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PROPERTIES OF ANICK'S SPACES

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ABSTRACT. We prove three useful properties of Anick's space $T^{2n-1}(p^r)$. First, at odd primes a map from $P^{2n}(p^r)$ into a homotopy commutative, homotopy associative H-space X can be extended to a unique H-map from $T^{2n-1}(p^r)$ into X. Second, at primes larger than 3, $T^{2n-1}(p^r)$ is itself homotopy commutative and homotopy associative. And third, the first two properties combine to show that the order of the identity map on $T^{2n-1}(p^r)$ is p^r.

1. INTRODUCTION

Let p be a prime. Throughout, all spaces are pointed, connected, topological spaces with the homotopy types of finite type CW-complexes. All spaces and maps have been localized at p. Let $E^2 : S^{2n-1} \longrightarrow \Omega^2 S^{2n+1}$ be the double adjoint of the identity map on S^{2n+1}. When $p \geq 5$, $r \geq 1$, and $n \geq 1$, Anick [A] constructed a homotopy fibration sequence of H-spaces

$$\Omega^2 S^{2n+1} \phi_r S^{2n-1} T^{2n-1}(p^r) \Omega S^{2n+1}$$

with the following two properties: $\phi_r \circ E^2 \simeq p^r$ and $E^2 \circ \phi_r \simeq \Omega^2 p^r$. In [AG] each map in this homotopy fibration sequence was shown to be an H-map. A new construction of this homotopy fibration sequence was given in [Th], which is also valid for the prime 3. The purpose of this paper is to prove two strong properties of the space $T^{2n-1}(p^r)$ conjectured by Anick and Gray [AG].

Theorem 1.1 (To be proven as Theorem 5.3)**.** *Let X be a homotopy commutative, homotopy associative H-space. Let $P^{2n}(p^r) \longrightarrow X$ be given. For $p \geq 3$, there exists an extension to an H-map $T^{2n-1}(p^r) \longrightarrow X$ which is unique up to homotopy.*

Theorem 1.2 (To be proven as Theorem 6.6)**.** *For $p \geq 5$, the H-space $T^{2n-1}(p^r)$ is homotopy commutative and homotopy associative.*

Theorem 1.1 is a universal property for the space $T^{2n-1}(p^r)$. It says that an H-map from $T^{2n-1}(p^r)$ to a homotopy commutative and homotopy associative space X is completely determined by its restriction to the bottom Moore space. This property has been used by Neisendorfer [N3] to show that $\Omega T^{2n-1}(p^r)$ is a retract of $\Omega^2 P^{2n+1}(p^r)$ provided $r \geq 2$ and $p \geq 5$. Gray [Gr2] anticipated Theorems 1.1 and 1.2 to carry through his unstable development of v_1-periodic homotopy theory.

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Theorems 1.1 and 1.2 combine to give an exponent result. To make this clear we give two definitions. A space X has *homotopy exponent* p^r if p^r is the least power of p which annihilates the p-primary torsion in $\pi_*(X)$. An H-space Y has *H-space exponent* p^r if p^r is the order of the identity map. Note that Y having H-space exponent p^r implies Y has homotopy exponent p^r, but the reverse need not be true—take for example S^3.

Now, Theorem 1.2 allows us to use $T^{2n-1}(p^r)$ as the range in Theorem 1.1 as well as the domain. Doing so proves an important application which Anick and Gray had in mind when they formulated their conjectures.

Theorem 1.3 (To be proven as Theorem 8.1). *For $p \geq 5$ and $r \geq 1$, $T^{2n-1}(p^r)$ has H-space exponent p^r.*

As for Theorems 1.2 and 1.3 when $p = 3$, Proposition 7.1 shows that if a certain homotopy class of S^{2n-1} is nontrivial and not divisible by 3^r, then $T^{2p-1}(3^r)$ is not homotopy associative. Examples are given where this is the case. On the other hand, Proposition 8.4 shows that $T^5(3)$ is both homotopy commutative and homotopy associative. If $T^{2n-1}(3^r)$ is homotopy commutative and homotopy associative, then the proof of Theorem 1.3 applies and $T^{2n-1}(3^r)$ has H-space exponent 3^r. Otherwise, a discussion of the known homotopy exponent results for $T^{2n-1}(3^r)$ is contained at the end of Section 7.

Finally, we mention that one of the tools used to prove Theorems 1.1 and 1.2 may have application elsewhere. In Section 4 we discuss homotopy action maps in general and homotopy fibration connecting maps in particular. Proposition 4.8 gives a condition which determines when a homotopy fibration connecting map satisfies the stronger property of being an H-map. Proposition 4.12 goes a bit further and gives conditions under which a homotopy fiber has an H-structure which is both homotopy commutative and homotopy associative.

This paper is organized as follows. Section 2 reviews the information about Anick's spaces we will require. Section 3 records some facts about H-spaces and co-H spaces. Section 4 is a general discussion about homotopy action maps and gives a criterion for when certain homotopy fibers are homotopy commutative, homotopy associative H-spaces. Section 5 uses a long induction to prove Theorem 1.1. Section 6 proves Theorem 1.2, while Section 7 gives a criterion for when $T^{2n-1}(3^r)$ is homotopy commutative and homotopy associative. Finally, Section 8 gives applications of Theorems 1.1 and 1.2; in particular, Theorem 1.3 is proven.

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Preliminary Global Definitions and Notation

For a co-H space A, let $A \xrightarrow{\underline{p}^r} A$ be the degree p^r map given by the co-H structure of A. For an H-space X, let $X \xrightarrow{p^r} X$ be the p^r-power map. In the case of an odd sphere localized at a prime $p \geq 3$, the maps $\underline{p}^r$ and p^r are homotopic and are commonly denoted by $S^{2n+1} \xrightarrow{p^r} S^{2n+1}$. In this case, let $S^{2n+1}\{p^r\}$ be the homotopy fiber of the degree p^r map on S^{2n+1}.

For $m \geq 2$ and $r \geq 1$, the *mod p^r Moore space* $P^m(p^r)$ is the cofiber of the degree p^r map on S^{m-1}. The m^{th} mod p^r homotopy set of a space X is the set of based homotopy classes $\pi_m(X; \mathbf{Z}/p^r\mathbf{Z}) = [P^m(p^r), X]$. An element $f \in \pi_m(X; \mathbf{Z}/p^r\mathbf{Z})$ has *order* n if n is the least positive integer such that $nf = 0$. We write $\mathcal{W}_r^{r+k}$ for the collection of spaces with the homotopy type of a finite type wedge of mod p^t Moore spaces, $r \leq t \leq r+k$.

For $j \geq 1$, the cyclic group of order p^j will be written as $\mathbf{Z}/p^j\mathbf{Z}$. The integers localized at the prime p will be written as $\mathbf{Z}_{(p)}$. Unless otherwise indicated, the ring of coefficients in homology will be $\mathbf{Z}/p\mathbf{Z}$, and $H_*(X;\mathbf{Z}/p\mathbf{Z})$ will be written as $H_*(X)$. For any coefficient ring R, the reduced homology of X with coefficients in R will be written as $\tilde{H}_*(X;R)$. We will write $\Sigma^{-1}\tilde{H}_*(X;R)$ for the graded R-module whose suspension is $\tilde{H}_*(X;R)$.

Let R be a commutative ring with 2 as a unit. A differential graded Lie algebra (dgL) L is a positively graded R-module with a bilinear pairing $[\ ,\ ] : L_n \times L_m \to L_{n+m}$ and a map $d : L \longrightarrow L$ of degree -1 such that: (i) $[\ ,\ ]$ satisfies graded anti-symmetry, Jacobi, and triple product identities, and (ii) d is a differential on L which is a derivation with respect to $[\ ,\ ]$.

The pair $[x,y]$ in L is also denoted by $ad(x)(y)$. For L included into an associative algebra A, we have $[x,y] = xy-(-1)^{|x||y|}yx$ in A. For $x, y \in L$, define $ad^0(x)(y) = y$, and for $k \geq 1$, inductively define $ad^k(x)(y) = ad(x)(ad^{k-1}(x)(y))$.

Let $\mathcal{S}$ be a positively graded set. Let $L\langle\mathcal{S}\rangle$ be the free graded Lie algebra over R generated by $\mathcal{S}$. Let $L_{ab}\langle\mathcal{S}\rangle$ be the free graded abelian Lie algebra over R generated by $\mathcal{S}$. The universal enveloping algebra over R of $L\langle\mathcal{S}\rangle$ is denoted by $UL\langle\mathcal{S}\rangle$. It is isomorphic over R to the free tensor algebra $T(\mathcal{S})$ generated by $\mathcal{S}$. Let $S(\mathcal{S})$ be the free commutative algebra over R generated by $\mathcal{S}$.

2. Review of Anick's Spaces

This section reviews some constructions in [Th]. Let p be an odd prime and $0 \leq k \leq \infty$. There exists an atomic co-H space $G_k^{2n}(p^r) = G_k$ described as follows. First, $G_0 = P^{2n+1}(p^r)$. For $k \geq 1$, G_k is defined as a homotopy cofiber

$$P^{2np^k}(p^{r+k}) \xrightarrow{\underline{p}^{r+k-1}\alpha} G_{k-1} \longrightarrow G_k$$

of a co-H map $\underline{p}^{r+k-1}\alpha$, where α is a homotopy class of order p^{r+k}. Let G_∞ be the homotopy colimit of $\{G_k\}_{k\geq 0}$.

Before listing more properties of G_k, we establish some notation and make two definitions. Let $M_k = \bigvee_{i=0}^{k} P^{2np^i}(p^{r+i})$. Note that $H_*(G_k) \cong H_*(\Sigma M_k)$, but the two spaces are not homotopy equivalent. However, as indicated by Theorem 2.1, they do tend to share many of the same properties involving suspensions and smashes. This analogy is useful in providing intuition for G_k.

Definition. The *universal Whitehead product* of a space X is the composite

$$\Sigma\Omega X \wedge \Omega X X \vee X \nabla X,$$

where the left-hand map is the homotopy fiber map determined by the inclusion of the wedge into the product and ∇ is the folding map.

Definition. Suppose there are spaces and maps $X \xleftarrow{f} Y \xrightarrow{g} Z$. Let $A \xrightarrow{h} X$ be given. If h lifts through f to a map $A \xrightarrow{h'} Y$, then the composite gh' is said to be an *indirect lift* of h from X to Z.

Theorem 2.1. *For $0 \leq k \leq \infty$, the atomic co-H space G_k satisfies the following properties:*

(a) *There are isomorphisms of Hopf algebras and homology Bockstein spectral sequences*

$$H_*(\Omega G_k) \cong T(u_0, v_0, \dots, u_k, v_k) \cong H_*(\Omega\Sigma M_k),$$

where $|u_i| = 2np^i - 1$, $|v_i| = 2np^i$, and $\beta^{r+i}(v_i) = u_i$.

(b) $\Sigma\Omega G_k/G_k \in \mathcal{W}_r^{r+k}$ *and* $\Sigma^2\Omega G_k \in \mathcal{W}_r^{r+k}$.

(c) $\Sigma\Omega G_k \wedge \Omega G_k \in \mathcal{W}_r^{r+k}$

(d) *There is an indirect lift of the universal Whitehead product of ΣM_k to G_k which factors through the universal Whitehead product of G_k. It determines a homotopy equivalence*

$$\Sigma\Omega\Sigma M_k \wedge \Omega\Sigma M_k \simeq \Sigma\Omega G_k \wedge \Omega G_k.$$

In particular, the indirect lifts can be chosen so that: (i) when $p \geq 5$, the dgL identities satisfied by the p-primary Whitehead products on ΣM_k are also satisfied by their indirect lifts to G_k, and (ii) when $p = 3$, the anti-symmetry and Jacobi identities satisfied by the subcollection of p-primary Whitehead products on ΣM_k of orders p^t, $t \geq 2$, are also satisfied by their indirect lifts to G_k.

Remark. The isomorphism in Thoerem 2.1 (a) is not realized by a map between spaces. If it were, then the map would be a homotopy equivalence, say $\Omega G_k \xrightarrow{\simeq} \Omega\Sigma M_k$ (for a map in the other direction the same argument applies). Using the co-H structure on G_k, we would then have a composite $f : G_k \longrightarrow \Sigma\Omega G_k \xrightarrow{\simeq} \Sigma\Omega\Sigma M_k \xrightarrow{ev} \Sigma M_k$. The increasing orders of the Bocksteins of the generators v_i in Theorem 2.1 (a) then imply f_* must be an isomorphism, and so f is a homotopy equivalence. But this cannot be the case, since the attaching maps constructing G_k as a CW-complex are nontrivial.

Certain important maps arise as byproducts of the definition of G_k as a homotopy cofiber. For each $k \geq 1$, there is a homotopy pushout diagram

$$\begin{array}{ccccc}
P^{2np^k}(p^{r+k}) & \xrightarrow{\underline{p}^{r+k-1}} & P^{2np^k}(p^{r+k}) & \longrightarrow & P^{2np^k}(p^{r+k-1}) \vee P^{2np^k+1}(p^{r+k-1}) \\
\| & & \downarrow{\alpha} & & \downarrow{a_k \vee c_k} \\
P^{2np^k}(p^{r+k}) & \xrightarrow{\underline{p}^{r+k-1}\alpha} & G_{k-1} & \longrightarrow & G_k.
\end{array}$$

which defines a_k and c_k. Including G_i into G_k gives a collection of maps $\{a_i, c_i\}_{i=1}^k$ to G_k.

As we will require them later, we record the homology images of the adjoints of a_k and c_k. In order to describe them we need some notation. For $i \geq 1$ and $0 \leq j \leq k$, define elements $\tau_j^i, \sigma_j^i \in H_*(\Omega G_k)$ by setting $\tau_j^i(v_j) = ad^{p^i-1}(v_j)(u_j)$ and $\sigma_j^i(v_j) = \frac{1}{p} \cdot \beta^{r+j}(\tau_j^i(v_j))$. The latter equation makes sense by [CMN1, 4.4]. Let b and a respectively denote the degree $2np^k - 1$ and $2np^k - 2$ generators of $H_*(P^{2np^k-1}(p^{r+k-1}))$, and d and c respectively denote the degree $2np^k$ and $2np^k-1$ generators of $H_*(P^{2np^k}(p^{r+k-1}))$. Denote the adjoint of a map by placing a tilde over its name.

Lemma 2.2. *In homology, the map* $P^{2np^k-1}(p^{r+k-1}) \vee P^{2np^k}(p^{r+k-1}) \xrightarrow{\tilde{a}_k \vee \tilde{c}_k} \Omega G_k$ *satisfies:*

(a) $(\tilde{c}_k)_*(d) = v_{k-1}^p - v_k + \zeta_1$,
(b) $(\tilde{c}_k)_*(c) = \tau_{k-1}^1$,
(c) $(\tilde{a}_k)_*(b) = w_{k-1}^1 + u_k + \zeta_2$,
(d) $(\tilde{a}_k)_*(a) = \sigma_{k-1}^1$,

where $\zeta_1, \zeta_2 \in H_*(\Omega G_{k-1})$ *are in the kernel of the Bockstein* β^{r+k-1}.

Remarks. (1) Let $q_{k-1} : G_{k-1} \longrightarrow P^{2np^{k-1}+1}(p^{r+k-1})$ be the pinch map. Lemma 2.2, as it appears in [Th, 6.5] has $(\tilde{c}_k)_*(c) = \tau_{k-1}^1 + \eta_1$ and $(\tilde{a}_k)_*(a) = \sigma_{k-1}^1 + \eta_2$, where $\eta_1, \eta_2 \in H_*(\Omega G_{k-1})$ are in the kernel of $(\Omega q_{k-1})_*$. But by Theorem 2.1 (a), Ωq_{k-1} determines an isomorphism in the (E_{r+k-1})-stage of the Bockstein spectral sequence. Thus the kernel of $(\Omega q_{k-1})_*$ has no torsion of order p^{r+k-1} or higher. But if η_1, η_2 are nonzero then they must equal $\beta^{r+k-1}(\zeta_1)$ and $\beta^{r+k-1}(\zeta_2)$ respectively, a contradiction.

(2) Note that the mod-p Hurewicz image in Lemma 2.2 (a) is a p^{th}-power plus some additional terms. The additional terms avoid contradiction with Hopf invariant one mod-p, while the presence of the p^{th}-power indicates that something interesting is going on with the map c_k. One way in which this is seen is in terms of the homotopy decomposition in Theorem 2.3, where the elements v_{k-1}^p and v_k in $H_*(\Omega G_k)$ become identified in $H_*(T_k)$ via the map ∂_k.

We next describe a homotopy decomposition of ΩG_k in terms of spaces T_k and R_k which are defined by the properties in Theorem 2.3. For $k = 0$, let $A_0 = C_0 = *$. For $k \geq 1$, let $A_k = \bigvee_{i=1}^k P^{2np^i}(p^{r+i-1})$ and $C_k = \bigvee_{i=1}^k P^{2np^i+1}(P^{r+i-1})$. Note the $k = 0$ case of Theorem 2.3 is that of the Moore space $G_0 = P^{2n+1}(p^r)$, and these statements were proven, using different notation, in [CMN3]. Also, the space $T^{2n-1}(p^r)$ described in the introduction is defined to be the space T_∞ in the $k = \infty$ case of Theorem 2.3.

Theorem 2.3. *For* $0 \leq k \leq \infty$, *there is a homotopy fibration sequence*

$$\Omega G_k \xrightarrow{\partial_k} T_k \xrightarrow{*} R_k \xrightarrow{\iota_k} G_k$$

satisfying the following properties:

(a) $\Omega G_k \simeq T_k \times \Omega R_k$.
(b) *As coalgebras,* $H_*(T_k) \cong H_*(S^{2n-1} \times \Omega S^{2n+1} \times \prod_{j=1}^\infty S^{2np^{j+k}-1}\{p^{r+k+1}\})$.
(c) $R_k \simeq A_k \vee C_k \vee B_k \in \mathcal{W}_r^{r+k}$, *where* ι_k *restricted to* $A_k \vee C_k$ *is* $\bigvee_{i=1}^k (a_i \vee c_i)$ *and* ι_k *restricted to* B_k *factors through the universal Whitehead product of* G_k.

3. H-Spaces and Co-H Spaces

This section records several standard properties of H-spaces and co-H spaces which will be needed subsequently.

Let X^k be the k-fold product of X. The James construction on X is the direct limit $J(X) = \varinjlim J_k(X)$, where $J_k(X) = X^k / \sim$ with

$$(x_1, \ldots, x_{j-1}, *, x_{j+1}, \ldots, x_k) \sim (x_1, \ldots, x_{j-1}, x_{j+1}, *, \ldots, x_k).$$

The space $J(X)$ is a topological monoid with multiplication defined by concatenation of sequences and $*$ as the identity.

Theorem 3.1 (James)**.** *There is a natural homotopy equivalence $J(X) \simeq \Omega\Sigma X$ in which the adjoint $j : X \longrightarrow \Omega\Sigma X$ of the identity map on ΣX is naturally homotopic to the inclusion $X = J_1(X) \longrightarrow J(X)$.*

Proof. See [J]. □

A common use of James spaces is to extend certain maps.

Lemma 3.2. *Let $X \xrightarrow{f} Y$ be a map of spaces where Y is a homotopy associative H-space. Then there is an extension of f to an H-map $\Omega\Sigma X \xrightarrow{\bar{f}} Y$ which is unique up to homotopy.*

Proof. See [W, VII, 2.5]. □

The uniqueness assertion of Lemma 3.2 is powerful. It is often used to show two H-maps $\Omega\Sigma X \longrightarrow Y$ into a homotopy associative H-space Y are homotopic by comparing their restrictions to X. One example is the following lemma, which shows that retractions with respect to homotopy associative spaces satisfy a naturality property.

Lemma 3.3. *Let $X \xrightarrow{f} Y$ be an H-map between homotopy associative spaces. Then there is a homotopy commuting diagram of H-spaces and H-maps*

$$\begin{array}{ccc} \Omega\Sigma X & \xrightarrow{\Omega\Sigma f} & \Omega\Sigma Y \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} \\ X & \xrightarrow{f} & Y. \end{array}$$

where both maps labelled r are left homotopy inverses for the canonical inclusions.

Proof. Applying Lemma 3.2 to the identity map $X \xrightarrow{1_X} X$ gives r. Similarly for Y. The two composites $f \circ r$ and $r \circ \Omega\Sigma f$ are now H-maps extending f, so the uniqueness assertion of Lemma 3.2 shows they are homotopic. □

Dually, there is a homotopy commutative diagram for co-H spaces which is analogous to that for H-spaces in Lemma 3.3. The dual is in fact a little stronger, as the H-spaces in Lemma 3.3 need to be homotopy associative while the co-H spaces in the following lemma do not need to be homotopy coassociative.

Recall that A is a co-H space if and only if there is a map $s : A \longrightarrow \Sigma\Omega A$ which is a right homotopy inverse of the standard evaluation map $ev : \Sigma\Omega A \longrightarrow A$. By [Ga], there is a bijection between the homotopy classes of maps s as above and the co-H structures of A.

Lemma 3.4. *Let $A \xrightarrow{f} B$ be a co-H map between co-H spaces. Then there is a homotopy commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle t} \\ \Sigma\Omega A & \xrightarrow{\Sigma\Omega f} & \Sigma\Omega B \end{array}$$

where s and t correspond to the co-H structures on X and Y respectively.

Proof. See [Gr2, 3.6]. □

One more property of H-spaces we need to investigate is how an H-space arises as a homotopy fiber. Recall that for spaces X and Y, the *join* of X and Y is the space $X * Y = X \times I \times Y/\!\sim$, where $(x,0,y) \sim (x,0,y')$, $(x,0,y) \sim (x',0,y)$. There is a natural homotopy equivalence $X * Y \simeq \Sigma X \wedge Y$. Now suppose X is an H-space with multiplication m. The *Hopf construction* on (X,m) is the map $X * X \xrightarrow{u} \Sigma X$ defined by $(x,t,y) \longrightarrow (t, m(x,y))$.

Lemma 3.5. *Let X be an H-space with multiplication m. Then the Hopf construction on (X,m) gives a homotopy fibration sequence*

$$\Omega\Sigma X \xrightarrow{r} X \xrightarrow{*} X * X \xrightarrow{u} \Sigma X,$$

where r is a left homotopy inverse to the canonical inclusion $X \xrightarrow{j} \Omega\Sigma X$. In particular, if (X,m) is homotopy associative then r can be chosen to be an H-map.

Proof. The existence of a homotopy fibration sequence for some retraction r is proven, for example, in [St, 1.10]. The H-map assertion is not difficult to prove; for example, see [Th, 2.6]. □

An application which uses several of the preceeding lemmas is the following. Let A be a co-H space. Then there is a map $s : A \longrightarrow \Sigma\Omega A$ which is a right homotopy inverse for the standard evaluation map. Let X be a homotopy associative H-space. Then by Lemma 3.5 there is an H-map $r : \Omega\Sigma X \longrightarrow X$.

Lemma 3.6. *Let A be a co-H space and X a homotopy associative H-space. Let $f : \Omega A \longrightarrow X$ be an H-map. Then with e and r defined as above, we have $f \simeq r \circ \Omega e$.*

Proof. Consider the diagram

$$\begin{array}{ccccc} \Omega A & \xrightarrow{\Omega s} & \Omega\Sigma\Omega A & \xrightarrow{\Omega\Sigma f} & \Omega\Sigma X \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \Omega(ev)} & & \downarrow{\scriptstyle r} \\ & & \Omega A & \xrightarrow{f} & X. \end{array}$$

Since f is an H-map between homotopy associative H-spaces, the right hand square homotopy commutes by Lemma 3.3. The left hand triangle homotopy commutes by the definition of s. Now simply observe that by definition the top row is Ωe. □

4. Homotopy Action Maps

This goal of this section is to prove Propositions 4.1 and 4.12. The first deals with the factorization of certain maps and will be used in section 5. The second gives conditions under which certain H-spaces have multiplications which are homotopy commutative and homotopy associative; it will be used in section 6. While the results in this section are probably known (or at least are easy to prove), to the author's knowledge they do not appear in the literature.

Definition. Let $t : Y \longrightarrow X$, where Y is an H-space. A *left homotopy action* on the triple (t, Y, X) is a map $\theta_L : Y \times X \longrightarrow X$ which fits into a homotopy commutative diagram

$$\begin{array}{ccc} Y \times Y & \xrightarrow{\mu} & Y \\ \downarrow{\scriptstyle Y\times t} & & \downarrow{\scriptstyle t} \\ Y \times X & \xrightarrow{\theta_L} & X. \end{array}$$

There are two prototypical examples of left homotopy actions. First, suppose X is an H-space with multiplication m and t is an H-map. Then the composite

$$\theta_L : Y \times X \xrightarrow{t \times X} X \times X \xrightarrow{m} X$$

defines a left homotopy action. Second, suppose $\Omega B \xrightarrow{\partial} F \longrightarrow E \longrightarrow B$ is a homotopy fibration sequence. It is classical that $(\partial, \Omega B, F)$ has a left homotopy action.

Our first result uses left homotopy actions to factor certain maps.

Proposition 4.1. *Let $Z \xrightarrow{u} Y \xrightarrow{t} X$ be a homotopy fibration where Y is an H-space, $\theta_L : Y \times X \longrightarrow X$ is a left homotopy action, and t has a right homotopy inverse s. Suppose R is an H-space and $Y \xrightarrow{f} R$ is an H-map such that fu is null homotopic. Let $\bar{f} : X \xrightarrow{s} Y \xrightarrow{f} R$. Then f factors as $Y \xrightarrow{t} X \xrightarrow{\bar{f}} R$, $\bar{f}$ is the unique map such that $f \simeq \bar{f}t$, and if X has the H-space structure determined by the given retraction off Y then $\bar{f}$ is an H-map.*

Proof. Let v be a left homotopy inverse for u. Consider the homotopy commutative diagram

$$\begin{array}{ccccccccc}
Y & \xrightarrow{\Delta} & Y \times Y & \xrightarrow{t \times v} & X \times Z & \xrightarrow{s \times u} & Y \times Y & \xrightarrow{\mu} & Y \\
& \searrow^{Y \times *} & \downarrow & & \downarrow & & \downarrow{\scriptstyle Y \times t} & & \downarrow{\scriptstyle t} \\
& & Y \times * & \xrightarrow{t \times *} & X \times * & \xrightarrow{s \times *} & Y \times X & \xrightarrow{\theta_L} & X.
\end{array}$$

Since $\mu(s \times u)$ and $(t \times v)\Delta$ are both homotopy equivalences, the top row is a homotopy equivalence e, possibly not the identity. After identifying $X \times *$ with X, note that the homotopy action implies $\theta(s \times *) \simeq 1_X$. The diagram therefore shows that $t \simeq te$, or equivalently, $t \simeq te^{-1}$. From the homotopy commutative diagram

$$\begin{array}{ccccccccc}
Y & \xrightarrow{\Delta} & Y \times Y & \xrightarrow{t \times v} & X \times Z & \xrightarrow{s \times u} & Y \times Y & \xrightarrow{\mu} & Y \\
& \searrow^{Y \times *} & \downarrow & & \downarrow & & \downarrow{\scriptstyle f \times f} & & \downarrow{\scriptstyle f} \\
& & Y \times * & \xrightarrow{t \times *} & X \times * & \xrightarrow{fs \times *} & R \times R & \xrightarrow{\mu} & R
\end{array}$$

we obtain $fe \simeq fst$. Applying e^{-1} to both sides then gives $f \simeq fste^{-1} \simeq fst$.

Set $\bar{f} = fs$. To show uniqueness, suppose $\tilde{f} : X \longrightarrow R$ is a second map such that $f \simeq \tilde{f}t$. Then $\bar{f}t \simeq \tilde{f}t$ implies $\bar{f}ts \simeq \tilde{f}ts$, which gives $\bar{f} \simeq \tilde{f}$. Finally, if X is given the H-structure determined by the retraction off Y, then the fact that $\bar{f}$ is an H-map follows from f being an H-map and the homotopy $f \simeq \bar{f}t$. □

Remark. If the homotopy action hypothesis of Proposition 4.1 is omitted, then there is a factorization of f as $Y \xrightarrow{t'} X \xrightarrow{\bar{f}} R$ for some map t' which is a left homotopy inverse for s (see, for example, [Gr2, 3.5]), but there is no guarantee that t' is homotopic to t.

Besides a left homotopy action we could equally well consider a right homotopy action. That is, suppose there is a map $t : Y \longrightarrow X$ where Y is an H-space. A *right homotopy action* is a map $\theta_R : X \times Y \longrightarrow X$ which fits in a homotopy commutative diagram

$$\begin{CD} Y \times Y @>\mu>> Y \\ @V{Y\times t}VV @VV{t}V \\ X \times Y @>\theta_R>> X. \end{CD}$$

Notation. Let $T_{A,B} : A \times B \longrightarrow B \times A$ be the map which interchanges the factors. We write simply T when the spaces A and B are understood.

As in the case of a left homotopy action, if $Y \xrightarrow{t} X$ is an H-map between H-spaces, then the composite

$$\theta_R : X \times Y \xrightarrow{X\times t} X \times X \xrightarrow{m} X$$

defines a right homotopy action. If X is homotopy commutative, then the two canonical homotopy actions are related by the formula $\theta_R = \theta_L \circ T_{Y,X}$. We would like an analogue of this formula in the case of a homotopy fibration sequence $\Omega B \xrightarrow{\partial} F \longrightarrow E \longrightarrow B$. We need, then, an analogue of the commutativity condition in the H-space case. This is provided by the following lemma.

Notation. For the rest of this section, μ will denote the loop space multiplication on ΩB. If in the homotopy fibration sequence $\Omega B \xrightarrow{\partial} F \longrightarrow E \longrightarrow B$ the homotopy fibration connecting map ∂ has a right homotopy inverse, then the multiplication determined on F by this retraction off ΩB will be denoted by μ_F.

Lemma 4.2. *Let $\Omega B \xrightarrow{\partial} F \longrightarrow E \longrightarrow B$ be a homotopy fibration sequence. Suppose $\partial\mu \circ T_{\Omega B,\Omega B} \simeq \partial\mu : \Omega B \times \Omega B \longrightarrow F$. Define $\theta_R = \theta_L \circ T_{\Omega B,F}$. Then θ_R is a right homotopy action.*

Proof. We need to show that $\theta_R(\partial \times 1_{\Omega B}) \simeq \partial\mu$. But

$$\begin{aligned} \theta_R(\partial \times 1_{\Omega B}) &= \theta_L \circ T_{\Omega B,F} \circ (\partial \times 1_{\Omega B}) \\ &\simeq \theta_L \circ (1_{\Omega B} \times \partial) \circ T_{\Omega B,\Omega B} \\ &\simeq \partial\mu \circ T_{\Omega B,\Omega B} \simeq \partial\mu. \end{aligned}$$

This proves the lemma. □

The commutativity condition of Lemma 4.2 begins to be more suggestive when ∂ has a right homotopy inverse so F has an H-space structure determined by the retraction off ΩB. We will show in Lemma 4.5 that under these circumstances the multiplication μ_F on F is both homotopy commutative and homotopy associative. We first require two preparatory lemmas.

Lemma 4.3. *Let $\Omega B \xrightarrow{\partial} F \longrightarrow E \longrightarrow B$ be a homotopy fibration sequence where ∂ has a right homotopy inverse s. Then the multiplication μ_F on F is homotopic to $\theta_L(s \times 1_F)$.*

Proof. The homotopy commutative diagram

$$\begin{array}{ccccc} F \times F & \xrightarrow{s\times s} & \Omega B \times \Omega B & \xrightarrow{\mu} & \Omega B \\ & \searrow^{s\times F} & \downarrow{\scriptstyle \Omega B\times\partial} & & \downarrow{\scriptstyle \partial} \\ & & \Omega B \times F & \xrightarrow{\theta_L} & F \end{array}$$

proves the lemma. □

Lemma 4.4. *Let A be an abelian monoid with multiplication μ. Then A is associative if $\mu(1_A \times \mu) = \mu(1_A \times \mu) \circ (T_{A,A} \times 1_A) \circ (1_A \times T_{A,A})$.*

Proof. Trivial. □

Proposition 4.5. *Let $\Omega B \xrightarrow{\partial} F \longrightarrow E \longrightarrow B$ be a homotopy fibration sequence, where ∂ has a right homotopy inverse s. Let F have the H-structure determined by this retraction off ΩB. Suppose $\partial\mu \simeq \partial\mu \circ T_{\Omega B,\Omega B} : \Omega B \times \Omega B \longrightarrow F$. Then F is homotopy commutative and homotopy associative.*

Proof. Homotopy commutativity follows immediately from preceding each side of the homotopy $\partial\mu \simeq \partial\mu \circ T_{\Omega B,\Omega B}$ by $s \times s$.

To prove Y is homotopy associative it now suffices by Lemma 4.4 (applied in the homotopy category) to show that

$$\mu_F(1_F \times \mu_F) \simeq \mu_F(1_F \times \mu_F) \circ (T_{F,F} \times 1_F) \circ (1_F \times T_{F,F}).$$

The homotopy commutative diagram

$$\begin{array}{ccccccc}
F \times F \times F & \xrightarrow{s\times s\times s} & \Omega B \times \Omega B \times \Omega B & \xrightarrow{\Omega B\times\mu} & \Omega B \times \Omega B & \xrightarrow{\mu} & \Omega B \\
\downarrow{\scriptstyle s\times\mu_F} & & & & \downarrow{\scriptstyle \Omega B\times\partial} & & \downarrow{\scriptstyle \partial} \\
\Omega B \times F & = & = & = & \Omega B \times F & \xrightarrow{\theta_L} & F.
\end{array}$$

shows that $\theta_L(s \times \mu_F) \simeq \partial\mu \circ (1_{\Omega B} \times \mu) \circ (s \times s \times s)$. But by Lemma 4.3,

$$\mu_F \circ (1_F \times \mu_F) \simeq \theta_L(s \times 1_F) \circ (1_F \times \mu_F) \simeq \theta_L(s \times \mu_F).$$

Hence $\mu_F(1_F \times \mu_F) \simeq \partial\mu \circ (1_{\Omega B} \times \mu) \circ (s \times s \times s)$. If this homotopy is preceded by $(T_{F,F} \times 1_F) \circ (1_F \times T_{F,F})$, then we have $\mu_F \circ (1_F \times \mu_F) \circ (T_{F,F} \times 1_F) \circ (1_F \times T_{F,F})$ and $\partial\mu \circ (1_{\Omega B} \times \mu) \circ (T_{\Omega B,\Omega B} \times 1_{\Omega B}) \circ (1_{\Omega B} \times T_{\Omega B,\Omega B}) \circ (s \times s \times s)$. Thus if there were a homotopy $\partial\mu \circ (1_{\Omega B} \times \mu) \simeq \partial\mu \circ (1_{\Omega B} \times \mu) \circ (T_{\Omega B,\Omega B} \times 1_{\Omega B}) \circ (1_{\Omega B} \times T_{\Omega B,\Omega B})$, we would be done.

For the rest of the proof, we will use T and 1 to denote $T_{\Omega B,\Omega B}$ and $1_{\Omega B}$ respectively. To show that $\partial\mu(1 \times \mu) \simeq \partial\mu(1 \times \mu)(T \times 1)(1 \times T)$, we proceed with two chains of homotopies. The first uses the homotopy associativity of μ, the hypothesis $\partial\mu \simeq \partial\mu T$, and the right homotopy action of Lemma 4.2:

$$\begin{aligned}
\partial\mu(1 \times \mu)(T \times 1) &\simeq \partial\mu(\mu \times 1)(T \times 1) \\
&\simeq \partial\mu(\mu T \times 1) \\
&\simeq \theta_R(\partial \times 1)(\mu T \times 1) \\
&\simeq \theta_R(\partial\mu T \times 1) \\
&\simeq \theta_R(\partial\mu \times 1) \\
&\simeq \theta_R(\partial \times 1)(\mu \times 1) \\
&\simeq \partial\mu(\mu \times 1) \\
&\simeq \partial\mu(1 \times \mu).
\end{aligned}$$

Thus $\partial\mu(1\times\mu)(T\times 1)(1\times T)\simeq\partial\mu(1\times\mu)(1\times T)$. The second chain of homotopies uses the hypothesis $\partial\mu\simeq\partial\mu T$ and the left homotopy action:

$$\begin{array}{rcl}\partial\mu(1\times\mu)(1\times T) & \simeq & \partial\mu(1\times\mu T)\\ & \simeq & \theta_L(1\times\partial)(1\times\mu T)\\ & \simeq & \theta_L(1\times\partial\mu T)\\ & \simeq & \theta_L(1\times\partial\mu)\\ & \simeq & \theta_L(1\times\partial)(1\times\mu)\\ & \simeq & \partial\mu(1\times\mu).\end{array}$$

This completes the proof. □

More can be said given the conditions of Proposition 4.5. We will prove in Proposition 4.8 that $\Omega B\xrightarrow{\partial}F$ is an H-map. First we require a general lemma which shows that a homotopy action, while not left distributive in general, does satisfy a partial distributivity formula.

Lemma 4.6. *Let $\Omega B\xrightarrow{\partial}F\longrightarrow E\longrightarrow B$ be a homotopy fibration sequence. Let $X\xrightarrow{f}\Omega B$ and $Y\xrightarrow{g}\Omega B$ be maps such that $\partial g\simeq *$. Then $\partial\mu(f\times g)\simeq\partial f\pi_1$, where π_1 is the projection onto the first factor.*

Proof. The homotopy commutative diagram

$$\begin{array}{ccccc} X\times Y & \xrightarrow{f\times g} & \Omega B\times\Omega B & \xrightarrow{\mu} & \Omega B\\ & \searrow^{f\times *} & \downarrow{\scriptstyle \Omega B\times\partial} & & \downarrow{\scriptstyle\partial}\\ & & \Omega B\times F & \xrightarrow{\theta_L} & F\end{array}$$

proves the lemma. □

If $\theta_R=\theta_L\circ T$ is a right homotopy action, then Lemma 4.6 can be made symmetrical.

Corollary 4.7. *Let $\Omega B\xrightarrow{\partial}F\longrightarrow E\longrightarrow B$ be a homotopy fibration sequence. Suppose $\theta_R=\theta_L\circ T_{\Omega B,F}$ is a right homotopy action. Let $X\xrightarrow{f}\Omega B$ and $Y\xrightarrow{g}\Omega B$ be maps such that $\partial g\simeq *$. Then $\partial\mu(f\times g)\simeq\partial g\pi_2$, where π_2 is the projection onto the second factor.*

Proposition 4.8. *Let $\Omega B\xrightarrow{\partial}F\longrightarrow E\xrightarrow{h}B$ be a homotopy fibration sequence, where ∂ has a right homotopy inverse s. Let F have the H-structure determined by this retraction off ΩB. Suppose $\partial\mu\simeq\partial\mu\circ T_{\Omega B,\Omega B}:\Omega B\times\Omega B\longrightarrow F$. Then ∂ is an H-map.*

Proof. The proof proceeds by constructing several chains of homotopies. Throughout, π_i will denote the projection onto the i^{th} factor of a product. First consider the diagram

$$
\begin{array}{ccccc}
\Omega E\times\Omega E\times F & \xrightarrow{\mu\times\Omega B} & \Omega E\times F & \xrightarrow{\pi_2} & F \\
\downarrow{\scriptstyle \Omega h\times\Omega h\times s} & & \downarrow{\scriptstyle \Omega h\times s} & & \| \\
\Omega B\times\Omega B\times\Omega B & \xrightarrow{\mu\times\Omega B} & \Omega B\times\Omega B & & \| \\
\downarrow{\scriptstyle \Omega B\times\mu} & & \downarrow{\scriptstyle \mu} & & \| \\
\Omega B\times\Omega B & \xrightarrow{\mu} & \Omega B & & \| \\
\downarrow{\scriptstyle \Omega B\times\partial} & & \downarrow{\scriptstyle \partial} & & \| \\
\Omega B\times F & \xrightarrow{\theta_L} & F & = & F.
\end{array}
$$

It is clear that each internal square in the diagram homotopy commutes, except possibly the rightmost. Since $\partial\mu\simeq\partial\mu\circ T_{\Omega B,\Omega B}$, Lemma 4.2 shows that $\theta_R=\theta_L\circ T_{\Omega B,F}$ is a right homotopy action. Thus the rightmost square above does homotopy commute by Corollary 4.7, and so the diagram as a whole homotopy commutes. Note that the top horizontal row is the projection π_3 onto the third factor. Precomposing with the map $\Omega E\times T_{\Omega E,F}$ and again using the fact that $\partial\mu\simeq\partial\mu\circ T_{\Omega B,\Omega B}$ gives a homotopy commutative diagram

$$
\begin{array}{ccc}
\Omega E\times F\times\Omega E & \xrightarrow{\Omega h\times s\times\Omega h} & \Omega B\times\Omega B\times\Omega B \\
\downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \theta_L\circ(\Omega B\times\partial\mu)} \\
F & = & F.
\end{array}
$$

This diagram implies the commutativity of the left hand side in the following homotopy commutative diagram (where $\lambda=s\times\Omega h\times s\times\Omega h$):

$$
\begin{array}{ccccccc}
F\times\Omega E\times F\times\Omega E & \xrightarrow{\lambda} & \Omega B\times\Omega B\times\Omega B\times\Omega B & & \xrightarrow{\mu\times\mu} & & \Omega B\times\Omega B \\
\downarrow & & \downarrow{\scriptstyle \Omega B\times\Omega B\times\mu} & & & & \downarrow{\scriptstyle \mu} \\
{\scriptstyle \pi_1\times\pi_3} & & \Omega B\times\Omega B\times\Omega B & \xrightarrow{\Omega B\times\mu} & \Omega B\times\Omega B & \xrightarrow{\mu} & \Omega B \\
\downarrow & & \downarrow{\scriptstyle \Omega B\times\theta_L\circ(\Omega B\times\partial)} & & \downarrow{\scriptstyle \Omega B\times\partial} & & \downarrow{\scriptstyle \partial} \\
F\times F & \xrightarrow{s\times F} & \Omega B\times F & = & \Omega B\times F & \xrightarrow{\theta_L} & F.
\end{array}
$$

Observe that $e=\mu\circ(s\times\Omega h)$ is a homotopy equivalence and the top row in the diagram is homotopic to $e\times e$. By Lemma 4.3 the bottom row is homotopic to μ_F, the multiplication on F. Thus $\partial\circ\mu\circ(e\times e)\simeq\mu_F\circ(\pi_1\times\pi_3)$.

By Lemma 4.6, the composite $F\times\Omega E\xrightarrow{e}\Omega B\xrightarrow{\partial}F$ is homotopic to the projection onto the first factor. Thus $\Omega B\xrightarrow{e^{-1}}F\times\Omega E\xrightarrow{\pi_1}F$ is homotopic to ∂. Now we have

$$
\begin{aligned}
\partial\mu &\simeq \partial\mu(e\times e)(e^{-1}\times e^{-1}) \\
&\simeq \mu_F(\pi_1\times\pi_3)(e^{-1}\times e^{-1}) \\
&\simeq \mu_F(\partial\times\partial),
\end{aligned}
$$

which says exactly that ∂ is an H-map. ∎

We now turn our attention to finding conditions implying the supposition $\partial\mu \simeq \partial\mu T$ of Propositions 4.5 and 4.8. The first arises from the partial distributivity formula of a homotopy action in Lemma 4.6.

Lemma 4.9. *Let $\Omega B \xrightarrow{\partial} F \longrightarrow E \longrightarrow B$ be a homotopy fibration. If $\Omega B \times \Omega B \xrightarrow{\mu\cdot(\mu T)^{-1}} \Omega B \xrightarrow{\partial} F$ is null homotopic, then $\partial\mu \simeq \partial\mu T$.*

Proof. We begin by rearranging. Applying both

$$\mu \cdot (\mu T)^{-1} \quad \text{and} \quad ((\mu T)^{-1} \cdot \mu) \circ (1_{\Omega B}^{-1} \times 1_{\Omega B}^{-1})$$

to a pair (a, b) of elements in $\Omega B \times \Omega B$ shows the two maps are homotopic. Since $1_{\Omega B}^{-1} \times 1_{\Omega B}^{-1}$ is a homotopy equivalence, we have $\partial(\mu \cdot (\mu T)^{-1}) \simeq *$ if and only if $\partial((\mu T)^{-1} \cdot \mu) \simeq *$. Next, the associativity of the group $[\Omega X \times \Omega X, \Omega X]$ implies $\mu = ((\mu T) \cdot (\mu T)^{-1})) \cdot \mu \simeq (\mu T) \cdot ((\mu T)^{-1} \cdot \mu)$. Recall that the product of two maps $f, g \in [Y, \Omega B]$ is given by the composite $\mu \circ (f \times g) \circ \Delta$, where $Y \xrightarrow{\Delta} Y \times Y$ is the diagonal. Thus, with $Y = \Omega B \times \Omega B$, $f = \mu T$, and $g = (\mu T)^{-1} \cdot \mu$, Lemma 4.6 shows that $\partial(f \cdot g) = \partial\mu(f \times g)\Delta \simeq (\partial f \pi_1)\Delta \simeq \partial f$. Hence $\partial\mu \simeq \partial(\mu T) \cdot ((\mu T)^{-1} \cdot \mu) \simeq \partial\mu T$, as required. □

Next, we give a slightly weaker condition which implies that $\partial(\mu \cdot (\mu T)^{-1}) \simeq *$ and hence that $\partial\mu \simeq \partial\mu T$. It is based on the identification of $\mu \cdot (\mu T)^{-1}$ as a commutator. We digress momentarily to discuss commutators.

Given $X \xrightarrow{f} \Omega Z$ and $Y \xrightarrow{g} \Omega Z$, then their commutator is the map $[f, g] : \Omega X \times \Omega Y \longrightarrow \Omega Z$ defined pointwise by $[f, g](x, y) = f(x)g(y)f^{-1}(x)g^{-1}(y)$. In terms of maps, it is the composite

$$[f, g] \simeq \mu \circ (1_{\Omega Z} \times 1_{\Omega Z}^{-1}) \circ (\mu \times \mu) \circ (f \times g \times f \times g) \circ (1_{X\times Y} \times T_{X\times Y}) \circ \Delta.$$

For example, consider $\Omega X \xrightarrow{\Omega e_j} \Omega(X \vee X)$, where e_j is the inclusion into the j^{th} wedge summand. The composite of $[\Omega e_1, \Omega e_2]$ with the loop of the canonical inclusion $\Omega(X \vee X) \xrightarrow{\Omega i} \Omega X \times \Omega X$ is determined by the left and right projections onto ΩX. It is trivial to see the projections are null homotopic. Thus there is a lift of $[\Omega e_1, \Omega e_2]$ to the homotopy fiber $\Omega(\Sigma\Omega X \wedge \Omega Y)$ of Ωi. Furthermore, composing $[\Omega e_1, \Omega e_2]$ with the loop of the fold map $X \vee X \xrightarrow{\nabla} X$ gives the commutator $[1_{\Omega X}, 1_{\Omega X}]$. Thus the adjoint of $[1_{\Omega X}, 1_{\Omega X}]$ factors through the universal Whitehead product of X,

$$\Sigma\Omega X \wedge \Omega X \xrightarrow{\Psi} X \vee X \xrightarrow{\nabla} X.$$

Lemma 4.10. *In $[\Omega B \times \Omega B, \Omega B]$ the map $\mu \cdot (\mu T)^{-1}$ is homotopic to the commutator $[1_{\Omega B}, 1_{\Omega B}]$.*

Proof. This follows from the definition of a commutator and the group structure in $[\Omega X \times \Omega X, \Omega X]$. Explicitly, the formula defining a commutator gives

$$[1_{\Omega B}, 1_{\Omega B}] \simeq \mu \circ (1_{\Omega B} \times 1_{\Omega B}^{-1}) \circ (\mu \times \mu) \circ (1_{\Omega B\times\Omega B} \times T_{\Omega B,\Omega B}) \circ \Delta.$$

But this is exactly the composition defining the sum $\mu \cdot (\mu T)^{-1}$. □

Lemma 4.11. *Let $\Omega B \xrightarrow{\partial} F \longrightarrow E \longrightarrow B$ be a homotopy fibration sequence. Suppose there is a homotopy commutative diagram*

$$\begin{array}{ccc} \Sigma\Omega B \wedge \Omega B & \longrightarrow & B \vee B \\ \big\downarrow & & \big\downarrow \nabla \\ E & \longrightarrow & B \end{array}$$

where the upper composite in the square is the universal Whitehead product of B. Then $\partial(\mu \cdot (\mu T)^{-1}) \simeq *$.

Proof. Suppose the square homotopy commutes. By Lemma 4.10 and the comments preceding the statement of this lemma, the adjoint of $\mu \cdot (\mu T)^{-1}$ factors through the upper composite in this square. Looping then shows that $\partial(\mu \cdot (\mu T)^{-1})$ factors through the homotopy fiber of ∂, and so is null homotopic. □

The following proposition summarizes Proposition 4.5, Proposition 4.8, Lemma 4.9, and Lemma 4.11.

Proposition 4.12. *Let $\Omega B \xrightarrow{\partial} F \longrightarrow E \longrightarrow B$ be a homotopy fibration sequence in which ∂ has a right homotopy inverse s. Let F have the H-structure determined by this retraction off ΩB. Suppose there is a homotopy commutative diagram*

$$\begin{array}{ccc} \Sigma\Omega B \wedge \Omega B & \longrightarrow & B \vee B \\ \big\downarrow & & \big\downarrow \nabla \\ E & \longrightarrow & B \end{array}$$

where the upper composite in the square is the universal Whitehead product of B. Then ∂ is an H-map, and F is homotopy commutative and homotopy associative.

5. A Universal Property of T_k

This section is concerned with proving that under certain conditions $P^{2n}(p^r)$ is a generating space for T_∞. That is, given a space X and a map $P^{2n}(p^r) \longrightarrow X$, we would like to know when it is possible to obtain an extension to a map $T_\infty \longrightarrow X$. Anick and Gray [AG, 4.7] showed that if X is an H-space with $p^{r+i-1} \cdot \pi_{2np^i-1}(X; \mathbf{Z}/p^{r+i}\mathbf{Z}) = 0$ for $i \geq 1$, then such an extension does exist, but it may not be unique. They conjectured [AG, 5.5(b)] that if X were a homotopy commutative, homotopy associative H-space, then there exists an extension to a unique H-map. We show that this is indeed the case.

The proof inductively constructs an extension of a given map $P^{2n}(p^r) \xrightarrow{\gamma} X$ to an H-map $\Omega G_k \xrightarrow{\gamma_k} X$. By Theorem 2.3 there is a homotopy fibration $\Omega R_k \xrightarrow{\Omega\iota_k} \Omega G_k \xrightarrow{\partial_k} T_k$, where ∂_k has a right homotopy inverse $d_k : T_k \longrightarrow \Omega G_k$. We thus have a map $\overline{\gamma}_k : T_k \xrightarrow{d_k} \Omega G_k \xrightarrow{\gamma_k} X$, but this need not be an H-map. To obtain the stronger result we show that $\gamma_k \circ \Omega\iota_k \simeq *$ and γ_k factors as $\Omega G_k \xrightarrow{\partial_k} T_k \xrightarrow{\overline{\gamma}_k} X$. A direct limit argument is then applied to obtain the case $k = \infty$.

The crux of the proof is really in the reason behind why γ_k factors as $\bar{\gamma}_k \circ \partial_k$. This stems from the strong assertion in Lemma 5.1.

Lemma 5.1. *For $p \geq 3$, there is a choice of the map $P^{2np^k}(p^{r+k}) \xrightarrow{\alpha} G_{k-1}$ in Theorem 2.1 such that the composite*

$$\Omega P^{2np^k}(p^{r+k}) \xrightarrow{\Omega\alpha} \Omega G_{k-1} \xrightarrow{\gamma_{k-1}} X$$

is null homotopic.

Proof. Consider the homotopy fibration $\Omega M \xrightarrow{\Omega\lambda} \Omega G_k \xrightarrow{\gamma_{k-1}} X$. As in [N3], since X is a homotopy commutative, homotopy associative H-space and γ_{k-1} is an H-map, Samelson products on ΩG_{k-1} lift through $\Omega\lambda$ to ΩM. Furthermore, the Lie algebra identities satisfied by the Samelson products in $\pi_*(\Omega G_{k-1}; \mathbf{Z}/p^t\mathbf{Z})$, $t \geq 1$, are also satisfied by their lifts to ΩM. The same applies to Whitehead products on G_{k-1} by adjointing. In particular (see [Th]), α is a mod p^{r+k} homotopy defined as an extension of a mod p^{r+k-1} homotopy class θ which factored through the universal Whitehead product of G_{k-1}. The extension existed because of Jacobi and anti-symmetry identities satisfied by the collection of indirect lifts in Theorem 2.1 (d). Thus, given X, the extension defining α could have been chosen to factor through M, and this would prove the lemma. ☐

Remark. Note that while the choice of α in Lemma 5.1 depends on the given homotopy commutative, homotopy associative H-space X, the mod p^{r+k-1} homotopy class θ from which α was obtained as an extension factors through the universal Whitehead product of G_{k-1}, and so $\underline{p}\alpha$ factors through the universal Whitehead product of G_{k-1}. Thus the attaching map $\underline{p}^{r+k-1}\alpha$ defining G_k is independent of X.

Another aid in obtaining the factorizations we desire is the following lemma.

Lemma 5.2. *Suppose there is an H-map $\Omega G_k \xrightarrow{f} R$ such that $f\Omega \circ \iota_k$ is null homotopic. Let $\bar{f} : T_k \xrightarrow{d_k} \Omega G_k \xrightarrow{f} R$. Then f factors as $\Omega G_k \xrightarrow{\partial_k} T_k \xrightarrow{\bar{f}} R$, $\bar{f}$ is the unique map such that $f \simeq \bar{f} \circ \partial_k$, and if T_k has the H-structure determined by the given retraction off ΩG_k then $\bar{f}$ is an H-map.*

Proof. Apply Proposition 4.1 to the homotopy fibration sequence in Theorem 2.3. ☐

We now proceed to the main theorem. Let $i_{k-1} : G_{k-1} \longrightarrow G_k$ be the inclusion.

Theorem 5.3. *Let X be a homotopy commutative, homotopy associative H-space. Let $P^{2n}(p^r) \xrightarrow{\gamma} X$ be given. Then for $p \geq 3$ and $0 \leq k \leq \infty$:*

(a) *There exists an H-map $\Omega G_k \xrightarrow{\gamma_k} X$ extending γ. Moreover, γ_k can be chosen so that $\gamma_k \circ \Omega i_{k-1} \simeq \gamma_{k-1}$.*
(b) *Let $\overline{\gamma}_k : T_k \xrightarrow{d_k} \Omega G_k \xrightarrow{\gamma_k} X$. Then γ_k factors as $\Omega G_k \xrightarrow{\partial_k} T_k \xrightarrow{\overline{\gamma}_k} X$, $\overline{\gamma}_k$ is the unique map such that $\gamma_k \simeq \overline{\gamma}_k \circ \partial_k$, and if T_k has the H-structure determined by the given retraction off ΩG_k, then $\overline{\gamma}_k$ is an H-map.*
(c) *The H-maps in* (a) *and* (b) *are unique up to homotopy.*

Proof of Theorem 5.3: The induction occupies several pages. Part (a) comes fairly quickly, but parts (b) and (c) require more effort.

First let us consider the base case $k = 0$. Here, G_0^{2n} is the Moore space $P^{2n+1}(p^r)$. Since X is homotopy associative, the James construction gives an extension of γ to a unique H-map $\Omega P^{2n+1}(p^r) \xrightarrow{\gamma_0} X$. In the homotopy fibration

$$T_0^{2n-1} \longrightarrow R_0 \xrightarrow{\iota_0} P^{2n+1}(p^r)$$

of Theorem 2.3, $R_0 \in \mathcal{W}_r^r$ and ι_0 is a wedge sum of p-primary Whitehead products. Thus, as X is homotopy commutative and γ_0 is an H-map, the composite

$$\Omega R_0 \xrightarrow{\Omega\iota_0} \Omega P^{2n+1} \xrightarrow{\gamma_0} X$$

is null homotopic. Applying Lemma 5.2 completes this case.

Now assume that $k < \infty$ and the theorem has been proven for $k-1$.

Proof of (a). We begin by constructing a map from G_k to ΣX. The inductive hypothesis gives an H-map $\Omega G_{k-1} \xrightarrow{\gamma_{k-1}} X$. Since G_{k-1} is a co-H space, there is a map $s_{k-1} : G_{k-1} \longrightarrow \Sigma\Omega G_{k-1}$ which is a right homotopy inverse for the standard evaluation map. Let $e_{k-1} : G_{k-1} \longrightarrow \Sigma X$ be the composite $\Sigma\gamma_{k-1} \circ s_{k-1}$.

Lemma 5.4. *The composite* $P^{2np^k}(p^{r+k}) \xrightarrow{\underline{p}^{r+k-1}\alpha} G_{k-1} \xrightarrow{e_{k-1}} \Sigma X$ *is null homotopic.*

Proof. Since $\underline{p}^{r+k-1}\alpha$ is a co-H map, by Lemma 3.4 there is a homotopy commutative diagram

$$\begin{array}{ccc} P^{2np^k}(p^{r+k}) & \xrightarrow{\underline{p}^{r+k-1}\alpha} & G_{k-1} \\ \big\downarrow & & \big\downarrow{\scriptstyle s_{k-1}} \\ \Sigma\Omega P^{2np^k}(p^{r+k}) & \xrightarrow{\Sigma\Omega(\underline{p}^{r+k-1}\alpha)} & \Sigma\Omega G_{k-1}. \end{array}$$

Thus to prove the lemma it suffices to show that the composite $\Sigma\gamma_{k-1}\circ\Sigma\Omega(\underline{p}^{r+k-1}\alpha)$ is null homotopic. But this follows from Lemma 5.1 which says that $\gamma_{k-1} \circ \Omega\alpha$ is null homotopic. □

Since $P^{2np^k}(p^{r+k}) \xrightarrow{\underline{p}^{r+k-1}\alpha} G_{k-1} \xrightarrow{i_{k-1}} G_k$ is a homotopy cofibration, Lemma 5.4 implies there is an extension of e_{k-1} to a map $e_k : G_k \longrightarrow \Sigma X$. Since X is homotopy associative, by Lemma 3.5 there is an H-map $r : \Omega\Sigma X \longrightarrow X$ which is a right homotopy inverse to the standard inclusion. By Lemma 3.6, the composite $\Omega G_{k-1} \xrightarrow{\Omega e_{k-1}} \Omega\Sigma X \xrightarrow{r} X$ is homotopic to γ_{k-1}. Let $\gamma_k : \Omega G_k \xrightarrow{\Omega e_k} \Omega\Sigma X \xrightarrow{r} X$. Then γ_k is an H-map and $\gamma_k \circ \Omega i_{k-1} \simeq \gamma_{k-1}$.

Proof of (b). To prove (b) it suffices by Lemma 5.2 to show that the composite $\Omega R_k \xrightarrow{\Omega\iota_k} \Omega G_k \xrightarrow{\gamma_k} X$ is null homotopic.

By Theorem 2.3, $R_k \simeq A_k \vee C_k \vee B_k$, where the restriction of R_k to $A_k \vee C_k$ maps to G_k by $(\bigvee_{i=1}^k a_i) \vee (\bigvee_{i=1}^k c_i)$ and the restriction to B_k maps to G_k through the universal Whitehead product of G_k. Write $R_k \simeq \Sigma\overline{R}_k$, $A_k \vee C_k = \Sigma(\overline{A}_k \vee \overline{C}_k)$, and $B_k = \Sigma\overline{B}_k$ for each of $\overline{R}_k$, $\overline{A}_k$, $\overline{C}_k$, and $\overline{B}_k$ in $\mathcal{W}_r^{r+k}$. By Lemma 3.2, the H-map $\gamma_k \circ \Omega\iota_k$ is determined by its restriction to the inclusion of $\overline{R}_k$ into ΩR_k. Therefore to show $\gamma_k \circ \Omega\iota_k$ is null homotopic it suffices to show it is null homotopic when restricted to each Moore space summand of $\overline{R}_k$.

By the definition of $\overline{B}_k$, the composite $\overline{B}_k \longrightarrow \Omega R_k \xrightarrow{\Omega\iota_k} \Omega G_k$ factors through the loop of the universal Whitehead product of G_k. Hence this composite composes trivially with γ_k into the homotopy commutative H-space X.

The maps a_i and c_i into G_k for $1 \le i \le k-1$ were defined using the inclusion $i_{k-1} : G_{k-1} \longrightarrow G_k$, and their domains $A_{k-1} \vee C_{k-1}$ are Moore space summands of R_{k-1}. Thus, since $\gamma_{k-1} \simeq \gamma_k \circ \Omega i_{k-1}$ by part (a), the restriction of $\gamma_k \circ \Omega\iota_k$ to $\overline{A}_{k-1} \vee \overline{C}_{k-1}$ is null homotopic.

It therefore remains to prove that the adjoints of a_k and c_k compose trivially with γ_k. More accurately, there was some choice in defining a_k and c_k in Section 2, but any choice would have sufficed to fulfil the requirements of the homotopy decomposition of Theorem 2.3. What we intend to show is that there is some choice of a_k and c_k whose adjoints compose trivially with γ_k. These choices depend on the given homotopy commutative, homotopy associative H-space X, but as they affect ΩG_k and T_k by self-equivalences, the stated outcome of Theorem 5.3 is not affected. The remainder of this subsection describes how to make the desired choices of a_k and c_k.

By Lemma 3.5 the Hopf construction on X gives a homotopy fibration sequence

$$\Omega\Sigma X \xrightarrow{r} X \xrightarrow{*} X * X \longrightarrow X,$$

where r is an H-map. Define spaces M and N by homotopy pullback diagrams

$$\begin{CD} M @>>> G_{k-1} \\ @VVV @VV{e_{k-1}}V \\ X * X @>>> \Sigma X \end{CD} \qquad\qquad \begin{CD} N @>>> G_k \\ @VVV @VV{e_k}V \\ X * X @>>> \Sigma X \end{CD}$$

where the lower row in each diagram is the Hopf construction on X. Since e_k restricted to G_{k-1} is e_{k-1}, there is also a homotopy pullback diagram

$$\begin{CD} M @>>> N \\ @VVV @VVV \\ G_{k-1} @>>> G_k. \end{CD}$$

We wish to show that properties of the maps α and $\underline{p}^{r+k-1}\alpha$ in Lemmas 5.1 and 5.4 imply there is a choice of the map $P^{2np^k}(p^{r+k-1}) \vee P^{2np^k+1}(p^{r+k-1}) \xrightarrow{a_k \vee c_k} G_k$ which factors through N. This would imply that for this choice, $\Omega(a_k \vee c_k)$ composes trivially through γ_k in the homotopy fibration $\Omega N \longrightarrow \Omega G_k \xrightarrow{\gamma_k} X$, and so would complete the proof of part (b).

We begin with any choice of $a_k \vee c_k$ and expand the homotopy pushout diagram of Section 2,

$$\begin{CD} P^{2np^k}(p^{r+k}) @>{\underline{p}^{r+k-1}}>> P^{2np^k}(p^{r+k}) @>>> P^{2np^k}(p^{r+k-1}) \vee P^{2np^k+1}(p^{r+k-1}) \\ @| @VV{\alpha}V @VV{a_k \vee c_k}V \\ P^{2np^k}(p^{r+k}) @>{\underline{p}^{r+k-1}\alpha}>> G_{k-1} @>>> G_k \\ @. @VV{f}V @VV{g}V \\ @. Z @= Z. \end{CD}$$

From this we obtain a homotopy pullback diagram

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & D \\ \| & & \downarrow & & \downarrow \\ F & \longrightarrow & G_{k-1} & \longrightarrow & G_k \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ & & Z & = & Z. \end{array}$$

By the definitions of Z and G_k as cofibers, α lifts to E and $\underline{p}^{r+k-1}\alpha$ lifts to F. Observe that both E and F are $(2np^k-2)$-connected. If $n>1$, then the actions of ΩZ and ΩG_k on E and F respectively imply the next cell in either E or F occurs in dimension greater than $2np^k$. Thus the lifts of α and $\underline{p}^{r+k-1}\alpha$ are inclusions of bottom Moore spaces. If $n=1$, then both ΩZ and ΩG_k have a single cell in dimension 1, and so their respective actions on E and F imply the $2p^k$-skeleton of each is homotopy equivalent to $P^{2p^k}(p^{r+k-1})\vee S^{2p^k}$. Thus there are choices as to how α and $\underline{p}^{r+k-1}\alpha$ lift to E and F, but any choice must be the homotopic to the identity on $P^{2p^k}(p^{r+k-1})$ and some multiple of the pinch map $P^{2p^k}(p^{r+k-1})\longrightarrow S^{2p^k}$.

We see then that for all $n\geq 1$ (making compatible choices when $n=1$), there is a homotopy commutative diagram

$$\begin{array}{ccc} P^{2np^k}(p^{r+k}) & \xrightarrow{\underline{p}^{r+k-1}} & P^{2np^k}(p^{r+k}) \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle i} \\ F & \longrightarrow & E, \end{array}$$

where h and i are inclusions and are lifts, respectively, of $\underline{p}^{r+k-1}\alpha$ and α.

By Lemma 5.1 there is a lift of $\Omega\alpha$ to ΩM. Including $P^{2np^k-1}(p^{r+k})$ into $\Omega P^{2np^k}(p^{r+k})$ and adjointing gives a lift of α to M. We now return to the homotopy pullback diagram

$$\begin{array}{ccccc} F & \longrightarrow & M & \longrightarrow & N \\ \| & & \downarrow & & \downarrow \\ F & \longrightarrow & G_{k-1} & \longrightarrow & G_k. \end{array}$$

The question is whether the lift of α to M can be chosen to be p^{r+k-1} times the lift of $\underline{p}^{r+k-1}\alpha$ to F. But a lift of α to M is really a lift of the $2np^k$-skeleton of E to M provided $n>1$, and when $n=1$, a lift of the Moore space summand of the $2p^k$-skeleton of E to M. Hence there is a homotopy commutative diagram

$$\begin{array}{ccc} P^{2np^k}(p^{r+k}) & \xrightarrow{\underline{p}^{r+k-1}} & P^{2np^k}(p^{r+k}) \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \bar{i}} \\ F & \longrightarrow & M, \end{array}$$

where $\bar{i}$ factors through E. Mapping through the cofiber of $\underline{p}^{r+k-1}$ then gives a homotopy commutative diagram

$$\begin{array}{ccc} P^{2np^k}(p^{r+k}) & \longrightarrow & P^{2np^k}(p^{r+k-1}) \vee P^{2np^k+1}(p^{r+k-1}) \\ \downarrow{\scriptstyle \bar{i}} & & \downarrow{\scriptstyle j} \\ M & \longrightarrow & N \end{array}$$

for some map j.

We can therefore refine our choice of $a_k \vee c_k$ by defining it as the composition $P^{2np^k}(p^{r+k-1}) \vee P^{2np^k+1}(p^{r+k-1}) \xrightarrow{j} N \longrightarrow G_k$. This choice of $a_k \vee c_k$ which factors through N is exactly what we required in order to complete the proof of part (b).

One more fact coming out of the proof should be noted, as it plays a part in the proof of uniqueness in part (c).

Lemma 5.5. *The map $a_k \vee c_k$ could be chosen to factor through N independently of the choice of extension $G_k \xrightarrow{e_k} \Sigma X$ of e_{k-1}. In other words, with the given choice of $a_k \vee c_k$, the composition*

$$\Omega(P^{2np^k}(p^{r+k-1}) \vee P^{2np^k+1}(p^{r+k-1})) \xrightarrow{\Omega(a_k \vee c_k)} \Omega G_k \xrightarrow{r \circ \Omega e_k} X$$

is trivial regardless of the choice of the extension e_k.

Proof of (c). At this point we have shown the existence of an H-map $\Omega G_k \xrightarrow{\gamma_k} X$ extending $P^{2n}(p^r) \xrightarrow{\gamma} X$, and proven it factors through T_k. To prove the uniqueness of both of these H-maps, it suffices by the uniqueness assertion of Lemma 5.2 to prove the uniqueness of the H-map γ_k.

Let γ_k and γ'_k be any two H-maps extending γ. Since G_k is a co-H space, there is a map $s_k : G_k \longrightarrow \Sigma\Omega G_k$ which is a right homotopy inverse for the standard evaluation map. Let e_k be the composite $G_k \xrightarrow{s_k} \Sigma\Omega G_k \xrightarrow{\Sigma\gamma_k} \Sigma X$, and define e'_k similarly using γ'_k. By Lemma 3.6, $\gamma_k \simeq r \circ \Omega e_k$ and $\gamma'_k \simeq r \circ \Omega e'_k$, where $\Omega\Sigma X \xrightarrow{r} X$ is the H-map given by the Hopf construction in Lemma 3.5. By induction, there is a unique H-map $\Omega G_{k-1} \xrightarrow{\gamma_{k-1}} X$ extending γ, and so $\Omega i_{k-1} \circ \gamma_k \simeq \Omega i_{k-1} \circ \gamma'_k$. Let e_{k-1} be the composite $G_{k-1} \xrightarrow{s_{k-1}} \Sigma\Omega G_{k-1} \xrightarrow{\Sigma\gamma_{k-1}} \Sigma X$.

Lemma 5.6. $e_{k-1} \simeq e_k \circ i_{k-1}$ *and* $e_{k-1} \simeq e'_k \circ i_{k-1}$.

Proof. Consider the diagram

$$\begin{array}{ccccc} G_{k-1} & \xrightarrow{s_{k-1}} & \Sigma\Omega G_{k-1} & \xrightarrow{\Sigma\gamma_{k-1}} & \Sigma X \\ \downarrow{\scriptstyle i_{k-1}} & & \downarrow{\scriptstyle \Sigma\Omega i_{k-1}} & & \| \\ G_k & \xrightarrow{s_k} & \Sigma\Omega G_k & \xrightarrow{\Sigma\gamma_k} & \Sigma X. \end{array}$$

Since G_k is contructed by attaching a Moore space to G_{k-1} by a co-H map, the skeletal inclusion i_{k-1} is a co-H map. Thus the left hand square homotopy commutes by Lemma 3.4. The right hand triangle homotopy commutes because $\gamma_{k-1} \simeq \gamma_k \circ \Omega i_{k-1}$. Thus $e_{k-1} \simeq e_k \circ i_{k-1}$. Since $\gamma_{k-1} \simeq \gamma'_k \circ \Omega i_{k-1}$ as well, the same diagram with γ_k replaced by γ'_k shows that we also have $e_{k-1} \simeq e'_k \circ i_{k-1}$. ☐

Now consider the homotopy pullback diagram

$$\begin{array}{ccccccc}
\Omega G_k & \longrightarrow & X & \longrightarrow & W_k & \xrightarrow{\lambda_k} & G_k \\
\Big\downarrow{\scriptstyle \Omega(e_k - e'_k)} & & \Big\| & & \Big\downarrow & & \Big\downarrow{\scriptstyle e_k - e'_k} \\
\Omega\Sigma X & \xrightarrow{r} & X & \xrightarrow{*} & X * X & \longrightarrow & \Sigma X,
\end{array}$$

where the lower row is again the homotopy fibration resulting from the Hopf construction on X. Since X is homotopy commutative and r is an H-map, the additive difference $e_k - e'_k$ in the co-H structure of ΣX, when looped and composed with r, is homotopic to the difference $(r \circ \Omega e_k) - (r \circ \Omega e'_k)$ in the H-structure of X. Thus $r \circ \Omega(e_k - e'_k) \simeq \gamma_k - \gamma'_k$.

If γ_k were the unique H-map extending the inclusion of the bottom Moore space into ΩG_k, then $\gamma_k - \gamma'_k \simeq *$, and so $\Omega\lambda_k$ has a right homotopy inverse. Since G_k is a co-H space, we equivalently obtain a right homotopy inverse of λ_k. Thus to prove the uniqueness assertion of Theorem 5.3 it suffices to prove that λ_k has a right homotopy inverse.

Define W_{k-1} by the homotopy pullback diagram

$$\begin{array}{ccccc}
 & & F & = = = & F \\
 & & \Big\downarrow & & \Big\downarrow \\
X & \longrightarrow & W_{k-1} & \xrightarrow{\lambda_{k-1}} & G_{k-1} \\
\Big\| & & \Big\downarrow & & \Big\downarrow{\scriptstyle i_{k-1}} \\
X & \longrightarrow & W_k & \xrightarrow{\lambda_k} & G_k.
\end{array}$$

By the definition of W_k, we see that W_{k-1} is also the homotopy pullback of the composite $G_{k-1} \xrightarrow{i_{k-1}} G_k \xrightarrow{e_k - e'_k} \Sigma X$ and the Hopf construction $X * X \longrightarrow \Sigma X$. By Lemma 5.6, $(e_k - e'_k) \circ i_{k-1} \simeq *$, and so λ_{k-1} has a right homotopy inverse, $\epsilon_{k-1} : G_{k-1} \longrightarrow W_{k-1}$. Thus $\Omega W_{k-1} \simeq \Omega G_{k-1} \times \Omega X$.

Next consider the homotopy pullback diagram

$$\begin{array}{ccccc}
F & \longrightarrow & Q & \longrightarrow & X \\
\Big\| & & \Big\downarrow & & \Big\downarrow \\
F & \longrightarrow & W_{k-1} & \longrightarrow & W_k \\
 & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \lambda_k} \\
 & & G_k & = = = & G_k,
\end{array}$$

where f is defined as the composition. When looped, $\Omega W_{k-1} \simeq \Omega G_{k-1} \times \Omega X$ and the composite $\Omega X \longrightarrow \Omega W_{k-1} \xrightarrow{\Omega f} \Omega G_k$ is trivial by definition of f. Since Ωf is multiplicative, this implies $\Omega Q \simeq \Omega F \times \Omega X$.

Now, since the attaching map $P^{2np^k}(p^{r+k}) \xrightarrow{\underline{p}^{r+k-1}\alpha} G_{k-1}$ lifts to F, we can use the splittings of ΩW_{k-1} and ΩQ to obtain a homotopy commutative diagram

$$\begin{array}{ccccc} \Omega P^{2np^k}(p^{r+k}) & \xrightarrow{i_1} & \Omega F \times \Omega X & \xrightarrow{\pi_2} & \Omega X \\ \downarrow{\scriptstyle \Omega(\underline{p}^{r+k-1}\alpha)} & & \downarrow & & \downarrow \\ \Omega G_{k-1} & \xrightarrow{i_2} & \Omega G_{k-1} \times \Omega X & \longrightarrow & \Omega W_k, \end{array}$$

where i_1 is the loop of the lift of $\underline{p}^{r+k-1}\alpha$ to F composed with the inclusion into the product, i_2 is the inclusion into the product, and π_2 is the projection onto the second factor. Thus the composition $\Omega\Theta : \Omega P^{2np^k}(p^{r+k}) \xrightarrow{\Omega(\underline{p}^{r+k-1}\alpha)} \Omega G_{k-1} \xrightarrow{\Omega\epsilon_{k-1}} \Omega W_{k-1} \longrightarrow \Omega W_k$ is null homotopic. Since $\underline{p}^{r+k-1}\alpha$ is a co-H map, Lemma 3.4, together with the standard evaluation map applied to the constituent maps in $\Sigma\Omega\Theta$, shows that Θ is null homotopic. Thus there is an extension through the homotopy cofiber of $\underline{p}^{r+k-1}\alpha$; that is, there is a homotopy commutative diagram

$$\begin{array}{ccc} G_{k-1} & \xrightarrow{\epsilon_{k-1}} & W_{k-1} \\ \downarrow{\scriptstyle i_{k-1}} & & \downarrow \\ G_k & \xrightarrow{\epsilon_k} & W_k, \end{array}$$

which defines ϵ_k. In particular, this implies the composite $G_k \xrightarrow{\epsilon_k} W_k \xrightarrow{\lambda_k} G_k$ is an isomorphism in $H_{2n}(\ )$, and so is a homotopy equivalence as G_k is atomic. Thus λ_k has a right homotopy inverse, which is what we required to complete the proof.

The Case of $k = \infty$. It remains to pass to the limit. As γ_k is an extension of γ_{k-1} for $k \geq 1$ and all spaces are of finite type, we can define $\gamma_\infty = \varinjlim \gamma_k$. Parts (a) and (b) now follow immediately. The only bone of contention is whether γ_∞ is the unique H-map extending γ. That is, informally speaking, it is necessary to check that there are no phantom H-maps. This will be shown by the following lemma.

We are considering the directed system determined by the maps $\Omega G_{k-1} \xrightarrow{\Omega i_{k-1}} \Omega G_k$, where i_{k-1} is the inclusion.

Lemma 5.7. *Let Y be an H-space. In the short exact sequence of groups*

$$0 \longrightarrow \varprojlim{}^1[\Omega G_k, \Omega Y] \longrightarrow [\Omega G_\infty, Y] \xrightarrow{t} \varprojlim\, [\Omega G_k, Y] \longrightarrow 0$$

the map t is an isomorphism.

Proof. Since Y is an H-space, it suffices to consider the same short exact sequence with spaces and maps replaced by their suspensions. Consider the universal case. The directed system gives a homotopy cofibration

$$\bigvee_{k=0}^{\infty} \Sigma\Omega G_k \longrightarrow \Sigma\Omega G_\infty \longrightarrow C.$$

Continuing the homotopy cofibration, we have

$$C \longrightarrow \bigvee_{k=0}^{\infty} \Sigma^2\Omega G_k \longrightarrow \Sigma^2\Omega G_\infty.$$

By Theorem 2.1 (b), $\Sigma^2\Omega G_k \in \mathcal{W}_r^{r+k}$. Further, Theorem 2.1 (a) implies Ωi_k is $(2np^{k+1}-2)$-connected. Thus any Moore space summand $P^m(p^{r+j})$ of $\Sigma^2\Omega G_\infty$ of dimension $m \leq 2np^{k+1}-2$ is a retract of $\Sigma^2\Omega G_k$. Hence $\Sigma^2\Omega G_\infty$ is a retract of $\bigvee_{k=0}^{\infty}\Sigma^2\Omega G_k$. Therefore C is also a retract of $\bigvee_{k=0}^{\infty}\Sigma^2\Omega G_k$, and so the map $\Sigma\Omega G_\infty \longrightarrow C$ is null homotopic. This implies $\underleftarrow{\lim}^1[\Omega G_k, \Omega Y] = 0$. □

6. H-space Properties of T_∞ when $p \geq 5$

When $p \geq 5$ we intend to apply Proposition 4.12 to the homotopy fibration $\Omega G_\infty \xrightarrow{\partial_\infty} T_\infty \xrightarrow{*} R_\infty \xrightarrow{\iota_\infty} G_\infty$ of Theorem 2.3. Since ∂_∞ has a section, it remains to prove the appropriate diagram homotopy commutes. This will be shown in Lemma 6.5. We leave the case of $p = 3$ until Section 7.

We begin with two lemmas on a wedge of Moore spaces. Recall the notation $M_k = \bigvee_{i=0}^{k} P^{2np^i}(p^{r+i})$.

Lemma 6.1. *In the homotopy fibration*

$$\Sigma\Omega\Sigma M_k \wedge \Omega\Sigma M_k \xrightarrow{\Psi} \Sigma M_k \vee \Sigma M_k \longrightarrow \Sigma M_k \times \Sigma M_k,$$

Ψ is homotopic to a sum of p-primary Whitehead products.

Proof. First note that $\Sigma\Omega\Sigma M_k \wedge \Omega\Sigma M_k \in \mathcal{W}_r^{r+k}$. Write $\Sigma\Omega\Sigma M_k \wedge \Omega\Sigma M_k$ as ΣW for $W \in \mathcal{W}_r^{r+k}$. Since $\Omega\Psi$ has a left homotopy inverse, it is an inclusion in homology. Furthermore, the Hurewicz image of each Moore space summand P of W under the composite

$$W \xrightarrow{j} \Omega\Sigma W \longrightarrow \Omega(\Sigma M_k \vee \Sigma M_k)$$

is a bracket in the generators of $H_*(\Omega(\Sigma M_k \vee \Sigma M_k)) \cong T(\tilde{H}_*(M_k \vee M_k))$. Using the identity and Bockstein maps on each Moore space summand of $\Sigma M_k \vee \Sigma M_k$, it is clear there exists a p-primary Samelson product on $\Omega(\Sigma M_k \vee \Sigma M_k)$ which has the same Hurewicz image as P. Summing these p-primary Samelson products, one for each Moore space summand of W, gives a map $W \xrightarrow{\lambda} \Omega(\Sigma M_k \vee \Sigma M_k)$. Each p-primary Samelson product factors through the loop of the universal Whitehead product, so λ lifts to a map $\lambda' : W \longrightarrow \Sigma\Omega\Sigma M_k \wedge \Omega\Sigma M_k$ with $\lambda \simeq \Psi\lambda'$. By construction, when λ' is extended to its James space, the resulting map is a homology isomorphism and hence a homotopy equivalence. Taking adjoints then proves the lemma. □

We next want to consider the behavior of the p-primary Whitehead products of Lemma 6.1 when composed with the fold map. That is, we are considering the universal Whitehead product of ΣM_k. To do so, we require some notation.

Let $L\langle u_0, v_0, \cdots, u_k, v_k\rangle$ be a free $\mathbf{Z}_{(p)}$-dgL where for $0 \leq i \leq k$ we have $|u_i| = 2np^i - 1$, $|v_i| = 2np^i$, $d(v_i) = p^{r+i}u_i$, and $d(u_i) = 0$. By Theorem 2.1 (a) there are isomorphisms of Hopf algebras and Bockstein spectral sequences

$$H_*(\Omega G_k) \cong UL\langle u_0, v_0, \cdots, u_k, v_k\rangle \cong H_*(\Omega\Sigma M_k).$$

Define L_k as a dgL kernel,

$$L_k \longrightarrow L\langle \mu_0, \nu_0, \cdots, \mu_k, \nu_k\rangle \longrightarrow L_{ab}\langle \mu_0, \nu_0, \cdots, \mu_k, \nu_k\rangle.$$

Since L_k is a free sub-Lie algebra of a Lie algebra, it is free. Suppose $L_k = L\langle V_k\rangle$ for some graded $\mathbf{Z}_{(p)}$-module V_k. Write $V_k = \{x_\gamma\}_{\gamma\in\Gamma}$. For $x_\gamma \in V_k$ let r_γ be the maximal degree for which the image of x_γ in $E_H^{r_\gamma}(\Omega\Sigma M_k)$ is nonzero. Let

$P_\gamma(p^{r_\gamma}) \longrightarrow \Omega\Sigma M_k$ be the p-primary Samelson product whose Hurewicz image is x_γ. Let $\overline{Y}_k = \bigvee_\gamma P_\gamma(p^{r_\gamma})$, and define $\bar{g} : \overline{Y}_k \longrightarrow \Omega\Sigma M_k$ by taking the coproduct over $\gamma \in \Gamma$. Let $Y_k = \Sigma\overline{Y}_k$, and let g be the adjoint of $\bar{g}$.

Lemma 6.2. *For $p \geq 5$, there exists a homotopy commutative diagram*

$$\begin{array}{ccc} \Sigma\Omega\Sigma M_k \wedge \Omega\Sigma M_k & \xrightarrow{\Psi} & \Sigma M_k \vee \Sigma M_k \\ \downarrow & & \downarrow \nabla \\ Y_k & \xrightarrow{g} & \Sigma M_k \end{array}$$

where the upper composite is the universal Whitehead product of ΣM_k. In particular, g factors through $\Delta \circ \Psi$.

Remark. It should be pointed out that $Y_k \xrightarrow{g} \Sigma M_k$ is not optimal in the sense that many of the p-primary Whitehead products in the sum are Bocksteins of other p-primary Whitehead products in the sum. Optimally, L_k would be free not just as a Lie algebra but as a dgL. Then Y_k could be chosen to correspond to the dgL generators. However, L_k need not be free as a dgL, but Y_k as above is sufficient for the intended purpose.

Proof. The previous lemma shows the universal Whitehead product on ΣM_k is composed of p-primary Whitehead products. If $p \geq 5$ then the p-primary Whitehead products on ΣM_k form a Lie algebra, and the lemma follows. □

Since M_k is a finite type wedge of Moore spaces, the above two lemmas also hold for the homotopy colimit, M_∞.

We now tie in G_∞. First, we require a lemma concerning the map $\Omega G_\infty \xrightarrow{\partial_\infty} T_\infty$. While it is not yet known that this is an H-map, it is possible now to show that it behaves like an H-map in homology.

Lemma 6.3. *In homology, $\Omega G_\infty \xrightarrow{\partial_\infty} T_\infty$ determines a Hopf algebra map*

$$UL\langle u_0, v_0, u_1, v_1, \cdots\rangle \xrightarrow{(\partial_\infty)_*} UL_{ab}\langle u_{2n-1}, v_{2n}\rangle.$$

Proof. First, keep in mind that by Theorem 2.1 (a) there is a Hopf algebra isomorphism $H_*(\Omega G_\infty) \cong UL\langle u_0, v_0, u_1, v_1 \cdots\rangle$, and by Theorem 2.3 there is a coalgebra isomorphism $H_*(T_\infty) \cong UL_{ab}\langle u_0, v_0\rangle = S(u_0, v_0)$.

Let $E[x_m]$ and $\mathbf{Z}/p\mathbf{Z}[x_m]$ respectively be exterior and polynomial algebras generated by an element of degree m. A Serre spectral sequence calculation shows that there is a Hopf algebra isomorphism

$$H_*(\Omega S^{2n+1}\{p^r\}) \cong \bigotimes_{i=0}^{\infty} E[a_{2np^i-1}] \otimes \bigotimes_{i=1}^{\infty} \mathbf{Z}/p\mathbf{Z}[b_{2np^i-2}] \otimes \mathbf{Z}/p\mathbf{Z}[c_{2n}],$$

where $\beta^r(c_{2n}) = a_{2n-1}$ and $\beta^1(a_{2np^i-1}) = b_{2np^i-2}$. This Hopf algebra isomorphism can be rewritten as

$$H_*(\Omega S^{2n+1}\{p^r\}) \cong H_*(T_\infty) \otimes \bigotimes_{i=1}^{\infty} E[a_{2np^i-1}] \otimes \bigotimes_{i=1}^{\infty} \mathbf{Z}/p\mathbf{Z}[b_{2np^i-2}].$$

Let $\pi : H_*(\Omega S^{2n+1}\{p^r\}) \longrightarrow H_*(T_\infty)$ be the Hopf algebra projection. By Theorem 5.3, the homotopy commutativity and homotopy associativity of $\Omega S^{2n+1}\{p^r\}$ imply there is an H-map $\Omega G_\infty \xrightarrow{\gamma_\infty} \Omega S^{2n+1}\{p^r\}$ which factors as the composite

$\Omega G_\infty \xrightarrow{\partial_\infty} T_\infty \xrightarrow{\bar\gamma_\infty} \Omega S^{2n+1}\{p^r\}$, where $\bar\gamma_\infty$ is an H-map. Note that $\pi\circ(\bar\gamma_\infty)_*$ equals the identity map on $H_*(T_\infty)$, and this defines a Hopf algebra structure on $H_*(T_\infty)$. Further, the composite

$$H_*(\Omega G_\infty) \xrightarrow{(\Omega\gamma_\infty)_*} H_*(\Omega S^{2n+1}\{p^r\}) \xrightarrow{\pi} H_*(T_\infty)$$

is a Hopf algebra map equal to $(\partial_\infty)_*$. □

We introduce some notation. Let X_∞ be the space defined by pinching each Moore space summand in $A_\infty \vee C_\infty$ to its top cell. Let R be the graded $\mathbf{Z}_{(p)}$-module defined by $R = \tilde H_*(R_\infty; \mathbf{Z}_{(p)})$. Similarly let $AC = \tilde H_*(A_\infty \vee C_\infty; \mathbf{Z}_{(p)})$ and $X = \tilde H_*(X_\infty; \mathbf{Z}_{(p)})$. Since $A_\infty \vee C_\infty$ is a retract of R_∞, there is a projection $\pi : L\langle R\rangle \longrightarrow L_{ab}\langle X\rangle$.

Lemma 6.4. *There is a pullback diagram of Hopf algebras*

$$\begin{array}{ccccc}
UL_\infty & = & UL_\infty & & \\
\downarrow{\scriptstyle U(j)} & & \downarrow{\scriptstyle U(i)} & & \\
UL\langle R\rangle & \xrightarrow{(\Omega\iota_\infty)_*} & UL\langle u_0, v_0, u_1, v_1, \cdots\rangle & \xrightarrow{(\partial_\infty)_*} & UL_{ab}\langle u_{2n-1}, v_{2n}\rangle \\
\downarrow{\scriptstyle U(\pi)} & & \downarrow{\scriptstyle U(\pi)} & & \| \\
UL_{ab}\langle X\rangle & \xrightarrow{\lambda} & UL_{ab}\langle u_0, v_0, u_1, v_1, \cdots\rangle & \xrightarrow{\gamma} & UL_{ab}\langle u_{2n-1}, v_{2n}\rangle
\end{array}$$

which defines γ and λ.

Remark. The important thing to note in Lemma 6.4 is that the underlying Lie algebra map i lifts through $(\Omega\iota_\infty)_*$ to a map j of Lie algebras.

Proof. The factorization of the lower right square holds because $(\partial_\infty)_*$ is multiplicative by Lemma 6.3 and $UL_{ab}\langle u_{2n-1}, v_{2n}\rangle$ is commutative. To show the lower left square commutes, recall by Theorem 2.3 that $R_\infty \simeq A_\infty \vee C_\infty \vee B_\infty$, where $R_\infty \xrightarrow{\iota_\infty} G_\infty$ restricted to B_∞ factors through the universal Whitehead product of G_∞. Thus, as $UL_{ab}\langle u_0, v_0, u_1, v_1, \cdots\rangle$ is commutative, the composite $U(\pi)\circ(\Omega\iota_\infty)_*$ factors through a map $UL_{ab}\langle AC\rangle \longrightarrow UL_{ab}\langle u_0, v_0, u_1, v_1, \cdots\rangle$. But by Lemma 2.2 the Bockstein of each Moore space summand in $A_\infty \vee C_\infty$ has Hurewicz image in $UL\langle u_0, v_0, u_1, v_1, \cdots\rangle$ projecting to zero under $U(\pi)$, and so $U(\pi)\circ(\Omega\iota_\infty)_*$ factors through $UL_{ab}\langle X\rangle$. The factorizations of the two lower squares define γ and λ and imply they are Hopf algebra maps. Finally, the pullback assertion now follows because $UL_{ab}\langle X\rangle$ lifts (as an algebra) to the Hopf algebra kernel of γ and the two have identical Euler-Poincaré series. □

Lemma 6.5. *For primes $p \geq 5$, there is a homotopy commutative diagram*

$$\begin{array}{ccc}
\Sigma\Omega G_\infty \wedge \Omega G_\infty & \xrightarrow{\Upsilon} & G_\infty \vee G_\infty \\
\downarrow & & \downarrow{\scriptstyle \nabla} \\
R_\infty & \xrightarrow{\iota_\infty} & G_\infty
\end{array}$$

where the upper composite is the universal Whitehead product on G_∞.

Proof. By Theorem 2.1 (d), the Lie algebra identities satisfied by the p-primary Whitehead products on ΣM_∞ are also satisfied by their indirect lifts to G_∞. Thus the homotopy commuting diagram of Lemma 6.2 for $k = \infty$ implies there is a homotopy commutative diagram

$$\begin{array}{ccc} \Sigma\Omega G_\infty \wedge \Omega G_\infty & \xrightarrow{\Upsilon} & G_\infty \vee G_\infty \\ \big\downarrow{\scriptstyle t} & & \big\downarrow{\scriptstyle \nabla} \\ Y_\infty & \xrightarrow{\quad f \quad} & G_\infty \end{array}$$

where the upper composite is the universal Whitehead product of G_∞ and f factors through $\Delta \circ \Upsilon$. In particular, $(\Omega f)_*$ has image contained in $UL_\infty \subset UL\langle u_0, v_0, u_1, v_1, \cdots\rangle$ because $(\Omega(\nabla \circ \Upsilon))_*$ does. Further, by construction the Hurewicz images of the wedge summands in the adjoint of $Y_\infty \xrightarrow{g} \Sigma M_\infty$ are linearly independent. This property is preserved by the indirect lift to G_∞, so $(\Omega f)_*$ has image onto UL_∞ by definition of Y_∞. Thus the lemma will be proven if it can be shown that f factors through ι_∞.

We begin by constructing a new map $Y_\infty \xrightarrow{f'} G_\infty$ and then compare it to f. As we used $L_\infty \xrightarrow{i} L\langle u_0, v_0, u_1, v_1, \cdots\rangle$ to define the p-primary Samelson products comprising $Y_\infty \xrightarrow{g} \Sigma M_k$, use $L_\infty \xrightarrow{j} L\langle R\rangle$ and ι_∞ to define p-primary Samelson products comprising a map $f' : Y_\infty \longrightarrow G_\infty$. By definition, f' factors through: (i) the universal Whitehead product of G_∞, and (ii) R_∞. That is, there is a homotopy commutative diagram

$$\begin{array}{ccc} Y_\infty & \longrightarrow & R_\infty \\ \big\downarrow{\scriptstyle s} & \searrow^{f'} & \big\downarrow{\scriptstyle \iota_\infty} \\ \Sigma\Omega G_\infty & \xrightarrow{\nabla\circ\Upsilon} & G_\infty \end{array}$$

for some map s.

The similarity of the definitions of f' and g together with the indirect lift of g to the map f and Lemma 6.4 shows that $(\Omega f')_*$ and $(\Omega f)_*$ have the same image in $H_*(\Omega G_\infty)$. The linear independence of the Hurewicz images of the wedge summands in the adjoints of f and f' implies the adjoint of $s \circ t$ is an inclusion onto the generating set of $H_*(\Omega Y_\infty)$, and hence $s \circ t$ is a homotopy equivalence. Thus $f \simeq f' \circ (t \circ s)^{-1}$, and so f factors through ι_∞, as required. □

Remark. There can be no version of Lemma 6.5 for finite k. The loop of such a potential diagram does not even commute in homology. Calculations in [Th] show that the element $\tau_k^j \in H_*(\Omega G_k)$ is sent nontrivially to $H_*(T_k)$ by $(\partial_k)_*$, so τ_k^j cannot factor through $(\Omega\iota_k)_*$.

Theorem 6.6. *For $p \geq 5$, the H-space T_∞ is homotopy commutative and homotopy associative, and the map $\Omega G_\infty \xrightarrow{\partial_\infty} T_\infty$ is an H-map.*

Proof. Apply Proposition 4.12 to Lemma 6.5 and the homotopy fibration sequence $\Omega G_\infty \xrightarrow{\partial_\infty} T_\infty \longrightarrow R_\infty \xrightarrow{\iota_\infty} G_\infty$. □

7. The Case of $p = 3$

When $p = 3$ it is not as clear whether there is a homotopy commutative diagram as in Lemma 6.2. The proof of that lemma relied on the dgL structure of p-primary Whitehead products for $p \geq 5$. When $p = 3$, the triple product identity fails. If

$r = 1$ as well, then the Jacobi identity also fails. In what follows, we show that in certain cases the failure of the triple product when $p = 3$ implies that T_∞ fails to be homotopy associative.

Let ι be the generator of $\pi_{2n-1}(\Omega S^{2n})$. By [L], [To1] the integral Samelson product $[\iota, [\iota, \iota]]$ is a nontrivial homotopy class of order 3. Consider the homotopy fibration sequence

$$\Omega S^{2n} \xrightarrow{s} S^{2n-1} \longrightarrow S^{4n-1} \xrightarrow{\omega} S^{2n},$$

where ω is the integral Whitehead product of the identity map on S^{2n} with itself. Localized at 3, s has a right homotopy inverse. We wish to consider the composite $s \circ [\iota, [\iota, \iota]] : S^{6n-3} \longrightarrow S^{2n-1}$.

Proposition 7.1. *If $s \circ [\iota, [\iota, \iota]]$ is nontrivial and not divisible by 3^r, then $T_\infty^{2n-1}(3^r)$ is not homotopy associative.*

Proof. Suppose $T_\infty^{2n-1}(3^r)$ is homotopy associative. As we will see in Corollary 8.2, the homotopy fibration $S^{2n-1} \xrightarrow{i} T_\infty^{2n-1}(3^r) \xrightarrow{\pi_\infty} \Omega S^{2n+1}$ is of H-spaces and H-maps. Thus by [Gr1, Prop. 3], $T_\infty^{2n-1}(3^r)$ is homotopy commutative as well.

Applying Lemma 3.2 to the identity map of the homotopy associative space $T_\infty^{2n-1}(3^r)$ gives an H-map $\Omega\Sigma T_\infty^{2n-1}(3^r) \xrightarrow{r} T_\infty^{2n-1}(3^r)$ which is a left homotopy inverse of the canonical inclusion. Consider the composite

$$\theta : S^{4n-2} \xrightarrow{j} \Omega S^{4n-1} \xrightarrow{\Omega\omega} \Omega S^{2n} \xrightarrow{\Omega\Sigma i} \Omega\Sigma T_\infty^{2n-1}(3^r) \xrightarrow{r} T_\infty^{2n-1}(3^r),$$

where j is the inclusion. Observe that $\Omega\omega \circ j$ is a Samelson product. Since $r \circ \Omega\Sigma i$ is an H-map, θ is a Samelson product. The homotopy commutativity of $T_\infty^{2n-1}(3^r)$ then implies $\theta \simeq *$. By Lemma 3.2 there is a unique H-map $\Omega S^{4n-1} \longrightarrow T_\infty^{2n-1}(3^r)$ extending θ. Hence $r \circ \Omega\Sigma i \circ \Omega\omega \simeq *$. This null homotopy, combined with the fact that s has a right homotopy inverse, implies by Proposition 4.1 that there is a homotopy commuting diagram

$$\begin{array}{ccc} \Omega S^{2n} & \xrightarrow{\Omega\Sigma i} & \Omega\Sigma T_\infty^{2n-1}(3^r) \\ \big\downarrow{\scriptstyle s} & & \big\downarrow{\scriptstyle r} \\ S^{2n-1} & \xrightarrow{i} & T_\infty^{2n-1}(3^r). \end{array}$$

The diagram shows that $s \circ [\iota, [\iota, \iota]]$ composes trivially with i, so it lifts through the homotopy fibration connecting map ϕ to $\Omega^2 S^{2n+1}$. By checking homology, the $(6n-3)$-skeleton of $\Omega^2 S^{2n+1}$ equals S^{2n-1}. Thus, since ϕ is of degree 3^r, $r \geq 1$, this would imply 3^r divides $s \circ [\iota, [\iota, \iota]]$. The proposition now follows contrapositively. □

While the author does not know for exactly which n and r the composite $r \circ [\iota, [\iota, \iota]]$ is nontrivial and not divisible by 3^r, two examples are known.

Theorem 7.2. *If $r \geq 2$ then $T_\infty^5(3^r)$ is not homotopy associative. If $r \geq 1$ then $T_\infty^9(3^r)$ is not homotopy associative.*

Proof. Localized at 3, $\Omega S^{2n} \simeq S^{2n-1} \times \Omega S^{4n-1}$. Checking the tables of homotopy groups of spheres in [To2] shows that there are isomorphisms in the 3-primary components between $\pi_{6n-3}(\Omega S^{2n})$ and $\pi_{6n-3}(S^{2n-1})$ for $n = 3$ and $n = 5$. When $n = 3$ there is a single summand of order 9, and when $n = 5$ there is a single summand of order 3. □

On the other hand, by Proposition 8.4 we immediately have the following.

Proposition 7.3. *$T^5_\infty(3)$ is homotopy associative and homotopy commutative.*

8. Applications

Our most important application is to combine Theorems 5.3 and 6.6 to determine an H-space exponent for $T^{2n-1}_\infty(p^r)$ when $p \geq 5$.

Theorem 8.1. *For $p \geq 5$ and $r \geq 1$, the space $T^{2n-1}_\infty(p^r)$ has H-space exponent p^r.*

Proof. By Theorem 6.6, $T^{2n-1}_\infty(p^r)$ is a homotopy commutative, homotopy associative H-space when $p \geq 5$. Consider the composite $\epsilon : P^{2n}(p^r) \xrightarrow{\underline{p}^r} P^{2n}(p^r) \xrightarrow{i} T^{2n-1}_\infty(p^r)$, where i is the inclusion. Note that $\underline{p}^r \simeq *$. Applying Theorem 5.3, there is a unique extension of ϵ to an H-map $T^{2n-1}_\infty(p^r) \longrightarrow T^{2n-1}_\infty(p^r)$. But the p^r-power map and the trivial map are both H-maps extending ϵ, and hence are homotopic. □

Remark. Using different methods, [N4, §6] shows that $\Omega T^{2n-1}_\infty(p^r)$ has H-space exponent p^r for $p \geq 3$ and $r \geq 2$. Thus, as far as homotopy exponents go, the remaining open case is $T^{2n-1}_\infty(3)$. We do know a special case. The homotopy equivalence between $T^{2p-1}_\infty(3)$ and $\Omega S^3\langle 3\rangle$ in Proposition 8.4 together with the fact from [Se] that $\Omega S^3\langle 3\rangle$ has homotopy exponent 3 shows that $T^{2p-1}_\infty(3)$ has homotopy exponent 3. Otherwise, it is only known [Th, 9.5] that $T^{2n-1}(3)$ has homotopy exponent bounded above by 3^2.

Our other applications are of a lesser order, but fit within the scheme of the whole program by sharpening some known results. One immediate corollary of Theorem 5.3 is to show that for $r \geq 1$, there are H-maps $\Omega^2 S^{2n+1} \xrightarrow{\phi_r} S^{2n-1}$ satisfying $\phi_r \circ E^2 \simeq p^r$ and $E^2 \circ \phi_r \simeq \Omega^2 p^r$.

Corollary 8.2. *The homotopy fibration sequence*

$$\Omega^2 S^{2n+1} \xrightarrow{\phi_r} S^{2n-1} \longrightarrow T^{2n-1}_\infty(p^r) \longrightarrow \Omega S^{2n+1}$$

is of H-spaces and H-maps.

Another application is to determine the existence of the homotopy fibration whose questionability in [Th] made the proof of the atomicity of T_k more circuitous. Note, however, that Proposition 8.3 is not a new proof of the atomicity of T_k, for it relies on Theorem 5.3 which in turn relied on the description of R_k in Theorem 2.3 (c), which was formulated in [Th, §8] using the atomicity of T_k. Let $V_k = \prod_{j=1}^{\infty} S^{2np^{k+j}-1}\{p^{r+k+1}\}$.

Proposition 8.3. *For $k \geq 0$, there is a homotopy fibration sequence*

$$S^{2n-1} \times V_k \longrightarrow T_k \longrightarrow \Omega S^{2n+1}.$$

Proof. Consider the pinch map $G_k \xrightarrow{q_k} P^{2np^k+1}(p^{r+k})$. Let ν and μ be the identity and Bockstein maps respectively on $P^{2np^k+1}(p^{r+k})$. By [CMN1], anti-symmetry and Jacobi identities imply the mod p^{r+k} Whitehead products $ad^{p^{k+j}-1}(\nu)(\mu)$, $1 \leq j \leq \infty$, have Bocksteins divisible by p. By Theorem 2.1 (d) the same is true of the indirect lifts $x_j : P^{2np^{k+j}}(p^{r+k}) \longrightarrow G_k$ of the maps $ad^{p^{k+j}-1}(\nu)(\mu)$ to G_k. Now as in Lemma 5.1, the homotopy commutativity and homotopy associativity of ΩS^{2n+1} allows us to choose extensions of the adjoints of the indirect lifts x_j to mod p^{r+k+1} homotopy classes, $y_j : P^{2np^{k+j}-1}(p^{r+k+1}) \longrightarrow \Omega G_k$, which compose trivially to ΩS^{2n+1}. Applying the James construction to each of the maps y_j and restricting $\Omega P^{2np^{k+j}}(p^{r+k+1})$ to its least connected indecomposable factor $S^{2np^{k+j}-1}\{p^{r+k+1}\}$ then gives a map $\Theta : V_k \longrightarrow \Omega G_k$ which composes trivially to ΩS^{2n+1}.

By Theorem 5.3, the H-map $\Omega G_k \xrightarrow{\gamma_k} \Omega S^{2n+1}$ factors as a composite, $\Omega G_k \xrightarrow{\partial_k} T_k \xrightarrow{\bar{\gamma}_k} \Omega S^{2n+1}$, where $\bar{\gamma}_k$ is an H-map. Thus $\partial_k \circ \Theta$ lifts to the homotopy fiber of $\bar{\gamma}_k$. That $(\partial_k \circ \Theta)_*$ is an injection follows as in [Th, §7]. Combining this with the inclusion of S^{2n-1} into the fiber of $\bar{\gamma}_k$ then proves the proposition. $\square$

Finally, we prove that the known homotopy equivalence between $T^{2p-1}_\infty(p)$ and $\Omega S^3\langle 3\rangle$ is an equivalence of H-spaces. Here, $S^3\langle 3\rangle$ is the three-connected cover of S^3, defined as the homotopy fiber of the canonical map $S^3 \longrightarrow K(\mathbf{Z}/p\mathbf{Z}, 3)$, where $K(\mathbf{Z}/p\mathbf{Z}, 3)$ is an Eilenberg-Mac Lane space having as homotopy a single $\mathbf{Z}/p\mathbf{Z}$ summand in dimension 3.

Proposition 8.4. *For $p \geq 3$, there is a homotopy equivalence of H-spaces between $T^{2p-1}_\infty(p)$ and $\Omega S^3\langle 3\rangle$.*

Proof. For $p \geq 3$, S^3 is an H-space. Since $K(\mathbf{Z}/p\mathbf{Z}, 3)$ is an infinite loop space, it too is an H-space. The H-deviation of the canonical map $S^3 \xrightarrow{i} K(\mathbf{Z}/p\mathbf{Z}, 3)$ is the difference $D(i) = \mu(i \times i) - i\mu$. It is null when restricted to $S^3 \vee S^3$, so $D(i)$ is homotopic to a map $S^6 \longrightarrow K(\mathbf{Z}/p\mathbf{Z}, 3)$. But $\pi_6(K(\mathbf{Z}/p\mathbf{Z}, 3)) = 0$, so i is an H-map. Thus the homotopy fiber $S^3\langle 3\rangle$ of i is an H-space, and so $\Omega S^3\langle 3\rangle$ is a homotopy commutative, homotopy associative H-space. Moreover, by [Se] or [CMN2, 1.3], S^3 has homotopy exponent 3, and hence so does $S^3\langle 3\rangle$. The inclusion of the bottom two cells, $P^{2p}(p) \longrightarrow \Omega S^3\langle 3\rangle$, can therefore be extended to an H-map $T^{2p-1}_\infty(p) \xrightarrow{f} \Omega S^3\langle 3\rangle$ by Theorem 5.3. It is easy to check that $\Omega S^3\langle 3\rangle$ has the same homology as $T^{2p-1}_\infty(p)$, so the multiplicative map f determines a homology isomorphism and hence a homotopy equivalence of H-spaces. $\square$

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