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ON THE ASYMPTOTIC GEOMETRY OF NONPOSITIVELY CURVED GRAPHMANIFOLDS

S. BUYALO AND V. SCHROEDER

Abstract. In this paper we study the Tits geometry of a 3-dimensional graph-manifold of nonpositive curvature. In particular we give an optimal upper bound for the length of nonstandard components of the Tits metric. In the special case of a $\pi/2$-metric we determine the whole length spectrum of the nonstandard components.

0. Introduction

The asymptotic behavior of geodesics on a closed Riemannian manifold M of nonpositive curvature may be described in terms of the boundary at infinity $\partial_\infty X$ of its universal covering X and the action of the fundamental group $\Gamma = \pi_1(M)$ on $\partial_\infty X$. The points of $\partial_\infty X$ are the classes of asymptotic geodesic rays in X. Usually one considers two topologies on $\partial_\infty X$: the standard and the metric ones. In the standard topology, points $z,\ z' \in \partial_\infty X$ are close if they are visible from a fixed point $x \in X$ under a small angle. The metric topology is associated with the angle metric $\angle$ on $\partial_\infty X$, where $\angle(z, z')$ is defined as the supremum of the angles under which the points $z,\ z' \in \partial_\infty X$ are visible from the points of X. The corresponding intrinsic metric on $\partial_\infty X$ is called *the Tits metric*, and the boundary at infinity with the metric topology is denoted by $\partial_T X$. The metric topology is finer than the standard one.

The group Γ acts on $\partial_\infty X$ by homeomorphisms with respect to the standard topology and by isometries of the Tits metric.

A typical example of a connected subset in $\partial_T X$ is the boundary at infinity $\partial_\infty E$ of a flat $E \subset X$ of dimension $k \geq 2$, i.e. a geodesic subspace isometric to $\mathbb{R}^k$. In that case $\partial_\infty E$ is isometric to the unit sphere $S^{k-1} \subset \mathbb{R}^k$. Such subspaces E are often associated with subgroups in Γ isomorphic to $\mathbb{Z}^k$.

In other words, there are connected components in $\partial_T X$ whose combinatorial structure reflects the combinatorial structure of the simplicial complex $\mathcal{A}$ of (maximal) abelian subgroups in Γ, whose simplices are collections of subgroups ordered by inclusion. Such components are called *standard* (for precise definition see §2).

At the same time, there are *nonstandard* components in $\partial_T X$, i.e. such that any point of them is not a boundary point of any k-flat in X with $k \geq 2$.

The very existence of nondegenerate (i.e. different from a point) nonstandard components in $\partial_T X$ is not obvious, and the first examples appeared only recently

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(see [CK], [HS]; in 1992 in a conversation with the first author, B. Leeb had mentioned a possibility of existence of fat triangles in a graphmanifold, which are not contained in a block, this is equivalent to the existence of nonstandard components; in 1996 C. Croke and B. Kleiner gave the first examples of nondegenerate nonstandard components by constructing certain nonsmooth metrics of nonpositive curvature on graphmanifolds).

Nonstandard connected components of $\partial_T X$ are the main subject of the present work. Here we restrict to the case of 3-dimensional graphmanifolds, i.e. the simplest case, where nonstandard components exist. For a graphmanifold M with nonpositively curved metric every nonstandard component is a segment of length $< \pi$ (see Proposition 2.12).

Our first result gives an optimal estimate for the length of a nonstandard component from above.

0.1. Theorem. *For a nonpositively curved metric g on a graphmanifold M the length of any nonstandard component $\mathcal{I}_w$ of $\partial_T X$ satisfies*

$$L_g(w) \leq \omega(g),$$

where $\omega(g) \in [\pi/2, \pi)$ is the maximal angle between singular directions of adjacent blocks of M.

Next, we show that there exists a component $\mathcal{I}_w$ of the maximal length $L_g(w) = \omega(g)$. Moreover, we obtain a slightly stronger result bringing *primitive* components into the game. A primitive nonstandard component $\mathcal{I}_w$ is associated with a gluing torus T_u: every geodesic ray c in X, $c(\infty) \in \mathcal{I}_w$ descends to a geodesic $\overline{c}$ in M, which starting from some moment lives only in two adjacent maximal blocks of M skipping from one to another through T_u.

0.2. Theorem. *For any gluing torus $T_u \subset M$ there exists an associated primitive component $\mathcal{I}_w \subset \partial_T X$ of the maximal length $L_g(w) = \omega_u(g)$, where $\omega_u(g) \in [\pi/2, \pi)$ is the maximal angle between singular directions on T_u.*

The set of nonstandard components $\mathcal{W}$ is independent of the choice of the nonpositively curved metric and defined only by the fundamental group (see §2). The metric defines only lengths $L(w)$, $w \in \mathcal{W}$ of the components $\mathcal{I}_w \subset \partial_T X$.

0.3. Corollary. *If metrics g_0, g_1 of nonpositive curvature on a graphmanifold M have the same marked length spectrum $(\mathcal{W}, L_{g_0}) = (\mathcal{W}, L_{g_1})$ of nonstandard connected components of the Tits boundaries $\partial_T X_0$, $\partial_T X_1$, then $\partial_T X_0$ and $\partial_T X_1$ are Γ-equivariantly isometric, $\Gamma = \pi_1(M)$.* □

In the case $\omega_u(g) = \pi/2$ we have a much better understanding of the length spectrum for the associated primitive components.

0.4. Theorem. *Assume that the angle between the singular directions of a gluing torus $T_u \subset M$ is $\omega_u = \omega_u(g) = \pi/2$. Then the length spectrum of the associated primitive components $\mathcal{I}_w$, $w \in \mathcal{W}_u$ of $\partial_T X$ coincides with $[0, \omega_u]$.*

For the definition of graphmanifolds and the notions mentioned in the results above see §1. In this work we consider C^1-smooth metrics on M. The condition for a metric to be nonpositively curved is understood is the sense of Alexandrov, i.e. in the sense of the angle comparison. Probably, our results are true for general nonpositively curved metrics on graphmanifolds.

Theorem 0.4 is motivated by the question, what is the role of nonstandard components in the asymptotic geometry of nonpositively curved metrics? In particular, by the questions, how does the marked length spectrum $L_g(w)$, $w \in \mathcal{W}$ of nonstandard components depend on the metric g, and in which degree this spectrum defines the metric itself?

Metrics of nonpositive curvature on graphmanifolds are typical examples of metrics of rank 1 in the sense of [BBE]. Here the situation is drastically distinct from that which takes place for the spaces of rank ≥ 2 or hyperbolic spaces. While for higher rank spaces there are no nontrivial metric deformations in the class of nonpositive curvature, in the hyperbolic case it is impossible to change the equivariant topology on $\partial_\infty X$ by changing the metric in the class considered. In the rank one case there are, in general, metric deformations but the asymptotic geometry is very sensitive to the change of the metric.

There are two types of deformations g_t of nonpositively curved metrics on a graphmanifold M. First, there are *rigid* type deformations, when the angle between S^1-factors of some adjacent blocks is changed (see §1) and, correspondingly, the geometry of the principal connected component $\mathcal{F}$ (see §2) of $\partial_T X$ is changed. In [Bu], an example of the rigid type deformation is given. We show (see Theorem 2.10) that any change of the geometry of $\mathcal{F}$ under the metric change $g_0 \mapsto g_1$ always leads to the result that there is no *continuous* Γ-equivariant map $\partial_\infty X_0 \to \partial_\infty X_1$, where $\Gamma = \pi_1(M)$ (recall that the spaces $\partial_\infty X_0$, $\partial_\infty X_1$ are homeomorphic to the sphere S^2).

Second, a metric deformation g_t might appear in the *soft* type, when the geometry of the principal component $\mathcal{F}$ is kept the same. Such deformations are, for example, all deformations in the class of $\pi/2$-*metrics*.

If the angle between the singular directions $\omega_u(g) = \pi/2$ for every gluing torus T_u in M, then the metric g is called a $\pi/2$-metric (see §1 for the precise definition). Not every graphmanifold admitting a nonpositively curved metric, carries a $\pi/2$-metric. At the same time, there are graphmanifolds on which any nonpositively curved metric is a $\pi/2$-metric. In §1 we indicate necessary and sufficient conditions for a graphmanifold to possess a $\pi/2$-metric.

The change of the equivariant topology $(\partial_\infty X, \Gamma)$ in the soft type deformation is not so obvious, however, it also takes place. A corresponding example was given by C. Croke and B. Kleiner. This happens because some (nondegenerate) nonstandard components degenerate even in the soft type deformation. Probably, the behavior of the length spectrum of nonstandard components is highly sensitive to any (nontrivial) metric deformation.

Structure of the paper. §1 contains some background material on graphmanifolds and metrics of nonpositive curvature on them. In particular, we give there a necessary and sufficient condition for a graphmanifold to carry a $\pi/2$-metric.

In §2, after recalling the definitions of the standard topology and Tits metric on $\partial_\infty X$, we describe the decomposition of the Tits boundary $\partial_T X$ for a nonpositively curved graphmanifold into connected components, showing that this decomposition can defined by the fundamental group Γ. Next, we show that each nonprincipal connected component of $\partial_T X$ is an interval of length $< \pi$ and may be degenerate (Proposition 2.12). We also discuss here the question of how a deformation of a metric effects the Γ-equivariant topology of $\partial_\infty X$ (Theorem 2.10 and Proposition 2.11).

Theorems 0.1, 0.2 and 0.4 are proved in §3.

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1. Preliminaries on metrics of nonpositive curvature on graphmanifolds

Here we collect necessary facts about nonpositively curved metrics on graphmanifolds. The proofs can be found in or extracted from [BK1], [BB]. For general reference we also refer to [BGS], and the books of Ballmann [B] and Eberlein [E].

1.1. In the present work a graphmanifold is understood as a closed 3-dimensional manifold M, for which there exists a nonempty minimal collection $\mathcal{E}$ of disjointly embedded incompressible tori and Klein bottles such that the abstract completion M_v of each connected component of the complement to $\mathcal{E}$ is a Seifert fiber space. The manifold M_v is called *a maximal block* of M. Furthermore, we assume that each maximal block admits a geometric structure of type $H^2\times\mathbb{R}$, i.e. a Riemannian metric such that the universal covering $\widetilde{M_v}$ is isometric to the metric product $Y_v\times\mathbb{R}$, where $Y_v\subset H^2$ is a closed convex subset bounded by infinite geodesics.

Any finite covering M' for M is also a graphmanifold of that type. It is known (see [Ka]) that there exists a finite cover $M'\to M$ such that every maximal block $M'_v\subset M'$ is homeomorphic to $F'_v\times S^1$, where F'_v is a compact surface with nonempty boundary. In that case one can assume that M' and, consequently, all F'_v are orientable. The condition of the existence of a $H^2\times\mathbb{R}$ structure excludes the possibility for F'_v to be the disc or the annulus.

Since the asymptotic geometry of M does not change, while taking a finite covering, in the sequel we assume for simplicity that M is orientable and every maximal block M_v is homeomorphic to $F_v\times S^1$, where F_v is a compact orientable surface with nonempty boundary different from the disc and the annulus. In that case the separating collection $\mathcal{E}$ consists of tori.

1.2. In the present work, we consider C^1-smooth Riemannian metrics of nonpositive curvature on M. An important example of such a metric is *a geometrization* of M, i.e. a metric which induces on every maximal block a geometric structure of type $H^2\times\mathbb{R}$. A geometrization exists iff the graphmanifold possesses a nonpositively curved metric (see [Le]). Let g be such a metric. Then each torus T of the collection $\mathcal{E}$ can be chosen flat and geodesic. The curves on T representing the factor S^1 from the decomposition $M_{v_i}=F_{v_i}\times S^1$, $i=0,1$ for two blocks adjacent along T are closed geodesics representing (due to the minimality of $\mathcal{E}$) independent elements of the homology group $H_1(T;\mathbb{Z})\simeq\mathbb{Z}^2$. Thus, if an orientation of the factors S^1 is fixed, the angle ω between these geodesics is well defined, $0<\omega<\pi$. The metric g is said to be a $\pi/2$-*metric*, if $\omega=\pi/2$ for all gluing tori $T\in\mathcal{E}$.

1.3. Here we give a necessary and sufficient condition for a graphmanifold M to possess a $\pi/2$-metric. It is formulated in terms of topological invariants of M introduced in [BK1], [BK2] and called *charges*.

Let V be the set of all maximal blocks of M. For $v\in V$ let ∂v be the set of the boundary components of the block M_v, $U=\bigcup_{v\in V}\partial v$. V is the vertex set of the graph $G=G_M$ of M, whose set of oriented edges is U. A vertex $v\in V$ is initial for an edge $u \in U$ if and only if $u \in \partial v$. The set of gluing tori $\mathcal{E}$ can be identified with the set of nonoriented edges of G, i.e. pairs $(u, -u)$, $u \in U$.

To define charges k_v, $v \in V$ we fix an orientation of M. This defines an orientation of every block M_v, for which we also fix an orientation of the factor S^1 in the decomposition $M_v = F_v \times S^1$.

For $u \in \partial v$ let $L_u \simeq \mathbb{Z}^2$ be the homology group $H_1(T_u; \mathbb{Z})$ of the torus $T_u = (\partial M_v)_u$. Next we choose a basis $\{(z_u, f_u) \mid u \in \partial v\}$ of the group $H_1(\partial M_v; \mathbb{Z}) = \bigoplus_{u \in \partial v} L_u$ such that the basis (z_u, f_u) of the lattice L_u is compatible with the orientation induced on ∂M_v, the element f_u represents the oriented factor S^1 of the block M_v, and the sum $\bigoplus_{u \in \partial v} z_u$ lies in the kernel of the inclusion homomorphism

$$H_1(\partial M_v; \mathbb{Z}) \to H_1(M_v; \mathbb{Z}),$$

i.e. represents the boundary ∂F_v of the surface F_v from the decomposition $M_v = F_v \times S^1$. The collection $(z, f) = \{(z_u, f_u) \mid u \in \partial v, v \in V\}$ is called *a Waldhausen basis*. In general, the choice of a Waldhausen basis is not unique even when the orientations as above are fixed.

To each oriented edge u of the graph G, there is a corresponding gluing map of boundary components of adjacent blocks, which induces an isomorphism $g_u : L_{-u} \to L_u$; with respect to the chosen bases, g_u has the matrix

$$\begin{pmatrix} a_u & b_u \\ c_u & d_u \end{pmatrix} \in GL(2, \mathbb{Z}),$$

i.e.

$$g_u(z_{-u}) = a_u z_u + c_u f_u,$$
$$g_u(f_{-u}) = b_u z_u + d_u f_u.$$

We have $\det g_u = a_u d_u - b_u c_u = -1$ since M is orientable, and $g_{-u} = g_u^{-1}$, $b_{-u} = b_u \neq 0$ because the elements f_u and $g_u(f_{-u})$ representing the factors S^1 of adjacent blocks are independent.

The charge of a vertex $v \in V$ is defined as

$$k_v = \sum_{u \in \partial v} d_u / b_u$$

and is an invariant of the oriented manifold M, i.e. it does not depend on the choice of orientations of factors S^1 and the Waldhausen basis. Charges change sign if the orientation of M is changed (see [BK1]).

1.4. Theorem. *A graphmanifold M admits a $\pi/2$-metric if and only if $k_v = 0$ for all $v \in V$.*

This fact easily follows from the decomposition principle (see [BK1, §13]). Let us give several examples illustrating Theorem 1.4.

1.5. Examples. 1. Assume that the graph of M consists of two vertices v_0, v_1 connected by an edge $(u, -u)$. The gluing map is given by the matrix

$$g_u = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

which might be an arbitrary element of $GL(2, \mathbb{Z})$ with $\det g_u = -1$ and $b \neq 0$. Then for the charges of vertices we have $k_0 = d/b$, $k_1 = -a/b$. The manifold M admits a nonpositively curved metric iff $k_0 = k_1 = 0$. This is equivalent to the condition $a = d = 0$, $b = c = \pm 1$ (see [Le], [BK1]). Notice that for such M there is only one Waldhausen basis, provided an orientation of M and orientations of the block fibers are fixed. Any nonpositively curved metric on M is a $\pi/2$-metric.

2. Assume that the graph of M consists of one vertex v and one edge $(u, -u)$, which is therefore a loop. As in the previous example, there is only one gluing map given by the matrix

$$g_u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z})$$

(this time the choice of Waldhausen bases is not unique). Then $k_v = (d - a)/b$. The manifold M admits a nonpositively curved metric iff $|d - a| < 2$. There is a $\pi/2$-metric on M iff $d = a$. In the last case any nonpositively curved metric on M is a $\pi/2$-metric. For $a = c = d = 1$, $b = 2$ we obtain an example of M, on which every nonpositively curved metric is a $\pi/2$-metric and which cannot be obtained by the switching generators gluing. This is because the number $|b|$ is a topological invariant of M (an intersection index, see [BK1]), and it is equal to 1, if the generators switch.

3. Assume that the graph of M is a circle with an *odd* number $|V| \geq 1$ of edges, and $k_v = 0$ for all $v \in V$. Then any nonpositively curved metric on M is a $\pi/2$-metric. In particular, every deformation in the class of nonpositively curved metrics on M is soft (and there are plenty of such nontrivial deformations).

4. Assume that the graph of M is a circle with an *even* number $|V| \geq 2$ of edges, and $k_v = 0$ for all $v \in V$. Then M possesses $\pi/2$-metrics as well as non-$\pi/2$-metrics of nonpositive curvature, for which the angle ω between singular directions of some adjacent blocks can take any value from the interval $(0, \pi)$. In particular, there are nontrivial rigid deformations in the class of nonpositively curved metrics. In the simplest case $|V| = 2$ the angles ω_0, ω_1 for the corresponding edges u_0, u_1 are related by

$$\cos \omega_0 + \cos \omega_1 = 0,$$

if $b_0 = b_1$, and a rigid deformation can be described with a parameter $\phi \in [0, \pi/2)$ as $\omega_0 = \pi/2 - \phi$, $\omega_1 = \pi/2 + \phi$; see [Bu].

2. The boundary at infinity of a graphmanifold

2.1. Let X be a Hadamard space, i.e. a complete simply connected metric space of nonpositive curvature. Geodesic rays c, $c' : [0, \infty) \to X$ are called *asymptotic*, if the distances $|c(t)c'(t)|$ are bounded as $t \to \infty$ (here and in the sequel we always assume that geodesic rays are parametrized by arc length). The boundary at infinity $\partial_\infty X$ of X consists of classes of asymptotic geodesic rays in X. Recall the definitions of the standard topology and the Tits metric on $\partial_\infty X$.

2.1.1. Fix a point $x_0 \in X$. Then any point $z \in \partial_\infty X$ can be identified with the geodesic ray $z : [0, \infty) \to X$, $z(0) = x_0$, $z(\infty) = z$. A base of the standard topology consists of the sets

$$U_{x_0,t}(z) = \{z' \in \partial_\infty X \mid |z(t)z'(t)| < 1\}, \quad z \in \partial_\infty X,\ t > 0.$$

2.1.2. For z,$z' \in \partial_\infty X$ and t, $t' > 0$ let $\overline{x}_0\overline{z}(t)\overline{z}'(t') \subset \mathbb{R}^2$ be the comparison triangle for the triangle $x_0 z(t) z'(t') \subset X$. Then the limit

$$\angle(z, z') = \lim_{t,t' \to \infty} \angle \overline{z}(t)\overline{x}_0\overline{z}'(t')$$

exists and is independent of the choice of $x_0 \in X$. This defines *the angle metric* $\angle(z, z') \in [0, \pi]$ on $\partial_\infty X$. The corresponding intrinsic metric is called *the Tits metric* on $\partial_\infty X$. We use the notation $\partial_\infty X$ for the boundary at infinity with the standard topology and $\partial_T X$ for the boundary at infinity with the Tits metric.

2.2. From now on we assume that X is a universal metric covering of a graphmanifold $M = \bigcup_{v \in V} M_v$ with a nonpositively curved metric g. Then $X = \bigcup_{\alpha \in \mathcal{A}} X_\alpha$, where X_α is a universal covering of some maximal block $M_v = F_v \times S^1$. The set X_α is also called *a block* of X.

2.2.1. Every block X_α, $\alpha \in \mathcal{A}$ is a closed convex subset in X isometric to the metric product $Y_\alpha \times \mathbb{R}$, where the surface Y_α is a universal covering of F_α with a nonpositively curved metric and a geodesic boundary. In particular, every connected component of ∂X_α is a 2-flat in X (covering some gluing torus in M). If different blocks X_α, $X_{\alpha'}$ intersect, then their intersection $X_\alpha \cap X_{\alpha'}$ is the common boundary component of these blocks, i.e. a flat in X.

2.2.2. The surface Y_α is cocompact, thus its boundary components are pairwise separated by a distance $\geq \rho$, where $\rho > 0$ depends only on the metric g and is independent of $\alpha \in \mathcal{A}$, because the manifold M consists of a *finite* number of maximal blocks M_v.

2.2.3. The metric on Y_α is hyperbolic in the sense of Gromov, i.e. for some $\delta > 0$ each side of any triangle in Y_α lies in a δ-neighborhood of the union of two other sides. Again, $\delta = \delta(g)$ depends only on the metric g and is independent of $\alpha \in \mathcal{A}$.

The last two properties of the surfaces Y_α will systematically be used in the sequel.

2.2.4. For instance, it follows from 2.2.2 that $\operatorname{width}(X_\alpha) \geq \rho$ for all $\alpha \in \mathcal{A}$, where $\operatorname{width}(X_\alpha) < \infty$ is the shortest distance in X between the points of different components of ∂X_α.

2.2.5. Lemma. *There exists a constant $\rho_1 > 0$, which depends only on the metric g and is independent of $\alpha \in \mathcal{A}$ such that the following is true.*

If geodesic segments xy, $x'y' \subset Y_\alpha$ connect a boundary component $C \subset \partial Y_\alpha$ (x, $x' \in C$) with components D, $D' \subset \partial Y_\alpha$, $\angle yxx' + \angle y'x'x \geq \pi$ and $|xx'| \geq \rho_1$, then $D \neq D'$.

Proof. One can take as ρ_1 the minimal displacement of a nontrivial isometry $\gamma \in \Gamma_v = \pi_1(F_v)$, which leaves C invariant. Here Γ_v is the deck transformation group of the covering $Y_\alpha \to F_v$. Since C is the unique boundary component of Y_α, invariant for γ, the claim follows. $\square$

We shall refer to this lemma as the *periodicity argument*.

2.2.6. The boundary at infinity $\partial_\infty X_\alpha$ is the suspension over the Cantor set $\partial_\infty Y_\alpha$, and the space $\partial_T X_\alpha$ is a graph with two vertices corresponding to the ends of the factor $\mathbb{R}$ and connected by edges of length π. These edges one-to-one correspond to the points of $\partial_\infty Y_\alpha$. In particular, $\partial_T X_\alpha$ is connected and has diameter equal to π.

Since $X_\alpha \subset X$ is convex, we have $\partial_\infty X_\alpha \subset \partial_\infty X$, $\partial_T X_\alpha \subset \partial_T X$.

The vertices of the graphs $\partial_T X_\alpha$ are called *singular points* of $\partial_T X$. Every singular point $z \in \partial_T X$ uniquely defines the corresponding block X_α, $\alpha = \alpha(z)$, and $\alpha(-z) = \alpha$, where $-z$ is the opposite to z vertex of the graph $\partial_T X_\alpha$.

2.2.7. Let α, $\alpha' \in \partial_T X$ be singular points, for which the blocks X_α, $X_{\alpha'}$ are adjacent, M_v, $M_{v'}$ the corresponding maximal blocks of M. The points α, α' represent the oriented S^1-factors of M_v, $M_{v'}$, and $\omega = \angle(\alpha, \alpha')$, $0 < \omega < \pi$ is the angle between the last on the gluing torus T, which is covered by $X_\alpha \cap X_{\alpha'}$.

Encoding of the connected components of $\partial_T X$

2.3. Let W be the set of infinite (in one direction) strings w consisting of letters of the alphabet $\mathcal{A}$; we require

(a) any letter $\alpha \in \mathcal{A}$ enters w at most one time;
(b) if letters α, $\alpha' \in w$ are neighboring, then $X_\alpha \cap X_{\alpha'}$ is a flat in X for the corresponding blocks X_α, $X_{\alpha'}$.

The strings w, $w' \in W$ are equivalent if they have a common tail; notation: $w \sim w'$, $\mathcal{W} = W/\sim$.

Any geodesic ray $c : [0, \infty) \to X$ defines, obviously, a string (finite or infinite) w_c of letters of $\mathcal{A}$.

2.3.1. Lemma. *For asymptotic rays c, $c' \in z \in \partial_\infty X$ we have*

(i) *w_c is finite if and only if $w_{c'}$ is finite;*
(ii) *if w_c, $w_{c'}$ are infinite, then $w_c \sim w_{c'}$.*

Proof. (i) If w_c is finite, then we can assume that $w_c = \{\alpha\}$ consists of one letter $\alpha \in \mathcal{A}$, i.e. $c \subset X_\alpha$. If $w_{c'}$ is infinite, then $\operatorname{dist}(X_\alpha, X_{\alpha'}) \to \infty$ as $\alpha' \to \infty$, $\alpha' \in w_{c'}$, because the number of blocks between X_α and $X_{\alpha'}$ tends to infinity, while $\operatorname{width}(X_\beta) \geq \rho > 0$ for any $\beta \in \mathcal{A}$. Thus $\operatorname{dist}(c'(t), c) \to \infty$ as $t \to \infty$. This contradicts the assumption that the rays c, c' are asymptoptic.

(ii) If the strings w_c, $w_{c'} \in W$ are not equivalent, then, obviously, some tails of them have no common letter. It follows that $\operatorname{dist}(X_\alpha, X_{\alpha'}) \to \infty$ as α, $\alpha' \to \infty$, $\alpha \in w_c$, $\alpha' \in w_{c'}$. Furthermore,

$$\operatorname{dist}\left(\bigcup_{\alpha \in w_c} X_\alpha, X_{\alpha'}\right) \to \infty \quad \text{as} \quad \alpha' \to \infty.$$

On the other hand, $c \subset \bigcup_{\alpha \in w_c} X_\alpha$, hence $\operatorname{dist}(c'(t), c) \to \infty$ as $t \to \infty$, a contradiction. $\square$

2.3.2. It follows from Lemma 2.3.1 that the set $\partial_\infty X$ is a disjoint union of two subsets $\mathcal{F}$ and $\mathcal{I}$, where

$z \in \mathcal{F}$ iff for any ray $c \in z$ the string w_c is finite;
$z \in \mathcal{I}$ iff for any ray $c \in z$ the string w_c is infinite.

2.3.3. On the structure of $\mathcal{F}$. This set coincides with the union

$$\mathcal{F} = \bigcup_{\alpha \in \mathcal{A}} \partial_\infty X_\alpha.$$

Indeed, for any point $z \in \partial_\infty X_\alpha$ there is a ray $c \in z$ with $w_c = \{\alpha\}$, thus $z \in \mathcal{F}$. Conversely, for $z \in \mathcal{F}$ any ray $c \in z$ generates a finite string w_c. Thus there is a subray $c' \subset c$, $c' \in z$ with $w_{c'} = \{\alpha\}$. Hence $z \in \partial_\infty X_\alpha$ for some $\alpha \in \mathcal{A}$.

In particular, $\mathcal{F} \subset \partial_T X$ is a connected subset, because $\bigcup_{\alpha \in \mathcal{A}} \partial_\infty X_\alpha$ is connected.

2.4. By Lemma 2.3.1, there is a map $f : \mathcal{I} \to \mathcal{W}$, where $f(z)$ is the class of the string w_c for a ray $c \in z$.

2.4.1. Lemma. *The map f is surjective.*

Proof. For a string $w \in W$ let us consider a sequence $x : \mathbb{N} \to X$, $x(n) \in X_{\alpha_n}$, where $w = \alpha_1\alpha_2 \dots$. Then the sequence of geodesic segments $x(1)x(n)$ subconverges to a ray $c : [0,\infty) \to X$, because $|x(1)x(n)| \to \infty$ as $n \to \infty$ by the same argument as above. We claim that $w_c = w$. It suffices for the proof of the lemma, because then $f([c]) = [w]$.

The set $\bigcup_{\alpha\in w} X_\alpha =: X_w \subset X$ is closed and convex. Thus $c \subset X_w$, and it suffices to show that the string w_c is infinite.

Assume that it is not the case, and let $\alpha \in w$ be the last letter. For the flat $E_2 = X_{\alpha+2} \cap X_{\alpha+3}$ its boundary at infinity $\partial_\infty E_2$ does not intersect $\partial_\infty E_0$ for $E_0 = X_\alpha \cap X_{\alpha+1}$. Thus for a point $x \in E_2$ the ray $c' : [0,\infty) \to X$, $c'(0) = x$, asymptotic to c forms an angle $\geq \phi > 0$ with any vector from T_xX tangent to E_2 (recall, we assume that the metric g is C^1-smooth). But then the segments $xx_{\alpha+k}$ for $k \geq 3$ have angles $\geq \phi$ with c' and thus cannot converge to it, a contradiction. □

2.4.2. In fact, it is proven that if $z \in \partial_\infty X$ is a limit of a sequence $x(n) \in X_{\alpha_n}$, where $w = \alpha_1\alpha_2 \dots \in W$, then $z \in \mathcal{I}$ and $f(z) = [w]$.

2.5. For $w \in \mathcal{W}$ we denote by $\mathcal{I}_w = \{z \in \mathcal{I} \,|\, f(z) = w\}$. By Lemma 2.4.1, the set $\mathcal{I}_w \subset \partial_\infty X$ is nonempty, and by Lemma 2.3.1(ii) we have

$$\mathcal{I} = \bigcup_{w\in\mathcal{W}} \mathcal{I}_w.$$

So, we have a decomposition (disjoint union)

$$\partial_\infty X = \mathcal{F} \cup \bigcup_{w\in\mathcal{W}} \mathcal{I}_w,$$

no one element of which is nonempty. Furthermore, $\mathcal{F} \subset \partial_T X$ is connected.

It will be shown that this decomposition is exactly the decomposition of the Tits boundary into its connected components (see Corollary 2.9).

It follows from 2.4.2 that if a sequence $\{z_n\} \subset \mathcal{I}_w$ converges to some point $z \in \partial_\infty X$ with respect to the standard topology, then $z \in \mathcal{I}_w$, i.e. $\mathcal{I}_w$ is closed both in the standard and the metric topologies.

2.6. Proposition. *Each set $\mathcal{I}_w \subset \partial_T X$, $w \in \mathcal{W}$ is connected and has diameter $< \pi$.*

Proof. The first assertion follows from the second one. For z, $z' \in \mathcal{I}_w$ we consider rays $c \in z$, $c' \in z'$ with a common vertex x. By the definition of $\mathcal{I}_w$ we have that $w_c = w_{c'}$ represent the same class $w \in W$.

Assume that the Proposition is not true. Then for any $\epsilon > 0$ there are z, $z' \in \mathcal{I}_w$ and a point $x \in X$ with $\angle_x(z, z') > \pi - \epsilon$.

For $\alpha \in w_c$ let $c_\alpha = c \cap X_\alpha$, $c'_\alpha = c' \cap X_\alpha$, $\overline{c}_\alpha$, $\overline{c}'_\alpha \subset Y_\alpha$ be the projections of segments c_α, c'_α on the factor Y_α from the decomposition $X_\alpha = Y_\alpha \times \mathbb{R}$. We represent the segments c_α, c'_α as $c_\alpha = x_\alpha y_\alpha$, $c'_\alpha = x'_\alpha y'_\alpha$, where x_α, x'_α are the points where the rays c, c' enter the block X_α, and y_α, y'_α are the points where c, c' leave X_α.

By the assumption, x_α, x'_α lies in the same connected component of the boundary ∂X_α. Since $\angle_x(z, z') > \pi - \epsilon$, the angles at the vertices x_α, x'_α of the triangle $xx_\alpha x'_\alpha$

are small, their sum $< \epsilon$. Thus the directions of c, c' at x_α, x'_α are almost opposite and close to the corresponding opposite directions of $x_\alpha x'_\alpha$. It is also clear that $|x_\alpha x'_\alpha| \to \infty$ as $\alpha \to \infty$, $\alpha \in w_c$. If the direction of $x_\alpha x'_\alpha$ forms the angle $\geq \epsilon_0 > \epsilon$ with the factor $\mathbb{R} = \mathbb{R}_\alpha$ in its component of ∂X_α, then $|\overline{x}_\alpha \overline{x}'_\alpha| \to \infty$ as $\alpha \to \infty$, and the segments $\overline{c}_\alpha$, $\overline{c}'_\alpha$ form the angles close to π with $\overline{x}_\alpha \overline{x}'_\alpha \subset \partial Y_\alpha$. Now the periodicity argument (Lemma 2.2.5) shows that the segments $\overline{c}_\alpha$, $\overline{c}'_\alpha$ cannot end in the same component of ∂Y_α, a contradiction.

Thus the direction of $x_\alpha x'_\alpha$ is ϵ-close to the direction of $\mathbb{R}_\alpha$. Let $\underline{x}_\alpha$, $\underline{x}'_\alpha$, $\underline{y}_\alpha$, $\underline{y}'_\alpha$ be the projections on $\mathbb{R}_\alpha$ (in its components of ∂X_α), $T_\alpha = |c_\alpha|$, $T'_\alpha = |c'_\alpha|$ the life times of the rays c, c' in X_α. Then

$$\begin{aligned} |\underline{y}_\alpha \underline{y}'_\alpha| &\geq |\underline{x}_\alpha \underline{x}'_\alpha| + \cos 2\epsilon (T_\alpha + T'_\alpha) \\ &\geq (|x_\alpha x'_\alpha| + T_\alpha + T'_\alpha) \cos 2\epsilon, \end{aligned}$$

while

$$\begin{aligned} |\overline{y}_\alpha \overline{y}'_\alpha| &\leq |\overline{x}_\alpha \overline{x}'_\alpha| + |\overline{c}_\alpha| + |\overline{c}'_\alpha| \\ &\leq (|x_\alpha x'_\alpha| + T_\alpha + T'_\alpha) \sin 2\epsilon. \end{aligned}$$

Thus

$$|\overline{y}_\alpha \overline{y}'_\alpha| / |\underline{y}_\alpha \underline{y}'_\alpha| \leq \tan 2\epsilon,$$

i.e. the segment $y_\alpha y'_\alpha$ forms the angle at most 2ϵ with $\mathbb{R}_\alpha$. Let $\beta \in w_c$ be the next letter after α. Then $x_\beta = y_\alpha$, $x'_\beta = y'_\alpha$, and the angle between $\mathbb{R}_\beta$ and $\mathbb{R}_\alpha$ is uniformly separated from zero by $\omega > 0$. Thus $x_\beta x'_\beta$ forms the angle $\geq \omega/2$ for $\epsilon < \omega/4$ with the direction of $\mathbb{R}_\beta$, and we obtain a contradiction as above. $\square$

2.6.1. It follows that our decomposition

$$\partial_\infty X = \mathcal{F} \cup \bigcup_{w \in \mathcal{W}} \mathcal{I}_w$$

consists of connected (in the Tits metric) subsets, and each $\mathcal{I}_w$ is closed and nonempty.

2.7. Proposition. *The set $\mathcal{F}$ is a connected component of $\partial_T X$.*

Proof. It suffices to show that any geodesic segment $\gamma \subset \partial_T X$ with ends in $\mathcal{F}$ actually lies in $\mathcal{F}$.

Assume that it is not the case, and for any $\epsilon > 0$ there is a point $z \in \gamma \cap \mathcal{I}$ with $|zz_0| < \epsilon$, where $z_0 \in \mathcal{F}$ is the initial point of γ.

Let $w \in \mathcal{W}$ with $z \in \mathcal{I}_w$. The interval $(z_0 z] \subset \gamma$ cannot lie in $\mathcal{I}_w$, since otherwise by 2.4.2 we have $z_0 \in \mathcal{I}_w$, a contradiction. Furthermore, by closedness of $\mathcal{I}_w$ we can assume that $z' \neq \mathcal{I}_w$ for any $z' \in [z_0 z)$. Then $(z_0 z)$ necessarily contains a point $z' \in \mathcal{F}$ (cp. the first argument in Lemma 2.8). It suffices to show that the segment $z_0 z'$ lies in $\mathcal{F}$.

By definition, the strings w_{c_0}, $w_{c'}$ are finite for any $x \in X$, where the rays $c_0 \in z_0$, $c' \in z'$ emanate from x. We can always choose $x \in X$ such that w_{c_0}, $w_{c'}$ have at most one common letter. For a pair α, α' of consecutive letters of $w = w_{c_0} \cup w_{c'}$ let $S_{\alpha\alpha'} \subset \partial_T X$ be the boundary circle of the separating flat $X_\alpha \cap X_{\alpha'}$. For three consecutive letters $\alpha_0 \alpha_1 \alpha_2 \subset w$ the circles $S_{\alpha_0 \alpha_1}$, $S_{\alpha_1 \alpha_2}$ have only two common points, which are singular, and for the letter $\alpha_3 \in w$ next after α_2 the circles $S_{\alpha_0 \alpha_1}$, $S_{\alpha_2 \alpha_3}$ are disjoint. The singular points of $\partial_T X$ are pairwise separated by a distance $\geq \omega > \epsilon$, every noncontractible loop in $\partial_T X$ has length at least 2π, and $\dim \partial_T X = 1$ by a result of B. Kleiner [Kl].

All this implies that w has at most three letters, and $z_0 z' \subset S_{\alpha_0\alpha_1} \cup S_{\alpha_1\alpha_2} \subset \mathcal{F}$. $\square$

2.8. Lemma. *If points z, $z' \in \mathcal{I}$ are in the same connected component of $\partial_T X$, then z, $z' \in \mathcal{I}_w$ for some $w \in \mathcal{W}$.*

Proof. If the classes $f(z)$, $f(z') \in \mathcal{W}$ are different, then any continuous curve $\gamma \subset \partial_\infty X$ between z, z' intersects the boundary $\partial_\infty E$ of at least one separating flat $E \subset X$, hence, $\gamma \cap \mathcal{F} \neq \emptyset$.

By the assumption, z, z' are connected by a geodesic $\gamma \subset \partial_T X$. Then γ is continuous in $\partial_\infty X$. It follows that if $f(z) \neq f(z')$, then γ intersects $\mathcal{F}$ and $\gamma \subset \mathcal{F}$. Thus z, $z' \in \mathcal{F}$, a contradiction. Hence, $f(z) = f(z')$ and z, $z' \in \mathcal{I}_w$ for some $w \in \mathcal{W}$. $\square$

From 2.6–2.8 we obtain

2.9. Corollary. *The decomposition*

$$\partial_T X = \mathcal{F} \cup \bigcup_{w \in \mathcal{W}} \mathcal{I}_w$$

coincides with the decomposition of $\partial_T X$ into connected components.

2.9.1. We shall call $\mathcal{F}$ *the principal component* of $\partial_T X$ and $\mathcal{I}_w$, $w \in \mathcal{W}$ *nonprincipal components*. By 2.4.1, 2.6 and 2.8, the map $f : \mathcal{I} \to \mathcal{W}$ induces a bijection between the set of nonprincipal components and $\mathcal{W}$. In the sequel, we identify $\mathcal{W}$ with the set of nonprincipal connected components of $\partial_T X$. Therefore, the last depends only on the fundamental group of M.

2.9.2. The principal component $\mathcal{F}$ of $\partial_T X$ is also independent of the metric g on M in the sense that for any other nonpositively curved metric g' on M there is a canonical homeomorphism $\mathcal{F} \to \mathcal{F}'$. This follows from the next two obvious facts.

(a) The vertices of the graph $\mathcal{F}$ are fixed points of the stabilizers St_α of blocks $X_\alpha \subset X$.

(b) Any edge of $\partial_T X_\alpha$ has length π and corresponds to a point of $\partial_\infty Y_\alpha$, $\alpha \in \mathcal{A}$. Since the metric on Y_α is hyperbolic, $\partial_\infty Y_\alpha$ is a quasi-isometric invariant of Y_α.

Now we consider consequences of a rigid deformation of nonpositively curved metric on M. The next result generalizes an example from [Bu].

2.10. Theorem. *If for metrics g_0, g_1 of nonpositive curvature on a graphmanifold M there is a continuous Γ-equivariant map $\partial_\infty X_0 \to \partial_\infty X_1$, $\Gamma = \pi_1(M)$, then the principal components $\mathcal{F}_0$, $\mathcal{F}_1$ of the Tits boundaries $\partial_T X_0$, $\partial_T X_1$ are canonically isometric.*

Proof. If $\mathcal{F}_0$, $\mathcal{F}_1$ are not isometric, then for some adjacent blocks X_α, $X_{\alpha'} \subset X$ we have $\angle_0(\alpha, \alpha') \neq \angle_1(\alpha, \alpha')$ for the Tits distances between corresponding singular points α, $\alpha' \in \partial_T X$. The boundary circle S of the separating flat $E = X_\alpha \cap X_{\alpha'}$ is the union of two edges $S = e_0 \cup e_1$ of length π between α and $-\alpha$ (in any metric g_0, g_1), and α', $-\alpha' \in S$. We can assume that α' is an interior point of e_0.

We take $\beta \in e_1$ with $\angle_0(\beta, \alpha) = \angle_0(\alpha, \alpha')$ and consider $\gamma \in \Gamma$, which represents a nonperipheral element of $\pi_1(F_v) \subset \Gamma$, where $M_v = F_v \times S^1$ is the maximal block covered by X_α. Then $\gamma(\alpha)=\alpha$, $\gamma(-\alpha)=-\alpha$, and $\gamma^n(e_0)$, $\gamma^n(e_1)$ converge in $\partial_\infty X$ to the same edge $e\subset\partial_T X$ between α and $-\alpha$ (the edge e is defined by one of the ends $z\in\partial_\infty Y_\alpha$ of an axis of γ in Y_α). Furthermore, the sequences $\gamma^n(\alpha')$, $\gamma^n(\beta)$ converge in $\partial_\infty X_0$ to the *same* point of e by the choice of β. However, for the metric g_1 we have $\angle_1(\beta,\alpha)\neq\angle_1(\alpha,\alpha')$, and the sequences $\gamma^n(\alpha')$, $\gamma^n(\beta)$ converge in $\partial_\infty X_1$ to *different* points of e (the point $\beta\in e_1$ being fixed can be approximated by ends of axis of elements from the stabilizer $\mathrm{St}_E\subset\Gamma$ of E, since $\mathrm{St}_E\simeq\mathbb{Z}^2$, hence it is well defined and for the metric g_1).

Therefore, there is no continuous Γ-equivariant map $\partial_\infty X_0\to\partial_\infty X_1$. □

We have the same sort of effects for a soft deformation, if the last results in a degeneration of a nonprincipal component.

2.11. Proposition. *Assume that for nonpositively curved metrics g_0, g_1 on a graphmanifold M there is a $w\in\mathcal{W}$ such that the connected component $\mathcal{I}_w$ is degenerate with respect to g_0 and distinct from a point w.r.t. g_1. Then there exists no continuous Γ-equivariant map $\partial_\infty X_0\to\partial_\infty X_1$ for $\Gamma=\pi_1(M)$.*

Proof. Fix some string $w_c\in w$ and consider the sequence $S_\alpha=\partial_\infty E_\alpha$, $\alpha\in w_c$ of circles on $\partial_\infty X_0$, $\partial_\infty X_1$, where E_α is the initial w.r.t. w_c separating flat of the block X_α. If a letter $\beta\in w_c$ follows α and is not the next one, then $S_\beta\subset\mathcal{D}_\alpha$, where $\mathcal{D}_\alpha\subset\partial_\infty X$ is an open subset homeomorphic to the open disc and bounded by S_α (we suppress the subsrcipts 0, 1).

The collection $\{\mathcal{D}_\alpha\}_{\alpha\in w_c}$ is a basis of neighborhoods in $\partial_\infty X$ of the component $\mathcal{I}_w$, in particular, $\bigcap_{\alpha\in w_c}\mathcal{D}_\alpha=\mathcal{I}_w$.

Therefore, for g_0 the sets S_α converge in $\partial_\infty X_0$ to the point $\mathcal{I}_w$ as $\alpha\to\infty$. On the other hand, since $\mathcal{I}_w$ is nondegenerate for g_1, we can find x_α, $y_\alpha\in S_\alpha$ such that

$$\lim_{\alpha\to\infty}x_\alpha\neq\lim_{\alpha\to\infty}y_\alpha$$

in $\partial_\infty X_1$, while $\lim_{\alpha\to\infty}x_\alpha=\lim_{\alpha\to\infty}y_\alpha=\mathcal{I}_w$ in $\partial_\infty X_0$. One can always assume that the points x_α, y_α are rational, i.e. correspond to some cyclic subgroups in Γ. Thus there is no continuous Γ-equivariant map $\partial_\infty X_0\to\partial_\infty X_1$. □

2.12. Proposition. *Each nonprincipal component $\mathcal{I}_w$, $w\in\mathcal{W}$ is a segment in $\partial_\infty X$ of length $<\pi$ and may be degenerate.*

For the proof we consider the following invariant of a point $z\in\mathcal{I}_w$. For a ray $c\in z$ and $\alpha\in w_c$ let $\eta_c(\alpha)$ be the angle between c and the boundary component of X_α in the entering point x_α of c into X_α. We define

$$\eta_c=\limsup_{\alpha\to\infty}\eta_c(\alpha).$$

2.12.1. Lemma. *The number η_c is independent on the ray $c\in z$, i.e. $\eta_c=\eta_z$ for all $c\in z$.*

Proof. Let c, $c'\in z$ be some rays. One can assume that $w_c=w_{c'}$. It easily follows from properties of asymptotic rays that for sufficiently large $\alpha\in w_c$ the tangent vectors $\dot c_\alpha$, $\dot c'_\alpha$ to c, c' at the entering points x_α, x'_α in X_α are almost parallel, i.e. by applying the parallel translation along $x_\alpha x'_\alpha$ they differ by a vector of size ϵ_α, where $\epsilon_\alpha\to0$ as $\alpha\to\infty$. It gives $\eta_c=\eta_{c'}$. □

The following lemma is the key one for the proof of Proposition 2.12.

2.12.2. Lemma. *If $\eta_z \neq 0$ for some point $z \in \mathcal{I}_w$, then the component $\mathcal{I}_w$ is degenerate, $\mathcal{I}_w = \{z\}$.*

Proof. Assume that it is not the case and the component $\mathcal{I}_w$ is nondegenerate. Then for each sufficiently small $\epsilon > 0$ there is a point $z' \in \mathcal{I}_w$, $z' \neq z$ with $\angle(z, z') = \epsilon$. We take ϵ on a scale of sizes arbitrarily smaller than $\eta = \eta_z$, $\epsilon \prec \eta$.

For any $\sigma > 0$, $\sigma \prec \epsilon$ there is a point $x \in X$ such that $\angle_x(z, z') \geq \angle(z, z') - \sigma$. Let $c \in z$, $c' \in z'$ be the rays with the common vertex x. Then for any t, $t' > 0$ the triangle $xc(t)c'(t')$ bounds in X a ruled σ-almost flat surface, i.e. whose absolute integral curvature is $\leq \sigma$. In particular, the vectors $\dot{c}(t)$, $\dot{c}'_{\|}(t') \in T_{c(t)}X$ form the angle σ-close to ϵ and lie σ-almost in the one 2-subspace with the direction of the segment $c(t)c'(t')$. Here $\dot{c}'_{\|}(t')$ is the parallel translate of $\dot{c}'(t')$ along $c'(t')c(t)$.

By the assumption, there is an infinite sequence of letters $\alpha \in w_c$, for which the direction of c at the entering point x_α into X_α forms the angle $\eta_c(\alpha) \geq \eta$ with the corresponding separating flat E_α. We denote $x'_\alpha = c' \cap E_\alpha$.

The segment $x_\alpha x'_\alpha$ forms the angle $\phi_\alpha \geq \omega/2$ with one of the $\mathbb{R}$-factors $\mathbb{R}_\alpha$ of X_α or $\mathbb{R}_\beta$ of the preceding block X_β, where ω is the minimal angle between singular directions of adjacent blocks in X.

First, consider the case when it is the factor $\mathbb{R}_\alpha$. Let $\overline{x}_\alpha \overline{x}'_\alpha$ be the projection of $x_\alpha x'_\alpha$ on the factor Y_α from the decomposition $X_\alpha = Y_\alpha \times \mathbb{R}$. Then $|\overline{x}_\alpha \overline{x}'_\alpha| \to \infty$ as $\alpha \to \infty$. Furthermore, the segments $\overline{c}_\alpha$, $\overline{c}'_\alpha$ form the angles ψ_α, ψ'_α with $\overline{x}_\alpha \overline{x}'_\alpha$, which are $(\eta - \epsilon)$-separated from zero and π, where $\overline{c}_\alpha$, $\overline{c}'_\alpha$ are the projections of the segments $c_\alpha = c \cap X_\alpha$, $c'_\alpha = c' \cap X_\alpha$ on Y_α.

Using the conditions $\eta_c(\alpha) \geq \eta$, $\phi_\alpha \geq \omega/2$ and the above remark about the triangles $xc(t)c'(t')$, it is not difficult to see that $\psi_\alpha + \psi'_\alpha \geq \pi + \lambda \cdot \epsilon - \sigma$, where the coefficient $\lambda > 0$ depends only on η and ω. Thus $\psi_\alpha + \psi'_\alpha > \pi$. In that case the periodicity argument immediately gives a contradiction.

Assume now that $x_\alpha x'_\alpha$ forms the angle $\phi_\alpha \geq \omega/2$ with the $\mathbb{R}$-factor $\mathbb{R}_\beta$ of the preceding block X_β. Then repeating the arguments above and replacing α by β, we obtain that the angles ψ_β, ψ'_β are $(\eta - \epsilon)$-separated from 0 and π. This time we only have $\pi \geq \psi_\beta + \psi'_\beta \geq \pi - \lambda \cdot \epsilon - \sigma$, where the coefficient $\lambda < \infty$ depends only on η and ω. Choosing ϵ sufficiently small (depending only on η and ω), letting $\alpha \to \infty$ and applying the periodicity argument, we find in Y_β σ-almost flat triangles with arbitrary large size of the inscribed disc. This is, obviously, impossible. □

Proof of Proposition 2.12. Assume that a component $\mathcal{I}_w \subset \partial_T X$ is nondegenerate. Then by Lemma 2.12.2, $\eta_z = 0$ for all $z \in \mathcal{I}_w$. Now let z, z', $z'' \in \mathcal{I}_w$ be pairwise different sufficiently close points, and let $\epsilon > 0$ be the minimal distance between them. For any $\sigma > 0$, $\sigma \prec \epsilon$ we can find $x \in X$ such that for the rays $c \in z$, $c' \in z'$, $c'' \in z''$ emanating from x the measure of nonflatness of any sectors cc', cc'', $c'c''$ is less than σ. Moving forward along c and using the condition $\eta_z = \eta_{z'} = \eta_{z''} = 0$, we find that for sufficiently large $\alpha \in w_c$ the direction of c at the entering point x_α into X_α is σ-close to the separating flat E_α, and this is also true for the rays $x_\alpha z'$, $x_\alpha z''$. It immediately follows that one of the points z, z', $z'' \in \mathcal{I}_w$ lies between two others. □

2.13. Action of the fundamental group Γ on the set of connected components of $\partial_T X$.

The principal component $\mathcal{F}$, obviously, is invariant for any $\gamma \in \Gamma$. No one component $\mathcal{I}_w$ is invariant for whole Γ, however there always exist nonprincipal components $\mathcal{I}_w$, which are invariant for some nontrivial $\gamma \in \Gamma$. If it is the case, then γ is a rank 1 isometry in the sense of [BB], and $\mathcal{I}_w$ is degenerate for any nonpositively curved metric on M. Such γ represents a closed geodesic in M, which is not contained in one block M_v of M.

A connected component of $\partial_T X$ is said to be *standard*, if it is invariant at least for one nontrivial $\gamma \in \Gamma$. The other components are called *nonstandard*.

3. The length spectrum of nonstandard components

We use notation $L(w)$ for the length of a nonstandard connected component $\mathcal{I}_w \subset \partial_T X$, $w \in \mathcal{W}$.

We start with the proof of Theorem 0.1. For that we need the following simple lemma (its proof we leave to the reader).

3.1. Lemma. *Assume that unit vectors v, v', α, $\beta \in \mathbb{R}^2$ satisfy the conditions $\angle(v, v') > \omega \geq \pi/2$ and $\pi/2 \leq \angle(\alpha, \beta) \leq \omega$.*

If the projections $\overline{v}_\alpha$, $\overline{v}'_\alpha$ of v, v' on the line $\mathbb{R} \cdot \alpha$ are codirected, then the projections $\overline{v}_\beta$, $\overline{v}'_\beta$ on $\mathbb{R} \cdot \beta$ are nonzero and opposite directed.

Proof of Theorem 0.1. Assume to the contrary that for some $w \in \mathcal{W}$ there exist z, $z' \in \mathcal{I}_w$ with $\angle(z, z') > \omega$. We put $\epsilon = \angle(z, z') - \omega$. Then $\epsilon < \pi - \omega$.

For any $\sigma > 0$ there is $x \in X$ with $\angle_x(z, z') \geq \omega + \epsilon - \sigma$. Then for the rays $c \in z$, $c' \in z'$ emanating from x and any t, $t' > 0$ the triangle $xc(t)c'(t')$ bounds in X a ruled σ-almost flat surface with total curvature $\leq \sigma$. Furthermore, by Lemma 2.12.2, one can assume that the rays c, c' form angles $\leq \sigma$ with the separating flat E_α at the entering points $x_\alpha = c \cap E_\alpha$, $x'_\alpha = c' \cap E_\alpha$ into the block X_α for all $\alpha \in w_c$.

Moving forward along the string w_c one can make σ arbitrary small comparing to ϵ; thus we assume that $\sigma \prec \epsilon$. It means, in particular, that we denote by the same symbol σ the constants of the scale of σ.

Let $\overline{c}_\alpha$, $\overline{c}'_\alpha$ be the projections of the segments $c_\alpha = c \cap X_\alpha$, $c'_\alpha = c' \cap X_\alpha$ on the factor Y_α from the decomposition $X_\alpha = Y_\alpha \times \mathbb{R}$. We denote by $\mathbb{R}_\alpha$ the $\mathbb{R}$-factor of that decomposition. In the sequel, we use the affine structure of the flat E_α. Speaking about the angle between a vector $v \in E_\alpha$ and $\mathbb{R}_\alpha$ we mean the angle between v and its projection on $\mathbb{R}_\alpha$ (such an angle is always $\leq \pi/2$).

Let v_α, v'_α be the unit vectors in E_α, which are the projections of the directions of the rays c, c' at x_α, x'_α. We say that there is *a sufficient separation* (from the $\mathbb{R}$-factor) for $\alpha \in w_c$, if $\angle(v_\alpha, \mathbb{R}_\alpha)$, $\angle(v'_\alpha, \mathbb{R}_\alpha) \geq \epsilon/4$.

Let $\overline{v}_\alpha$, $\overline{v}'_\alpha$ be the projections of v_α, v'_α on the factor Y_α. We say that we have *opposite projections for* $\alpha \in w_c$, if the vectors $\overline{v}_\alpha$, $\overline{v}'_\alpha$ are nonzero and opposite directed; otherwise we have *codirected projections* for $\alpha \in w_c$.

Step 1. Assume that for $\alpha \in w_c$ there is a sufficient separation from $\mathbb{R}_\alpha$ and opposite projections. We shall show that this leads to a contradiction.

The vectors v_α, v'_α form the angles ϕ_α, ϕ'_α with the segment $x_\alpha x'_\alpha$, whose sum differs from $\pi + \omega + \epsilon$ at most by σ. The angle $\angle(v_\alpha, v'_\alpha)$ also differs from $\omega + \epsilon$ at most by σ. It easily follows from this that

3.1.1. If ϕ_α, $\phi'_\alpha \leq \pi - \epsilon/8$, then the vectors v_α, v'_α are directed in the same half-plane in E_α with respect to the line of the segment $x_\alpha x'_\alpha$.

Now, if $x_\alpha x'_\alpha$ forms the angle $< \epsilon/8$ with $\mathbb{R}_\alpha$, then ϕ_α, $\phi'_\alpha \leq \pi - (\epsilon/4 - \epsilon/8) = \pi - \epsilon/8$. Thus by 3.1.1, the vectors v_α, v'_α are on one side with respect to $x_\alpha x'_\alpha$.

Using the sufficient separation from $\mathbb{R}_\alpha$, we obtain that the projections $\overline{v}_\alpha$, $\overline{v}'_\alpha$ are codirected, which contradicts our assumption.

Hence, $x_\alpha x'_\alpha$ forms the angle $\geq \epsilon/8$ with $\mathbb{R}_\alpha$. Let $\overline{x}_\alpha$, $\overline{x}'_\alpha$ be the projections of x_α, x'_α on Y_α. Since $|x_\alpha x'_\alpha| \to \infty$ as $\alpha \to \infty$, we can assume that $|\overline{x}_\alpha \overline{x}'_\alpha| \geq \rho_1$ (see Lemma 2.2.5). The segments $\overline{c}_\alpha$, $\overline{c}'_\alpha \subset Y_\alpha$ form the angles $\leq \sigma$ with the boundary component of Y_α corresponding to E_α. Using again 3.1.1, we obtain that $\overline{c}_\alpha$, $\overline{c}'_\alpha$ form the angles $\geq \pi - \sigma$ with the segment $\overline{x}_\alpha \overline{x}'_\alpha$. Then the periodicity argument immediately gives a contradiction.

Step 2. Assume that there is no sufficient separation from $\mathbb{R}_\alpha$ for $\alpha \in w_c$. Let $\beta \in w_c$ be the next letter after α. We shall show that for β we have a sufficient separation from $\mathbb{R}_\beta$ and opposite projections.

Without loss of generality, we assume that $\angle(v_\alpha, \mathbb{R}_\alpha) < \epsilon/4$. Then, obviously, $\angle(v_\beta, \mathbb{R}_\alpha) \leq \epsilon/4 + \sigma$ (here the vector v_β and the factor $\mathbb{R}_\alpha$ are considered in the flat E_β). Thus $\angle(v_\beta, \mathbb{R}_\beta) \geq \pi - \omega - \epsilon/4 - \sigma \geq \epsilon/4$. Furthemore, $\angle(v'_\beta, \mathbb{R}_\beta) \geq \angle(v_\beta, v'_\beta) - \angle(v_\beta, \mathbb{R}_\alpha) - \omega \geq \epsilon/4$. Moreover, the vectors v_β, v'_β are directed into different half-planes in E_β with respect to $\mathbb{R}_\beta$, i.e. $\overline{v}_\beta$, $\overline{v}'_\beta$ are opposite directed.

Therefore, for β we have a sufficient separation from $\mathbb{R}_\beta$ and opposite projections. By Step 1, this leads to a contradiction.

Step 3. Hence, for $\alpha \in w_c$ we have a sufficient separation from $\mathbb{R}_\alpha$ and codirected projections. Let $\beta \in w_c$ be the next letter after α. We shall show that for β we have opposite projections.

The angles $\angle(v_\alpha, \mathbb{R}_\alpha)$, $\angle(v_\beta, \mathbb{R}_\alpha)$ differ at most by σ, and the same is true for the pairs of angles $\angle(v'_\alpha, \mathbb{R}_\alpha)$, $\angle(v'_\beta, \mathbb{R}_\alpha)$ and $\angle(v_\alpha, v'_\alpha)$, $\angle(v_\beta, v'_\beta)$. Since for α we have codirected projections, it means that

$$\angle(v_\alpha, \mathbb{R}_\alpha) + \angle(v_\alpha, v'_\alpha) + \angle(v'_\alpha, \mathbb{R}_\alpha) = \pi.$$

Assume that the projections of v_β, v'_β on Y_α (in the flat E_β) are opposite directed. Then, without loss of generality, we have

$$\angle(v_\beta, \mathbb{R}_\alpha) + \angle(v_\beta, v'_\beta) - \angle(v'_\beta, \mathbb{R}_\alpha) = \pi.$$

Thus

$$\begin{aligned}\angle(v_\beta, \mathbb{R}_\alpha) - \angle(v_\alpha, \mathbb{R}_\alpha) &= \angle(v'_\alpha, \mathbb{R}_\alpha) + \angle(v'_\beta, \mathbb{R}_\alpha) + \angle(v_\alpha, v'_\alpha) - \angle(v_\beta, v'_\beta) \\ &\geq \epsilon/2 - \sigma,\end{aligned}$$

which contradicts the estimate $\angle(v_\beta, \mathbb{R}_\alpha) - \angle(v_\alpha, \mathbb{R}_\alpha) \leq \sigma$.

Therefore, the projections of v_β, v'_β on Y_α are codirected. It follows from Lemma 3.1 that for β we have opposite projections. By Step 2, we also have a sufficient separation for β. By Step 1, this leads to a contradiction, which completes the proof of Theorem 0.1. ∎

Nonstandard components of maximal length

We show here that the estimate of Theorem 0.1 is optimal, in particular, that nondegenerate nonstandard components do exist.

3.2. A nonstandard component $\mathcal{I}_w$ is said to be *primitive*, if the separating flats E_α for all $\alpha \in w_c$ belong to the same orbit of the action of $\Gamma = \pi_1(M)$ for some $w_c \in w$.

In other words, it means that any geodesic ray $c : [0, \infty) \to X$, $c(\infty) \in \mathcal{I}_w$ descends to a geodesic $\overline{c}$ in M, which starting from some moment intersects only one gluing torus $T_u \subset M$ and hence $\overline{c}$ lives only in two adjacent maximal blocks M_v, $M_{v'}$ of M skipping from one to another. We use the notation $\mathcal{W}_u$ for the set of the nonprincipal connected components of $\partial_T X$ associated with T_u.

3.2.1. *Remark.* We have actually proved that for any primitive component $w \in \mathcal{W}_u$ its length satisfies $L(w) \leq \omega_u$, where $\omega_u \in [\pi/2, \pi)$ is the angle between S^1-factors on T_u.

3.3. Here we prove Theorem 0.2. To each singular point $\alpha \in \mathcal{F}$ it corresponds exactly one block X_α with singular direction α. Any singular point $\beta \in \mathcal{F}$ is connected with α by a minimizer $\alpha\beta \subset \mathcal{F}$. If some minimizer $\alpha\beta$ contains no other singular point, then the blocks X_α, X_β are adjacent, $\alpha\beta$ lies in the boundary at infinity $S_{\alpha\beta}$ of the separating flat $E_{\alpha\beta} = X_\alpha \cap X_\beta$ and $\angle(\alpha, \beta) \leq \omega(g)$. In that case the points α, β are called *neighboring*. We say that a pair of neighboring singular points (α, β) *belongs to the class* $\mathcal{W}_u$, if the separating flat $E_{\alpha\beta}$ covers the torus T_u and $\angle(\alpha, \beta) = \omega_u$.

3.3.1. Lemma. *Let (α, β) be a pair of neighboring points of a class $\mathcal{W}_u$. Then there exists a sequence of singular points $\beta_n \in \mathcal{F}$ such that*

(i) *$\beta_n \to \alpha$ in the standard topology;*
(ii) *the pairs (β_n, α) belong to the class $\mathcal{W}_u$.*

Proof. Applying to X_α the isometries from $\mathrm{St}_\beta \backslash \mathrm{St}_\alpha$, where $\mathrm{St}_\beta \subset \Gamma$ is the stabilizer of the block X_β, we obtain infinitely many blocks $X_{\beta'}$ adjacent to X_β, for which (β, β') belong to the class $\mathcal{W}_u$. Choosing an isometry $\gamma \in \mathrm{St}_\alpha \cap \mathrm{St}_\beta$, whose axis l_γ has the positive direction $l_\gamma(\infty) = \alpha$, and applying γ^n, $n \geq 1$ to β' we obtain a required sequence β_n. □

3.3.2. Corollary. *Let (α, β) be a pair of singular points of a class $\mathcal{W}_u$. Then for any neighborhood U of α in $\partial_\infty X$ there exists a singular point $\alpha' \in U \cap \mathcal{F}$ such that (β, α') belongs to the class $\mathcal{W}_u$ and $X_\alpha \neq X_{\alpha'}$.*

3.4. Proof of Theorem 0.2. We fix an $\epsilon > 0$ and a sequence ϵ_i with $\sum_{i \geq 0} \epsilon_i \leq \epsilon$. The required component $\mathcal{I}_w \subset \partial_T X$ will be obtained by the following inductive construction of a sequence of letters $\alpha_0, \beta_0, \alpha_1, \beta_1 \ldots \in \mathcal{A}$ with $w = \alpha_0\beta_0\alpha_1\beta_1 \ldots$.

Step 0. Take a pair (α_0, β_0) of the class $\mathcal{W}_u$ and $x_0 \in X$ such that the block X_{α_0} separates x_0 with X_{β_0}. Notation: $\alpha_0 < \beta_0$.

We can identify any $z \in \partial_\infty X$ with the ray $c \in z$ emanating from x_0. In particular, α_0, $\beta_0 : [0, \infty) \to X$ are geodesic rays with $\alpha_0(0) = \beta_0(0) = x_0$ and $\alpha_0(\infty) = \alpha_0$, $\beta_0(\infty) = \beta_0$.

One can find $t_0 > 0$ such that for the comparison triangle $\overline{x}_0\overline{\alpha}_0(t_0)\overline{\beta}_0(t_0) \subset \mathbb{R}^2$ we have $\angle\overline{\alpha}_0(t_0)\overline{x}_0\overline{\beta}_0(t_0) \geq \omega_u - \epsilon_0$.

Moving forward along the rays α_0, β_0 if necessary, we additionally assume that $\alpha_0(t_0)$ belongs to the block, preceding X_{α_0} (with respect to x_0), and $\beta_0(t_0) \in X_{\alpha_0}$.

Step 1. By Corollary 3.3.2, there exists a singular point $\alpha_1 \in \mathcal{F}$ such that $\beta_0 < \alpha_1$, (β_0, α_1) belongs to the class $\mathcal{W}_u$ and $|\alpha_0(t_0)\alpha_1(t_0)| \leq \epsilon_0$.

Having chosen this point we can find $t_1 > t_0$ such that for the comparison triangle $\overline{x}_0\overline{\alpha}_1(t_1)\overline{\beta}_0(t_1)$ we have $\angle\overline{\alpha}_1(t_1)\overline{x}_0\overline{\beta}_0(t_1) \geq \omega_u - \epsilon_1$. Again, one can additionally assume that $\alpha_1(t_1) \in X_{\beta_0}$ and $\beta_0(t_1) \in X_{\alpha_0}$.

Repeating this argument, we can find a singular point $\beta_1 \in \mathcal{F}$ such that $\beta_1 > \alpha_1$, the pair (α_1, β_1) belongs to the class $\mathcal{W}_u$ and $|\beta_0(t_1)\beta_1(t_1)| \leq \epsilon_1$. Then we find $t_2 > t_1$ with $\angle\overline{\alpha}_1(t_2)\overline{x}_0\overline{\beta}_1(t_2) \geq \omega_u - \epsilon_2$, $\alpha_1(t_2) \in X_{\beta_0}$, $\beta_1(t_2) \in X_{\alpha_1}$.

Step 2. As in Step 1, we find a singular point $\alpha_i > \beta_{i-1}$, for which the pair (β_{i-1}, α_i) belongs to the class $\mathcal{W}_u$, such that

$$\left|\alpha_i(t_{2(i-1)})\alpha_{i-1}(t_{2(i-1)})\right| \le \epsilon_{2(i-1)}. \tag{3.4.1}$$

Moving forward along the rays α_i, β_{i-1}, we find $t_{2i-1} > t_{2(i-1)}$ such that

$$\angle\overline{\alpha}_i(t_{2i-1})\overline{x}_0\overline{\beta}_{i-1}(t_{2i-1}) \ge \omega_u - \epsilon_{2i-1}, \tag{3.4.2}$$

$$\alpha_i(t_{2i-1}) \in X_{\beta_{i-1}},\ \beta_{i-1}(t_{2i-1}) \in X_{\alpha_{i-1}}. \tag{3.4.3}$$

Likewise, we find $\beta_i > \alpha_i$, for which (α_i, β_i) belongs to the class $\mathcal{W}_u$, such that

$$|\beta_i(t_{2i-1})\beta_{i-1}(t_{2i-1})| \le \epsilon_{2i-1}. \tag{3.4.4}$$

Then we find $t_{2i} > t_{2i-1}$ with

$$\angle\overline{\alpha}_i(t_{2i})\overline{x}_0\overline{\beta}_i(t_{2i}) \ge \omega_u - \epsilon_{2i}, \tag{3.4.5}$$

$$\alpha_i(t_{2i}) \in X_{\beta_{i-1}},\ \beta_i(t_{2i}) \in X_{\alpha_i}. \tag{3.4.6}$$

This procedure generates an infinite string $w \in W$, which is primitive by the construction and its class belongs to $\mathcal{W}_u$. It follows from (3.4.1) and (3.4.4) that

$$|\alpha_i(t_j)\alpha_{i-1}(t_j)| \le \epsilon_{2(i-1)} \quad \text{and} \quad |\beta_i(t_j)\beta_{i-1}(t_j)| \le \epsilon_{2i-1} \tag{3.4.7}$$

for all $0 \le j \le 2(i-1)$, which implies by the choice of $\{\epsilon_i\}$ that $\{\alpha_i\}$, $\{\beta_i\}$ are Cauchy sequences of rays in X. For the limit rays $\alpha = \lim \alpha_i$ and $\beta = \lim \beta_i$ we have

$$w_\alpha = w_\beta = w, \tag{3.4.8}$$

i.e. α, β intersect the same infinite sequence X_{α_0}, X_{β_0}, X_{α_1}, $X_{\beta_1}, \dots$ of blocks. This follows from (3.4.3) and (3.4.6). Hence, α, $\beta \in \mathcal{I}_w$.

Furthermore, by (3.4.7) we have

$$|\alpha(t_{2i})\alpha_i(t_{2i})| \le \epsilon \quad \text{and} \quad |\beta(t_{2i})\beta_i(t_{2i})| \le \epsilon$$

for every $i \ge 1$. Together with (3.4.5) this gives

$$\angle\overline{\alpha}(t_{2i})\overline{x}_0\overline{\beta}(t_{2i}) \ge \omega_u - \epsilon_{2i} - \epsilon/t_{2i}.$$

Hence, $\angle(\alpha, \beta) \ge \omega_u$ and by Remark 3.2.1, $L(w) = \omega_u$.

The length spectrum of primitive components

Theorem 0.4 can be reformulated as follows.

3.5. Theorem. *Assume that for a gluing torus $T_u \subset M$ the angle between the S^1-factors of the adjacents maximal blocks of M (which can coincide) is $\omega_u = \pi/2$. Then given $l \in [0, \omega_u]$ there exists a nonstandard component $\mathcal{I}_w$, $w \in \mathcal{W}_u$ with the length $L(w) = l$.*

Theorem 3.5 is already proved for $l = \omega_u$ in Theorem 0.2 even without the condition $\omega_u = \pi/2$. Thus in the sequel, we assume that $0 \le l < \omega_u$. This is essential for the proof, because it uses an approach distinct from that of Theorem 0.2, which does not work for $l = \omega_u$. The key step in the proof is Proposition 3.6 below (the Collapsing Proposition). The restriction $\omega_u = \pi/2$ is basically related to it.

Recall that a pair (α, β) of neighboring singular points of $\mathcal{F}$ belongs to the class $\mathcal{W}_u$, if the separating flat $E_{\alpha\beta}$ covers the torus T_u and $\angle(\alpha, \beta) = \omega_u$. The flat $E_{\alpha\beta}$ is also called (α, β)-*window.*

Fix $x_0 \in X$ and consider an (ordered) pair (β, α) of the class $\mathcal{W}_u$ so that $\beta < \alpha$ w.r.t. x_0. Let $R > 0$. The (β, α)-window *is located in a R-restricted direction,* if $|x_0 x_\alpha| \le R$, where the ray c_α in X emanating from x_0 intersects $E_{\alpha\beta}$ orthogonally and $x_\alpha \in c_\alpha \cap X_\beta$ is the entering point of c_α into the block X_β.

Furthermore, the (β, α)-window is said to be *τ-thin,* $\tau > 0$, if $\underline{\alpha}$, $\underline{\alpha}' \in U_{x_\alpha, 10\tau}(c_\alpha)$, where x_α, c_α are as above, $\underline{\alpha}$, $\underline{\alpha}'$ are the midpoints of the circle $S_{\alpha\beta} = \partial_\infty E_{\alpha\beta}$ between β and $-\beta$ (in the case of a $\pi/2$-metric, $\underline{\alpha} = \alpha$, $\underline{\alpha}' = -\alpha$).

Of course, it is assumed that then τ is larger, the window is thinner.

Recall that

$$U_{x_0,t}(z) = \{z' \in \partial_\infty X \mid |z(t)z'(t)| < 1\}, \quad z \in \partial_\infty X,\ t > 0,$$

is a neighborhood of $z \in \partial_\infty X$, where a point $z' \in \partial_\infty X$ is identified with the ray $z' : [0, \infty) \to X$, $z'(0) = x_0$, $z'(\infty) = z'$.

3.6. Proposition. *Assume that $\omega_u = \pi/2$. Given $x_0 \in X$, a pair (α_0, β_0) of the class $\mathcal{W}_u$, $\alpha_0 < \beta_0$ w.r.t. x_0, $s \in [\omega_u/2, \omega_u)$, R, $t > 0$, then there exists $\tau > 0$ such that if for $\alpha > \beta_0$ the (β_0, α)-window belongs to the class $\mathcal{W}_u$, is τ-thin and located in a R-restricted direction, then for any $\rho > 0$ there is a singular point $\beta' \in \mathcal{F}$ with*

- (i) *$\beta' > \alpha$ and (α, β') belongs to the class $\mathcal{W}_u$;*
- (ii) *the (α, β')-window is ρ-thin and located in a R_α-restricted direction, where $R_\alpha > 0$ depends only on α;*
- (iii) *if $b_\alpha \in \alpha\beta_0$ is the point with $\angle(\alpha, b_\alpha) = s$ and $b'_\alpha \in \alpha\beta'$ is the point with $\angle(\alpha, b'_\alpha) = s$, then the segment $\beta' b'_\alpha \subset U_{x_0,t}(b_\alpha)$;*
- (iv) *$a' \in U_{x_0,t}(a)$ for any $0 \le s' \le s$, where $a \in \alpha\beta_0$, $a' \in \alpha\beta'$ with $\angle(\alpha, a) = s' = \angle(\alpha, a')$.*

The properties (iii), (iv) are the most important ones for constructing a component $w \in \mathcal{W}_u$ with prescribed length. They mean, roughly, that the subsegment $b'_\alpha \beta' \subset \alpha\beta'$ collapses when viewing from x_0 (while $\alpha b'_\alpha$ is observed almost at the given angle s). For this reason we refer to Proposition 3.6 as to the Collapsing Proposition.

We start the proof of Proposition 3.6 with the following

3.6.1. Lemma. *Under the condition of Proposition 3.6, there exists $\tau' > 0$ such that if for $\alpha > \beta_0$ the (β_0, α)-window belongs to the class $\mathcal{W}_u$, is τ'-thin, located in a R-restricted direction and $a \in \alpha\beta_0$, $\angle(\alpha, a) = s'$, $0 \le s' \le s$, then*

$$U_{x_\alpha, \underline{\tau}'}(a) \subset U_{x_0, 2t}(a)$$

for $\underline{\tau}' = \tau' / \cos s'$, where $x_\alpha \in \partial X_{\beta_0}$ as in the definition of a τ-thin window.

Proof. If $x_0 \in X_{\beta_0}$, then $x_\alpha = x_0$, and the lemma is trivial. Thus we suppose that $x_0 \notin X_{\beta_0}$. In that case the entering points x_α's into the block X_{β_0} are contained in a compact subset K of the corresponding boundary component of X_{β_0} by the condition of the R-restricted direction. Assuming that the lemma is not true and using the condition $s' \le s < \pi/2$, we find corresponding limit points $a_\infty \in \partial_\infty X_{\beta_0}$, $x_\infty \in K$, for which a required neighborhood exists by properties of the standard topology. This is incompatible with our assumption. □

In the sequel, we use notation $[xy\rangle$ for the ray in X emanating from $x \in X$ and passing through $y \in X \cup \partial_\infty X$.

Proof of Proposition 3.6. We pick $\tau' > 0$ provided by Lemma 3.6.1 and take $\epsilon > 0$ with $\tan s - \tan(s-\epsilon) = 1/4\tau'$. In other words, if $xyz \subset \mathbb{R}^2$ is a triangle with $|xy| = \tau'$, $\angle xyz = \pi/2$, $\angle yxz = s$, and $h \in yz$ satisfies $|hz| = 1/4$, then $\angle zxh = \epsilon$. Notice that $\epsilon \to 0$ as s is fixed and $\tau' \to \infty$. In particular, we assume that $\epsilon < s$, $\pi/2 - s$.

Since the factors Y_α of the decompositions $X_\alpha = Y_\alpha \times \mathbb{R}$, $\alpha \in \mathcal{A}$ are hyperbolic (see 2.2.3), there is $H > 0$ such that if $|pq| \geq H/16$ for an infinite triangle pqr in Y_α, $p, q \in Y_\alpha$, $r \in \partial_\infty Y_\alpha$, $\angle pqr = \pi/2$, then $\angle qpr \leq \epsilon^3$. Now we take $\tau = H \cdot \tau'$.

Step 1: choosing a disc D_α. Let $\alpha \in \mathcal{F}$ be a singular point such that $\alpha > \beta_0$ w.r.t. x_0, (β_0, α)-window belongs to the class $\mathcal{W}_u$, is τ-thin and let $b_\alpha \in \alpha\beta_0$ with $\angle(\alpha, b_\alpha) = s$. For the ray c_α in X, which emanates from x_0 and meets the window $E_{\alpha\beta_0}$ orthogonally at y_α let $x_\alpha \in c_\alpha \cap X_{\beta_0}$ be the entering point into X_{β_0}.

We take $d_\alpha \in E_{\alpha\beta_0}$ with $\angle_{x_\alpha}(d_\alpha, y_\alpha) = s$, which projects along the $\mathbb{R}$-factor of X_{β_0} in y_α. Let $D_\alpha \subset E_{\alpha\beta_0}$ be the disc of radius $H_\alpha = H \cdot |x_\alpha y_\alpha|/2\tau$ centered at d_α (since (β_0, α)-window is τ-thin, we have $|x_\alpha y_\alpha| > \tau$).

Let $d_\alpha^\pm \in D_\alpha$ be the ends of the vertical diameter of D_α, i.e. the segment $d_\alpha^- d_\alpha^+$ is parallel to the $\mathbb{R}$-factor of X_{β_0}. We assume that d_α^- lies between d_α and y_α. Then by the choice of H_α, τ and τ' we have $[x_\alpha d_\alpha^\pm\rangle \in U_{x_\alpha, 2\underline{\tau}'}([x_\alpha d_\alpha\rangle)$, where $\underline{\tau}' = \tau'/\cos s$. Moreover, $[x_\alpha x\rangle \in U_{x_\alpha, 2\underline{\tau}'}([x_\alpha d_\alpha\rangle)$ for the remaining points $x \in D_\alpha$, since the curvature of the factor Y_α is nonpositive. Recalling the choice of d_α, b_α and the definition of a τ-thin window, we obtain that

$$U_{x_\alpha, 2\underline{\tau}'}([x_\alpha d_\alpha\rangle) \subset U_{x_\alpha, \underline{\tau}'}(b_\alpha).$$

Now by the choice of τ' and Lemma 3.6.1, we have

$$z_x \in U_{x_0, 2t}(b_\alpha) \tag{3.6.2}$$

for any $x \in D_\alpha$, where $z_x = [x_\alpha x\rangle(\infty)$.

The last property allows one to replace the point of view from x_0 to x_α while considering a τ-thin window $E_{\alpha\beta_0}$. The advantage is that the initial segments $x_\alpha x$ of rays $[x_\alpha x\rangle$, $x \in D_\alpha$ are contained in the block X_{β_0}, and we can use the splitting $X_{\beta_0} = Y_{\beta_0} \times \mathbb{R}$ to make necessary calculations.

Step 2: choosing β'. Let h_α be the midpoint of $d_\alpha^- d_\alpha$, $J_\alpha = [h_\alpha - \rho_1, h_\alpha]$ the subsegment of $d_\alpha^- h_\alpha$ of length ρ_1 (with obvious notation), where the constant ρ_1 is provided by Lemma 2.2.5. Since $H_\alpha \to \infty$ as $\tau \to \infty$, while ρ_1 is independent of any choice made in the proposition, we can assume that $\rho_1 \leq H_\alpha/8$.

Since $\omega_u = \pi/2$, the interval $X_\alpha \cap [x_\alpha x\rangle$ lies in a horizontal slice $Y_\alpha \times \{\mathrm{pt}\}$ of $X_\alpha = Y_\alpha \times \mathbb{R}$ for any $x \in d_\alpha^- d_\alpha^+$ (here we use the fact that the ray c_α meets $E_{\alpha\beta_0}$ orthogonally). Hence, by the periodicity argument, there are points d, $d' \in J_\alpha$ such that the segments $X_\alpha \cap [x_\alpha d\rangle$, $X_\alpha \cap [x_\alpha d'\rangle$ connect $E_{\alpha\beta_0}$ with different boundary components of X_α. It immediately follows that for any $\rho > 0$ we can find a ρ-thin window $E_{\alpha\beta'}$ in the corridor $(dd') \subset J_\alpha$ (i.e. the rays $[x_\alpha \underline{\beta}\rangle$, $[x_\alpha \underline{\beta}'\rangle$ intersect $E_{\alpha\beta_0}$ in (dd'), where $\underline{\beta} = \beta'$, $\underline{\beta}' = -\beta' \in S_{\alpha\beta'}$ are the midpoints between α and $-\alpha$) with $\beta' > \alpha$ w.r.t. x_0, the pair (α, β') belongs to the class $\mathcal{W}_u$ and $E_{\alpha\beta'}$ is located in a R_α-restricted direction, where R_α depends only on α. Therefore, the properties (i), (ii) of the proposition are proved. Moreover, it follows from (3.6.2) that $\beta' \in U_{x_0, 2t}(b_\alpha)$.

Step 3: shifting vertically. The rays $[x_\alpha d_\alpha\rangle$, $[x_\alpha h_\alpha\rangle$ cut out a segment of length $\tau'\cdot H_\alpha/2|x_\alpha y_\alpha| = \tau'\cdot H/4\tau = 1/4$ on the vertical line $\mathbb{R}\times\{\mathrm{pt}\}\subset X_{\beta_0}$ projecting in the point $[x_\alpha y_\alpha\rangle(\tau')$. Then by the choice of ϵ we have $\angle_{x_\alpha}(d_\alpha, h_\alpha) = \epsilon$.

Consider the points $h'_\alpha = [x_\alpha\beta'\rangle\cap J_\alpha$ and $h''_\alpha\in d^-_\alpha h_\alpha$ with $|h_\alpha h''_\alpha| = H_\alpha/4$. By the choice of $H_\alpha > H/2$ and H we see that h''_α is below of the interval J_α and $|h'_\alpha h''_\alpha|\geq H/16$.

The infinite triangle $h''_\alpha h'_\alpha\beta'\subset Y_\alpha\times\{\mathrm{pt}\}$, $\beta'\in\partial_\infty Y_\alpha$ has the angle $\angle h''_\alpha h'_\alpha\beta'\geq \pi/2$. Hence $\angle h'_\alpha h''_\alpha\beta'\leq\epsilon^3$ by the choice of H. On the other hand, $\angle d_\alpha h'_\alpha\beta'\geq \angle_{x_\alpha}(d_\alpha, h_\alpha) = \epsilon$. Thus there exists $h_0\in(h''_\alpha h'_\alpha)$ with

$$\tan\overline{\psi}_0 = \frac{2}{\pi}\epsilon^2$$

for $\overline{\psi}_0 = \angle h'_\alpha h_0\beta'$.

Step 4: shifting horizontally. On the ray $[h_0\alpha\rangle\subset E_{\alpha\beta_0}$ we consider the segment h_0h_1 of length $r_1 = H_\alpha/8$. Then $h_0h_1\subset D_\alpha$. We denote by $h_r\in h_0h_1$ the point with $|h_0h_r| = r\cdot r_1$ for $0\leq r\leq 1$.

On the unit sphere of the tangent space $T_{h_r}X$ consider the outgoing direction $v(r)$ of the segment $x_\alpha h_r$ and its projections $v_\alpha(r)$ on Y_α and $v_{\beta_0}(r)$ on Y_{β_0} (in the obvious sense). We denote by $\phi_r = \angle(v_{\beta_0}(r),\alpha)$, $\psi_r = \angle(v_\alpha(r),\beta_0)$ and $\gamma_r = \angle(v(r),\beta_0)$. Clearly, $\gamma_r\geq\gamma_0\geq\pi/2-(s-\epsilon)\geq\epsilon$ for each $0\leq r\leq 1$.

Using the facts that $[x_\alpha h_r\rangle(\infty)\in U_{x_0,2t}(b_\alpha)$ by (3.6.2) and $s\geq\omega_u/2$, one can easily see that $\pi/2-\gamma_r$ is separated from 0 by a constant $c>0$ independently of $t\to\infty$. We can assume that $\epsilon^3/c < 2\epsilon^2/\pi$.

A simple exercise in the spherical trigonometry shows that

$$\sin\phi_r = \tan\psi_r/\tan\gamma_r,\quad 0\leq r\leq 1.$$

Since $|h_0h_1| = H_\alpha/8$, it follows from hyperbolicity of Y_{β_0} that $\phi_1\leq\epsilon^3$ by the choice of H. Thus

$$\tan\psi_1 = \sin\phi_1\tan\gamma_1\leq\epsilon^3/c < 2\epsilon^2/\pi.$$

Hence, we can find $r\in(0,1)$ with $\psi_r = \overline{\psi}_0$, i.e. $\tan\psi_r = \tan\overline{\psi}_0 = 2\epsilon^2/\pi$, because $\psi_0 = \angle h_0x_\alpha\beta_0\geq\epsilon$.

This implies that the part of the ray $[x_\alpha h_r\rangle$ which lies in X_α projects onto the ray $[h_0\beta'\rangle$ in the corresponding slice $Y_\alpha\times\{\mathrm{pt}\}$. Therefore we have $b''_\alpha = [x_\alpha h_r\rangle(\infty)\in \alpha\beta'$. For such r we have

$$\frac{2}{\pi}\phi_r\leq\sin\phi_r\leq\frac{1}{\epsilon}\tan\psi_r,$$

because $\gamma_r\geq\epsilon$. Hence, $\phi_r\leq\epsilon$.

The spherical triangle $v(r)v_{\beta_0}(r)\alpha$ has a right angle at the vertex v_{β_0}. It follows that

$$(\angle(\alpha, v(r)))^2\leq\phi_r^2+(\pi/2-\gamma_r)^2\leq\epsilon^2+(s-\epsilon)^2 < s^2$$

by the choice of ϵ.

Using the splitting $X_\alpha = Y_\alpha\times\mathbb{R}$, we obtain that $\angle(\alpha, v(r)) = \angle(\alpha, b''_\alpha)$. Therefore, we have found the point b''_α on the segment $\alpha\beta'\subset\partial_T X$, for which $\angle(\alpha, b''_\alpha) < s$ and $b''_\alpha\in U_{x_0,2t}(b_\alpha)$ by (3.6.2). Since $\angle(\alpha, b'_\alpha) = s$ and $\beta'\in U_{x_0,2t}(b_\alpha)$, an easy argument shows that $\beta' b'_\alpha\subset U_{x_0,2t}(b_\alpha)\subset U_{x_0,t}(b_\alpha)$. This gives (iii).

It remains to prove (iv). For $z \in \partial_T X$ let $\xi(z)$ be the angle between the ray $[x_\alpha z\rangle$ and a horizontal slice $Y_{\beta_0} \times \{\mathrm{pt}\}$, $\xi(z) = \pi/2 - \angle_{x_\alpha}(\beta_0, z)$. Then we have $\xi(a') \le \xi(a) = s'$, $\xi(b'_\alpha) \le \xi(b_\alpha) = s$ and $\xi(b'_\alpha) - \xi(a') \le s - s'$.

If the claim is false, then $\xi(a) - \xi(a') > \xi(b_\alpha) - \xi(b'_\alpha)$. This follows from Lemma 3.6.1, the already proved condition $b'_\alpha \in U_{x_0,2t}(b_\alpha) \subset U_{x_0,t}(b_\alpha)$ and that the horizontal projections of the considered rays emanating from x_α diverge negligibly small when compared to their vertical divergence while t is fixed, because (β_0, α)-window is τ-thin. Thus

$$s - s' \ge \xi(b'_\alpha) - \xi(a') = \xi(b_\alpha) - \xi(a) + \xi(a) - \xi(a') - (\xi(b_\alpha) - \xi(b'_\alpha)) > s - s',$$

a contradiction. This completes the proof of Proposition 3.6. □

3.7. Lemma. *Given $x_0 \in X$, a pair (α_0, β_0) of the class $\mathcal{W}_u$, $\alpha_0 < \beta_0$ w.r.t. x_0, $s \in [\omega_u/2, \omega_u)$, R, $\epsilon > 0$, there exists $\tau > 0$ such that if for $\alpha > \beta_0$ the (β_0, α)-window belongs to the class $\mathcal{W}_u$, is τ-thin and located in an R-restricted direction, and a_α, $b_\alpha \in \alpha\beta_0$, $\angle(\alpha, b_\alpha) = \angle(a_\alpha, \beta_0) = s$, then*

$$l - \epsilon \le \angle \overline{a_\alpha}(\tau)\overline{x}_0\overline{b_\alpha}(\tau) \le l,$$

where $l = 2s - \omega_u$, $\overline{x}_0\overline{a_\alpha}(\tau)\overline{b_\alpha}(\tau) \subset \mathbb{R}^2$ is the comparison triangle for $x_0 a_\alpha(\tau) b_\alpha(\tau)$ (as usual, we identify a point of $\partial_\infty X$ with the ray emanating from x_0).

Proof. The assertion is trivial, if $x_0 \in X_{\beta_0}$, i.e. $R = 0$, because of the splitting $X_{\beta_0} = Y_{\beta_0} \times \mathbb{R}$. Now assume that $x_0 \notin X_{\beta_0}$. Let x_α be the entering point in X_{β_0} of the ray $c_\alpha = [x_0 y_\alpha\rangle$, where $y_\alpha \in E_{\beta_0\alpha}$ is the point closest to x_0. It follows from the condition of the R-restricted direction and the periodicity argument that the entering points of the rays a_α, b_α into the block X_{β_0} are at the distance from x_α bounded by a constant depending only on R, $\omega_u - s$ and ρ_1 (the constant from Lemma 2.2.5) for any τ and τ-thin window $E_{\beta_0\alpha}$.

Thus the distances dist $(a_\alpha(\tau), [x_\alpha a_\alpha\rangle)$, dist $(b_\alpha(\tau), [x_\alpha b_\alpha\rangle)$ are bounded independently of τ and τ-thin window $E_{\beta_0\alpha}$. Hence, the claim. □

3.8. Proof of Theorem 3.5. Recall some standard notations, which will be used in the proof.

For $x_0 \in X$ and a pair of neighboring singular points α, $\beta \in \mathcal{F}$, $\alpha < \beta$ w.r.t. x_0 we denote $S_{\alpha\beta} = \partial_\infty E_{\alpha\beta}$. The complement $\partial_\infty X \setminus S_{\alpha\beta}$ consists of two open discs; let $\mathcal{D}_{\alpha\beta}$ be that, for which every ray $[x_0 z\rangle$, $z \in \mathcal{D}_{\alpha\beta}$ meets the flat $E_{\alpha\beta}$.

For $l \in [0, \omega_u)$ let $s = l + (\omega_u - l)/2$. Then $s \in [\omega_u/2, \omega_u)$. We fix $\epsilon > 0$, a sequence $\epsilon_i > 0$ with $\sum_i \epsilon_i \le \epsilon$, a point $x_0 \in X$, a pair (α_0, β_0) of the class $\mathcal{W}_u$, $\alpha_0 < \beta_0$ w.r.t. x_0.

Step 0. We choose two neighboring singular points $\beta_\#$ and α_0 in $\mathcal{F}$, such that the separating flat $E_{\beta_\#\alpha_0}$ covers T_u. We choose our starting point x_0 in $E_{\beta_\#\alpha_0}$ and (by slight abuse of notation) we consider the pair $(\beta_\#, \alpha_0)$ as a pair of class $\mathcal{W}_u$ with $\beta_\# < \alpha_0$ which is located in an R-restricted direction for $R = 0$. By applying the Collapsing Proposition to x_0, $(\beta_\#, \alpha_0)$, s, $R = 0$, $t_0 = 1/\epsilon_0$ we obtain a $\tau_0 \ge t_0$ such that if a window (α_0, β_0) belongs to the class $\mathcal{W}_u$, is τ_0-thin and located in an R-restricted direction, then for every $\rho > 0$ there is a singular point $\alpha' \in \mathcal{F}$, $\alpha' > \beta_0$, for which the conditions (i)–(iv) of the Collapsing Proposition are fulfilled for $t = t_0$. Clearly, such a τ_0-thin window (α_0, β_0) exists.

Let $J_0 = a_0 b_0 \subset \alpha_0\beta_0$ be the middle segment of length l, i.e. $\angle(\alpha_0, b_0) = \angle(a_0, \beta_0) = s$.

Since $\angle(a_0,b_0)=l$ and the rays a_0, b_0 (emanating from x_0) stay in X_{α_0}, we have $\angle_{x_0}(a_0(t),b_0(t))=l$ for any $t>0$. We put

$$V_0=\bigcup_{c\in J_0}U_{x_0,t_0}(c).$$

For any $z\in\mathcal{D}_{\alpha_0\beta_0}$ with $(\omega_u-l)/2\le\angle_{x_0}(z,\alpha_0)\le s$ we have $z\in V_0$, since (α_0,β_0)-window is τ_0-thin and $\tau_0\ge t_0$.

Step 1. We put $R_1=R_{\beta_0}$ (see condition (ii) of the Collapsing Proposition obtained in Step 0), $t_1=1/\epsilon_1$, and for x_0, (α_0,β_0), s, R_1, t_1 using the Collapsing Proposition we find $\tau_1\ge t_1$ such that if for $\alpha_1>\beta_0$ the (β_0,α_1)-window belongs to the class $\mathcal{W}_u$, is τ_1-thin, located in an R_1-restricted direction, and a_{α_1}, $b_{\alpha_1}\in\alpha_1\beta_0$, $\angle(\alpha_1,b_{\alpha_1})=\angle(a_{\alpha_1},\beta_0)=s$, then for any $\rho>0$ there is a singular point $\beta'\in\mathcal{F}$, $\beta'>\alpha_1$, for which the conditions (i)–(iv) of the Collapsing Proposition are fulfilled for $t=t_1$.

By the conditions (i), (ii) of the Collapsing Proposition obtained in Step 0, such α_1 does exist. Furthermore, it follows from (iii), (iv) of Step 0 that $U_{x_0,t_1}(a_{\alpha_1})\subset U_{x_0,t_0}(a_0)$, $U_{x_0,t_1}(b_{\alpha_1})\subset U_{x_0,t_0}(b_0)$ and α_1, $-\alpha_1\in U_{x_0,t_0}(a_0)$.

Step 2. We put $t_2=1/\epsilon_2$, and for x_0, (β_0,α_1), s, R_{α_1}, t_2, ϵ_2 using the Collapsing Proposition and Lemma 3.7 we find $\tau_2\ge t_2$ such that if for $\beta_1>\alpha_1$ the (α_1,β_1)-window belongs to the class $\mathcal{W}_u$, is τ_2-thin, located in the R_{α_1}-restricted direction, and a_1, $b_1\in\alpha_1\beta_1$, $\angle(\alpha_1,b_1)=\angle(a_1,\beta_1)=s$, then

$$l-\epsilon_2\le\angle\overline{a}_1(\tau_2)\overline{x}_0\overline{b}_1(\tau_2)\le l$$

and for any $\rho>0$ there is a singular point $\alpha'\in\mathcal{F}$, $\alpha'>\beta_1$, for which the conditions (i)–(iv) of the Collapsing Proposition are fulfilled for $t=t_2$.

By Step 1, such β_1 does exist, and we put $J_1=a_1b_1$,

$$V_1=\bigcup_{c\in J_1}U_{x_0,t_2}(c).$$

By the condition (iii) of the Collapsing Proposition obtained in Step 1 we have $b_1\beta_1\in U_{x_0,t_1}(b_{\alpha_1})\subset U_{x_0,t_0}(b_0)$, and by (iv) $a_1\in U_{x_0,t_0}(a_0)$. Furthermore, we can assume by Step 1 that $\mathcal{D}_{\alpha_1\beta_1}\subset V_0$.

Continuing in this way, we obtain a string $w=\alpha_0\beta_0\alpha_1\beta_1\cdots\in W$, the class of which belongs to $\mathcal{W}_u$. For the middle segments $J_i=a_ib_i\subset\alpha_i\beta_i$ of length l, where $\angle(\alpha_i,b_i)=\angle(a_i,\beta_i)=s$, we have $a_{i+1}\in U_{x_0,t_{2i}}(a_i)$, $b_{i+1}\in U_{x_0,t_{2i}}(b_i)$, $t_i=1/\epsilon_i$,

$$l-\epsilon_{2i}\le\angle\overline{a}_i(\tau_{2i})\overline{x}_0\overline{b}_i(\tau_{2i})\le l$$

for $\tau_i\ge t_i$. Thus $a_i\to a$, $b_i\to b$ as $i\to\infty$ in the standard topology, and $\angle(a,b)=l$. Furthermore, $ab\subset\mathcal{I}_w$ by the construction.

For the neighborhoods $V_i=\bigcup_{c\in J_i}U_{x_0,t_{2i}}(c)$ we have (very important point)

$$\mathcal{D}_{\alpha_i\beta_i}\subset V_{i-1}.$$

This implies $\mathcal{I}_w\subset\bigcap_{i\ge0}V_i$. Therefore, it remains to show that $\bigcap_{i\ge0}V_i=ab$.

Let $z\in\bigcap_{i\ge0}V_i$. Then for each $i\ge0$ there is $z_i\in a_ib_i$ with $z\in U_{x_0,t_{2i}}(z_i)$ and, hence, $z_i\in U_{x_0,t_{2i}}(z)$. Thus $z_i\to z$ in the standard topology. Now we use the following obvious fact.

Assume that for a sequence of segments $a_ib_i\subset\partial_\infty X$ of length $\angle(a_i,b_i)\le\omega_u<\pi$ their ends a_i, b_i converge in the standard topology to a, b and $\lim\angle(a_i,b_i)=\angle(a,b)$. Then their midpoints $c_i\in a_ib_i$ converge in the standard topology to the midpoint $c\in ab$. More generally, the segments a_ib_i (parametrized on $[0,1]$) converge in the standard topology pointwise to the segment ab. This easily follows from the lower semi-continuity of the Tits distance with respect to the standard topology and that $\partial_T X$ contains no bigon of length $< 2\pi$.

Therefore, $z \in ab$, and the primitive component $\mathcal{I}_w = ab$ has the length l. For $l > 0$ the component $\mathcal{I}_w$ is nonstandard. If $l = 0$, then the freedom of choice given by the construction above allows us to produce uncountably many components $\mathcal{I}_w$ with $L(w) = 0$. One of them is necessarily nonstandard, because there are only countable many standard components. This completes the proof of Theorem 3.5.

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St. Petersburg Branch, Steklov Mathematical Institute, Fontanka 27, 191011 St. Petersburg, Russia

E-mail address: buyalo@pdmi.ras.ru

Institut für Mathematik, Universität Zürich, Winterthurer Str. 190, CH-8057 Zürich, Switzerland

E-mail address: vschroed@math.unizh.ch