-zero projection $p' \le p$ with $d = p - p' \neq 0$ such that $n[q] < n[p']$ in $D(A)$. Write $Q = diag(q, q, ..., q)$ and $P' = diag(p', p', ..., p')$ in $M_n(A)$. There exists $V = (v_{ij}) \in M_n(A)$ (with $v_{ij} \in A$) such that

$$V^*V = Q \quad \text{and} \quad VV^* \le P'.$$

For $0 < \delta < 1/2$ and $\mathcal{F} = \{q, p', v_{ij}, i, j = 1, 2, ..., n\}$, since A is TAF, there exist a projection $e \in A$ and a finite dimensional C^*-subalgebra $F \subset A$ with $1_F = e$ such that

(1) $\|[e, x]\| < \delta$ for all $x \in \mathcal{F}$,
(2) $exe \in_\delta F$ for all $x \in \mathcal{F}$,
(3) $1 - e \preceq d$.

So, there are $q_1, p_1' \in eFe$ and $q_2, p_2' \in (1 - e)A(1 - e)$ such that $q_1 + q_2$ is unitarily equivalent to q and $p_1' + p_2'$ is unitarily equivalent to p. Let $B = M_n(F)$ and $E = diag(e, e, ..., e)$. Then we also have, for any $\varepsilon > 0$, with sufficiently small δ,

(4) $\|[E, P']\| < \varepsilon$, $\|[E, Q]\| < \varepsilon$ and $\|[E, V]\| < \varepsilon$, and
(5) $EVE \in_\varepsilon B$.

Since $\|[E, V]\| < \varepsilon$, it is standard that (with sufficiently small ε) there exists $W \in B$ such that

$$W^*W = diag(q_1, q_1, ..., q_1) \quad \text{and} \quad WW^* \le diag(p_1', p_1', ..., p_1').$$

Since B is finite dimensional, we have $q_1 \preceq p_1'$. Since $(1 - e) \preceq d$, we obtain $q_2' \le d$. So we conclude that $q \preceq p$. □

3.7. Proposition. *Let A be a unital separable simple C^*-algebra satisfying the following condition of Popa:*

For any $\varepsilon > 0$ and any finite subset $\mathcal{F}$ of A, there exists a non-zero finite dimensional C^*-subalgebra $F \subset A$ with $p = 1_F$ such that

(1) $\|[p, x]\| < \varepsilon$ for all $x \in \mathcal{F}$;
(2′) $pxp \in_\varepsilon F$ for all $x \in \mathcal{F}$.

Then A is MF ([BK1]*).*

Proof. Let A be such a C^*-algebra and let $\{x_n\}$ be a dense sequence in the unit ball of A. By (1) and (2′) above, there are projections $p_n \in A$ and non-zero finite dimensional C^*-subalgebras B_n with $1_{B_n} = p_n$ such that

(1) $\|[p_n, x_i]\| < 1/n$,
(2) $px_ip \in_{1/n} B_n$ for $i = 1, 2, ..., n$.

Let $\mathrm{id}_n : B_n \to B_n$ be the identity map and let $j : B_n \to M_{K(n)}$ be a unital embedding. We note that such j exists provided that $K(n)$ is large enough. By 5.2 in [Pa], there exists a completely positive map $L_n' : p_nAp_n \to M_{K(n)}$ such that $L_n'|_{B_n} = j \circ \mathrm{id}_n$. Since L_n' is unital, by 5.9 and 5.10 in [Pa], L_n' is a contraction. We define $L_n : A \to M_{K(n)}$ by $L_n(a) = L_n'(p_nap_n)$. Let $y_{i,n} \in B_n$ be such that $\|p_nx_ip_n - y_{i,n}\| < 1/n$, $n = 1, 2, ...$ Then

$$\|L_n(x_i) - p_nx_ip_n\| \le \|L_n(x_i - y_{i,n}) - (y_{i,n} - p_nx_ip_n)\| < 2/n \to 0$$

as $n \to \infty$. Combining this with (1) above, we see that

$$\|L_n(ab) - L_n(a)L_n(b)\| \to 0$$

as $n \to \infty$. Define $\Phi : A \to \prod_{n=1}^{\infty} M_{K(n)}$ by sending a to $\{L_n(a)\}$. Then Φ is a completely positive map. Denote $\pi : \prod_{n=1}^{\infty} M_{K(n)} \to \prod_{n=1}^{\infty} M_{K(n)}/\bigoplus_{n=1}^{\infty} M_{K(n)}$. Then

$$\pi \circ \Phi : A \to \prod_{n=1}^{\infty} M_{K(n)}/\bigoplus_{n=1}^{\infty} M_{K(n)}$$

is a nonzero homomorphism. Since A is simple, $\pi \circ \Phi$ is injective. It follows from 3.22 in [BK1] that A is an MF-algebra. $\square$

3.8. Proposition. *Let A be a unital separable simple C^*-algebra. Then A is TAF if and only if the following hold:*

For any $\varepsilon > 0$, any finite subset $\mathcal{F}$ of A, and any nonzero positive element $a \in A$, there exists a finite dimensional C^*-subalgebra $F \subset A$ with $p = 1_F$ such that

(1) $\|[p, x]\| < \varepsilon$ for all $x \in \mathcal{F}$,

(2′) $pxp \in_\varepsilon F$ for all $x \in \mathcal{F}$, and

(3′) $1 - p$ is unitarily equivalent to a projection in $Her(a)$.

Proof. For the "only if" part, we note, by 3.3, A has cancellation of projections. So $(3')$ holds.

Now we consider the "if" part. We only need to obtain (3) in 2.1 and the second part of (2) in 2.1. We may assume that A is a non-elementary simple C^*-algebra. First we claim that A has property (SP) (see 2.12). Otherwise, there is an $a \neq 0$, $a \in A_+$, such that $Her(a)$ has no non-zero projection. By $(3')$ we see that $1-p = 0$, which implies that A is AF. So A does have (SP) (see 2.12). From 3.7, A is MF. By 3.3.8 in [BK1], A is stably finite.

Let $\varepsilon > 0$, let $\mathcal{F} \subset A$ be a finite subset, let $n > 2$ be an integer, let $x_1 \in \mathcal{F}$ be a non-zero positive element, and let $a \in A$ be a nonzero positive element. We will show that (1), (2) and (3) in 2.1 hold for ε, $\mathcal{F}$, n, x_1 and a. We may assume that $0 < \varepsilon < \|x_1\| \leq 1$.

Let $f(t) \in C_0((0, 1])_+$ be such that $f \leq 1$, $f(t) = 1$ for $t \in [\|x_1\| - \varepsilon/8, 1]$, and $f(t) = 0$ for $t \in [0, \|x_1\| - \varepsilon/4]$. By what we have proved, there exists a non-zero projection $e_1 \in Her(f(x_1))$. By 3.1 and 3.2, there exists a projection $e_1' \leq e_1$ such that e_1' is equivalent to a non-zero subprojection q of $Her(a)$. To save notation, we may assume that $e_1 = e_1'$.

Set $\mathcal{G} = \mathcal{F} \cup \{e_1, x_1^{1/2}\}$. Suppose now that (1), $(2')$ and (3) hold for $\varepsilon/128$, $\mathcal{G}$ and a.

By 3.2, we obtain $n+1$ non-zero mutually orthogonal projections $\{q_1, q_2, ..., q_{n+1}\}$ in qAq. So, by $(3')$, we can choose $1 - p$ so that it is unitarily equivalent to a subprojection of q_1 $(\preceq q)$. This implies that $q_2 + q_3 + \cdots + q_{n+1} \preceq p$. Therefore (3) in 2.1 holds. To show that the second part of (2) holds, we note that $pe_1p \neq 0$. Otherwise $e_1 \leq (1 - p)$. However, $2[1 - p] \leq [e_1]$, and A is stably finite. This is impossible. Thus, by (1) and $(2')$ (for $\varepsilon/128$ and $\mathcal{G}$), there exist a nonzero projection $e_2 \in F$ and a non-zero projection $e_3 \in e_1Ae_1$ such that

$$\|pe_1p - e_2\| < \varepsilon/32 \quad \text{and} \quad \|e_1pe_1 - e_3\| < \varepsilon/32.$$

Hence

$$\|e_2 - e_3\| < \varepsilon/16.$$

Since $x_1^{1/2} \in \mathcal{G}$, we estimate that

$$\|x_1^{1/2} e_2 x_1^{1/2} - x_1^{1/2} e_3 x_1^{1/2}\| < \varepsilon/16.$$

Since $e_3 \le e_1$, by the construction of e_1, we have

$$\|x_1^{1/2} e_3 x_1^{1/2}\| = \|e_3 x_1 e_3\| \ge \|x_1\| - \varepsilon/4.$$

Therefore,

$$\|x_1^{1/2} e_2 x_1^{1/2}\| \ge \|x_1\| - \varepsilon/2.$$

Since $x_1^{1/2} \in \mathcal{G}$, by (1) (for $\varepsilon/128$),

$$\|px_1p - x_1^{1/2} p x_1^{1/2}\| < \varepsilon/64.$$

Therefore,

$$\|px_1p\| \ge \|x_1^{1/2} p x_1^{1/2}\| - \varepsilon/64 \ge \|x_1^{1/2} e_2 x_1^{1/2}\| - \varepsilon/64 \ge \|x_1\| - \varepsilon/2 - \varepsilon/64.$$

This proves the second part of (2) in 2.1. So A is TAF. □

3.9. *Remark.* For unital separable simple C^*-algebras, we may use (1), (2′) and (3′) in 3.8 for the definition of TAF. From the proof of 3.8, the condition that A is separable can be replaced by A is finite. Moreover, from the proof of 3.8, for general unital simple C^*-algebras, condition (3) can be replaced by (3') of 3.8 in the definition of TAF.

3.10. Theorem. *Let A be a unital simple TAF C^*-algebra. Then, for any integer n, $M_n(A)$ is TAF.*

Proof. By 3.4 and 3.6, A has real rank zero, stable rank one and weakly unperforated $K_0(A)$. So does $M_n(A)$. Let $\varepsilon > 0$, $\sigma > 0$, and let $\mathcal{F} \subset M_n(A)$ be a finite subset. By 2.4, it suffices to show that there exists a finite dimensional C^*-subalgebra $F \subset M_n(A)$ with $1_F = p$ such that (1), (2′) and (3′) in 2.4 are satisfied. It is clear that we may assume that elements in $\mathcal{F}$ have the form $x \otimes y$, where $x \in \mathcal{G}_1 \subset A$, $y \in \mathcal{G}_2 \subset M_n$ and both $\mathcal{G}_1$ and $\mathcal{G}_2$ are finite. Applying 2.4 to A, we obtain a finite dimensional C^*-subalgebra $F_2 \subset A$ with $q_1 = 1_{F_2}$ such that (with $\eta = \varepsilon/(\max\{\|y\| : y \in \mathcal{G}_2\} + 1)$)

(i) $\|[x, p_1]\| < \eta$ for $x \in \mathcal{G}_1$,

(ii) $p_1 x p_1 \in_\eta F_2$ for $x \in \mathcal{G}_1$, and

(iii) $\tau(1 - p_1) < \sigma$ for all $\tau \in QT(A)$.

Let $p = p_1 \otimes \mathrm{id}_{M_n}$ and $F = F_1 \otimes M_n$. We note that F is finite dimensional and $1_F = p$. We have

(1) $\|[x \otimes y, p]\| < \varepsilon$ for all $x \in \mathcal{G}_1$ and $y \in \mathcal{G}_2$,

(2) $p(x \otimes y)p = p_1 x p \otimes y \in_\varepsilon F$.

Furthermore, $t(1 - p) = \tau(1 - p_1)$ for any $t \in QT(M_n(A))$ with the form $\tau \otimes tr$, where tr is the normalized trace on M_n. So we also have

(3) $t(1 - p) < \sigma$ for all $t \in QT(A)$.

By 2.4, $M_n(A)$ if TAF. □

3.11. Definition. A C^*-algebra A is said to be *locally TAF* if for any finite subset $\mathcal{F}$ of A and $\varepsilon > 0$ there exists a TAF C^*-subalgebra B such that $\mathcal{F} \in_\varepsilon B$.

3.12. Theorem. *(1) Every hereditary C^*-subalgebra of a unital simple TAF C^*-algebra is TAF.*

(2) A unital simple locally TAF C^*-algebra is TAF. In particular, a unital simple direct limit of unital TAF C^*-algebras is TAF.

(3) The tensor product of a unital TAF simple C^*-algebra with an AF-algebra is TAF.

Proof. Part (1) follows from 3.8, 3.3 and 2.13.

For part (2), from the definition of locally TAF, we see that we do have (1) and (2) as described in 2.1. To obtain (3), we note that $1-p \preceq a$ for any positive element $a \in A$ that we want to check. By 2.12, every hereditary C^*-subalgebra has a nonzero projection. So we can always assume that a is a projection. Since A is locally TAF, there is a C^*-subalgebra $B \subset A$ such that B contains a projection e which is (unitarily) equivalent to a. By working inside this C^*-subalgebra we see that we can make $1-p \preceq a$.

For part (3), we let B be a unital simple TAF C^*-algebra. By 3.10, we know that $B \otimes C$ is TAF for every finite dimensional C^*-subalgebra. Then part (3) follows from part (2). ∎

4. Direct limits of residually finite dimensional C^*-algebras

In this section we will give a class of unital simple TAF C^*-algebras constructed from a unital residually finite dimensional C^*-algebra. In particular, we present a class of non-nuclear TAF C^*-algebras. The construction is the same as that of Dadarlat ([D2]; see also [Go]). Recall that a C^*-algebra B is said to be residually finite dimensional if for any $x \in B$ with $x \neq 0$, there exists a finite dimensional irreducible representation π of B such that $\pi(x) \neq 0$. When B is separable, B has a sequence of finite dimensional irreducible representations $\{\pi_n\}$ which separates elements in B.

4.1. Let B be a unital residually finite dimensional separable C^*-algebra. Let $\{\pi_n\}$ be a sequence of finite dimensional irreducible representations of B such that, for any non-zero element $b \in B$, there exists π_n with $\pi_n(b) \neq 0$. Suppose that π_n has dimension $k(n)$. We assume that in the sequence $\{\pi_n\}$ each irreducible representation repeats infinitely many times. For each n, define $\psi_n : B \to M_{k(n)}(B)$ by the composition $B \xrightarrow{\pi_n} M_{k(n)} \xrightarrow{\mathrm{id}\otimes 1_B} M_{k(n)}(B)$. Consider the following sequence of homomorphisms. Fix $\{\pi_1, ..., \pi_{l(1)}\}$. We define a homomorphism $h_1 : B \to M_{I(2)}(B)$, where $I(2) = 1 + \sum_{i=1}^{l(1)} k(i)$, by sending b to $diag(b, \psi_1(b), ..., \psi_{l(1)}(b))$ for all $b \in B$. Suppose that $h_m : M_{I(m)}(B) \to M_{I(m+1)}(B)$ has been defined. Choose $\{\pi_1, ..., \pi_{l(m+1)}\}$ and denote $\bar{\psi}_{n,m+1} = \psi_n \otimes 1_{I(m+1)} : M_{I(m+1)}(B) \to M_{k(n)I(m+1)}(B)$. Define $h_{m+1} : M_{I(m+1)}(B) \to M_{I(m+2)}(B)$ by sending b to

$$diag((\mathrm{id}_B \otimes \mathrm{id}_{I(m+1)})(b), \bar{\psi}_{1,m+1}(b), ..., \bar{\psi}_{l(m+1),m+1}(b))$$

for all $b \in M_{I(m+1)}(B)$, where $I(m+2) = I(m+1)(1 + \sum_{i=1}^{l(m+1)} k(i))$. We consider the C^*-algebra $A = \lim_{n\to\infty}(M_{I(m)}(B), h_m)$. In what follows, we will use $h_{n,m} : M_{I(n)}(B) \to M_{I(m)}(B)$ and $h_\infty : M_{I(n)}(B) \to A$ for the monomorphisms induced by this inductive system. Furthermore, we let $B_n = h_\infty(M_{I(n)}(B))$.

4.2. Proposition. *A is always a unital simple C^*-algebra.*

Proof. Let I be an ideal of A and $\pi : A \to A/I$ the quotient map. Since $\pi(B_n) \cong B_n/I \cap B_n$ and $\bigcup_n B_n$ is dense in A, it suffices to show that $I \cap B_n = \{0\}$ or $I \cap B_n = B_n$. Let $x \in I \cap B_n$ be a nonzero element. There is $y \in M_{I(n)}(B)$ such that $h_\infty(y) = x$. By the construction of $\{\pi_m\}$, there is π_m such that $\pi_m \otimes \mathrm{id}_{M_{I(n)}}(y) \neq 0$. Let $n + l \geq m$. By considering the composition $h_{n+l} \circ h_{n+l-1} \circ \cdots \circ h_{n+1}$, we may write

$$h_{n,n+l}(y) = H(y) \oplus \pi_m \otimes \mathrm{id}_{M_{I(n)}} \otimes \mathrm{id}_{M_J}(y)$$

for some positive integer J and some homomorphism H. Since $\psi_m \otimes \mathrm{id}_{M_{I(n)}} \otimes \mathrm{id}_{M_J}(y)$ is a nonzero element in $M_{I(n+l)}$, the ideal generated by $\psi_m \otimes \mathrm{id}_{M_{I(n)}} \otimes \mathrm{id}_{M_J}(y)$ contains $M_{I(n+l)}$. Therefore it contains $M_{I(n+l)}(B)$. So the ideal generated by $h_{n,n+l}(y)$ is $M_{I(n+l)}(B)$. This implies that the ideal generated by x is A. □

4.3. Proposition. *A is always TAF.*

Proof. Let $\mathcal{F}$ be a finite subset of A with a nonzero positive element x_1. Without loss of generality, we may assume that $\|x_1\| = 1$. Fix $1 > \varepsilon > 0$, and let $f \in C_0((0,1])$ with $0 \leq f \leq 1$ be such that $f(t) = 1$ for $t \in [1-\varepsilon/2, 1]$ and $f(t) = 0$ for $t \in (0, 1-\varepsilon]$. Set $\mathcal{F}_1 = \mathcal{F} \cup \{f(x_1)\}$.

Without loss of generality, we may assume that $\mathcal{F}_1 \subset h_\infty(M_{I(m)}(B))$. There is a finite subset $\mathcal{G} \subset M_{I(m)}(B)$ such that $h_\infty(\mathcal{G}) = \mathcal{F}$.

Since

$$I(m+2) = I(m+1)(1 + \sum_{i=1}^{l(m+1)} k(i)) \geq 2I(m+1),$$

for any integer $n > 0$, we can choose l large enough so that $I(m+l)/I(m) > n+1$. Let $h_{m,m+l} = h_{m+l-1} \circ h_{m+l-2} \circ \cdots \circ h_m$. Then we have

$$h_{m,m+l}(x) = \sigma(x) \oplus \Phi(x)$$

for all $x \in M_{I(m)}(B)$, where $\sigma : M_{I(m)}(B) \to (1-p)M_{I(m+l)}(B)(1-p)$ is a unital injective homomorphism such that $1 - p = \sum_i^{I(m)} e_{ii}$, where $\{e_{ij}\}$ is the matrix system for $M_{I(m+l)}$, and $\Phi : M_{I(m)}(B) \to F$, where F is a finite dimensional subalgebra of $pM_{I(m+l)}(B)p$ with $1_F = p$. Moreover, for sufficiently large l, we may assume that $\Phi(f(x_1)) \neq 0$. Since Φ is a homomorphism by the spectral theorem, we must have $\|\Phi(x_1)\| \geq \|x_1\| - \varepsilon$. Note that $I(m+l) \geq (n+1)I(m)$. So

$$n[1-p] < [p] \quad \text{in} \quad D(M_{I(m+l)}).$$

We see that $C = h_\infty(F)$ is a finite dimensional subalgebra with $1_C = h_\infty(p)$ and

$$n[h_\infty(1-p)] < [h_\infty(p)] \quad \text{in} \quad D(A).$$

We also have

$$[y, h_\infty(p)] = 0$$

for all $y \in \mathcal{F}_1$ and

$$\|h_\infty(p)x_1h_\infty(p)\| \geq \|\Phi(x_1)\| \geq \|x_1\| - \varepsilon.$$

From the above, we have shown that A satisfies the local approximation properties (1), (2) and part of (3) in 2.1. Therefore, by 2.12, every hereditary C^*-subalgebra of A contains a non-zero projection. Thus, to obtain the rest of part (3), it suffices to show (see 3.8) that, for any non-zero projection e, we can choose p so that $h_\infty(1-p)$ is unitarily equivalent to a subprojection of e. To do this, without loss of generality, we may assume that $e \in B_k$. So there is a projection $q \in B \otimes M_{I(k)}$ such that $h_\infty(q) = e$ (this is possible because the $h_{n,n+1}$ are all injective). Without loss of generality, we may assume that $m > k$. We may write

$$h_{k,m}(q) = diag(q', q''),$$

where $q'' \in M_{I(m)}$ and $q'' \neq 0$. Let tr be the normalized trace on $M_{I(m)}$ and let $d = tr(q'') > 0$. In the above we could choose l so large that, also,

$$I(m)/I(m+l) < d.$$

Then it is clear that $1 - p$ is unitarily equivalent to a subprojection of $h_{m,m+l}(q'')$. Since $h_{m,m+l}(q'') \leq h_{k,m+l}(q)$, we conclude that $h_\infty(1-p)$ is unitarily equivalent to a subprojection of e. This implies that A is TAF. □

4.4. Proposition (see [D2]). *There are separable unital simple quasidiagonal TAF C^*-algebras with unique normalized trace which are not nuclear.*

Proof. In 4.1, choose B to be the full C^*-algebra of the free group of two generators. From [Ch], B is residually finite dimensional. Note that B is not exact (see [K1], and also Corollary 3.5 in [W]). Then A is not nuclear, since no non-exact C^*-algebra can be embedded into a nuclear C^*-algebra (see [K2]). From the above, A is a separable unital simple TAF C^*-algebra. By 3.4, A has real rank zero and stable rank one. Since each B_n is quasidiagonal, A is also quasidiagonal (this also follows from [Po], since A is TAF). It remains to show that A admits a unique normalized trace. Since each B_n is isomorphic to $M_{I(m)}(B)$, by [Ch], there is a tracial state τ_n on each B_n. Denote again by τ_n a state extension of τ_n on A. Let T be a weak-* limit of $\{\tau_n\}$. Then T is a state on A. To verify that T is a tracial state, we take $b \in B_n$ for some n. Then $T(b^*b) = \lim \tau_{k(n)}(b^*b)$ for some subsequence $\{k(n)\}$. Since $B_n \subset B_m$ if $m > n$, we obtain that $\tau_{k(n)}(b^*b) = \tau_{k(n)}(bb^*)$. This implies that $T(b^*b) = T(bb^*)$ for each $b \in B_n$, and so for each $b \in \bigcup_n B_n$. This implies that T is a tracial state on A. To show that there is only one normalized trace, we let $t_1, t_2 \in T(A)$ and take $a \in B_m$. In what follows, to save notation, we will identify B_n with $B \otimes M_{I(n)}$. For any integer $k > 0$, choose a large l so that

$$I(m)/I(m+l) < 1/(k+1).$$

Then we may write

$$a = diag(a', a'')$$

in $B \otimes M_{I(m+l)}$ such that $a' \in qB \otimes M_{I(m+1)}q$ and $a'' \in M_{I(m+l)}$, where $q = \sum_{i=1}^{I(m)} e_{ii}$ and $\{e_{ij}\}$ is the matrix unit for $M_{I(m+l)}$. Note that $t_i|_{M_{I(m+l)}} = tr$ $(i = 1, 2)$, where tr is the normalized trace on $M_{I(m+l)}$. We also have $t_i = s_1 \otimes tr$, where $s_i = t_i|_B$ $(i = 1, 2)$. Therefore we have

$$|t_1(a) - t_2(a)| \leq |s_1(a') - s_2(a')| < 2/(k+1).$$

Since this holds for any $k > 0$, we conclude that $t_1(a) = t_2(a)$. Thus there is only one normalized trace on A. □

4.5. We acknowledge that the construction in this section is the same as that in [D2] and the example in Proposition 4.4 is the example that Dadarlat used ([D2]) to answer a question of S. Popa in [Po]. We present these examples here only for the purpose of showing that this construction actually gives TAF C^*-algebras. It is clear that one can make more general direct limits of residually finite dimensional C^*-algebras which are simple and TAF. For example, one can repeat id_B, and one could also have a sequence of residually finite dimensional C^*-algebras instead of one fixed C^*-algebra B. The simple TAF C^*-algebras constructed in the next section are nuclear. On the other hand, if we start with a unital residually finite dimensional C^*-algebra B which is not exact (such as in 4.4), then the direct limit A is not nuclear for the same reason as in 4.4. The example in 4.4 also provides an example of a unital simple non-nuclear quasidiagonal C^*-algebra of real rank zero, stable rank one and with a unique normalized trace.

5. Simple quasidiagonal C^*-algebras

5.1. Let A be a unital strong NF algebra. By 6.1.1 in [BK1], we may write $A = \lim_{n\to\infty}(A_n, \phi_{n,m})$, where the inductive system is the *generalized* one in the sense in [BK1]: each A_n is of finite dimension and $\phi_{n,m} : A_n \to A_m$ $(m > n)$ is a complete order embedding. Note that $\phi_{n,m}$ is *not multiplicative* in general. By 7.1 in [CE2], we may write $\phi_{n,n+1} = h_n \oplus \psi_{n,n+1}$, where the decomposition has the following meaning: there exists a projection $p_{n+1} \in A_{n+1}$ such that $h_n : A_n \to p_{n+1}A_{n+1}p_{n+1}$ is a monomorphism and $\psi_{n,n+1} : A_n \to (1 - p_{n+1})A_{n+1}(1 - p_{n+1})$ is a completely positive contraction. Put $h_{n,m} = h_m \circ h_{m-1} \circ \cdots \circ h_{n+1}$ $(m > n)$ and $q_{n,m} = 1_{A_m} - h_{n,m}(1_{A_n})$. Let $a_n = \phi_{n,\infty}(1_{A_{n+1}} - p_{n+1})$, where $\phi_{n,\infty} : A_n \to A$ is the completely positive embedding induced by the generalized inductive system.

Let $QT(A)$ be the normalized quasitrace space of A. Then each a_n is a continuous function on $QT(A)$. We are interested in the infinite series $\sum_{n=1}^{\infty} a_n$. In particular, we are interested in when it converges. So we consider those simple strong NF algebras with $\sum_{n=1}^{\infty} a_n(t)$ converging uniformly on $QT(A)$. Since $a_n \geq 0$, $\sum_{n=1}^{\infty} a_n$ converges uniformly on $QT(A)$ if and only if $\sum_{n=1}^{\infty} a_n$ is a continuous function on $QT(A)$.

5.2. Lemma. *Let A be a unital C^*-algebra of real rank zero and B a hereditary C^*-subalgebra. Suppose that C is a unital C^*-subalgebra of A which contains B as an ideal. Suppose that C/B has real rank zero. Then, for every projection $\bar{e} \in C/B$, there exists a projection $e \in C$ such that $\pi(e) = \bar{e}$, where $\pi : C \to C/B$ is the quotient map. Consequently, C has real rank zero.*

Proof. There exists a positive element $a \in C$ with $0 \leq a \leq 1$ such that $\pi(a) = \bar{e}$.

Let $r_1 = 1/16, r_2 = 1/4, r_3 = 1/32$. Define continuous functions f_1, f_2, f_3 and g as follows. Put

$$f_i = \begin{cases} 1 & \text{if } r_i \leq t \leq 1, \\ \text{linear} & \text{if } r_i/2 \leq t < r_i, \\ 0 & \text{if } 0 \leq t < r_i/2, \end{cases}$$

$i = 1, 2$, and put

$$g = \begin{cases} 0 & \text{if } 1/4 \le t \le 1, \\ \text{linear} & \text{if } 1/8/ \le t < 1/4, \\ 1 & \text{if } 1/32 \le t < 1/8, \\ \text{linear} & \text{if } 1/64 \le t < 1/32, \\ 0 & \text{if } 0 \le t \le 1/64. \end{cases}$$

Let p be the open projection of A (in A^{**}) corresponding to the (relative) open subset $(1/8, 1]$ ($p = \lim_n (f_2)^{1/n}$), and let $1 - q$ be the open projection of A (in A^{**}) corresponding to the (relative) open subset $(1/32, 1]$ ($1 - q = \lim_n (f_1)^{1/n}$). Since $p \le f_1(a)$, p is compact. Note that q is a closed projection and $pq = 0$. Since A has real rank zero, by Brown's interpolation theorem ([Bn]), there is a projection $e \in A$ such that

$$p \le e \le 1 - q.$$

We have

$$f_2(a) \le e \le f_3(a).$$

So $e - f_2(a) \le g(a)$. Since $a \in C$, $g(a) \in C$. However, $\pi(g(a)) = g(\pi(a)) = 0$. Therefore $g(a) \in B$. Consequently $e - f_2(a) \in B$, since B is hereditary. But $f_2(a) \in C$. We finally conclude that $e \in C$. The inequality $f_2(a) \le e \le f_3(a)$ implies that $\pi(e) = \bar{e}$. It follows from [Zh3] that C has real rank zero. □

5.3. Theorem. *Let A be a unital strong NF C^*-algebra of real rank zero. Then, for any $\varepsilon > 0$ and any finite subset $\mathcal{F}$ of A, there exists a finite dimensional C^*-subalgebra $F \subset A$ with $p = 1_F$ such that*

(1) $\|[p, x]\| < \varepsilon$ for all $x \in \mathcal{F}$;

(2) $pxp \in_\varepsilon F$ for all $x \in \mathcal{F}$.

Proof. Let $\varepsilon > 0$ and a finite subset $\mathcal{F} = \{x_1, x_2, ..., x_n\}$ be given. From the definition of strong NF (see also [BK1]), there exists a finite dimensional C^*-algebra F and a completely positive order embedding $L : F \to A$ such that

$$\|x_i - L(b_i)\| < \varepsilon/2$$

for $x_i \in \mathcal{F}$ and some $b_i \in F$. Let D be the C^*-subalgebra generated by $L(F)$ and let $h : D \to F$ be the homomorphism (provided by 4.1 in [CE1]) such that h extends L^{-1}. Let $I = \ker h$, and let B be the hereditary C^*-subalgebra of A generated by I. We will show that for any $d \in D$ we have $bd, db \in B$ for all $b \in B$. Note that B is the closure of IAI. For any $\varepsilon > 0$, there is a positive element $e \in I$ such that

$$\|eb - b\| < \varepsilon.$$

We also have $de \in I \subset B$ for all $d \in D$. Therefore

$$\|db - deb\| < \varepsilon.$$

This implies that $db \in B$. Similarly, $bd \in B$. Let C be the closure of $D + B$. Then C is a C^*-subalgebra containing B as an ideal. Since $D/B = D/B \cap I = D/I \cong F$, we see that $C/B \cong F$. By 5.2, every projection in F lifts to a projection in C. Note that B has real rank zero, so it admits an approximate identity consisting of projections. Now a standard argument (see Lemma 9.8 in [Ef]) shows that C contains a finite dimensional C^*-subalgebra F_1 such that $\pi(F_1) = F$, where $\pi : C \to C/B$ is the quotient map. Let $q = 1_{F_1}$; then $(1-q)C(1-q) = (1-q)B(1-q)$. Write $F_1 = M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_k}$, and assume that $d_i \in M_{n_i}$ are minimal projections in M_{n_i}. Let $\{e_0^{(m)}\}$ be an approximate identity for $(1-q)B(1-q)$ consisting of projections. Let $\{c_i^{(m)}\}$ be an approximate identity for d_iBd_i consisting of projections, $i = 1, 2, ..., k$. Write the hereditary C^*-subalgebra of C generated by M_{n_i} as $M_{n_i}(d_iCd_i)$. Put $e_i^{(m)} = diag(c_i^{(m)}, c_i^{(m)}, ..., c_i^{(m)})$ and $E_m = e_0^{(m)} + \sum_{i=1}^k e_i^{(m)}$. Then it is clear that $\{E_m\}$ forms an approximate identity for B consisting of projections. Furthermore, by construction,

$$E_m x = x E_m$$

for each $x \in F_1$. For each $y \in \{L(b_1), L(b_2), ..., L(b_n)\}$, there exists $x \in F_1$ such that $x - y \in B$. Therefore, for sufficiently large m,

$$\|(1 - E_m)(x - y)(1 - E_m)\| < \varepsilon/2.$$

Moreover,

$$\|(1 - E_m)y - y(1 - E_m)\| < \varepsilon/2$$

for all $y \in \{L(b_1), L(b_2), ..., L(b_n)\}$. Let $p_1 = (1 - q - e_0^{(m)})$ and $p_2 = 1_{F_1} - E_m$. Set

$$F_2 = \mathbf{C} \cdot p_1 \oplus p_2 F_1 p_2.$$

Then F_2 is a finite dimensional C^*-subalgebra of A. Write $P = p_1 + p_2 = (1 - E_m)$. Then $1_{F_2} = P$. Note that $p_2F_1p_2 \subset F_2$ and $(1 - E_m)x(1 - E_m) \in p_2F_1p_2$ for all $x \in F_1$. Thus

$$PL(b_i)P \in_{\varepsilon/2} F_2, \;\; i = 1, 2, ..., n.$$

Furthermore, we have

(1) $\|[P, x_i]\| < \varepsilon$, $i = 1, 2, ..., n$;

(2) $Px_iP \in_\varepsilon F_2$, $i = 1, 2, ..., n$. □

5.4. Corollary (S. Popa [Po])**.** *Let A be a unital nuclear separable simple quasidiagonal C^*-algebra of real rank zero. Then there exists a finite dimensional C^*-subalgebra $F \subset A$ with $p = 1_F$ such that*

(1) $\|[p, x]\| < \varepsilon$ for all $x \in \mathcal{F}$;

(2) $pxp \in_\varepsilon F$ for all $x \in \mathcal{F}$.

Proof. It follows from 2.6 in [BK2] that every separable simple quasidiagonal C^*-algebra is inner quasidiagonal. By Theorem 4.5 in [BK2], A is strong NF. So the corollary follows from 5.3. □

5.5. Lemma. *Let A be as in 5.1. There exists a C^*-subalgebra C of A which contains $\phi_{1,\infty}(A_1)$ and a homomorphism $H : C \to A_1$ such that*

$$H \circ \phi_{1,\infty} = id_{A_1}$$

and

$$\sup\{t(a) : a \in \ker H, \, 0 \le a \le 1\} \le \sum_{n=1}^{\infty} a_n(t)$$

for all $t \in T(A)$.

Proof. Let $A_n' = h_{1,n}(A_1) \oplus q_{1,n} A_n q_{1,n} \subset A_n$, and let C_n' be the C^*-subalgebra generated by $\phi_{n,\infty}(A_n')$. Let $f_n : A_n' \to A_1$ be defined by $f_n(a \oplus b) = h_{1,n}^{-1}(a)$, where $a \in h_{1,n}(A_1)$ and $b \in q_{1,n} A_n q_{1,n}$. Note that $h_{1,n+1} = h_{n,n+1} \circ h_{1,n}$. Since each $h_{n,n+1}$ is a monomorphism, the map $\phi_{n,n+1}$ maps A_n' into A_{n+1}'. We have the following commutative diagrams:

$$\begin{tikzcd} A_n' \arrow[rr, "\phi_{n,n+1}"] & & A_{n+1}' \\ & A_1 \arrow[ul, "\phi_{1,n}"] \arrow[ur, "\phi_{1,n+1}"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} A_n' \arrow[rr, "\phi_{n,n+1}"] \arrow[dr, "f_n"'] & & A_{n+1}' \arrow[dl, "f_{n+1}"] \\ & A_1 & \end{tikzcd}$$

Let $j : C_n' \to C_{n+1}'$ be the embedding. Then we also have the following commutative diagram:

$$\begin{tikzcd} C_n' \arrow[r, "j"] & C_{n+1}' \\ A_n' \arrow[u, "\phi_{n,\infty}"] \arrow[r, "\phi_{n,n+1}"] & A_{n+1}' \arrow[u, "\phi_{n+1,\infty}"'] \end{tikzcd}$$

Note that $\phi_{n,\infty}$ is a complete order embedding. By 4.1 of [CE1] (see 5.1), there is a homomorphism $\psi_n' : C_n' \to A_n'$ such that

$$\psi_n'|_{\phi_{n,\infty}(A_n')} = \phi_{n,\infty}^{-1}|_{A_n'}$$

for each n. We obtain the following commutative diagram:

$$\begin{tikzcd} C_n' \arrow[r, "j"] \arrow[d, "\psi_n'"'] & C_{n+1}' \arrow[d, "\psi_{n+1}'"] \\ A_n' \arrow[r, "\phi_{n,n+1}"] & A_{n+1}' \end{tikzcd}$$

We then combine this to obtain the following commutative diagram:

$$\begin{tikzcd} C_n' \arrow[rr, "j"] \arrow[d, "\psi_n'"'] & & C_{n+1}' \arrow[d, "\psi_{n+1}'"] \\ A_n' \arrow[rr, "\phi_{n,n+1}"] \arrow[dr, "f_n"'] & & A_{n+1}' \arrow[dl, "f_{n+1}"] \\ & A_1 & \end{tikzcd}$$

Put $C = cl(\bigcup C_n')$. Define $H : C \to A_1$ by $H|_{C_n'} = f_n \circ \psi_n'$. The above commutative diagram shows that H is a well defined homomorphism. Furthermore, we have

$$H|_{\phi_{1,\infty}(A_1)} = \phi_{1,\infty}^{-1}.$$

It is easy to see that the closure of $\bigcup \ker f_n \circ \psi_n'$ is the kernel of H. Let $a \in \ker f_n \circ \psi_n'$ and $0 \le a \le 1$. Let $b \in A_n'$ be such that $\phi_{n,\infty}(b) = a$. So

$$b \le q_{1,n} \le \phi_{2,n}(q_{1,2}) + \phi_{3,n}(q_{2,3}) + \cdots + \phi_{n-1,n}(q_{n-1,n}).$$

Hence

$$a \le \sum_{k=1}^{n} a_k.$$

It follows that

$$\sup\{t(a) : 0 \le a \le 1,\ a \in \ker H\} \le \sum_{k=1}^{\infty} a_k(t)$$

for all $t \in T(A)$. □

5.6. Theorem. *Let A be a separable nuclear unital simple quasidiagonal C^*-algebra of real rank zero. Suppose that $\sum_{n=1}^{\infty} a_n(t)$ is a continuous function on $QT(A)$. Then A satisfies the following: For any $\varepsilon > 0, \sigma > 0$ and any finite subset $\mathcal{F} \subset A$, there exist a projection p and a finite dimensional C^*-subalgebra F with $1_F = p$ such that*
(1) $\|[p, x]\| < \varepsilon$ for all $x \in \mathcal{F}$;
(2) $pxp \in_\varepsilon F$;
(3) $t(1 - p) < \sigma$ for traces $t \in T(A)$.

Proof. We write $A = \lim(A_n, \phi_{n,n+1})$ as a generalized inductive system, where each A_n is a finite dimensional C^*-algebra and each $\phi_{n,n+1}$ is a complete order embedding. The proof uses a combination of 5.5 and the proof of 5.3. Let $\varepsilon > 0$, and let $\mathcal{F}$ be a finite subset of A. Choose n so that

$$\mathcal{F} \subset \phi_{n,\infty}(A_n)$$

and

$$\sum_{k=n}^{\infty} a_k(t) < \sigma \quad \text{for all } t \in T(A).$$

(Note that, since $0 \le a_n$, and $\sum_{n=1}^{\infty} a_n(t)$ is continuous, the above infinite series converges uniformly on $T(A)$.) So, by applying 5.3, we obtain p and F which satisfy (1) and (2). To see that (3) holds, we note that it suffices to show in the proof of 5.3 that

$$\sup\{t(a) : 0 \le a \le 1, a \in B\} < \sigma$$

for all $t \in T(A)$. Since (in the proof of 5.3) $B = IAI$, it suffices to show that

$$\sup\{t(a) : 0 \le a \le 1, a \in I\} < \sigma$$

for all $t \in T(A)$. But, by 5.5, we have

$$\sup\{t(a) : 0 \le a \le 1, a \in I\} \le \sum_{k=n}^{\infty} a_k(t) < \sigma.$$

□

5.7. Recall ([Bl]) that a unital simple C^*-algebra A is said to have *fundamental comparability* if $\tau(p) < \tau(q)$ for all $\tau \in QT(A)$ implies that $p \preceq q$.

5.8. Corollary. *Let A be a separable unital simple quasidiagonal C^*-algebra of real rank zero with fundamental comparability (for example, A also has stable rank one and $K_0(A)$ is weakly unperforated). Suppose that $\sum_{n=1}^{\infty} a_n(t)$ is a continuous function on $QT(A)$. Then A is TAF.*

5.9. Corollary. *Let A be a separable unital simple quasidiagonal C^*-algebra of real rank zero. Suppose that $\sum_{n=1}^{\infty} a_n(t)$ is a continuous function on $QT(A)$. Then $A \otimes B$ is TAF for any unital simple AF-algebra B.*

Proof. Write $A = \lim(F_n, h_n \oplus \psi_{n,n+1})$ as a generalized inductive limit as in 5.1, and write $B = \lim(B_n, \phi_n)$, where the B_n are finite dimensional C^*-algebras and the ϕ_n are homomorphisms. Then $A \otimes B = \lim(F_n \otimes B_n, (h_n \oplus \psi_{n,n+1}) \otimes \phi_n)$. It is then easy to check that the new inductive limit satisfies the condition in 5.6. It follows from 1.4 in [BKR] that $A \otimes B$ has fundamental comparability. So 5.9 follows from 5.8. □

5.10. Corollary. *Let A be a separable unital simple quasidiagonal C^*-algebra with unique normalized quasitrace. Suppose that $\sum_{n=1}^{\infty} a_n(t) < \infty$. Then $A \otimes B$ is TAF for any unital UHF-algebra B.*

Proof. By 7.2 in [Rr], since A has a unique normalized quasitrace, $A \otimes B$ has real rank zero. □

6. Characterization of rational simple AF-algebras

In this section we will show that a unital separable nuclear simple TAF C^*-algebra with the right K-theory is isomorphic to a unital UHF-algebra with rational K_0. We regard this result as the first step in classifying simple nuclear C^*-algebras of real rank zero and stable rank one without assuming any special inductive limit structure. The work in previous sections suggests that the class of TAF C^*-algebras is the right class of C^*-algebras to study. However, since K-theory will be the invariant, we need to assume that C^*-algebras satisfy the Universal Coefficient Theorem. All C^*-algebras in the so-called "bootstrap" class satisfy the UCT. This is already a sufficiently large class of C^*-algebras. Moreover, there are no known examples of nuclear C^*-algebras that do not satisfy the UCT.

In what follows, we use the notation Q for the unital UHF-algebra with $K_0(A) = \mathbf{Q}$ and $[1_Q] = 1$ in $K_0(A) = \mathbf{Q}$.

6.1. Definition. Let A and B be C^*-algebras, let $L : A \to B$ be a contractive completely positive linear morphism, let $\varepsilon > 0$, and let $\mathcal{F} \subset A$ be a subset. L is said to be *$\mathcal{F}$-ε-multiplicative* if

$$\|L(xy) - L(x)L(y)\| < \varepsilon$$

for all $x, y \in \mathcal{F}$.

6.2. Proposition. *Let A be a separable nuclear C^*-algebra. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exist $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ satisfying the following: for any C^*-algebra C, any C^*-subalgebra B of C and any δ-$\mathcal{G}$-multiplicative completely positive contraction $L : A \to C$ with* $\operatorname{dist}(L(a), B) < \delta$ *for all $a \in \mathcal{G}$,* *there exists a completely positive contraction $\phi : A \to B$ such that*

$$\|L(a) - \phi(a)\| < \varepsilon$$

for all $a \in \mathcal{F}$.

Proof. If the proposition failed, we would obtain a sequence of C^*-algebras C_n with C^*-subalgebras $B_n \subset C_n$ and a sequence of completely positive contractions $L_n : A \to C_n$ with

$$\operatorname{dist}(L_n(a), B_n) \to 0 \quad \text{and} \quad \|L_n(a)L_n(b) - L_n(ab)\| \to 0$$

for all a, $b \in A$, but

$$\inf\{\sup\{\|L_n(a) - \phi_n(a)\| : a \in \mathcal{F}\}\} \geq \varepsilon_0 > 0,$$

where the infimum is taken over all completely positive contractions $\phi_n : A \to B_n$.

Let $L : A \to \prod_n C_n$ be defined by $L(a) = \{L_n(a)\}$ for all $a \in A$. Let $B = \prod_n B_n$ be a C^*-subalgebra of $\prod_n C_n$, and let $\pi : \prod_n C_n \to \prod_n C_n / \bigoplus_n C_n$ be the quotient map. Note that $\pi(B) = B/(\bigoplus_n C_n \cap B) = B/\bigoplus_n B_n$. Consider $\bar{L} = \pi \circ L : A \to \prod_n C_n / \bigoplus_n C_n$. The condition that

$$\operatorname{dist}(L_n(a), B_n) \to 0 \quad \text{and} \quad \|L_n(a)L_n(b) - L_n(ab)\| \to 0$$

for all a, $b \in A$ implies that $\bar{L}$ actually is a homomorphism and maps A into $\pi(B)$. Since A is nuclear, there exists a completely positive contraction $\Phi : A \to B$ such that $\pi \circ \Phi = \bar{L}$. We write $\Phi = \{\phi_n\}$, where each $\phi_n : A \to B_n \subset C_n$ is a completely positive contraction. Therefore, we have

$$\|L_n(a) - \phi_n(a)\| \to 0$$

as $n \to \infty$. This completes the proof. □

6.3. Proposition. *Let A be a nuclear unital TAF C^*-algebra. Then, for any $\varepsilon > 0$, any integer $n > 0$, any finite subset $\mathcal{F}$ of A, and any full element $a \in A_+$, there exist a finite dimensional C^*-subalgebra $F \subset A$ with $p = 1_F$ and a completely positive contraction $L : A \to F$ which is ε-$\mathcal{F}$-multiplicative such that*

(1) $\|[p, x]\| < \varepsilon$ for all $x \in \mathcal{F}$,

(2) $\|x - ((1-p)x(1-p) + L(x))\| < \varepsilon$ for all $x \in \mathcal{F}$, and

(3) $n[1-p] \leq [p]$ in $D(A)$ and $1 - p \preceq a$.

Proof. This follows immediately from the definition of TAF, the nuclearity of A, and Proposition 6.2. □

6.4. Lemma. *Let A be a nuclear unital TAF C^*-algebra. Then, for any simple AF-algebra B, any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there is an $\mathcal{F}$-ε-multiplicative contractive completely positive linear map $L : A \to B$.*

Proof. We note that, for any finite dimensional C^*-algebra F, there exists a monomorphism $\psi : F \to B$. Then the lemma follows from 6.3. □

6.5. Let A be a C^*-algebra. Denote by $M_\infty(A)$ the union of the $M_m(A)$, $m = 1, 2, \ldots$. Let $\mathcal{P} \subset M_\infty(A)$ be a finite subset of projections in $M_\infty(A)$. There are a finite subset $\mathcal{G}(\mathcal{P}) \subset A$ and $\delta(\mathcal{P}) > 0$ such that if B is any unital C^*-algebra and $L : A \to B$ is a contractive completely positive linear morphism which is $\mathcal{G}(\mathcal{P})$-$\delta(\mathcal{P})$-multiplicative, then L (well) defines a map $[L]$ from $\mathcal{P}$ into $K_0(B)$. (see for example 1.4 in [Ln3]).

6.6. Lemma. *Let A be a unital separable nuclear TAF C^*-algebra such that $K_0(A)$ is a countable additive subgroup of $\mathbf{Q}$ (as an ordered group). Let $\alpha : K_0(A) \to \mathbf{Q}$ be an order homomorphism. Given a finite subset X of projections in $M_\infty(A)$, there exist a finite subset $\mathcal{G}$ and $\varepsilon > 0$ such that if $L : A \to Q$ is a $\mathcal{G}$-ε-multiplicative contractive completely positive linear map, then there is a homomorphism $h : Q \to Q$ such that*

$$h_*([L]|_X) = \alpha|_{\bar{X}},$$

where $\bar{X}$ is the image of X in $K_0(A)$.

Proof. By sending $[1_A]$ to 1 in $\mathbf{Q}$, we may identify $K_0(A)$ with a subgroup of $\mathbf{Q}$ in which $[1_A] = 1$ in $\mathbf{Q}$. We may assume that $\bar{X} = \{[p_1], [p_2], ..., [p_k]\}$, where $p_1, p_2, ..., p_k$ are projections in A. Suppose that $[p_i] = n_i/m_i$, where m_i and n_i are positive integers. Since A has stable rank one (and real rank zero), there is a partial isometry $v_i \in M_{m_i+n_i}(A)$ for each i such that

$$v_i^* v_i = diag(1_A, \cdots, 1_A, 0, \cdots, 0) \quad \text{and} \quad v_i v_i^* = diag(p_i, \cdots, p_i, 0, \cdots, 0),$$

where in the first diagonal there are n_i 1_A's and in the second diagonal there are m_i p_i's.

Let $l = \max_i\{m_i + n_i\}$. Choose a large $\mathcal{G}$ so that $M_l(\mathcal{G})$ contains p_i, $v_i^* v_i$, $v_i v_i^*$ and v_i, $1 \le i \le k$. Choose δ so small that $[L]$ is well-defined on 1_A, p_i, $v_i^* v_i$, and $v_i v_i^*$, and

$$[L]([v_i^* v_i]) = [L]([v_i v_i^*]) \text{ (see } \ 6.5).$$

One sees that $[L](p_i) = (m_i/n_i) \cdot [L]([1_A])$ in $K_0(Q)$, $i = 1, 2, ..., k$. There is a rational number $r \in \mathbf{Q}$ such that $r \cdot [L]([1_A]) = \alpha([1_A])$. Let $e \in M_d(Q)$ be such that $[e] = r$ for some integer $d > 0$. The homomorphism $\sigma : \mathbf{Q} \to \mathbf{Q}$ defined by $\sigma(x) = rx$ is an order homomorphism. From the classification theory of AF-algebras, we know that there is a homomorphism $h : Q \to Q$ such that $h([1_Q]) = r$. It is then easy to check that $h_*([L]|_X) = \alpha|_{\bar{X}}$. □

6.7. Let A and B be two C^*-algebras with A unital. Let $h_1, h_2 : B \to A$ be two homomorphisms and $\mathcal{F}$ a subset of B. For convenience, we write (for $\varepsilon > 0$)

$$h_1 \sim_\varepsilon h_2 \quad \text{on } \mathcal{F}$$

if there exists a unitary $u \in A$ such that $\|u^* h_1(b) u - h_2(b)\| < \varepsilon$ for all $b \in \mathcal{F}$; and

$$h_1 \approx_\varepsilon h_2 \quad \text{on } \mathcal{F}$$

if $\|h_1(b) - h_2(b)\| < \varepsilon$ for all $b \in \mathcal{F}$.

The following lemma is certainly known.

6.8. Lemma. *Let A be an AF-algebra and B a C^*-algebra. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there are a finite subset $\mathcal{P}(\mathcal{F}, \varepsilon)$ of projections in A, a finite subset $\mathcal{G}(\mathcal{F}, \varepsilon) \subset A$ and $\delta(\mathcal{F}, \varepsilon) > 0$ satisfying the following: If L_1, $L_2 : A \to B$ are two $\mathcal{G}$-δ-multiplicative contractive completely positive linear maps with $[L_1]|_{\mathcal{P}} = [L_2]|_{\mathcal{P}}$ (here we assume that $\mathcal{G}$ is large enough and δ is small enough so that they are well-defined—see 6.5), then*

$$L_1 \sim_\varepsilon L_2 \quad \textit{on } \mathcal{F}.$$

The following is an easy version of 5.3 in [Ln5] (see also 5.4 in [Ln5]), which we state here for convenience.

6.9. Theorem (5.3 and 5.4 in [Ln5])**.** *Let A be a unital simple nuclear C^*-algebra with torsion free $K_0(A)$ and trivial $K_1(A)$ which satisfies the Universal Coefficient Theorem. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$ there exist a positive number $\delta > 0$, a finite subset $\mathcal{G} \subset A$, a finite subset $\mathcal{P}$ of projections in $A \otimes \mathcal{K}$ and an integer $n > 0$ satisfying the following: for any unital C^*-algebra B of stable rank one, if $\phi, \psi, \sigma : A \to B$ are three $\mathcal{G}$-δ-multiplicative contractive completely positive linear maps with $[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}}$, and σ is unital, then there is a unitary $u \in M_{n+1}(B)$ such that*

$$\|u^* diag(\phi(a), \sigma(a), \cdots, \sigma(a))u - diag(\psi(a), \sigma(a), \cdots, \sigma(a))\| < \varepsilon$$

for all $a \in \mathcal{F}$, where $\sigma(a)$ repeats n times.

6.10. Lemma. *Let A be a nuclear simple unital TAF C^*-algebra. For any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$ and any positive integer $n > 0$, there exist projections p_1 and p_2 such that $(1-p_1)A(1-p_1) = M_n(p_2Ap_2)$ with $p_1 \preceq p_2$, and there are unital $\mathcal{F}$-ε-multiplicative contractive completely positive linear maps $\phi : A \to F_1$, where F_1 is a finite dimensional C^*-subalgebra of p_2Ap_2, such that*

(a) $\|[p_i, x]\| < \varepsilon$ for all $x \in \mathcal{F}$;

(b) $\|x - (p_1xp_1 \oplus diag(\phi(x), \phi(x), ..., \phi(x)))\| < \varepsilon$ for all $x \in \mathcal{F}$ (where $\phi(x)$ repeats n times).

Proof. We use a modification of 6.3. Let F be as in 6.3 with $(n+1)[1-p] \le [p]$ in $D(A)$. Write $F = M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_k}$. Let d_i be a (non-zero) minimal projection of M_{n_i}, $i = 1, 2, ..., k$. Choose m sufficiently large. Applying Zhang's halving theorem ([Zh2]), we obtain

$$d_i = d_i^{(1)} \oplus d_i^{(2)} \oplus \cdots \oplus d_i^{(2^m)} + r_i,$$

where the $d_i^{(j)}$ are mutually equivalent projections and $r_i \in d_iAd_i$ is a projection with $r_i \preceq d_i^{(1)}$. By results in Section 3, A is of real rank zero, stable rank one and has weakly unperforated $K_0(A)$. By comparing traces and by choosing large m, we may assume that

$$(1-p) \oplus \sum_i \left(\sum_{j=nl+1}^{2^m} d_i^{(j)} \right) + \sum_i r_i \preceq \sum_i \left(\sum_{j=1}^{l} d_i^{(j)} \right)$$

with $nl \le 2^m < n(l+1)$. Let

$$p_1 = (1-p) \oplus \sum_i diag \left(\sum_{j=nl+1}^{2^m} d_i^{(j)} + \sum_i r_i, ..., \sum_{j=nl+1}^{2^m} d_i^{(j)} + \sum_i r_i \right),$$

$$E = \sum_i diag \left(\sum_{j=1}^{nl} d_i^{(j)}, ..., \sum_{j=1}^{nl} d_i^{(j)} \right) \quad \text{and} \quad p_2 = \sum_i diag \left(\sum_{j=1}^{l} d_i^{(j)}, ..., \sum_{j=1}^{l} d_i^{(j)} \right),$$

where the ith diagonal has n_i entries. Note that

$$\left[diag \left(\sum_{j=1}^{l} d_i^{(j)}, ..., \sum_{j=1}^{l} d_i^{(j)} \right), y \right] = 0$$

for each $y \in M_{n_i}$. So E commutes with every element in F. We also estimate that

$$\|[p_1, x]\| < \varepsilon.$$

Let $F_1 = EFE$. Note that

$$[E, L(a)] = 0 \quad \text{and} \quad [p_2, L(a)] = 0$$

for every $a \in A$. Define $L' = ELE$ and $\phi(a) = p_2 L(a) p_2$ for $a \in A$. Then $L'(a) = diag(\phi(a), ..., \phi(a))$, where $\phi(a)$ repeats n times. It is easy to see that p_1, p_2, ϕ and F_1 satisfy the requirements. □

6.11. Theorem. *Let A be a separable nuclear simple TAF C^*-algebra with the UCT such that $K_1(A) = 0$ and $K_0(A)$ is the additive group of $\mathbf{Q}$ (as an ordered group) with $[1_A] = 1$ in $\mathbf{Q}$. Then $A \cong Q$.*

Proof. Let $\mathcal{F}$ be a finite subset of A and $\varepsilon > 0$. We will show that there is a C^*-subalgebra $B \subset A$ which is AF and such that $\mathcal{F} \subset_\varepsilon B$. This implies that A is an AF-algebra. Then, by Elliott's classification theorem ([Ell1]) for AF-algebras, $A \cong Q$.

We may assume that $\mathcal{F}$ is in the unit ball of A. Set $\mathcal{F}_1 = \mathcal{F} \cup \{1_A\}$. Let $\mathcal{G}_1 = \mathcal{G}(\varepsilon/4, \mathcal{F}_1)$, $\delta_1 = \delta(\mathcal{F}_1, \varepsilon/4)$, $\mathcal{P}_1 = \mathcal{P}(\mathcal{F}_1, \varepsilon/4)$ and the positive integer $n > 0$ be as required in Theorem 6.9. Let $\mathcal{G}_1'$ and δ_1' be as required by 6.6 for $X = \mathcal{P}_1$. Set $\delta_2 = \min\{\delta_1/2, \varepsilon/8, \delta_1'/2\}$ and $\mathcal{G}_2 = \mathcal{G}_1 \cup \mathcal{G}_1'$. By 6.10, we obtain two projections, p_1 and p_2, and a unital C^*-subalgebra $p_1Ap_1 \oplus M_n(p_2Ap_2)$ of A with $p_1 \preceq p_2$, and we obtain a δ_2-$\mathcal{G}_2$-multiplicative contractive completely positive linear map $L_1 : A \to F_1$, where F_1 is a finite dimensional C^*-subalgebra of p_2Ap_2, such that

$$\|x - (p_1xp_1 + diag(L_1(x), L_1(x), ..., L_1(x))\| < \delta_2 \quad \text{and} \quad \|[p_1, x]\| < \delta_2$$

for all $x \in \mathcal{G}_2$. Define $\psi_1 : A \to p_1Ap_1$ by $\psi_1(a) = p_1xp_1$, and define $\psi_2 : A \to (1-p_1)A(1-p_1)$ by $\psi_2(x) = diag(L_1(x), \cdots, L_1(x))$. Note that ψ_1 and ψ_2 are also $2\delta_2$-$\mathcal{G}_2$-multiplicative. We may assume that $[\psi_i(\mathcal{P}_1)]$ is well defined ($i = 1, 2$—see 6.5). Set $L_1^{(i)} = diag(0, ..., L_1, 0, ..., 0)$ (the ith place is L_1) and

$$\mathcal{G}_2' = \{diag(L_1, L_1, ..., L_1)(\mathcal{G}_2), p_1, (1-p_1), L_1^{(i)}(\mathcal{G}_2), i = 1, 2, ..., n\}.$$

Let $F_1' = \bigoplus_{i=1}^n F_1$ (n summands of F_1). Since F_1' is finite dimensional, there exists a monomorphism $\phi_1' : F_1' \to Q$ such that $[\phi_1'(p)] = [j_*(p)]$ for all $p \in \mathcal{P}_2'$, where $j_* : K_0(F_1') \to \mathbf{Q}$ is induced by the embedding $j : F_1' \to A$. Let $P = \phi_1'(1_{F_1'})$. We may assume that $P \le 1_Q$.

By applying 6.4 and 6.6, we obtain $\phi_1'' : p_1Ap_1 \to Q$, a δ_3-$\psi_1(\mathcal{G}_2)$-multiplicative contractive completely positive linear map such that

$$[\phi_1'' \circ \psi_1(p)] = [\psi_1(p)] \text{ in } \mathbf{Q}$$

for all $p \in \mathcal{P}_1$. We may assume that $\phi_1''(p_1) = 1_Q - P$. Set $\phi_1 = \phi_1'' \circ \psi_1 \oplus \phi_1' \circ \psi_2 : A \to Q$.

Let $h : Q \to A$ be a unital homomorphism with $[h] = \mathrm{id}_{\bar{Q}}$ given by 2.9 in [Ln1]. By 6.8, we obtain a unitary $u_1 \in A$ such that

$$ad(u_1) \circ h \circ \phi_1' \approx_{\delta_2/2} \mathrm{id}_A \quad \text{on } \psi_2(\mathcal{G}_1)$$

for each i. Put $h' = ad(u_1) \circ h$. So

$$h' \circ \phi_1' \circ diag(L_1, L_1, ..., L_1) \approx_{\delta_2/2} diag(L_1, L_1, ..., L_1) \quad \text{on } \mathcal{G}_1.$$

Now, by construction and Theorem 6.9, we have

$$\psi_1 \oplus diag(L_1, L_1, ..., L_1) \sim_{\varepsilon/4} h' \circ \phi_1'' \circ \psi_1 \oplus diag(L_1, L_1, ..., L_1) \quad \text{on } \mathcal{F}_1.$$

So

$$\begin{aligned}\mathrm{id}_A &\approx_{\varepsilon/4} \psi_1 \oplus diag(L_1, L_1, ..., L_1)\\ &\sim_{\varepsilon/4} h' \circ \phi_1'' \circ \psi_1 \oplus diag(L_1, L_1, ..., L_1) \sim_{\varepsilon/4} h' \circ \phi_1 \quad \text{on } \mathcal{F}_1.\end{aligned}$$

Therefore, there is a unitary $v_1 \in A$ such that

$$\mathrm{id}_A \approx_\varepsilon h_1 \circ \phi_1 \quad \text{on } \mathcal{F}_1,$$

where $h_1 = ad(v_1) \circ h'$.

Let $B = h_1(Q)$. Then $\mathcal{F} \subset_\varepsilon B$. □

The method used in this section is based on results in [Ln5]. A classification result for unital nuclear separable simple TAF C^*-algebras will appear in a subsequent paper.

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