TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 353, Number 3, Pages 1151–1167
S 0002-9947(00)02751-3
Article electronically published on November 8, 2000

THE FBI TRANSFORM ON COMPACT $\mathcal{C}^\infty$ MANIFOLDS

JARED WUNSCH AND MACIEJ ZWORSKI

ABSTRACT. We present a geometric theory of the Fourier-Bros-Iagolnitzer transform on a compact $\mathcal{C}^\infty$ manifold M. The FBI transform is a generalization of the classical notion of the wave-packet transform. We discuss the mapping properties of the FBI transform and its relationship to the calculus of pseudodifferential operators on M. We also describe the microlocal properties of its range in terms of the "scattering calculus" of pseudodifferential operators on the noncompact manifold T^*M.

1. INTRODUCTION

In this paper we discuss the Fourier-Bros-Iagolnitzer (FBI) transform on smooth, boundaryless manifolds. The purpose is to revisit the seminal paper of Córdoba and Fefferman [4] in a more geometric way, incorporating the point of view introduced and developed by Sjöstrand (see Sect. 1 of [25] and references given there). We also point out a connection to the scattering calculus of Melrose [18] arising through non-compactness of the cotangent bundle. By reviewing the basic constructions of the FBI transform theory, we hope to make this useful and elegant tool more accessible to a wider audience.

The origins of the FBI transform lie with with the Bargmann transform (see [1]) which intertwines the Schrödinger and Bargmann representations of the Heisenberg group. A microlocal account in the spirit of [22] is given by Sjöstrand in unpublished lecture notes, partly reproduced in Sect. 6 of [13] and Sect. 3 of [26] (see also Guillemin's article [10] and Folland's book [8] for different treatments and many references). It is related to the old tradition of "wave-packets" in quantum mechanics.

The basic Bargmann transform in $\mathbb{R}^n$ is given by

$$T_h u(z) = c_n h^{-3n/2} \int e^{-(x-z)^2/(2h)} u(x) dx \,, \quad z = \alpha_x - i\alpha_\xi \in \mathbb{C}^n \,. \tag{1.1}$$

The range of T_h on the space $L^2(\mathbb{R}^n, dx)$ is the space of holomorphic functions in $L^2(\mathbb{C}^n, e^{-(\operatorname{Im} z)^2} dz d\bar{z})$, and the orthogonal projector on the range is the well known Bergman projector. Its integral kernel (with respect to the Lebesgue measure on $\mathbb{C}^n$) is given by

$$\Pi(z,w) = \tilde{c}_n h^{-n} e^{(-\frac{1}{h}(z-\bar{w})^2 - (\operatorname{Im} w)^2)/h} \,. \tag{1.2}$$

The transformation (1.1) was used by Bros and Iagolnitzer (see [14]) to characterize the analytic wave front set, and it proved extremely useful in the study of

Received by the editors October 26, 1999.
2000 *Mathematics Subject Classification.* Primary 35A22; Secondary 58J40, 81R30.
Key words and phrases. FBI transform, Fourier-Bros-Iagolnitzer transformation, wave-packet.

analyticity of solutions for partial differential equations — see Sect. 9.6 of [12] for an introduction and Sjöstrand's book [22] for a general treatment. Roughly speaking, the behaviour of the holomorphic function $T_h u(z)$ as $h \to 0$ reflects the microlocal properties of u at $(\operatorname{Re} z, -\operatorname{Im} z) \in T^*\mathbb{R}^n \setminus 0$. The relation between the singularities of u and the properties of $T_h u$ can be obtained using the Lebeau inversion formula [15] (see (9.6.8) in [12]). We refer to Delort's book [6] for the development and applications of the theory of the holomorphic FBI transform in $\mathbb{R}^n$.

The definition (1.1) can be interpreted as the heat kernel at time h, applied to u and continued holomorphically in z. This suggests a natural geometric definition for an arbitrary real analytic manifold X: the FBI transform will take a function on the manifold to the holomorphic continuation of the heat kernel applied to the function. The holomorphic continuation lives on the Grauert tube of the manifold, which can be naturally identified with T^*X. This program has been carried out by Golse, Leichtnam and Stenzel [9].

The Córdoba-Fefferman point of view [4] differs by taking a "wave packet" integral kernel rescaled to adapt it to the usual pseudodifferential calculus. Thus the basic transform in $\mathbb{R}^n$ becomes

$$\begin{gathered} Tu(\alpha) = c_n \langle \alpha_\xi \rangle^{\frac{n}{4}} \int e^{i(\alpha_x - x)\alpha_\xi - \langle \alpha_\xi \rangle (\alpha_x - x)^2} u(x) dx \,, \\ \alpha = (\alpha_x, \alpha_\xi) \in T^*\mathbb{R}^n \,, \quad \langle \alpha_\xi \rangle = (1 + |\alpha_\xi|^2)^{\frac{1}{2}} \,. \end{gathered} \tag{1.3}$$

This type of transform has good properties when composed with pseudodifferential operators: it conjugates their action, approximately, to multiplication by symbols; this yields, for instance, a quick proof of the sharp Gårding inequality. The standard holomorphy of $T_h(z)$ in (1.1) has to be modified by replacing $\bar{\partial}_z$ by a pseudodifferential system, $\zeta(\alpha, D_\alpha)$,

$$\zeta(\alpha, \alpha^*) = \alpha_x^* - \alpha_\xi - 2i\langle \alpha_\xi \rangle \alpha_\xi^* + \mathcal{O}((\alpha_\xi^*)^2) \,, \tag{1.4}$$

where α^* denote the dual variables, the operators are microlocalized to a neighbourhood of $\alpha_\xi^* = 0$, and $\langle \alpha_\xi \rangle^{-1}$ plays the rôle of the Planck constant in all expansions (see Sect. 4.3 for a discussion in the language of the scattering calculus on T^*X).

Consequently the projector on the range of T has structure similar to that of the Bergman projector (1.2). That follows essentially from the arguments of Boutet de Monvel and Sjöstrand [3], and for more general transform in $\mathbb{R}^n$ it was investigated by Helffer and Sjöstrand [11], leading to generalized Toeplitz operators in the style of Boutet de Monvel and Guillemin [2].

The phase and the amplitude in (1.3) are real analytic in α, permitting deformations in $T^*\mathbb{C}^n$, the Grauert tube of $T^*\mathbb{R}^n$. This deformation allows the use of weights as in the holomorphic theory of Sjöstrand [22]—see [11] for $\mathbb{R}^n$ and [25] for compact real analytic manifolds. This theory has been extremely useful in the study of resonances (see [11],[24],[25],[29]) and of tunneling (see [16],[28]).

In the applications to tunneling and resonances, the "wave packet" transform (1.3) is modified by allowing the semiclassical parameter h in the phase, just as in (1.1). In this paper we discuss a generalization of such transforms to smooth compact manifolds.

Our setup is similar to that of [25], but we work without analyticity assumptions, and, since no weights are involved, construct the orthogonal projection onto the range of the transform in a direct manner. We further analyze the structure of the Schwartz kernel of this projection and of the associated Toeplitz operators in terms of the "scattering calculus" of pseudodifferential operators developed by Melrose [18]; the composition formula for Toeplitz operators follows from a crude description of the structure of the projection. The appearance of a calculus associated with a non-compact manifold is natural in view of non-compactness of T^*X. The wave front set of a distribution u on X is totally determined by the behaviour of its FBI transform as α_ξ tends to infinity, and we refine this statement to include "scattering wave front set" information on $T_h u$. Since the phase is close to being homogeneous in α_ξ, the more standard characterization of the wave front set obtained by letting h tend to 0 is immediate.

The proofs are all essentially well known (but not easy to locate in an accessible form!) and rely in the $\mathcal{C}^\infty$ case on the complex stationary phase method of Melin and Sjöstrand [17]. Since Sect. 7.7 of [12] contains one of the standard accounts of this theory, we follow it rather than the almost analytic extension method of the original paper.

Throughout the paper, $f \sim g$ means that there exists a fixed $\epsilon > 0$ with $\epsilon g < f < \epsilon^{-1} g$. The letter C will denote a large constant (different each time it appears).

Acknowledgments. The authors are grateful to András Vasy and Steve Zelditch for helpful comments on the manuscript. The first author gratefully acknowledges the support of a Postdoctoral Fellowship of the National Science Foundation of the U.S. The second author thanks the National Science and Engineering Research Council of Canada and the National Science Foundation of the U.S. for partial support.

2. The FBI transform

Let X be a compact, boundaryless, $\mathcal{C}^\infty$ manifold of dimension n with Riemannian metric g. We will let x denote a point in X and α a point in T^*X, in canonical coordinates (α_x, α_ξ). Let $\Delta \subset T^*X \times X$ denote the diagonal $\alpha_x = x$.

Definition 2.1. An *admissible phase function* is a complex-valued function $\phi \in \mathcal{C}^\infty(T^*X \times X)$ such that

1. ϕ is an elliptic polyhomogeneous symbol of order one in α_ξ,
2. $\operatorname{Im}\phi \geq 0$,
3. $d_x\phi|_\Delta = -\alpha_\xi \, dx$,
4. $d_x^2 \operatorname{Im}\phi|_\Delta \sim \langle \alpha_\xi \rangle$,
5. $\phi|_\Delta = 0$.

The Hessian $d_x^2 \operatorname{Im}\phi$ is well defined on the diagonal, since $d_x \operatorname{Im}\phi$ vanishes there.

Note that as a consequence of these properties, we find that near Δ, we can write

$$\phi = \alpha_\xi \cdot (\alpha_x - x) + \langle Q(x,\alpha)(\alpha_x - x), (\alpha_x - x)\rangle \tag{2.1}$$

with Q denoting a symmetric matrix-valued symbol of degree 1 in α_ξ with

$$\operatorname{Im} Q \restriction_{x=\alpha_x} \sim \langle \alpha_\xi \rangle I.$$

Example 2.2. Let exp be the exponential map with respect to the metric g on X, and d the distance function. Then

$$\phi(\alpha, x) = -\alpha_\xi \left(\exp_{\alpha_x}^{-1}(x)\right) + \frac{i}{2}\langle \alpha_\xi \rangle d(\alpha_x, x)^2$$

is an admissible phase function.

We remark that if we were interested in the classical case only (that is, pseudodifferential operators without the semiclassical parameter h), then the assumptions would only be necessary for large values of $|\alpha_\xi|$.

Definition 2.3. A function $a \in \mathcal{C}^\infty(T^*X \times X \times [0,\epsilon)_h)$ for some $\epsilon > 0$ is a symbol in $S^{m,k}_{\text{phg}}(T^*X \times X)$ (or just $S^{m,k}_{\text{phg}}$) if

$$a(\alpha, x; h) \sim h^{-m}(a_k(x,\alpha) + h a_{k-1}(x,\alpha) + \dots),$$

where $a_j(x,\alpha)$ is polyhomogeneous symbol of degree j in α_ξ and the asymptotic expansion is in both h and α_ξ, that is,

$$\left| a - h^{-m}(a_k + \dots + h^j a_{k-j}) \right| \leq C_j h^{-m+j+1} |\alpha_\xi|^{k-j-1} \quad \text{for } |\alpha_\xi| > 1.$$

Symbols on T^*X are defined analogously.

Such a symbol is said to be elliptic if $|a_m(\alpha)| \sim \langle \alpha_\xi \rangle$ uniformly with respect to other variables. We define the quantization of such a symbol as the operator

$$\mathrm{Op}(a)u(x) = \frac{1}{(2\pi h)^n} \int e^{i\alpha_\xi \exp^{-1}_{\alpha_x}(x)/h} a(\alpha, x; h) u(\alpha_x) \chi(x, \alpha_x)\, d\alpha_x\, d\alpha_\xi,$$

where $d\alpha_x\, d\alpha_\xi$ is shorthand for the Liouville volume element on T^*X and χ is a smooth cut-off function supported near $x = \alpha_x$. We note that by the standard results of pseudodifferential calculus, the operators of the form $\mathrm{Op}(a) + R$, for some symbol $a \in S^{m,k}$, and $R = \mathcal{O}(h^\infty) : \mathcal{C}^{-\infty}(X) \to \mathcal{C}^\infty(X)$, form an algebra with a well defined symbol map. We let $\Psi^{m,k}_h(X)$ denote the (bi-filtered) algebra of such operators and $H^{m,k}$ the associated family of Sobolev spaces (see, for example, [7] for further information). The symbol map, $\sigma_{m,k} : \mathrm{Op}(a) + R \mapsto a$, produces the usual short exact sequence:

$$0 \longrightarrow \Psi^{m-1,k-1}_h(X) \longrightarrow \Psi^{m,k}_h(X) \xrightarrow{\sigma_{m,k}} S^{m,k}(T^*X)/S^{m-1,k-1}(T^*X) \longrightarrow 0.$$

Definition 2.4. An *FBI transform* is a map $T_h : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(T^*X)$ given by

$$T_h u(\alpha) = \int_X e^{i\phi(\alpha,x)/h} a(\alpha, x; h) \chi(\alpha, x) dx,$$

where dx is the volume form with respect to g, χ is a cut-off function to a small neighborhood of Δ, $a \in S^{\frac{3n}{4}, \frac{n}{4}}_{\text{phg}}(T^*X \times X)$ is elliptic, and ϕ is an admissible phase function. The support of χ is small enough so that $\operatorname{Im} \phi \leq -d(x,y)^2/C$ on it.

We will often omit the parameter h in discussing T.

3. Basic properties

Since T^*X is equipped with a canonical volume form, we can define T_h^*.

Proposition 3.1. *Let $p \in S^{m,k}_{\text{phg}}(T^*X)$. Then $T_h^* p T_h \in \Psi^{m,k}_h(X)$, and has principal symbol $|a|^2 bp$, where $b \in S^{-\frac{3n}{2}, -\frac{n}{2}}_{\text{phg}}$ is a positive elliptic symbol depending on ϕ only. Furthermore,*

$$\mathrm{WF}' T_h^* p T_h = \text{ess supp}(p).$$

Proof. The Schwartz kernel of $T^* p T$ is given by

(3.1)
$$K(x,y) = \int_{T^*X} e^{i(\phi(\alpha,y) - \bar{\phi}(\alpha,x))/h} \bar{\chi}(\alpha,x) \chi(\alpha,y) p(\alpha;h) \bar{a}(\alpha,x;h) a(\alpha,y;h) p(\alpha)\, d\alpha,$$

where $d\alpha$ is the standard volume form.

We apply the method of complex stationary phase (see [12]) to the phase function $\Phi(\alpha,x,y)=\phi(\alpha,y)-\bar\phi(\alpha,x)$. We will apply stationary phase to the α_x integration in (3.1), leaving the α_ξ integration as the phase integral in the formula for the Schwartz kernel of a pseudodifferential operator. All considerations are local, so we use coordinates in $\mathbb{R}^n$. By (2.1), we can write

(3.2)

$$\Phi=(x-y)\cdot\alpha_\xi+\frac{1}{2}(\langle Q(y,\alpha)(y-\alpha_x),(y-\alpha_x)\rangle-\frac{1}{2}\big\langle\bar Q(x,\alpha)(x-\alpha_x),(x-\alpha_x)\big\rangle),$$

where

$$\operatorname{Im}Q(x,\alpha)|_{x=\alpha_x}\sim\langle\alpha_\xi\rangle I. \tag{3.3}$$

Φ thus satisfies the hypotheses of Theorem 7.7.12 of [12] (complex stationary phase). To apply this theorem, we let $\mathcal{I}$ be the ideal generated by $\partial\Phi/\partial\alpha_{x_i}$ $(i=1,\ldots,n)$. Thus $\mathcal{I}$ is generated by the entries in the vector-valued function

$$\begin{aligned}F&=Q(y,\alpha)\cdot(\alpha_x-y)-\bar Q(x,\alpha)\cdot(\alpha_x-x)\\&\quad+\frac{1}{2}\langle\nabla_{\alpha_x}Q(y,\alpha)(y-\alpha_x),(y-\alpha_x)\rangle\\&\quad-\frac{1}{2}\big\langle\nabla_{\alpha_x}\bar Q(x,\alpha)(x-\alpha_x),(x-\alpha_x)\big\rangle\end{aligned}$$

(recall that Q is a matrix).

Now change to variables $r=(x-y)/2$ and $s=(x+y)/2$. This gives

$$\begin{aligned}F&=Q(-r+s,\alpha)(\alpha_x+r-s)-\bar Q(r+s,\alpha)(\alpha_x-r-s)\\&\quad+\frac{1}{2}\langle\partial_{\alpha_x}Q(-r+s,\alpha)(-r+s-\alpha_x),(-r+s-\alpha_x)\rangle\\&\quad-\frac{1}{2}\big\langle\partial_{\alpha_x}\bar Q(r+s,\alpha)(r+s-\alpha_x),(r+s-\alpha_x)\big\rangle.\end{aligned}$$

Hence

$$\frac{\partial F_i}{\partial\alpha_{x_j}}\restriction_{r=0,s=\alpha_x}=(Q(\alpha_x,\alpha)-\bar Q(\alpha_x,\alpha))_{ij};$$

this expression is $\sim 2i\langle\alpha_\xi\rangle I$, by (3.3). We can now use the Malgrange preparation theorem to conclude that there exist functions X, defined for r and $s-\alpha_x$ small, such that

$$\alpha_x\equiv X(\alpha_\xi,r,s)\pmod{\mathcal{I}}.$$

Expanding in Taylor series, we obtain

$$X\equiv s+i(\operatorname{Im}Q)^{-1}(\operatorname{Re}Q)r+\mathcal{O}(r^2)\pmod{\mathcal{I}}. \tag{3.4}$$

We would now want to substitute this back into (3.2), but this is not meaningful, as the $\mathcal{O}(r^2)$ term is complex valued. However, expanding Φ into Taylor series around $\alpha_x=s$ and using (3.4) give

$$\Phi\equiv(x-y)\cdot\alpha_\xi+\frac{i}{4}\langle\operatorname{Im}P(x-y),(x-y)\rangle+\mathcal{O}(|x-y|^3)\pmod{\mathcal{I}}, \tag{3.5}$$

where

$$P=(I+i(\operatorname{Im}Q)^{-1}(\operatorname{Re}Q))^tQ(I+i(\operatorname{Im}Q)^{-1}(\operatorname{Re}Q))\restriction_{(y;y,\alpha_\xi)}.$$

We can now apply Theorem 7.7.12 of [12] to show that[1]

$$
\begin{aligned}
K \sim \int \left(\det \frac{\Phi''_{\alpha_x \alpha_x}}{2\pi i h} \right)^{-\frac{1}{2}} & e^{(i/h)[(x-y)\cdot\alpha_\xi + \frac{i}{4}\langle \operatorname{Im} P(x-y),(x-y)\rangle + \mathcal{O}(|x-y|^3)]} \\
& \times \sum_j L_j \Big(\bar\chi \chi\, \bar a((x+y)/2 + \mathcal{O}(|x-y|)^2, \alpha_\xi; x; h) \\
& \times\ a((x+y)/2 + \mathcal{O}(|x-y|)^2, \alpha_\xi; y; h) \\
& \times p((x+y)/2 + \mathcal{O}(|x-y|)^2, \alpha_\xi; h) \Big) (h/\langle \alpha_\xi \rangle)^j \, d\alpha_\xi
\end{aligned} \tag{3.6}
$$

where the L_j are differential operators of order $2j$ in α_x depending on the phase as described in [12], and Φ'' refers, by abuse of notation, to the Hessian of the right-hand side of (3.5). Thus we can sum the series asymptotically to $(2\pi h)^{-n}$ times a symbol $c(x, \alpha_\xi; y; h) \in S^{m,k}_{\text{phg}}$ and write, modulo an error in $\Psi_h^{-\infty,-\infty}(X)$,

$$
K = \frac{1}{(2\pi h)^n} \int e^{i(x-y)\cdot\alpha_\xi/h - (\langle \operatorname{Im} P(x-y),(x-y)\rangle + \mathcal{O}(|x-y|^3))/4h} c(x, \alpha_\xi; y; h)\, d\alpha_\xi. \tag{3.7}
$$

Note that the top-order term in c is simply $|a|^2 bp$, where b is a positive elliptic symbol of order $(-3n/2, -n/2)$, depending on ϕ. Note also that c is rapidly decreasing in α_ξ in a conic neighborhood of any (x, α_x, x) such that $(x,\xi) \notin \operatorname{ess\,supp} p$.

We now wish to use the Kuranishi trick to reduce the phase in (3.7) to the standard pseudodifferential phase. We work in local coordinates. Let $\tilde c$ be an almost-analytic extension of c, i.e. an extension into the complex whose antiholomorphic derivatives vanish to infinite order in $|\operatorname{Im} \alpha_\xi / \operatorname{Re} \alpha_\xi|$ (see [17]). Let ψ denote the phase in the integral (3.7). We can split ψ into a piece which does not vanish at $\alpha_\xi = 0$ and one supported in $|\alpha_\xi| < 1/2$, and thus write

$$
\psi = \langle F(x,y,\alpha_\xi)(x-y), \alpha_\xi \rangle + \langle G(x,y,\alpha_\xi)(x-y), \hat e_1 \rangle
$$

for some matrix-valued $F(x,y,\alpha_\xi)$ and $G(x,y,\alpha_\xi)$ whose entries are symbols of order 0 and 1 respectively, and where $\hat e_1 = (1,0,\dots,0) \in \mathbb{R}^n$. Note that $F\restriction_{x=y} = I$ and $G\restriction_{x=y} = 0$. We now set $\eta = F^t \alpha_\xi + G^t \hat e_1$. Thus η is a symbol of order 1, and

$$
\psi = (x-y)\cdot \eta;
$$

near $x = y$, the Jacobian $|\partial\eta/\partial\alpha_\xi|$ does not vanish. In general, η is a complex-valued function away from $\{x = y\}$. Note, however, that we do have

$$
\frac{|\operatorname{Im} \eta|}{|\operatorname{Re} \eta|} = \mathcal{O}(|x-y|).
$$

Let $\Gamma_{x,y}$ denote the image of $\mathbb{R}^n$ under the map $\alpha_\xi \to \eta(x,y,\alpha_\xi)$, so that

$$
K(x,y) = \frac{1}{(2\pi h)^n} \int_{\Gamma_{x,y}} e^{i(x-y)\cdot\eta/h} \tilde c(x, \eta, y; h)\, d\eta,
$$

[1] Technically, we need slightly more than the result quoted: we are performing stationary phase in the large parameter $h^{-1}\langle \alpha_\xi \rangle$; since ϕ was merely a phg symbol in α_ξ, we have a parameter-dependent phase. After we factor out $\langle \alpha_\xi \rangle$, the resulting phase is of course bounded in α_ξ, together with all of its derivatives, and the proof of Theorem 7.7.12 of [12] goes through under these hypotheses.

where we have now absorbed a Jacobian factor (again only dependent on ϕ) into the symbol (and then taken an almost analytic extension of the resulting symbol). Let

$$K_0(x,y) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\eta/h}\tilde{c}(x,\eta,y;h)\,d\eta.$$

For $x \neq y$, integration by parts allows us to write K and K_0 as

$$\frac{h^{|\alpha|}}{(2\pi h)^n}(x-y)^{-\alpha}\int e^{i(x-y)\cdot\eta/h}D_\eta^\alpha\tilde{c}(x,\eta;h)d\eta,$$

where D_η is $1/i$ times the *holomorphic* derivative in η, and the integral is over $\Gamma_{x,y}$ and $\mathbb{R}^n$ respectively.

A further application of Stokes's Theorem gives

(3.8)
$$K(x,y) - K_0(x,y) = \frac{h^{|\alpha|}}{(2\pi h)^n}(x-y)^{-\alpha}\int_\Omega \bar{\partial}_\eta\left(e^{i(x-y)\cdot\eta/h}D_\eta^\alpha\tilde{c}(x,\eta,y;h)\,d\eta\right),$$

where for each x,y, $\Omega_{x,y}$ is the manifold in $\mathbb{C}^n$ given by $\alpha_\xi t + \eta(x,y,\alpha_\xi)(1-t)$, $t \in [0,1]$, $\alpha_\xi \in \mathbb{R}^n$. (Because we may assume that α is large, hence $D_\eta^\alpha\tilde{c}$ has arbitrarily low order, there are no boundary terms at infinity.) Since

$$\left|\bar{\partial}D_\eta^\alpha\tilde{c}\right| \leq C_{N,\alpha}\left(\frac{|\operatorname{Im}\eta|}{|\operatorname{Re}\eta|}\right)^N \leq \tilde{C}_N|x-y|^N$$

for any integer $N \geq 0$, we can, by choosing $\alpha \in \mathbb{N}^n$ in (3.8), estimate

$$K(x,y) - K_0(x,y) = \mathcal{O}(h^\infty|x-y|^\infty);$$

hence the difference is in $\Psi_h^{-\infty,-\infty}(X)$. □

Corollary 3.2.

i. *$T_h : L^2(X) \to L^2(T^*X)$ is bounded for all $h \in [0,\epsilon)$.*
ii. *Let $|a|^2 = 1/b$, where a is the symbol of T_h and b is as in Proposition 3.1. Let $P = \operatorname{Op}(p) \in \Psi_h^{m,k}(X)$. Then*

$$T_h^* p T_h - P \in \Psi_h^{m-1,k-1}(X).$$

Corollary 3.3 (Sharp Gårding inequality)**.** *If $P \in \Psi_h^{m,k}(X)$ and $\operatorname{Re}p \geq 0$, there exists $C \in \mathbb{R}$ such that for all $u \in \mathcal{C}^\infty(M)$,*

$$\langle Pu,u\rangle \geq -C\|u\|^2_{\frac{m-1}{2},\frac{k-1}{2}}.$$

(The notation $\|\bullet\|_{m,k}$ indicates the norm in the semiclassical Sobolev space $H^{m,k}(X)$.)

Proof (*following* [4]). By the preceding corollary,

$$\operatorname{Re}\langle Pu,u\rangle = \langle T^*pTu,u\rangle - \langle Ru,u\rangle$$

with $R \in \Psi_h^{m-1,k-1}(X)$. Thus,

$$\begin{aligned}\langle Pu, u\rangle &= \langle pTu, Tu\rangle - \langle Ru, u\rangle \\ &\geq -\langle Ru, u\rangle \\ &\geq -C' h^{1-m} \Big\langle (I + h^2\Delta)^{-(k-1)/2} Ru, (I + h^2\Delta)^{(k-1)/2} u \Big\rangle \\ &\geq -C\|u\|^2_{\frac{m-1}{2}, \frac{k-1}{2}}\end{aligned}$$

by Corollary 3.2. ∎

Corollary 3.4. *For any FBI transform T_h, there exist an elliptic $D \in \Psi_h^{0,0}(X)$ and $R \in \Psi_h^{-\infty,-\infty}(X)$ such that*

$$T_h^* T_h = D^* D + R.$$

We also need to understand the composition with pseudodifferential operators:

Proposition 3.5. *Let $P \in \Psi_h^{0,0}(X)$ be elliptic with symbol p and let T be an FBI transform with symbol a and phase function ϕ. Then TP is an FBI transform with principal symbol ap and phase function ϕ.*

Proof. The kernel of TP has the form

$$\frac{1}{(2\pi h)^n} \iint e^{i\phi(\alpha,x)/h + i(x-y)\cdot\xi/h} a(\alpha, x; h) p(x, \xi; h) d\xi\, dx.$$

A complex stationary phase computation in ξ and x reduces the kernel to the desired form. The phase is clearly unchanged, as $x = y$ at the critical point. ∎

4. Orthogonal projection

We now discuss the range of T and the corresponding projection operator. Given an admissible phase function ϕ and an elliptic symbol $a \in S_{\mathrm{phg}}^{\frac{3n}{4}, \frac{n}{4}}$ defining an FBI transform T, by Corollary 3.4 we may write

$$T^* T = D^* D + R$$

with D elliptic and R residual. (Note that the principal symbol of D^*D is just $|a|^2 b$, in the notation of Proposition 3.1.) By elliptic regularity, there exists an elliptic operator $W \in \Psi_h^{0,0}(X)$ with $WD - I,\ DW - I \in \Psi_h^{-\infty,-\infty}(X)$. We now let B be the selfadjoint operator

$$B = (TW)(TW)^*.$$

Note that by Proposition 3.5, B is simply $\tilde{T}\tilde{T}^*$, for $\tilde{T}$ an FBI transform with the same phase as T; alternatively we can write $B = T(WW^*)T^*$; note that $\sigma(WW^*) = |a|^{-2} b^{-1}$, so that $\tilde{T}$ is the same as the T of Corollary 3.2, part ii.

We now require some simple functional analysis to show that the range of T is closed.

Lemma 4.1. *If $T : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded operator between Hilbert spaces and T^*T is Fredholm, then T has closed range.*

Proof. Since $\operatorname{Ran} T = \operatorname{Ran} T\restriction_{(\ker T)^\perp}$, we may assume that T is injective. Let $\{u_j\}$ be a sequence in $\mathcal{H}_1$ with $Tu_j \to \phi \in \mathcal{H}_2$. We wish to show that $\phi \in \operatorname{Ran} T$. First we deal with the case in which $\|u_j\|$ is bounded. Then we may replace $\{u_j\}$ by a subsequence converging weakly to $v \in \mathcal{H}_1$. Hence $Tu_j \rightharpoonup Tv$, so $\phi = Tv$.

Thus we may assume that $\|u_j\|$ is unbounded. Since T^*T is a selfadjoint Fredholm operator with no kernel, it is invertible, so there exists $\epsilon > 0$ such that $\|T^*Tu_j\| \geq \epsilon\|u_j\|$. But $T^*Tu_j \to T^*\phi$, i.e. $\|T^*Tu_j\|$ is bounded, a contradiction. □

Thus we can decompose

$$L^2(T^*X) = \operatorname{Ran} T \oplus \ker T^*.$$

Let Π denote the orthogonal projection operator onto $\operatorname{Ran} T$. We claim that Π and B differ only by a residual operator. In fact, on $(\operatorname{Ran} T)^\perp$, B vanishes identically, while

$$B(Tu) = TWW^*T^*Tu = TWW^*(D^*D + R)u = Tu + Su$$

where $S \in \Psi_h^{-\infty,-\infty}(X)$. Thus $B - \Pi$ vanishes on $(\operatorname{Ran} T)^\perp$ and is a residual operator on $\operatorname{Ran} T$ (it maps tempered distributions on T^*X to $h^\infty\mathcal{S}(T^*X)$). We record this result as

Lemma 4.2. *Let T be an FBI transform with symbol a and phase ϕ. There exists $Q \in \Psi_h^{0,0}(X)$ with $\sigma(Q) = |a|^{-2}b^{-1}$ (in the notation of Proposition 3.1), and there exists another FBI transform $\tilde{T}$ with phase ϕ such that*

$$\Pi - TQT^* = \Pi - \tilde{T}\tilde{T}^* = \mathcal{O}(h^\infty) : \mathcal{S}'(T^*X) \to \mathcal{S}(T^*X).$$

As an easy corollary, we obtain the composition formula for Toeplitz operators:

Proposition 4.3 (Composition of Toeplitz Operators)**.** *If $p, q \in S^{0,0}_{\mathrm{phg}}(T^*X)$, then*

$$\Pi pq\Pi - (\Pi p\Pi)(\Pi q\Pi) = \mathcal{O}(h) : L^2(T^*X) \to \langle\alpha_\xi\rangle^{-1}L^2(T^*X).$$

Proof. Let $\tilde{T}$ be as in Lemma 4.2. Then, letting $\equiv$ denote equivalence modulo $\mathcal{O}(h) : L^2(T^*X) \to \langle\alpha_\xi\rangle^{-1}L^2(T^*X)$,

$$\begin{aligned} \Pi pq\Pi &= \tilde{T}\tilde{T}^*pq\tilde{T}\tilde{T}^* \\ &\equiv \tilde{T}PQ\tilde{T}^* \text{ (Cor. 3.2)} \\ &\equiv \tilde{T}\tilde{T}^*p\tilde{T}\tilde{T}^*q\tilde{T}\tilde{T}^* \text{ (Cor. 3.2)} \\ &= \Pi p\Pi q\Pi \text{ (Lemma 4.2).} \quad \square \end{aligned}$$

By Lemma 4.2, to study the microlocal structure of Π it suffices to study the microlocal structure of $\tilde{T}\tilde{T}^*$ for $\tilde{T}$ an FBI transform with the same phase as T. Owing to the noncompactness of T^*X, it is convenient to precede such a study by recalling some of the notions of the "scattering calculus" introduced by Melrose [18] in the study of geometric scattering theory.

4.1. **The scattering calculus.** The scattering calculus of pseudodifferential operators has a long history. It was described and developed on $\mathbb{R}^n$ by Shubin [27], Parenti [20], Cordes [5], and on manifolds by Schrohe [21] and Melrose [18]. We now describe the point of view of Melrose, in which this calculus is a calculus of pseudodifferential operators on a manifold M with boundary, whose kernels can be described as conormal distributions on a blown-up version of $M \times M$. The following presentation is quite sketchy; for details, see [18].

For a manifold Y and a closed embedded submanifold $Z \subset Y$, we let $[Y; Z]$ denote the blowup of Y along Z, in effect obtained by introducing polar coordinates in Y about the submanifold Z (see for instance Sect. 5 of [19] for a self-contained description). In this notation, let $M_{\mathrm{b}}^2 = [M \times M; \partial M \times \partial M]$, and let $M_{\mathrm{sc}}^2 = [M_{\mathrm{b}}^2; \partial \Delta_{\mathrm{b}}]$, where Δ_{b} is the lift of the diagonal of M under the blowdown map $M_{\mathrm{b}}^2 \to M^2$, that is to say, it is the closure of the preimage of the interior of the diagonal under the blowdown. Let sf be the "scattering front face" i.e. the new boundary face introduced by blowing up M_{b}^2 to M_{sc}^2 (see Fig.1). The Schwartz kernels of elements of the scattering calculus $\Psi_{\mathrm{sc}}(M)$ of pseudodifferential operators (acting on half-densities) are then defined to be half-densities on M_{sc}^2 that are

1. conormal to sf,
2. conormal to Δ_{sc}, the lift of Δ_{b} to M_{sc}^2,
3. vanishing to infinite order at $\partial M_{\mathrm{sc}}^2 \backslash \mathrm{sf}$.

That such operators form a calculus (are closed under composition, adjoint, etc.) is shown in [18] by the construction of an appropriate "triple-space" obtained by blowing up M^3; one can also do this by computing "locally" on $\mathbb{R}^n$. On $\mathbb{R}^n$, the calculus can be obtained by quantization of symbols $a(z, \zeta)$ satisfying

$$\left| \partial_z^\alpha \partial_\zeta^\beta a(z, \zeta) \right| \leq C_{\alpha,\beta} \langle z \rangle^{m-|\alpha|} \langle \zeta \rangle^{-l-|\beta|} \tag{4.1}$$

for fixed m, l.

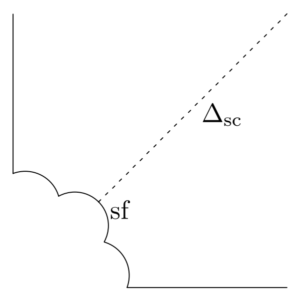

FIGURE 1. The scattering double-space M_{sc}^2, for a one-dimensional manifold M with boundary.

The scattering calculus is the natural calculus of pseudodifferential operators on a manifold with boundary endowed with a "scattering metric" with specified singularity at the boundary; the metric $\langle \alpha_\xi \rangle^2 d\alpha_x^2 + d\alpha_\xi^2$ is such a metric on $\overline{T}^* X$.

Let $\overline{T}^* X$ denote the radial compactification of the cotangent bundle of X. That means a stereographic compactification of each fiber—see [18] and Fig.2. The Schwartz kernel of B, lifted to a distribution on $(\overline{T}^* X)_{\mathrm{sc}}^2$, does not satisfy properties 1–3, but it comes close—it fails to satisfy Property 3 as it is not rapidly decreasing on sf at ∂sf.

Let ${}^{\mathrm{sc}}TM$ be the vector bundle whose sections are vector fields of the form ρV with V tangent to ∂M and ρ a defining function for ∂M. Let ${}^{\mathrm{sc}}T^*M$ denote its dual. Let ${}^{\mathrm{sc}}\overline{T}^* M$ denote the fiberwise radial compactification of ${}^{\mathrm{sc}}T^*M$, with fiber boundary-defining-function σ. Then the symbol map $\sigma_{\mathrm{sc}}^{m,l}$ for the algebra $\Psi_{\mathrm{sc}}^{m,l}(M)$ takes values in (equivalence classes of) conormal distributions on the manifold with corners ${}^{\mathrm{sc}}\overline{T}^* M$. The compactified space, ${}^{\mathrm{sc}}\overline{T}^* M$, has two types of boundary hypersurfaces: those in ${}^{\mathrm{sc}}\overline{T}^*_{\partial M} M$, and those in ${}^{\mathrm{sc}}S^*M$, the unit sphere bundle. The scattering symbol $\sigma_{\mathrm{sc}}(A)$ can be thought of as restriction of $\sigma^m \rho^{-l}$ times the Fourier transform of the kernel of A in ${}^{\mathrm{sc}}T^*M$ (identified with a tubular

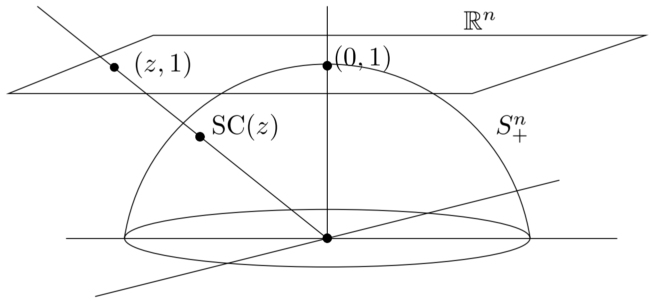

FIGURE 2. The stereographic compactification.

neighbourhood of Δ_{sc}) to these two components: the former gives the "reduced normal symbol"

$$\widehat{\mathrm{N}}_{\mathrm{sc}}(A) = \mathcal{F}(A\restriction_{\mathrm{sf}})$$

and the latter gives (a rescaled version of) the standard pseudodifferential symbol $\sigma(A)$.[2] In the simple case of $\mathbb{R}^n$, these two symbol components are merely the restrictions of the total symbol a in (4.1) to the spheres at infinity in z and ξ.

We can now define the microsupport $\mathrm{WF}'_{\mathrm{sc}}(A) \subset \partial({}^{\mathrm{sc}}\overline{T}^*M)$ as the complement of the set of points at which $\sigma_{\mathrm{sc}}(A)$ vanishes at all orders; the elliptic set $\mathrm{ell}(A) \subset \partial({}^{\mathrm{sc}}\overline{T}^*M)$ is the set of points at which $\sigma^{m,l}_{\mathrm{sc}}(A)$ is invertible.

4.2. **The Schwartz kernel of Π.** In the statement of the following theorem, the definitions of reduced normal symbol $\widehat{\mathrm{N}}_{\mathrm{sc}}$ and the microsupport are applied to the Schwartz kernel of Π, even though Π does not lie in $\Psi_{\mathrm{sc}}(\overline{T}^*X)$. We choose boundary-defining-function $\rho = \langle \alpha_\xi \rangle^{-1}$ for $M = \overline{T}^*X$. Then $d\alpha_x/\rho$ and $d\alpha_\xi$ form a basis of local sections of ${}^{\mathrm{sc}}\overline{T}^*M$. Let the canonical one-form on ${}^{\mathrm{sc}}\overline{T}^*M$ be

$$\mu \frac{d\alpha_x}{\rho} + \alpha^*_\xi \cdot d\alpha_\xi; \tag{4.2}$$

hence we can take $(\alpha_x,\ \alpha_\xi,\ \mu,\ \alpha^*_\xi)$ as coordinates on ${}^{\mathrm{sc}}\overline{T}^*M$; if we wish to work near the boundary of M, we replace α_ξ with ρ and $\hat{\alpha}_\xi = \rho\alpha_\xi$. Let X and Y be functions on M^2 given by

$$X = (\beta_x - \alpha_x)/\rho,$$
$$Y = \beta_\xi - \alpha_\xi.$$

Then X, Y lift to smooth coordinates on M^2_{sc} near sf, vanishing exactly at Δ_{sc}, the lifted diagonal; we use

$$(\rho, \alpha_x, \hat{\alpha}_\xi, X, Y)$$

as a coordinate system on M^2_{sc} valid near sf. Since the scattering vector fields $\rho\partial_{\beta_x}$ and ∂_{β_ξ} respectively lift to ∂_X and ∂_Y on $\mathrm{sf} \subset M^2_{\mathrm{sc}}$, we can identify the interior of sf with ${}^{\mathrm{sc}}T_{\partial M}M$ and hence $(\mathrm{sf}^\circ)^*$ with ${}^{\mathrm{sc}}\overline{T}^*_{\partial M}M$ in such a way that the μ, α^*_ξ coordinates defined in (4.2) are dual to X and Y respectively (see [18] for details).

[2]This is literally true only for the "polyhomogeneous" subcalculus; in general, one must take equivalence classes of conormal distributions, just as in the case of the non-polyhomogeneous calculus on a compact manifold, in which the principal symbol lies in a quotient of symbol classes rather than being a true function on the cosphere bundle.

Theorem 4.4. *The Schwartz kernel of the operator* Π *lifted to* ${}^{\mathrm{sc}}\overline{T}^* M$*, and in the coordinates on* ${}^{\mathrm{sc}}\overline{T}^* M$ *introduced above, is given by*[3]

$$h^{-n}\rho^n c(\rho, \alpha_x, \hat{\alpha}_\xi, \rho X, \rho Y; h) e^{i(X\cdot\hat{\alpha}_\xi + \rho^{-1}\tilde{\psi})/h} \bmod h^\infty \Psi_{\mathrm{sc}}^{-\infty,\infty}(M),$$

where c *is a classical symbol in* ρ *and* h*, and* $c_0(\alpha_x, \hat{\alpha}_\xi; h) = c(0, \alpha_x, \hat{\alpha}_\xi, 0, 0)$ *is a smooth, nonvanishing function on* $\partial M = S^*X$*, and*

$$\tilde{\psi} = \tilde{\psi}(\rho, \alpha_x, \hat{\alpha}_\xi, \rho X, \rho Y) = \mathcal{O}((\rho X)^2 + (\rho Y)^2)$$

satisfies

$$\operatorname{Im}\tilde{\psi} \geq ((\rho X)^2 + (\rho Y)^2)/C\,. \tag{4.3}$$

Hence $\mathrm{WF}'_{\mathrm{sc}}\Pi = \{\mu = \hat{\alpha}_\xi,\ \alpha_\xi^* = 0\}$*, and*

$$\widehat{\mathrm{N}}_{\mathrm{sc}}(\Pi) = (2\pi)^{2n}(\rho/h)^n c_0(\alpha_x, \hat{\alpha}_\xi)\delta(\mu - \hat{\alpha}_\xi)\delta(\alpha_\xi^*).$$

We remark that in the α, β variables, and in local coordinates, the estimate on the imaginary part of the phase can be written as

$$\operatorname{Im}\psi(\alpha,\beta) \sim \langle\alpha_\xi\rangle^{-1}(\alpha_\xi - \beta_\xi)^2 + \langle\alpha_\xi\rangle(\alpha_x - \beta_x)^2\,,$$

which is the homogenized behaviour of the phase of the Bergman projector (1.2). The point made in Theorem 4.4 is that the oscillatory part of the phase is very clear in the "scattering coordinates".

Proof. By Lemma 4.2, it suffices to compute the kernel of $B = \tilde{T}\tilde{T}^*$, where $\tilde{T}$ has the same phase as T. We drop the tilde henceforth, as we are not concerned with the precise form of the symbol.

Absorbing cut-off functions in the symbols for notational simplicity and identifying operators with their kernels, we have

$$B(\alpha,\beta)d\beta = TT^*(\alpha,\beta)d\beta = \int e^{i(\phi(\alpha,x) - \bar{\phi}(\beta,x))/h} a(\alpha, x; h)\bar{a}(\beta, x; h)\, dx\, d\beta$$

(the density factor $d\beta$ occurring since we are letting B act on functions). We write the phase $\Phi = \phi(\alpha, x) - \bar{\phi}(\beta, x)$. Then by (2.1),

$$\begin{aligned}\Phi = (x-\alpha_x)\cdot\alpha_\xi - (x-\beta_x)\cdot\beta_\xi + \frac{1}{2}\langle Q(x,\alpha)(x-\alpha_x), (x-\alpha_x)\rangle \\ - \frac{1}{2}\big\langle \bar{Q}(x,\beta)(x-\beta_x), (x-\beta_x)\big\rangle,\end{aligned}$$

where $\operatorname{Im} Q(x,\alpha)|_{x=\alpha_x} \sim \langle\alpha_\xi\rangle$ and Q is a symbol of degree 1 in α_ξ. Thus,

$$\begin{aligned}\partial_x\Phi = \alpha_\xi - \beta_\xi + Q(x,\alpha)(x-\alpha_x) - \bar{Q}(x,\beta)(x-\beta_x) \\ + \mathcal{O}(\langle\alpha_\xi\rangle|x-\alpha_x|^2 + \langle\beta_\xi\rangle|x-\beta_x|^2).\end{aligned}$$

Let $\vartheta = x - \alpha_x$ and $\varpi = \beta_x - \alpha_x$; we switch to coordinates $(\vartheta, \varpi, \alpha, \beta_\xi)$. We further set $\zeta = \langle\alpha_\xi\rangle^{-1}(\alpha_\xi - \beta_\xi)$, $s = \langle\beta_\xi\rangle/\langle\alpha_\xi\rangle$, $\rho = \langle\alpha_\mu\rangle^{-1}$; let $\hat{\alpha}_\xi = \alpha_\xi/\langle\alpha_\xi\rangle$. Thus $(\rho, \alpha_x, \hat{\alpha}_\xi, \varpi, \zeta)$ are smooth coordinates on $(\overline{T}^*X)^2_{\mathrm{b}}$, with $\rho = 0$ defining the b-front-face, and $\varpi = \zeta = 0$ defining Δ_{b}.

[3] $\Psi_{\mathrm{sc}}^{-\infty,\infty}(X)$ is the ideal of residual operators, in the indexing convention of [18].

In these new coordinates, we write

$$\begin{aligned}\partial_x\Phi = \frac{\zeta}{\rho} + \frac{1}{\rho}P(\rho,\vartheta+\alpha_x,\alpha_x,\hat\alpha_\xi)\vartheta - \frac{1}{\rho}\bar P(\rho,\vartheta+\alpha_x,\varpi+\alpha_x,\hat\alpha_\xi-\zeta)(\vartheta-\varpi)\\ + \rho^{-1}\mathcal{O}(\vartheta^2+\varpi^2)\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\Phi &= \vartheta\cdot\frac{\hat\alpha_\xi}{\rho} - (\vartheta-\varpi)\left(\frac{\hat\alpha_\xi-\zeta}{\rho}\right)\\ &\quad + \frac{1}{2\rho}\langle P(\rho,\vartheta+\alpha_x,\alpha_x,\hat\alpha_\xi)\vartheta,\vartheta\rangle\\ &\quad - \frac{1}{2\rho}\big\langle \bar P(\rho,\vartheta+\alpha_x,\varpi+\alpha_x,\hat\alpha_\xi-\zeta)(\vartheta-\varpi),(\vartheta-\varpi)\big\rangle,\end{aligned} \tag{4.5}$$

where $P(\rho,\dots,\hat\xi)=\rho Q(\dots,\xi)$ is a smooth complex valued function. Thus if we let $\mathcal{I}$ denote the ideal generated by $\partial\Phi/\partial x_i$ for all $i=1,\dots,n$, we apply the Malgrange Preparation Theorem to obtain functions $X_{\vartheta_i}(\rho,\zeta,\varpi,\alpha_x,\hat\alpha_\xi)$ such that

$$\vartheta_i \equiv X_{\varphi_i} \pmod{\mathcal{I}}\,.$$

Expanding (4.4) in Taylor series near the set $\vartheta=\varpi=\zeta=0$ shows that

$$X_\vartheta \equiv \frac{i}{2}(\operatorname{Im}P)^{-1}\left(\zeta+\bar P\varpi+\mathcal{O}(\zeta^2+\varpi^2)\right), \tag{4.6}$$

where P now denotes the restriction $P(\rho,\alpha_x,\alpha_x,\hat\alpha_\xi)$. We can now apply the method of stationary phase (specifically Theorem 7.7.12 of [12]) with large parameter $(h\rho)^{-1}$. To find the phase of B we proceed as in the proof of Proposition 3.1. Formally it amounts to inserting the expression for X_ϑ into (4.5). We write the resulting phase as

(4.7)

$$\begin{aligned}\psi(\rho,\alpha_x,\hat\alpha_\xi,\varpi,\zeta) = \frac{i}{2\rho}\big\langle(\operatorname{Im}P)^{-1}\zeta,\zeta\big\rangle + \frac{i}{2\rho}\big\langle(\operatorname{Im}P)^{-1}\bar P\varpi,\zeta\big\rangle + \frac{\varpi\cdot\hat\alpha_\xi}{\rho} - \frac{\varpi\cdot\zeta}{\rho}\\ - \frac{1}{4\rho}\operatorname{Im}\big\langle(\operatorname{Im}P)^{-1}P(\operatorname{Im}P)^{-1}(\zeta+\bar P\varpi),(\zeta+\bar P\varpi)\big\rangle\end{aligned}$$

where the inner product is still the *real* inner product. Let $P=S+iT$ with S, T real. The Hessian matrix of $\operatorname{Im}\psi$ with respect to ζ, ϖ is given by

$$\begin{aligned}\frac{1}{4\rho}\begin{pmatrix}2T^{-1} & T^{-1}S\\ ST^{-1} & 0\end{pmatrix} - \frac{1}{4\rho}\begin{pmatrix}T^{-1} & 0\\ 0 & -T^{-1}-ST^{-1}S\end{pmatrix}\\ = \frac{1}{4\rho}\begin{pmatrix}I & 0\\ S & I\end{pmatrix}\begin{pmatrix}T^{-1} & 0\\ 0 & T\end{pmatrix}\begin{pmatrix}I & S\\ 0 & I\end{pmatrix};\end{aligned}$$

by our nondegeneracy assumption on the phase, T is positive definite, hence the Hessian is as well, and

$$\operatorname{Im}\psi \ge C^{-1}\frac{\varpi^2+\zeta^2}{\rho} \tag{4.8}$$

for ϖ, ζ sufficiently small.

The symbol resulting from the stationary phase computation is h^{-n} times a smooth function in $(\alpha_x,\hat\alpha_\xi,\rho X,\rho Y)$ (i.e. a symbol of order 0), nonvanishing at $\rho=0$ (i.e. elliptic).

We are interested in the lift of $\psi(\alpha,\beta)$ to $M^2_{\rm sc}$, so we now let $X=\varpi/\rho$, $Y=\zeta/\rho$ be the smooth coordinates on the scattering front-face in $M^2_{\rm sc}$ introduced above. Since $\varpi=\zeta=0$ defines $\Delta_{\rm b}$, the inequality (4.8) holds for all sufficiently small ρ in the coordinate system $(\rho,\alpha_x,\hat\alpha_\xi,X,Y)$ on $(\overline{T}^*X)^2_{\rm sc}$.

By (4.7), we can now write (as usual, identifying the operator with its kernel)

$$B=h^{-n}c(\rho,\alpha_x,\hat\alpha_\xi,\rho X,\rho Y;h)e^{i(X\cdot\hat\alpha_\xi+\rho^{-1}\tilde\psi)/h}\rho^n\,dX\,dY$$

with $c\in\mathcal{C}^\infty$, with $\tilde\psi=\tilde\psi(\rho,\alpha_x,\hat\alpha_\xi,\rho X,\rho Y)=\mathcal{O}((\rho X)^2+(\rho Y)^2)$, and with $\operatorname{Im}\tilde\psi\geq((\rho X)^2+(\rho Y)^2)/C$ for some $C>0$ by (4.8). This permits us to write

$$\begin{aligned}&\mathcal{F}_{X,Y}(B)(\mu,\alpha^*_\xi)\\&\quad=h^{-n}\rho^n\int c(\rho,\alpha_x,\hat\alpha_\xi,\rho X,\rho Y;h)e^{i[(\hat\alpha_\xi-\mu)\cdot X-\alpha^*_\xi\cdot Y]/h}e^{i\tilde\psi(\rho X,\rho Y)/(\rho h)}\,dX\,dY\end{aligned}$$

where the integral is absolutely convergent. Since $h(\hat\alpha_\xi-\mu)^{-1}D_X$ and $-h(\alpha^*_\xi)^{-1}D_Y$ leave $e^{i[(\hat\alpha_\xi-\mu)\cdot X-\alpha^*_\xi\cdot Y]/h}$ invariant, integration by parts shows that $\mathcal{F}_{X,Y}(B)(\mu,\alpha^*_\xi)$ $=\mathcal{O}(h^\infty\rho^\infty)$ away from $\{\mu=\hat\alpha_\xi,\alpha^*_\xi=0\}$. On the other hand, the leading term of $\mathcal{F}_{X,Y}(B)$ at $\rho=0$ is given by

$$\begin{aligned}h^{-n}\rho^n\int c(\alpha_x,\hat\alpha_\xi;h)e^{i[(\hat\alpha_\xi-\mu)\cdot X-\alpha^*_\xi\cdot Y]/h}\,dX\,dY&\\=(2\pi)^{2n}(\rho/h)^nc(\alpha_x,\hat\alpha_\xi;h)\delta(\mu-\hat\alpha_\xi)\delta(\alpha^*_\xi)&\end{aligned}$$

where $c(\alpha_x,\hat\alpha_\xi;h)=c(0,\alpha_x,\hat\alpha_\xi,0,0;h)$ in our previous notation. Thus, we conclude that

$$\begin{aligned}&\mathrm{WF}'_{\rm sc}B=\{\mu=\hat\alpha_\xi,\ \alpha^*_\xi=0\},\\&\widehat{\mathrm{N}}_{\rm sc}(B)=(2\pi)^{2n}(\rho/h)^nc(\alpha_x,\hat\alpha_\xi;h)\delta(\mu-\hat\alpha_\xi)\delta(\alpha^*_\xi),\end{aligned}\tag{4.9}$$

where the second part has to be understood formally since B is not a scattering pseudodifferential operator (as explained above). □

As straightforward consequences of Theorem 4.4, we record the following results:

Proposition 4.5. *Let $p\in S^{0,0}_{\rm phg}(T^*X)$. Then*

$$\Pi p-p\Pi=\mathcal{O}(h^{\frac12}):L^2(d\alpha)\to\langle\alpha_\xi\rangle^{-\frac12}L^2(d\alpha).$$

Proof. Lifting p from the left and the right to $M^2_{\rm sc}$ and subtracting shows that

$$\Pi p-p\Pi=\left(\frac{\rho}{h}\right)^ne^{i(X\cdot\hat\alpha_\xi+\rho^{-1}\tilde\psi(\rho,\alpha_x,\hat\alpha_\xi,\rho X,\rho Y))/h}\mathcal{O}(\rho X,\rho Y)\,dX\,dY.$$

We can estimate this kernel using (4.3), and the conclusion follows from Schur's lemma. □

Corollary 4.6. *Let $p\in S^{0,0}_{\rm phg}(T^*X)$ and let $P\in\Psi_h(X)$ with $\sigma(P)=p$. Then*

$$TP-pT=\mathcal{O}(h^{\frac12}):L^2(dx)\to\langle\alpha_\xi\rangle^{-\frac12}L^2(d\alpha).$$

Proof. Let $Q\in\Psi^{0,0}_h(M)$ be the operator given in Lemma 4.2. In the notation of Proposition 3.1, its symbol is given by $|a|^{-2}b^{-1}$. Then $QT^*pT-P\in\Psi^{-1,-1}_h(X)$, and consequently

$$TP-TQT^*pT=\mathcal{O}(h):L^2(T^*X;d\alpha)\to\langle\alpha_\xi\rangle^{-1}L^2(T^*X;d\alpha)\,.\tag{4.10}$$

By Lemma 4.2, we can replace TQT^* by Π in (4.10), and Proposition 4.5 concludes the proof, as $TP - pT = (TP - \Pi pT) + (\Pi p - p\Pi)T$. $\square$

4.3. **An invariant intepretation.** There is an illuminating invariant way to think of $\mathrm{WF}'_{\mathrm{sc}}\Pi$, which we now discuss. Let $\Lambda \in \mathcal{C}^\infty({}^{\mathrm{sc}}T^*\overline{T}^*X)$ denote the canonical one-form on $M = T^*X$ (extended to the compactification); hence, in the coordinates used previously,

$$\Lambda = \alpha_\xi \cdot d\alpha_x = \hat{\alpha}_\xi \cdot \frac{d\alpha_x}{\rho}.$$

The scattering cotangent bundle of the cotangent bundle has its own canonical one-form; recall that we write it

$$\mu \frac{d\alpha_x}{\rho} + \alpha_\xi^* \cdot d\alpha_\xi,$$

and that this defines the canonical dual coordinates μ and α_ξ^* on ${}^{\mathrm{sc}}\overline{T}^*(M)$. Hence, regarding Λ as a *section* of ${}^{\mathrm{sc}}T^*(M)$, we see that

$$\operatorname{Graph}\Lambda = \left\{\mu = \hat{\alpha}_\xi,\ \alpha_\xi^* = 0\right\},$$

that is,

$$\mathrm{WF}'_{\mathrm{sc}}B = \operatorname{Graph}\Lambda \restriction_{\partial({}^{\mathrm{sc}}T^*M)}.$$

There is a relation between this description and the framework used by Boutet de Monvel and Guillemin [2]. A proper discussion would involve a generalization of the pseudodifferential family (1.4). Here we only point out that the microlocalization to a neighbourhood of $\alpha_\xi^* = 0$ is natural in view of the relation between $\operatorname{Graph}\Lambda$ and B.

The proof of the following proposition on the scattering wavefront set of a "Toeplitz operator" now follows the same lines as the proof of the composition formula for symbols and microsupports of scattering pseudodifferential operators [18], applied formally to the operators p and B:

Theorem 4.7. *Let $p \in S^{m,k}_{\mathrm{phg}}(T^*X)$. Then*

$$\mathrm{WF}'_{\mathrm{sc}}BpB = \mathrm{WF}'_{\mathrm{sc}}p \cap \operatorname{Graph}\Lambda \restriction_{\partial({}^{\mathrm{sc}}T^*M)} = \pi_\Lambda^{-1}(\operatorname{ess\,supp} p)$$

where $\pi_\Lambda : \operatorname{Graph}\Lambda \restriction_{\partial({}^{\mathrm{sc}}T^*M)} \to \partial M$ is the projection map from the graph of Λ, restricted to the fiber boundary of the compactified cotangent bundle; furthermore,

$$\widehat{\mathrm{N}}_{\mathrm{sc}}(BpB) = c_0(\alpha_x, \hat{\alpha}_\xi; h)\, \delta_{\operatorname{Graph}\Lambda}\, \pi_\Lambda^*(p),$$

where c_0 is as in Theorem 4.4.

(Note that multiplication by p is a scattering pseudodifferential operator on $\overline{T}^*X$ which leaves the wavefront set of a distribution unchanged, except where p vanishes.)

Theorem 4.8. *The wavefront set of $u \in \mathcal{D}'(X)$ can be determined from the behaviour of Tu as $\alpha_\xi \to \infty$* or *as $h \to 0$ as follows:*

$$\mathrm{WF}_{\mathrm{sc}}Tu = \pi_\Lambda^{-1}(\mathrm{WF}u) \subset {}^{\mathrm{sc}}\overline{T}^*_{\partial M}M,$$

hence

$$\mathrm{WF}u = \operatorname{ess\,supp} Tu \subset \partial\overline{T}^*X;$$

also,

$$\mathrm{WF}u = \complement\{(x,\xi)\ : \textit{there exists an open set } U \ni (x,\xi) \\ \textit{such that } T_h u(\alpha) = \mathcal{O}(h^\infty) \textit{ for } \alpha \in U\}\,.$$

Proof. The first part follows from the following sequence of equivalent statements: given an open set $S \subset \partial(\overline{T}^* X)$,

$$\begin{aligned} & S \subset \complement(\mathrm{WF}u) \\ & \iff T^* pTu \in \mathcal{C}^\infty(X)\ \forall p : \text{ess supp}\, p \subset S \text{ (Proposition 3.1)} \\ & \iff pTu \in \mathcal{S}(T^*X)\ \forall p : \text{ess supp}\, p \subset S \\ & \iff \mathrm{WF}_{\mathrm{sc}} Tu \cap \pi_\Lambda^{-1}\, \text{ess supp}\, p = \emptyset\ \forall p : \text{ess supp}\, p \subset S \\ & \iff \mathrm{WF}_{\mathrm{sc}} Tu \cap \pi_\Lambda^{-1} S = \emptyset. \end{aligned}$$

The second part follows from the first part applied to a new FBI transform $\widetilde{T}$ with phase

$$\tilde{\phi}(\alpha, x) = \phi(x, \alpha_x, \hat{\alpha}_\xi |\xi|)\,(|\alpha_\xi|/|\xi|)$$

and symbol

$$\tilde{a}(x,\alpha) = a(x, \alpha_x, \hat{\alpha_\xi}|\xi|)\,(|\alpha_\xi|/|\xi|)^{n/4},$$

extended arbitrarily to smooth functions near $\alpha_\xi = 0$. Then

$$\widetilde{T}_h u(\alpha) = T_{h|\xi|/|\alpha_\xi|} u(\alpha_x, \hat{\alpha}_\xi|\xi|),$$

and the limit of $T_h u$ as $h \to 0$ with α_ξ in a neighborhood of ξ coincides with the limit of $\widetilde{T}_1 u(\alpha)$ as $\alpha_\xi \to \infty$ in a positive conic neighborhood of ξ. ☐

References

[1] Bargmann, V. On a Hilbert space of analytic functions and an associated integral transform. *Comm. Pure Appl. Math.* **14**, (1961) 187–214. MR **28:**486

[2] Boutet de Monvel, L. and Guillemin, V., The spectral theory of Toeplitz operators. *Annals of Mathematics Studies*, **99**, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1981. MR **85j:**58141

[3] Boutet de Monvel, L. and Sjöstrand, J., Sur la singularité des noyaux de Bergman et de Szegö, *Journées: Équations aux Dérivées Partielles de Rennes (1975)*,123–164. *Asterisque*, **34-35** Soc. Math. France, Paris, 1976. MR **58:**28684

[4] Córdoba, A. and Fefferman, C., Wave packets and Fourier integral operators, *Comm. Partial Differential Equations*, **3** (1978), 979-1005. MR **80a:**35117

[5] Cordes, H. O., A global parametrix for pseudodifferential operators over $\mathbb{R}^n$ with applications, preprint No. 90, SFB 72, Bonn, 1976.

[6] Delort, J.-M., F.B.I. transformation. Second microlocalization and semilinear caustics. *Lecture Notes in Mathematics* **1522**, Springer-Verlag, Berlin, 1992. MR **93i:**35010

[7] Dimassi, M. and Sjöstrand, J., Spectral asymptotics in the semi-classical limit, *London Math. Soc. Lecture Note Ser.*, **268**, Cambridge Univ. Press, Cambridge, 1999. CMP 2000:07

[8] Folland, G., Harmonic analysis in phase space, *Ann. of Math. Studies*, **122**, Princton Univ. Press, Princeton, NJ, 1989. MR **92k:**22017

[9] Golse, F., Leichtnam, E., and Stenzel, M. Intrinsic microlocal analysis and inversion formulae for the heat equation on compact real-analytic Riemannian manifolds. *Ann. Sci. École Norm. Sup.* **29** (1996), 669–736. MR **97h:**58153

[10] Guillemin, V., Toeplitz operators in n dimensions. *Integral Equations and Operator Theory*, **7** (1984), 145-205. MR **86i:**58130

[11] Helffer, B. and Sjöstrand, J. Resonances en limite semi-classique. *Mém. Soc. Math. France (N.S.)* **24-25** (1986). MR **88i:**81025

[12] Hörmander, L., *Linear partial differential equations*, v.1, Springer Verlag, Berlin.

[13] Hörmander, L., Quadratic hyperbolic operators. *Microlocal analysis and applications (Montecatini Terme, 1989)*, 118–160, *Lecture Notes in Math.* **1495**, Springer, Berlin, 1991. MR **93k:**35187
[14] Iagolnitzer, D., Microlocal essential support of a distribution and decomposition theorems – an introduction. in *Hyperfunctions and theoretical physics (Rencontre, Nice, 1973)*, 121–132. *Lecture Notes in Math.*, **449**, Springer Verlag, Berlin, 1975. MR **52:**11583
[15] Lebeau, G., Fonctions harmoniques et spectre singulier. *Ann. Sci. École Norm. Sup.* **13** (1980), 269–291. MR **81m:**58072
[16] Martinez, A., Estimates on complex interactions in phase space. *Math. Nachr.* **167** (1994), 203–254. MR **95g:**81025
[17] Melin, A. and Sjöstrand, J., "Fourier integral operators with complex-valued phase functions," in *Fourier integral operators and partial differential equations* ed. J. Chazarain, Lecture Notes in Math. **459**, Springer-Verlag, Berlin, 1975, 120–223. MR **55:**4290
[18] Melrose, R. B., Spectral and scattering theory for the Laplacian on asymptotically Euclidean spaces, *Spectral and scattering theory* (M. Ikawa, ed.), Marcel Dekker, 1994, 85–130. MR **95k:**58168
[19] Melrose, R. B. and Zworski, M., Scattering metrics and geodesic flow at infinity. *Invent. Math.* **124** (1996), 389–436. MR **96k:**58230
[20] Parenti, C., Operatori pseudodifferentiali in $\mathbb{R}^n$ e applicazioni, *Ann. Math. Pura Appl.* **93** (1972), 359–389. MR **55:**10838
[21] Schrohe, E., Spaces of weighted symbols and weighted Sobolev spaces on manifolds, *Pseudodifferential operators, Proceedings, Oberwolfach 1986*, Lecture Notes in Mathematics **1256**, Springer-Verlag, 1987, 360–377. MR **89g:**58200
[22] Sjöstrand, J., Singularités analytiques microlocales. *Astérisque*, **95** (1982), 1–166. MR **84m:**58151
[23] Sjöstrand, J., Lecture Notes, *Lund University*, 1985-86.
[24] Sjöstrand, J., Geometric bounds on the density of resonances for semi-classical problems. *Duke Math. J.*, **60** (1990), 1-57. MR **91e:**35166
[25] Sjöstrand, J., Density of resonances for strictly convex analytic obstacles. With an appendix by M. Zworski, *Can. J. Math.,* **48** (1996), 397–447. MR **97j:**35117
[26] Sjöstrand, J. and Zworski, M., The complex scaling method for scattering by strictly convex obstacles, Ark, Mat. **33** (1995), 135-172. MR **96f:**35127
[27] Shubin, M. A., Pseudodifferential operators in $\mathbb{R}^n$, *Dokl. Akad. Nauk SSSR*, **196** (1971), 316–319, *Soviet Math. Dokl.* **12**, No.1 (1971), 147–151. MR **42:**8341
[28] Toth, J., Eigenfunction decay estimates in the quantum integrable case. *Duke Math. J.* **93** (1998), 231–255; **96** (1999), 469. MR **2000e:**58041a,b
[29] Zworski, M., Dimension of the limit set and the density of resonances for convex co-compact hyperbolic surfaces, *Invent. Math.* **136** (1999), 353-409. CMP 99:12

Department of Mathematics, Columbia University, New York, New York 10027
Current address: Department of Mathematics, SUNY at Stony Brook, Stony Brook, New York 11794-3651
E-mail address: `jwunsch@math.sunysb.edu`

Department of Mathematics, University of California, Berkeley, California 94720
E-mail address: `zworski@math.berkeley.edu`