TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 353, Number 2, Pages 723–731
S 0002-9947(00)02696-9
Article electronically published on October 11, 2000

LINEAR MAPS DETERMINING THE NORM TOPOLOGY

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Abstract. Let A be a Banach function algebra on a compact space X, and let $a \in A$ be such that for any scalar λ the element $a + \lambda e$ is not a divisor of zero. We show that any complete norm topology on A that makes the multiplication by a continuous is automatically equivalent to the original norm topology of A. Related results for general Banach spaces are also discussed.

1. Introduction

Let A be a Banach space, and let T be a linear map on A. In most automatic continuity problems (see, for example; [2], [6], [9], [11], [12]) we ask for the algebraic conditions on T and A that *automatically* force the continuity of T. Here we would like to reverse the problem; we assume that T is continuous and would like to know if that uniquely determines the complete norm topology of A. Certainly, not all maps T have this property; for example, if T is equal to the identity map on an infinite-dimensional, complemented subspace B of A, we can freely change the norm on B without disturbing the continuity of T. We will mostly investigate multiplication operators on commutative, semisimple Banach algebras, but we shall also prove that any separable Banach space has a continuous map that determines its complete norm topology. The same problem, primarily for $C(X)$ spaces and uniform algebras, has been very recently investigated by A. R. Villena [13]. Our result generalizes that of Villena by providing both necessary and sufficient conditions and by extending the results to a larger class of algebras.

There are a number of deep theorems concerning uniqueness of a *Banach algebra norm*; that is, a norm which is complete and submultiplicative. The results range from classical—the uniqueness of the complete Banach algebra norm for semisimple Banach algebras [8], [7]—to much more recent [1], [3]. In this paper we do not assume that the second norm makes A into a Banach algebra; that is, we *do not* assume that the multiplication remains continuous.

Our results are valid in both the real and the complex cases.

2. Notation

For a Banach algebra A we will denote by $\mathfrak{M}(A)$ the set of all non-zero linear and multiplicative functionals on A equipped with the weak * topology. If A is commutative and semisimple, the Gelfand transform $\widehat{\ }$, defined by $\hat{a}(x) = x(a)$, is a continuous homomorphism of A into the space $C(\mathfrak{M}(A))$ of continuous functions on $\mathfrak{M}(A)$ [5]. Hence, any such Banach algebra can be identified with an algebra of

Received by the editors May 14, 1998 and, in revised form, May 12, 1999.
2000 *Mathematics Subject Classification.* Primary 46B03, 46J10; Secondary 46E15.
Key words and phrases. Automatic continuity, uniqueness of norm.

continuous functions on a compact space; we will use the same symbol to denote an element of an algebra and its Gelfand transform. We say that an element a of an algebra A is a divisor of zero if there is a non-zero element b of A such that $ab = 0$.

For a Banach space $A = (A, \|\cdot\|)$ we denote by $B(A) = B((A, \|\cdot\|))$ the space of all continuous linear maps from A into itself. We say that $T \in B((A, \|\cdot\|))$ *determines the complete norm topology* of A if, for any complete norm $|\cdot|$ on A such that $T \in B((A, |\cdot|))$, the topologies defined by the norms $|\cdot|$ and $\|\cdot\|$ are the same. Notice that in general the requirement that the topologies are identical is stronger than the requirement that the Banach spaces $(A, \|\cdot\|)$ and $(A, |\cdot|)$ are isomorphic. For a Banach algebra A we say that $a \in A$ *determines the complete norm topology* if the operator T_a of multiplication by a on A determines the complete norm topology.

3. The Results

Theorem 1. *Let A be a unital, semisimple, commutative Banach algebra. Then an element a of A determines the complete norm topology of A if and only if, for each scalar λ such that $(a + \lambda e)$ is a divisor of zero, the codimension of $(a + \lambda e) A$ is finite.*

The assumption that the second norm is complete is crucial—we can easily define a noncomplete norm on a given algebra such that not one, but *all*, of the multiplication operators remain continuous. For example, the space $C([0,1])$ of all continuous functions on a unit segment is a Banach algebra when equipped with the sup norm. Any operator on $C([0,1])$ of multiplication by a continuous function is also continuous with respect to the L^1-norm, $\|f\|_1 = \int_0^1 |f|$, but the L^1-norm and the sup norm are obviously not equivalent. A very similar example can be constructed for any infinite-dimensional, commutative, semisimple Banach algebra.

Theorem 2. *Let A be a unital, semisimple, commutative Banach algebra such that the maximal ideal space of A does not have any isolated point. Then an element a of A determines the complete norm topology of A if and only if, for each scalar λ, the element $a + \lambda e$ is not a divisor of zero.*

Notice that one of the implications of Theorem 2 follows immediately from Theorem 1: if, for any scalar λ, the element $a + \lambda e$ is not a divisor of zero, then a determines the complete norm topology. For this implication we do not even need the assumption that $\mathfrak{M}(A)$ does not have any isolated points; however, this property of $\mathfrak{M}(A)$ follows from the other one since, if $x_0 \in \mathfrak{M}(A)$ is an isolated point, then $a - a(x_0)e$ must be a divisor of zero.

Corollary 3. *Let A be a unital, semisimple, commutative Banach algebra, and let $a \in A$ be such that, for any scalar λ, the subset $\{x : a(x) = \lambda\}$ of $\mathfrak{M}(A)$, has an empty interior. Then a determines the complete norm topology of A.*

Proof of the Corollary. Since the set $\{x : a(x) = \lambda\}$ has an empty interior, it does not contain any isolated point. Consequently, since λ is arbitrary, $\mathfrak{M}(A)$ has no isolated points. Also, $a + \lambda e$ is not a divisor of zero, since if $(a + \lambda e) b = 0$, then the support of b would be contained in the set $\{x : a(x) = \lambda\}$; however, the support of any nonzero continuous function has a nonempty interior. $\square$

In the last section of the paper we provide several examples answering questions posed by Villena in [13]. We show that the condition of the corollary that $\{x : a(x) = \lambda\}$ has empty interior is stronger than the condition that $a(x) - \lambda e$ is not a divisor of zero. We also prove there that a commutative, semisimple, separable Banach algebra contains an element a such that $\operatorname{int}\{x \in \mathfrak{M}(A) : a(x) = \lambda\} = \emptyset$ for any scalar λ if and only if its maximal ideal space does not contain isolated points.

For non-semisimple Banach algebras, multiplication operators may never determine the complete norm topology. For instance, this is so when the multiplication on the algebra is trivial; that is, $ab = 0$ for all a, b. It may be interesting to find a condition for an element of a noncommutative (semisimple) Banach algebra that guarantees the same topology.

The next result shows that for any separable Banach space there is a continuous linear map determining the complete norm topology of that space.

Theorem 4. *For any separable Banach space X there is a bounded linear map T on X that determines a complete norm topology of X.*

4. Proofs

4.1. **Preliminaries.** Before we can prove the theorems stated in the previous section we need to develop some preliminary results. The following lemma is a special case of what is perhaps the most basic result in automatic continuity theory [12].

Lemma 5. *Let S be a linear map between Banach spaces X and Y, and let T_n, R_n be continuous linear maps on X and on Y, respectively, such that $R_n S = ST_n$ for all $n \in \mathbb{N}$. Let $\mathfrak{S}_n$ be the norm closure of $R_n \circ R_{n-1} \circ \cdots \circ R_1(\mathfrak{S})$, where $\mathfrak{S}$ is the separating space of S. Then there is an integer N such that $\mathfrak{S}_n = \mathfrak{S}_N$ for each $n \geq N$.*

Recall that the separating space $\mathfrak{S}$ of a linear map S between Banach spaces X and Y is defined by

$$\mathfrak{S} = \{y \in Y : \text{ there is a sequence } (x_n)_{n=1}^{\infty} \text{ in } X \text{ such that } x_n \to 0 \text{ and } Sx_n \to y\}.$$

By the Closed Graph Theorem, S is continuous if and only if $\mathfrak{S} = \{0\}$.

We also need the following standard fact.

Lemma 6. *Assume T is a bounded linear map from a Banach space X into a Banach space Y. If the codimension of $T(X)$ is finite, then $T(X)$ is closed.*

A part of the proof of Theorem 1 will be based on the following proposition.

Proposition 7. *Let T be a continuous linear map from an infinite-dimensional Banach space A into itself. Assume that $\ker T \cap T(A) = \{0\} \neq \ker T$, and that $T(A)$ is of infinite codimension in A. Then T does not determine the complete norm topology of A.*

Proof of the Proposition. Assume to begin with that a linear complement B of $\operatorname{span}(\ker T \cup T(A))$ is infinite-dimensional. Since $\ker T \cap T(A) = \{0\}$, the vector space A can be identified with a direct linear product $\ker T \oplus T(A) \oplus B$. Since $\dim B = \infty$, there is a discontinuous linear map Q from B into $\ker T$. Let

$$P : \ker T \oplus T(A) \oplus B \to \ker T \oplus T(A) \oplus B$$

be defined by

$$P(a_1, a_2, a_3) = (a_1 + Qa_3, 0, 0).$$

The map P is a discontinuous linear projection from A onto $\ker T$ such that $P \circ T = 0$. We define a new norm $|\cdot|$ on A by

$$|a| = \left\| a_{/\ker T} \right\| + \|Pa\|, \quad \text{for } a \in A,$$

where $\left\| a_{/\ker T} \right\|$ is the norm of the equivalence class of a in the quotient space $A_{/\ker T}$. Notice that $|a|$ is a well-defined complete norm on A. Indeed, $(A, |\cdot|)$ is isometric with the direct product of $A_{/\ker T}$ and $\ker T$. Since P is discontinuous, the new norm and the original one are inequivalent (though the Banach spaces $(A, \|\cdot\|)$ and $(A, |\cdot|)$ may be isomorphic).

To show that T is continuous on $(A, |\cdot|)$, let $a \in A$ be such that $|a| < 1$. Since $\left\| a_{/\ker T} \right\| \leq |a|$, we have $\left\| a_{/\ker T} \right\| < 1$, and there is an element b in $\ker T$ such that $\|a + b\| < 1$. By our assumption $P \circ T = 0$, so

$$\begin{aligned} |Ta| &= \left\| Ta_{/\ker T} \right\| + \|PTa\| = \left\| Ta_{/\ker T} \right\| \\ &\leq \|Ta\| = \|T(a+b)\| \leq \|T\| \|a+b\| < \|T\|, \end{aligned}$$

where $\|T\|$ is the norm of T in $(A, \|\cdot\|)$. The above shows that the norm of T in $(A, |\cdot|)$ is not greater than $\|T\|$.

Assume now that $\operatorname{span}(\ker T \cup T(A))$ is of finite codimension in A. Since according to our assumptions the codimension of $T(A)$ is infinite, it follows that $\dim \ker T = \infty$. Since $\ker T \cap T(A) = \{0\}$, there is a linear projection P from A onto $\ker T$ such that $P \circ T = 0$. If P is discontinuous, then the previous part of the proof provides a desired nonequivalent complete norm, so we may assume that P is continuous. Since any infinite-dimensional Banach space has infinitely many inequivalent complete norms, there is a complete norm $q(\cdot)$ on $\ker T$ not equivalent to the original one. We define a new norm $|\cdot|$ on A by

$$|a| = \left\| a_{/\ker T} \right\| + q(Pa), \quad \text{for } a \in A.$$

The new norm is complete, and the identity map is not continuous on $\ker T$. Exactly the same argument as before shows that T is continuous in $(A, \|\cdot\|)$. ☐

4.2. **Proof of Theorem 1.** One part of the theorem follows immediately from Proposition 7. Indeed, suppose that there is a scalar λ such that $a + \lambda e$ is a divisor of zero and the codimension of $(a + \lambda e) A$ is infinite. Let $T_{a+\lambda e}$ be the operator of multiplication by $a + \lambda e$. Since $a + \lambda e$ is a divisor of zero, $\ker T_{a+\lambda e}$ is nontrivial; and since A is semisimple, $\ker T_{a+\lambda e} \cap T_{a+\lambda e}(A) = \{0\}$. So, by Proposition 7, $T_{a+\lambda e}$ does not determine the complete norm topology of A.

To prove the other part of the theorem, suppose that a satisfies the assumptions of Theorem 1, but there is another complete norm $|\cdot|$ on A such that the operator T_a of multiplication by a is continuous on $(A, |\cdot|)$. Let S be the identity map from $(A, |\cdot|)$ onto $(A, \|\cdot\|)$, and put

$$X = \{x \in \mathfrak{M}(A) : x \circ S \text{ is discontinuous}\}.$$

Let b be a nonzero element of $\mathfrak{S}$, the separating space of S. Notice that

$$\{x \in \mathfrak{M}(A) : b(x) \neq 0\} \subset X.$$

Assume first that there is an infinite sequence of distinct points $x_1, x_2, \ldots$ in $\mathfrak{M}(A)$ such that

$$((a - a(x_1)) \cdot \cdots \cdot (a - a(x_n)) \cdot b)(x_{n+1}) \neq 0, \quad \text{for } n \in \mathbb{N}. \tag{1}$$

Let us denote by R_n the continuous operator of multiplication by $a - a(x_n)$, defined on the space $(A, \|\cdot\|)$, and by T_n the continuous operator of multiplication by $a - a(x_n)$ on $(A, |\cdot|)$. By Lemma 5, there is an N such that $\mathfrak{S}_N = \mathfrak{S}_{N+1}$. However, the space $\mathfrak{S}_{N+1}$ is contained in the kernel of the continuous functional $x_{N+1} \in \mathfrak{M}(A)$, while $\mathfrak{S}_N$ contains an element $(a - a(x_1)) \cdot \cdots \cdot (a - a(x_N)) \cdot b$ which, by (1), is not in the kernel of x_{N+1}. This contradiction shows that

$$(a - a(x_1)) \cdot \cdots \cdot (a - a(x_n)) \cdot b = 0$$

for some finite set $\{x_1, \ldots x_n\}$ of distinct points in $\mathfrak{M}(A)$. Removing some of the points from the set $\{x_1, \ldots x_n\}$, if necessary, we may assume that, for any $j = 1, \ldots, n$, the product

$$(a - a(x_1)) \cdot \cdots \cdot (a - a(x_{j-1})) \cdot (a - a(x_{j+1})) \cdot \cdots \cdot (a - a(x_n)) \cdot b$$

is not equal to the zero function. Hence $a - a(x_j)$ is a divisor of zero for $j = 1, \ldots, n$. Put

$$K_j \stackrel{\text{df}}{=} \{x \in \mathfrak{M}(A) : a(x) = a(x_j)\}.$$

If one of the sets K_j were infinite, then the image of $T_{a-a(x_j)e}$ would be contained in the intersection of infinitely many distinct multiplicative functionals. Since multiplicative functionals are linearly independent, it would follow that the image of $T_{a-a(x_j)e}$ would have infinite codimension, contrary to the assumptions of the theorem. So all of the K_j are finite. Since the set $\{x \in \mathfrak{M}(A) : b(x) \neq 0\}$ is open and contained in $\bigcup_{j=1}^{n} K_j$, the set $\{x \in \mathfrak{M}(A) : b(x) \neq 0\}$ consists of finitely many isolated points.

We have proved that for each $b \in \mathfrak{S}$ the set $\{x \in \mathfrak{M}(A) : b(x) \neq 0\}$ is finite and consists of isolated points. Now we show that the set

$$Y \stackrel{\text{df}}{=} \bigcup_{b \in \mathfrak{S}} \{x \in \mathfrak{M}(A) : b(x) \neq 0\}$$

is also finite. Assume $y_1, y_2, \ldots$ is an infinite sequence of distinct points in Y. One way to get a contradiction is to select a sequence $(b_k)_{k=1}^{\infty}$ in $\mathfrak{S}$ such that $b_k(y_k) \neq 0$ for all $k \in \mathbb{N}$ and carefully construct a sequence of numbers $(\alpha_k)_{k=1}^{\infty}$ such that $b_0 = \sum_{k=1}^{\infty} \alpha_k b_k$ is in $\mathfrak{S}$, and does not vanish at all of the points y_j, $j \in \mathbb{N}$. Here is a less computational argument. Define

$$\Phi : \mathfrak{S} \to l^{\infty} : \Phi(b) = (b(y_k))_{k=1}^{\infty}.$$

The range of Φ is infinite-dimensional and is contained in the $\aleph_0$-dimensional space $\{(t_k)_{n=1}^{\infty} : t_k = 0 \text{ for all but finitely many } k\}$. Hence $\dim \Phi(\mathfrak{S}) = \aleph_0$. However, the separating space is closed [12] and Φ is linear and continuous, so the range of Φ is isomorphic, as a vector space, with the quotient Banach space $\mathfrak{S}/\ker \Phi$, which cannot have an infinite countable linear dimension.

For $f \in A$, let $P(f)$ be the restriction of f to $\mathfrak{M}(A) \setminus Y$. Since $\mathfrak{S}(P \circ S) = \{0\}$, $P \circ S$ is continuous; consequently X is equal to Y and is finite, say

$$X = \{y_1, ..., y_p\}.$$

Let $\widetilde{T}$ be the operator of multiplication by $(a - a(y_1)) \cdot \cdots \cdot (a - a(y_n))$, and put

$$\begin{aligned} A_0 &= \widetilde{T}(A) = (a - a(y_1)) \cdots (a - a(y_p))\, A, \\ X_j &= \{x \in \mathfrak{M}(A) : a(x) = a(y_j)\}, \\ A_1 &= \left\{ a \in A : a(x) = 0 \text{ for } x \in \bigcup_{j=1}^{p} X_j \right\}. \end{aligned}$$

According to our assumptions, for any $j = 1, \ldots, n$, the image of $T_{a-a(y_j)e}$ is finite-codimensional, so the same is true about the image of $\widetilde{T}$, and by Lemma 6, $A_0 = \widetilde{T}(A)$ is closed with respect to both norms. Since $A_0 \subset A_1$, it follows that

$$\operatorname{codim} A_1 < \infty, \text{ and } A_1 \text{ is both } |\cdot|\text{-closed and } \|\cdot\|\text{-closed.}$$

Let $S_{|A_1}$ be the restriction of the identity map S to a map from the Banach space $(A_1, |\cdot|)$ onto the Banach space $(A_1, \|\cdot\|)$. Since $\mathfrak{S}\left(S_{|A_1}\right) = \mathfrak{S}(P \circ S) = \{0\}$, the map $S_{|A_1}$ is continuous. By the Open Mapping Theorem $\left(S_{|A_1}\right)^{-1}$ is also continuous so the norms $|\cdot|$ and $\|\cdot\|$ are equivalent on the finite-codimensional closed subspace A_1, and consequently equivalent on the entire space A. □

4.3. **Proof of Theorem 2.** One of the implications follows immediately from Theorem 1. To prove the other implication, assume that for some scalar λ the element $a + \lambda e$ is a divisor of zero, and let $b \in A$, $b \neq 0$, be such that $(a + \lambda e)\, b = 0$. Since $\mathfrak{M}(A)$ does not have any isolated point, the set $G \overset{\text{df}}{=} \{x \in \mathfrak{M}(A) : b(x) \neq 0\}$ is open and infinite. Let $T_{a+\lambda e}$ be the operator of multiplication by $a + \lambda e$. The ideal $T_{a+\lambda e}(A) \subset A$ is infinite-codimensional, since it is contained in the intersection of kernels of all of the multiplicative functionals from G and multiplicative functionals are linearly independent. By Proposition 7, the element a does not determine the complete norm topology on A. □

4.4. **Proof of Theorem 4.** Let X be a separable Banach space, and let (x_n^*, x_n) be an M-bounded biorthogonal system for X:

$$\begin{aligned} x_n^*(x_k) &= \delta_{nk}, \quad \|x_k\| = 1, \quad \|x_n^*\| \leq M, \qquad \text{for all } n, k \in \mathbb{N}, \\ \bigcap_{j=1}^{\infty} \ker x_j^* &= \{0\}, \quad \text{and} \quad \operatorname{span}\{x_j : j \in \mathbb{N}\} \text{ is dense in } X. \end{aligned}$$

Such a system exists, with $M = 1 + \varepsilon$, for any $\varepsilon > 0$ and any separable Banach space [10]. We define $T : X \to X$ by

$$T(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} x_n^*(x)\, x_n \qquad \text{for } x \in X.$$

Notice that the series is absolutely convergent for any $x \in X$. We shall show that T determines the complete norm topology of X.

It is standard to check that the spectrum $\sigma(T)$ of T is equal to $\{0\} \cup \left\{\frac{1}{2^n} : n \in \mathbb{N}\right\}$.

Assume that $|\cdot|$ is another complete norm on X such that T is also continuous on $(X, |\cdot|)$. We shall denote by $B(X, |\cdot|)$ the Banach algebra of all $|\cdot|$-bounded linear maps, and by $B(X, \|\cdot\|)$ the Banach algebra of all $\|\cdot\|$-bounded maps.

The spectrum of T as an element of $B(X, |\cdot|)$ is also equal to $\{0\} \cup \left\{\frac{1}{2^n} : n \in \mathbb{N}\right\}$. Indeed, the spectrum of a continuous linear map T must always by the same regardless of the complete norm we consider, provided the map is continuous in both norms: if λ is not in the spectrum of $T \in B(X, |\cdot|)$, then the linear map $T - \lambda \operatorname{Id}$ has an inverse, since $T - \lambda \operatorname{Id}$ is also $\|\cdot\|$-continuous, by the Open Mapping Theorem, its inverse is $\|\cdot\|$-continuous as well, so λ is not in the spectrum of $T \in B(X, \|\cdot\|)$.

Let S be the identity map from $(X, |\cdot|)$ onto $(X, \|\cdot\|)$, let $\mathfrak{S}$ be the separating space of S, and let $\mathfrak{S}_n$ be the $\|\cdot\|$-norm closure of

$$\left(T - \frac{1}{2} \operatorname{Id}\right) \circ \left(T - \frac{1}{2^2} \operatorname{Id}\right) \circ \cdots \circ \left(T - \frac{1}{2^n} \operatorname{Id}\right)(\mathfrak{S}).$$

The maps $(T - \lambda \operatorname{Id})$ are continuous in both norms and commute with S, and so, by Lemma 5, there is an $N \in \mathbb{N}$ such that

$$\mathfrak{S}_n = \mathfrak{S}_N \quad \text{for } n \geq N.$$

Let $y_0 \in \mathfrak{S}$; we show that

$$x_n^*(y_0) = 0 \quad \text{for all } n > N. \tag{2}$$

Assume there is an $n_0 > N$ such that $x_{n_0}^*(y_0) \neq 0$. Then

$$\begin{aligned} &x_{n_0}^* \circ T_{1/2} \circ T_{1/2^2} \circ T_{1/2^3} \circ \cdots \circ T_{1/2^N}(y_0) \\ &\qquad = T_{1/2} \circ T_{1/2^2} \circ T_{1/2^3} \circ \cdots \circ T_{1/2^N}\left(x_{n_0}^*(y_0)\, x_{n_0}\right) \neq 0, \end{aligned}$$

but $x_{n_0}^* \circ T_{1/2^{n_0}} = 0$, so that

$$x_{n_0}^* \circ T_{1/2} \circ T_{1/2^2} \circ T_{1/2_3} \circ \cdots \circ T_{1/2^N} \circ \cdots \circ T_{1/2^{n_0}}(y) = 0 \quad \text{for any } y \in X.$$

Hence $\mathfrak{S}_{n_0}$ is contained in the kernel of $x_{n_0}^*$ while $\mathfrak{S}_N$ is not; this is a contradiction, since $x_{n_0}^*$ is $\|\cdot\|$-continuous.

Let f be a function defined on an open neighborhood of $\sigma(T)$ such that

$$\begin{aligned} f &\equiv 1 \quad \text{on a neighborhood of } \left\{1, ..., 2^{-N}\right\}, \\ f &\equiv 0 \quad \text{on a neighborhood of } \sigma(T) \setminus \left\{1, \ldots, 2^{-N}\right\}. \end{aligned}$$

Since f is analytic on a neighborhood of the spectrum of T, the map $P = f(T)$ is well defined and continuous [5] with respect to both topologies, and it is a linear projection onto $\bigcap_{j=1}^{N} \ker x_j^*$. We have

$$Px = x - \sum_{j=1}^{N} x_j^*(x)\, x_j$$

and by (2)

$$\mathfrak{S} \subset \operatorname{span}\left\{x_j^* : j \leq N\right\},$$

so the map P is continuous also as a map from $(X, |\cdot|)$ into $(X, \|\cdot\|)$. The above proves that PX is a closed, complemented, finite-codimensional subspace of both $(X, |\cdot|)$ and $(X, \|\cdot\|)$, and that S is continuous on PX. Since S is also continuous on the finite-dimensional space $\operatorname{span}\left\{x_j^* : j \leq N\right\}$, the map S is continuous on X. $\square$

5. Examples and other results

A. R. Villena [13] asked if any uniform algebra whose maximal ideal space has no isolated points has an element which satisfies the assumption of Corollary 3. We show that in the nonseparable case even the simplest uniform algebras, that is, the algebras of the form $C(K)$, may not have such elements. We show that any separable, commutative, semisimple Banach algebra has an element with the desired property.

Villena also asked if for a uniform algebra defined on a set without isolated points the condition on a in Corollary 3 is not only sufficient but also necessary for a to determine the complete topology. We give a counterexample (Example 2).

Example 1. Let ϖ_1 be the first uncountable ordinal number; ϖ_1 can be represented as a locally compact ordered space of all countable ordinals equipped with the standard order topology. Let $X = \varpi_1 \times [0,1]$ be the product of ϖ_1 and the closed unit segment $[0,1]$. Let $X^* = X \cup \{x_0\}$ be a one-point compactification of X. Finally let K be the compact set obtained from a disjoint union of X^* and $[0,1]$ by identifying the point x_0 and the point $0 \in [0,1]$.

Let f be an element of $C(K)$. We will show that f is constant on an open nonempty subset of K. For any rational number $q \in [0,1]$, let f_q be the restriction of f to $\varpi_1 \times \{q\} \subset K$. It is well known [4] that any continuous scalar-valued function on ϖ_1 is constant on all but a countable initial portion of ϖ_1. Consequently, there is a countable ordinal α_q such that $f_q(\varpi)$ is constant for $\varpi \succ \alpha_q$. Let α^* be the lowest upper bound for $\{\alpha_q : q \in [0,1] \cap Q\}$. Since Q is countable, so is α^*. All of the functions $f_q(\varpi)$ are constant for $\varpi \succ \alpha^*$; moreover, since f is continuous at the point x_0, all the functions are equal to the same constant, namely $f(x_0)$. Since $[0,1] \cap Q$ is dense in $[0,1]$, it follows that the function f is constant on $\{\varpi : \varpi \succ \alpha^*\} \times [0,1]$, which is an open subset of K. □

Example 2. Let $K = \mathbb{D} \times T$ be a product of a closed unit disc $\mathbb{D}$ and a unit circle T, and let $\mathbb{D}_0$ be another copy of the closed unit disc. Let X be the compact set obtained from the disjoint union of K and $\mathbb{D}_0$ by identifying the circle $\{0\} \times T$ of K with the boundary of $\mathbb{D}_0$. Define A to be the uniform algebra of all continuous functions f on X such that f restricted to $\mathbb{D}_0$, as well as the functions $f(\cdot, t)$ for any $t \in T$, are all in the disc algebra. The maximal ideal space of A is X. The function $Z \in A$ defined on K by $Z(z,t) = z$ is equal to zero on $\operatorname{int} \mathbb{D}_0$, which is an open subset of the maximal ideal space of A. However, if $Z \cdot f = 0$ for some $f \in A$, then $f = 0$ on the support of Z, that is, on K; consequently, $f = 0$. So f is not a divisor of zero, and, by Theorem 1, multiplication by Z determines the complete norm topology of A.

Theorem 8. *Any commutative, separable Banach algebra A contains an element a_0 such that, for any scalar λ, the set $\operatorname{int}\{x \in \mathfrak{M}(A) : a_0(x) = \lambda\}$ contains at most one point. Consequently, if the maximal ideal space of A has no isolated points, then $\operatorname{int}\{x : a_0(x) = \lambda\}$ is empty for all λ, and a determines the complete norm topology of A.*

Proof. Since A is separable, $\mathfrak{M}(A)$ is a metrizable compact space; let $d(\cdot,\cdot)$ be a metric on $\mathfrak{M}(A)$. There is [4] a family $\{K_n : n \in \mathbb{N}\}$ of finite, mutually disjoint subsets of $\mathfrak{M}(A)$ such that for any n and any point $x \in \mathfrak{M}(A)$ there is a point $y \in \bigcup_{j=1}^n K_n$ with $d(x,y) < \frac{1}{n}$. We shall define inductively a sequence of elements a_n in A. In the construction we will repeatedly use the fact that any finite set K of distinct multiplicative functionals is linearly independent, and therefore $\{a(K) : a \in A\} = \mathbb{C}^{\operatorname{card}(K)}$ (or $\mathbb{R}^{\operatorname{card}(K)}$ in the real case). Let a_1 be any element of A such that $\|a_1\| \leq 1$ and the restriction of a_1 to K_1 is injective. Assume the elements $a_1, a_2, a_3, \ldots a_n$ have been defined. Let $a_{n+1} \in A$ be such that

$$\|a_{n+1}\| \leq \frac{1}{2^n}, a_{n+1}(x) = 0 \text{ for } x \in \bigcup_{j=1}^n K_n, \text{ and } a_{n+1} \text{ is injective on } K_{n+1}.$$

Put $a = \sum_{n=1}^{\infty} a_n$. Since the multiplicative functionals are continuous, it follows that a is injective on $\bigcup_{j=1}^{\infty} K_n$. Let λ be a scalar. Since the set $\bigcup_{j=1}^{\infty} K_n$ is dense in $\mathfrak{M}(A)$ (in particular, it contains all the isolated points of $\mathfrak{M}(A)$), the set defined by $\operatorname{int}\{x \in \mathfrak{M}(A) : a(x) = \lambda\}$ is empty or is equal to a single isolated point. $\square$

The author would like to thank the referee for extensive comments of editorial nature.

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