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A UNIVERSAL CONTINUUM OF WEIGHT $\aleph$

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Abstract. We prove that every continuum of weight $\aleph_1$ is a continuous image of the Čech-Stone-remainder R^* of the real line. It follows that under CH the remainder of the half line $[0,\infty)$ is universal among the continua of weight $\mathfrak{c}$ — universal in the 'mapping onto' sense.

We complement this result by showing that 1) under MA every continuum of weight less than $\mathfrak{c}$ is a continuous image of R^*, 2) in the Cohen model the long segment of length ω_2+1 is not a continuous image of R^*, and 3) PFA implies that I_u is not a continuous image of R^*, whenever u is a $\mathfrak{c}$-saturated ultrafilter.

We also show that a universal continuum can be gotten from a $\mathfrak{c}$-saturated ultrafilter on ω, and that it is consistent that there is no universal continuum of weight $\mathfrak{c}$.

Introduction

In [21] Parovičenko proved that every compact Hausdorff space of weight $\aleph_1$ is a continuous image of the space ω^* (the Čech-Stone remainder of the discrete space ω). It follows that under CH the space ω^* is a universal compact space of weight $\mathfrak{c}$, universal in the sense of onto mappings rather than embeddings.

The purpose of this paper is to prove a similar result for $\mathbb{H}^*$, the Čech-Stone remainder of the half line $\mathbb{H}=[0,\infty)$. As $\mathbb{H}^*$ is compact and connected (a *continuum*), the following theorem is the proper analogue of Parovičenko's result.

Theorem 1. *Every continuum of weight $\aleph_1$ is a continuous image of $\mathbb{H}^*$.*

We present proofs of our theorem using each of three quite different approaches to Parovičenko's theorem from the literature that have proven valuable.

It is clear that new ideas are needed to prove Theorem 1, since the fact that ω^* is zero-dimensional appears to be so essential in each of the proofs of Parovičenko's theorem.

We now give brief outlines of the proofs of Parovičenko's theorem, and we indicate how our proofs parallel these arguments; more details, including definitions, will appear later in the paper. We shall finish this introduction with some remarks on universal spaces.

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Two algebraic proofs. The first two proofs of Parovičenko's theorem use Stone's duality, see [23], between compact zero-dimensional spaces and Boolean algebras. One first uses a theorem of Alexandroff, from [3], to find, given the compact space X of weight $\aleph_1$, a compact zero-dimensional space Y of weight $\aleph_1$ and a continuous map from Y onto X, thus reducing the problem to the case of a zero-dimensional range space. Stone's duality then tells us that finding a continuous map from ω^* onto Y is equivalent to finding an embedding of the Boolean algebra $\mathbb{C}$ of clopen subsets of Y into the Boolean algebra $\mathbb{B}$ of clopen subsets of ω^*.

This gives rise to two proofs of Parovičenko's theorem. In the first, one constructs the desired embedding by transfinite induction; this is Parovičenko's original proof. The second proof recognizes that Parovičenko's theorem is in fact a special case of a well-known result about $\aleph_1$-saturated models.

Our parallels of these proofs are naturally based on Wallman's generalization of Stone's duality to the class of distributive lattices, and amount to embedding, for the given continuum, a suitable sublattice of its lattice of closed sets into the lattice of closed subsets of $\mathbb{H}^*$. The key new idea is to extract the hidden model-theoretic aspects of the aforementioned proofs.

A topological proof. The third proof is of a more topological nature; it appears in [6]. It starts by writing the compact space X as an inverse limit, $\varprojlim X_\alpha$, of an ω_1-sequence of compact metrizable spaces. The continuous map from ω^* onto X is obtained as the limit of a family of continuous onto maps $f_\alpha : \omega^* \twoheadrightarrow X_\alpha$. These maps are constructed by transfinite induction and using the following lemma.

Lemma 1. *Let M and N be compact metrizable spaces and let $f : \omega^* \twoheadrightarrow N$ and $g : M \twoheadrightarrow N$ be continuous surjections. Then there is a continuous surjection $h : \omega^* \twoheadrightarrow M$ such that $g \circ h = f$.*

Our third proof will be along these lines, and the main problem is that the above lemma is false if we simply replace ω^* by $\mathbb{H}^*$ and assume that M and N are continua — see Example 3.2. Intriguingly, if g is induced by an appropriate elementary embedding then the analogue of Lemma 1 will hold; indeed, this is the key new feature of our proofs: the introduction of elementarity.

Universal spaces. As with Parovičenko's theorem, Theorem 1 becomes a result about universal spaces if one assumes the Continuum Hypothesis (CH).

Theorem 2 (CH)**.** *The space $\mathbb{H}^*$ is a universal continuum of weight $\mathfrak{c}$.*

It is of interest to note that in the class of compact metrizable spaces the situation is somewhat different from what we just found for spaces of weight $\mathfrak{c}$. On the one hand the Cantor set *is* a universal compact metrizable space; this is a theorem of Alexandroff [2] and Hausdorff [13]. On the other hand, in [25] Waraszkiewicz constructed a family of plane continua such that no metric continuum maps onto all of them, hence there is no universal metric continuum.

Part of the balance is restored by the following theorem, due to Aarts and van Emde Boas [1]; as we shall need this theorem and its proof later, we provide a short argument.

Theorem 3. *The space $\mathbb{H}^*$ maps onto every metric continuum.*

Proof. Consider a metric continuum M and assume it is embedded into the Hilbert cube $Q = [0,1]^\infty$. Choose a countable dense subset A of M and enumerate it

as $\{a_n : n \in \omega\}$. Next choose, for every n, a finite sequence of points $a_n = a_{n,0}, a_{n,1}, \ldots, a_{n,k_n} = a_{n+1}$ such that $d(a_{n,i}, a_{n,i+1}) < 2^{-n}$ for all i — this uses the connectivity of M. Finally, let e be the map from $\mathbb{H}$ to $(0,1] \times Q$ with first coordinate $e_1(t) = 2^{-t}$ and whose second coordinate satisfies $e_2(n + \frac{i}{k_n}) = a_{n,i}$ for all n and i and is (piecewise) linear otherwise.

It is clear that e is an embedding, and one readily checks that $\operatorname{cl} e[\mathbb{H}] = e[\mathbb{H}] \cup \big(\{0\} \times M\big)$; the Čech-Stone extension βe of e maps $\mathbb{H}^*$ onto M. □

Limitations and extensions. As with Parovičenko's results, ours are limited to $\aleph_1$: we show by examples similar to the ones used for ω^* that not necessarily every continuum of weight $\aleph_2$ is a continuous image of $\mathbb{H}^*$ and that Martin's Axiom is not strong enough to guarantee that every continuum of weight $\mathfrak{c}$ is a continuous image of $\mathbb{H}^*$.

On the other hand, in Section 4 we exhibit a few more continua with the same behaviour as $\mathbb{H}^*$, and toward the end of Section 5 we show that from a $\mathfrak{c}$-saturated ultrafilter one can construct a universal continuum of weight $\mathfrak{c}$.

The ultimate limitation occurs at the end of the paper, where we show it is consistent that no universal continuum of weight $\mathfrak{c}$ exists.

We thank the referee for a detailed and thoughtful report; it enabled us to improve both content and presentation of this paper.

1. Algebraic preliminaries

We already indicated that two of our proofs will involve the lattice of closed sets of our continua and that the basis for this is Wallman's generalization, to the class of distributive lattices, of Stone's representation theorem for Boolean algebras. Wallman's representation theorem is as follows.

Theorem 1.1 (Wallman [24]). *If L is a distributive lattice, then there is a compact T_1-space X with a base for its closed sets that is a homomorphic image of L. The homomorphism is an isomorphism iff L is* disjunctive*, which means: if $a \nleq b$ then there is $c \in L$ such that $c \leq a$ and $c \wedge b = 0$.*

The representing space X is Hausdorff iff the lattice L is *normal*; this means that, given $a, b \in L$ with $a \wedge b = 0$, there are $c, d \in L$ such that $c \vee d = 1$, $a \wedge d = 0$ and $b \wedge c = 0$. Note that this mimics the formulation of normality of topological spaces in terms of closed sets only.

Also, if one starts out with a compact Hausdorff space X and a base $\mathcal{C}$ for its closed sets that is closed under finite unions and intersections, then this base $\mathcal{C}$ is a normal lattice and its representing space is X.

The following theorem shows how one can create onto mappings from maps between lattices. In it we use 2^X to denote the family of closed subsets of the space X.

Theorem 1.2. *Let X and Y be compact Hausdorff spaces and let $\mathcal{C}$ be a base for the closed subsets of Y that is closed under finite unions and finite intersections. Then Y is a continuous image of X if and only if there is a map $\varphi : \mathcal{C} \to 2^X$ such that*

1. *$\varphi(\emptyset) = \emptyset$, and if $F \neq \emptyset$ then $\varphi(F) \neq \emptyset$;*
2. *if $F \cup G = Y$ then $\varphi(F) \cup \varphi(G) = X$; and*
3. *if $F_1 \cap \cdots \cap F_n = \emptyset$ then $\varphi(F_1) \cap \cdots \cap \varphi(F_n) = \emptyset$.*

Proof. Necessity is easy: given a continuous onto map $f : X \to Y$, let $\varphi(F) = f^{\leftarrow}[F]$. Note that φ is in fact a lattice-embedding.

To prove sufficiency, let $\varphi : \mathcal{C} \to 2^X$ be given and consider for each $x \in X$ the family $\mathcal{F}_x = \bigl\{F \in \mathcal{C} : x \in \varphi(F)\bigr\}$. We claim that $\bigcap \mathcal{F}_x$ consists of exactly one point. Indeed, by condition 3 the family $\mathcal{F}_x$ has the finite intersection property, so that $\bigcap \mathcal{F}_x$ is nonempty. Next assume that $y_1 \neq y_2$ in Y and take $F, G \in \mathcal{C}$ such that $F \cup G = Y$, $y_1 \notin F$ and $y_2 \notin G$. Then, by condition 2, either $x \in \varphi(F)$ and so $y_1 \notin \bigcap \mathcal{F}_x$, or $x \in \varphi(G)$ and so $y_2 \notin \bigcap \mathcal{F}_x$.

We define $f(x)$ to be the unique point in $\bigcap \mathcal{F}_x$.

To demonstrate that f is continuous and onto, we show that for every closed subset F of Y we have

$$(*) \qquad f^{\leftarrow}[F] = \bigcap \bigl\{\varphi(G) : G \in \mathcal{C}, F \subseteq \operatorname{int} G\bigr\}.$$

This will show that preimages of closed sets are closed and that every fiber $f^{\leftarrow}(y)$ is nonempty.

We first check that the family on the right-hand side has the finite intersection property. Even though F and the complement K of $\bigcap_i \operatorname{int} G_i$ need not belong to $\mathcal{C}$, we can still find G and H in $\mathcal{C}$ such that $G \cap K = H \cap F = \emptyset$ and $H \cup G = Y$. Indeed, apply compactness and the fact that $\mathcal{C}$ is a lattice to find C in $\mathcal{C}$ such that $F \subseteq C \subseteq \bigcap_i \operatorname{int} G_i$ and then $D \in \mathcal{C}$ with $K \subseteq D$ and $D \cap C = \emptyset$; then apply normality of $\mathcal{C}$ to C and D. Once we have G and H we see that for each i we also have $H \cup G_i = Y$, and so $\varphi(H) \cup \varphi(G_i) = X$; combined with $\varphi(G) \cap \varphi(H) = \emptyset$, this gives $\varphi(G) \subseteq \bigcap_i \varphi(G_i)$.

To verify $(*)$, first let $x \in X \setminus f^{\leftarrow}[F]$. As above we find G and H in $\mathcal{C}$ such that $f(x) \notin G$, $H \cup G = Y$ and $H \cap F = \emptyset$. The first property gives us $x \notin \varphi(G)$; the other two imply that $F \subseteq \operatorname{int} G$.

Second, if $F \subseteq \operatorname{int} G$, then we can find $H \in \mathcal{C}$ such that $H \cup G = X$ and $H \cap F = \emptyset$. It follows that if $x \notin \varphi(G)$ we have $x \in \varphi(H)$; hence $f(x) \in H$ and so $f(x) \notin F$. $\square$

An obvious corollary of this theorem is that Y is a continuous image of X iff there is an embedding of the lattice $\mathcal{C}$ into the lattice 2^X.

Theorem 1.2 is formulated so as to be applied in the following way: given the continuum K, we shall take a base $\mathcal{C}$ for its closed sets and define a map as in the theorem into the canonical base $\mathcal{L}$ for the closed sets of $\mathbb{H}^*$, rather than into the family $2^{\mathbb{H}^*}$. We do not know whether it is actually possible to get a lattice-embedding of $\mathcal{C}$ into $\mathcal{L}$.

The canonical base in question is just

$$\mathcal{L} = \{A^* : A \text{ is closed in } \mathbb{H}\}.$$

Here, as is common, A^* abbreviates $\operatorname{cl} A \cap \mathbb{H}^*$. Notice that this is exactly analogous to the collection of clopen subsets of ω^*, but that it also points up another quite stark difference, since unlike in the zero-dimensional case there doesn't appear to be a suitable internally defined base to choose. The base consisting of all zero sets is worth considering, but it lacks the model theoretic saturation properties that we will require.

The following lemma will be the first step in the constructions of both embeddings.

Lemma 1.3. *Let K be a metric continuum and let $x \in K$. There is a map φ from 2^K to $2^{\mathbb{H}}$ such that*

1. *$\varphi(\emptyset) = \emptyset$ and $\varphi(K) = \mathbb{H}$;*
2. *$\varphi(F \cup G) = \varphi(F) \cup \varphi(G)$;*
3. *if $F_1 \cap \cdots \cap F_n = \emptyset$ then $\varphi(F_1) \cap \cdots \cap \varphi(F_n)$ is compact; and*
4. *$\mathbb{N} \subseteq \varphi\big(\{x\}\big)$.*

In addition, if some countable family $\mathcal{C}$ of nonempty closed subsets of K is given in advance, then we can arrange that for every F in $\mathcal{C}$ the set $\varphi(F)$ is not compact.

Proof. As proved in Theorem 3, there is a map from $\mathbb{H}^*$ onto K.

The proof given in [1] is flexible enough to allow us to ensure that the embedding e of $\mathbb{H}$ into $(0,1] \times Q$ is such that $e(n) = \langle 2^{-n}, x\rangle$ for every $n \in \mathbb{N}$ and that for every element y of some countable set C the set $\{t : e_1(t) = y\}$ is cofinal in $\mathbb{H}$ — it is also easy to change the description of e in the proof we gave to produce another e with the desired properties. In our case we let C be a countable subset of K that meets every element of the family $\mathcal{C}$.

We now identify K and $\{0\} \times K$, and define a map $\psi : 2^K \to 2^{I_u \times Q}$ by

$$\psi(F) = \big\{y \in: \mathbb{I} \times Q : d(y, F) \leq d(y, K \setminus F)\big\}.$$

In [19, § 21 XI] it is shown that for all F and G we have

- $\psi(F \cup G) = \psi(F) \cup \psi(G)$;
- $\psi(K) = \mathbb{I} \times Q$ and $\psi(\emptyset) = \emptyset$ — by the fact that $d(y, \emptyset) = \infty$ for all y; and
- $\psi(F) \cap K = F$.

Note that for every $y \in K$ we have $d\big(\langle t, y\rangle, \{y\}\big) = d\big(\langle t, y\rangle, K \setminus \{y\}\big) = t$, and hence $\mathbb{I} \times \{y\} \subseteq \psi\big(\{y\}\big)$.

Now define $\varphi(F) = e^{\leftarrow}\big[\psi(F)\big]$—or rather, after identifying $\mathbb{H}$ and $e[\mathbb{H}]$, set $\varphi(F) = \psi(F) \cap e[\mathbb{H}]$. All desired properties are easily verified: 1 and 2 are immediate; to see that 3 holds, note that if $F_1 \cap \cdots \cap F_n = \emptyset$ then $\operatorname{cl}\varphi(F_1) \cap \cdots \cap \operatorname{cl}\varphi(F_n) \cap K = \emptyset$, so that $\operatorname{cl}\varphi(F_1) \cap \cdots \cap \operatorname{cl}\varphi(F_n)$ is a compact subset of $\mathbb{H}$. That 4 holds follows from the way we chose the values $e(n)$ for $n \in \mathbb{N}$.

Finally, if $F \in \mathcal{C}$ and $y \in C \cap F$, then the cofinal set $\{t : \pi\big(e(t)\big) = y\}$ is a subset of $\varphi(F)$, so that $\varphi(F)$ is not compact. □

Remark 1.4. If φ is as in Lemma 1.3, then the map $\psi : \mathcal{C} \to \mathcal{L}$ defined by

$$\psi : C \mapsto \varphi(C) \mapsto \varphi(C)$$

is as in Theorem 1.2. Observe that ψ is even a $\cup$-homomorphism; to get a lattice homomorphism the map φ would have to satisfy $3'$ instead of 3, where $3'$ reads:

$3'$. the symmetric difference of $\varphi(F \cap G)$ and $\varphi(F) \cap \varphi(G)$ is bounded in $\mathbb{H}$.

We do not know how to achieve $3'$, and at this point there is no extra benefit to be had from this condition, so we leave it be.

2. Two algebraic proofs

We shall now show how to construct, given a continuum K, a map φ from a base for the closed sets of K into the base $\mathcal{L}$ as in Theorem 1.2.

A proof using model theory. Our plan is to find the map φ promised above by an appeal to some machinery from model theory. The first step would be to show that $\mathcal{L}$ is an $\aleph_1$-saturated lattice and hence an $\aleph_2$-universal structure; this latter notion says that every structure of size $\aleph_1$ that is *elementarily equivalent* to $\mathcal{L}$ is embeddable into $\mathcal{L}$. The second step would then be to show that every lattice of size $\aleph_1$ is embeddable into a lattice of size $\aleph_1$ that itself is elementarily equivalent to $\mathcal{L}$.

There are two problems with this approach: 1) we were not able to show that $\mathcal{L}$ is $\aleph_1$-saturated, and 2) Lemma 1.3 does not give a lattice-embedding. We shall deal with these problems in turn. But first we give some definitions from model theory. We shall try to illustrate these definitions by means of linear orderings.

Saturation and universality. Our basic reference for model theory is Hodges' book [14], in particular Chapter 10. A structure (e.g., a field, a group, an ordered set, a lattice) is said to be $\aleph_1$-saturated if, loosely speaking, every countable consistent set of equations has a solution, where a set of equations is consistent if every finite subsystem has a solution. Thus, the ordered set of the reals is *not* $\aleph_1$-saturated because the following countable system of equations, though consistent, does not have a solution: $0 < x$ together with $x < \frac{1}{n}$ $(n \in \mathbb{N})$. On the other hand, any ultrapower $\mathbb{R}^\omega_u$ of $\mathbb{R}$ is $\aleph_1$-saturated as an ordered set — see [14, Theorem 9.5.4]. Such an ultrapower is obtained by taking the power $\mathbb{R}^\omega$, an ultrafilter u on ω and identifying points x and y if $\{n : x_n = y_n\}$ belongs to u. The ordering $<$ is defined in the obvious way: $x < y$ iff $\{n : x_n < y_n\}$ belongs to u. It is relatively easy to show that this gives an $\aleph_1$-saturated ordering; given a countable consistent set of equations $x < a_i$ and $x > b_i$ $(i \in \omega)$, one has to produce a single x that satisfies them all; the desired x can be constructed by a straightforward diagonalization.

Two structures are *elementarily equivalent* if they satisfy the same sentences (formulas without free variables); for example, any two dense linearly ordered sets without end points are elementarily equivalent as linear orders, see [14, Section 2.7]. From this, and the fact that $\aleph_1$-saturated structures are $\aleph_2$-universal, it follows at once that any ultrapower $\mathbb{R}^\omega_u$ contains an isomorphic copy of every $\aleph_1$-sized dense linear order without end points — a result that can also be established directly by a straightforward transfinite recursion. It also follows that $\mathbb{R}^\omega_u$ contains an isomorphic copy of *every* $\aleph_1$-sized linear order: simply make it dense by inserting a copy of the rationals between any pair of neighbours and attach copies of the rationals at the beginning and the end to get rid of possible end points; the resulting ordered set is still of cardinality $\aleph_1$ and can therefore be embedded into $\mathbb{R}^\omega_u$.

This then is how our proof shall run: we identify a certain $\aleph_1$-saturated structure that serves as a base for the closed sets of some quotient of $\mathbb{H}^*$, and we show how to expand every $\aleph_1$-sized structure into one of the same size but elementarily equivalent to our $\aleph_1$-saturated one.

An $\aleph_1$-saturated structure. As mentioned above, we do not know whether $\mathcal{L}$ is $\aleph_1$-saturated. We can however find an $\aleph_1$-saturated sublattice:

$$\mathcal{L}' = \{A^* : A \text{ is closed in } \mathbb{H}, \text{ and } \mathbb{N} \subseteq A \text{ or } \mathbb{N} \cap A = \emptyset\}.$$

This lattice is a base for the closed sets of the space $\boldsymbol{H}$, obtained from $\mathbb{H}^*$ by identifying ω^* to a point.

To see that $\mathcal{L}'$ is $\aleph_1$-saturated we introduce another space, namely $\mathbb{M} = \omega \times \mathbb{I}$, where $\mathbb{I}$ denotes the unit interval. The canonical base $\mathcal{M}$ for the closed sets of $\mathbb{M}^*$ is

naturally isomorphic to the reduced power $(2^{\mathbb{I}})^\omega$ modulo the cofinite filter. It is well-known that this structure is $\aleph_1$-saturated — see [15]. The following substructure $\mathcal{M}'$ is $\aleph_1$-saturated as well:

$$\mathcal{M}' = \{A^* : A \text{ is closed in } \mathbb{M}, \text{ and } \boldsymbol{N} \subseteq A \text{ or } \boldsymbol{N} \cap A = \emptyset\},$$

where $\boldsymbol{N} = \{0,1\} \times \omega$. Indeed, consider a countable set T of equations with constants from $\mathcal{M}'$ such that every finite subset has a solution in $\mathcal{M}'$. We can then add either $\boldsymbol{N} \subseteq x$ or $\boldsymbol{N} \cap x = \emptyset$ to T without losing consistency. Any element of $\mathcal{M}$ that satisfies the expanded T will automatically belong to $\mathcal{M}'$.

We claim that $\mathcal{L}'$ and $\mathcal{M}'$ are isomorphic. To see this, consider the map $q : \mathbb{M} \to \mathbb{H}$ defined by $q(n,x) = n + x$. The Čech-Stone extension of q maps $\mathbb{M}^*$ onto $\mathbb{H}^*$, and it is readily verified that $L \mapsto q^{-1}[L]$ is an isomorphism between $\mathcal{L}'$ and $\mathcal{M}'$. (In topological language: the space $\boldsymbol{H}$ is also obtained from $\mathbb{M}^*$ by identifying $\boldsymbol{N}^*$ to a point.)

A new language. The last point that we have to address is that Lemma 1.3 does not provide a lattice embedding, but rather a map that only partially preserves unions and intersections. This is where Theorem 1.2 comes in: we do not need a full lattice embedding, but only a map that preserves certain identities. We abbreviate these identities as follows:

$$\begin{aligned} J(x,y) &\equiv x \vee y = 1, \\ M_n(x_1,\dots,x_n) &\equiv x_1 \wedge \cdots \wedge x_n = 0. \end{aligned}$$

We can restate the conclusion in Theorem 1.2 in the following manner: Y is a continuous image of X if and only if there is an $\mathfrak{L}$-homomorphism from $\mathcal{C}$ to 2^X, where $\mathfrak{L}$ is the language that has J and the M_n as its predicates *and* where J and the M_n are interpreted as above.

Note that by considering a lattice with 0 and 1 as an $\mathfrak{L}$-structure we do not have to mention 0 and 1 anymore; they are implicit in the predicates. For example, we could define a normal $\mathfrak{L}$-structure to be one in which $M_2(a,b)$ implies $(\exists c,d)\big(M_2(a,d) \wedge M_2(c,b) \wedge J(c,d)\big)$. Then a lattice is normal iff it is normal as an $\mathfrak{L}$-structure.

The proof. Let $\mathcal{C}$ be a base of size $\aleph_1$ for the closed sets of the continuum K. We want to find an $\mathfrak{L}$-structure $\mathcal{D}$ of size $\aleph_1$ that contains $\mathcal{C}$ and that is elementarily equivalent to $\mathcal{L}'$. To this end we consider the diagram of $\mathcal{C}$; that is, we add the elements of $\mathcal{C}$ to our language $\mathfrak{L}$ and we consider the set $D_{\mathcal{C}}$ of all atomic sentences from this expanded language that hold in $\mathcal{C}$. For example, if $a \cap b = \emptyset$ and $c \cup d = K$, then $M_2(a,b) \wedge J(c,d)$ belongs to $D_{\mathcal{C}}$.

To $D_{\mathcal{C}}$ we add the theory $T_{\mathcal{L}'}$ of $\mathcal{L}'$, to get a theory $T_{\mathcal{C}}$. Let $\mathcal{C}'$ be any countable subset of $\mathcal{C}$ and assume, without loss of generality, that $\mathcal{C}'$ is a normal sublattice of $\mathcal{C}$. The Wallman space X of $\mathcal{C}'$ is metrizable, because $\mathcal{C}'$ is countable, and connected because it is a continuous image of K. We may now apply Lemma 1.3 to obtain an $\mathfrak{L}$-embedding of $\mathcal{C}'$ into $\mathcal{L}'$; indeed, condition 4 says that ω^* will be mapped onto a fixed point x of X.

This shows that, for every countable subset $\mathcal{C}'$ of $\mathcal{C}$, the union of $D_{\mathcal{C}'}$ and $T_{\mathcal{L}'}$ is consistent, and so, by the compactness theorem, the theory $T_{\mathcal{C}}$ is consistent. Let $\mathcal{D}$ be a model for $T_{\mathcal{C}}$ of size $\aleph_1$. This model is as required: it satisfies the same sentences as $\mathcal{L}'$ and it contains a copy of $\mathcal{C}$, to wit the set of interpretations of the constants from $\mathcal{C}$.

An extension. It is clear from the above reasoning and Lemma 2.2 below that the following theorem holds.

Theorem 2.1. *If X is a continuum with an $\aleph_1$-saturated base for its closed sets that maps onto every metric continuum, then it maps onto every continuum of weight $\aleph_1$.*

Lemma 2.2. *Suppose that X is a compact space which has a base $\mathcal{C}$ for the closed sets that is $\aleph_1$-saturated as a lattice. Then, if X maps onto a metric space M, there is an embedding of a base $\mathcal{D}$ for the closed subsets of M into $\mathcal{C}$ such as in Theorem 1.2, i.e., an $\mathfrak{L}$-embedding.*

Proof. Simply fix a map f from X onto M and let $\mathcal{D} = \{D_n : n \in \omega\}$ be a base for the closed subsets of M such that each D_n is a regular closed subset of M (that such a base, which in addition is closed under finite unions and intersections, exists is established in [7]). We inductively choose $C_n \in \mathcal{C}$ so that $f[C_n] = D_n$. To begin we may assume that $D_0 = \emptyset$ and $D_1 = M$, so, of course, we let $C_0 = \emptyset$ and $C_1 = X$ (note that since $\mathcal{C}$ is a lattice we may assume that $\emptyset \in \mathcal{C}$). Suppose that we have chosen C_{n-1} and we must choose C_n. Since we plan to guarantee that $f[C_m] = D_m$ for each m, we must have $\bigcap\{C_k : k \in F\} = \emptyset$ for each $F \in [\omega]^{<\omega}$ such that $\bigcap\{D_k : k \in F\} = \emptyset$, and that C_m will be empty whenever D_m is empty. Therefore our only concern is that $C_k \cup C_n$ should be X whenever $D_k \cup D_n$ is equal to M. To accomplish this we first prove that, for each closed $K \subseteq M$, there is a $C \in \mathcal{C}$ such that $f^{\leftarrow}\big[\int(K)\big] \subseteq C$ and $f[C] = K$. Indeed, let $\{F_i : i \in \omega\}$ be a family of closed subsets of M whose union is the interior of K. Also let $\{U_i : i \in \omega\}$ be a neighbourhood base for K in M. For each i, there are J_i and Y_i in $\mathcal{C}$ such that $f^{\leftarrow}[F_i] \subseteq J_i$, $J_i \subseteq f^{\leftarrow}\big[\int(K)\big]$, $f^{\leftarrow}[K] \subseteq Y_i$, and $Y_i \subseteq f^{\leftarrow}[U_i]$. The countable set of equations $\{J_i \subseteq x, x \subseteq Y_i : i \in \omega\}$ is clearly consistent; so, because $\mathcal{C}$ is $\aleph_1$-saturated, we can choose $L \in \mathcal{C}$ that contains each J_i and is contained in each Y_i. Clearly L has the desired properties.

Now for each $k < n$ such that $D_k \cup D_n = M$, let $J_k \in \mathcal{C}$ be such that

$$f^{\leftarrow}\big[\operatorname{int}(M \setminus D_k)\big] \subseteq J_k \quad \text{and} \quad f[J_k] = M \setminus \operatorname{int} D_k$$

(recall that D_k is regular closed). If $D_k \cup D_n$ (for $k < n$) is not equal to M, then let J_k be the empty set. Also let $J_n \in \mathcal{C}$ be any set such that $f(J_n) = D_n$. Now define C_n to be the union of J_n with all J_k for $k < n$. Clearly $C_n \cup C_k = X$ for all $k < n$ such that $D_n \cup D_k = M$, so it suffices to show that $f[C_n] = D_n$. Since $J_n \subseteq C_n$, it follows that $D_n \subseteq f[C_n]$. Let $k < n$ and assume that $D_k \cup D_n = M$. Then $M \setminus \operatorname{int} D_k \subseteq D_n$ (again using that D_k is regular closed). Since $f[J_k] = M \setminus \operatorname{int} D_k$, it follows that $f[C_n] \subseteq D_n$. □

A direct construction. Rather than relying on general model-theoretic results we can also give a direct construction of an $\mathfrak{L}$-embedding of a base for the closed sets of a continuum K into the lattice $\mathcal{L}$. We do this in order to point out where the zero-dimensionality in Parovičenko's theorem hides an important model theoretic step.

Let K be a continuum of weight $\aleph_1$, and fix a base $\mathcal{C}$ for the closed sets of K of size $\aleph_1$ and closed under finite unions and intersections. We also assume $\{x\} \in \mathcal{C}$ for some fixed $x \in K$.

To construct an $\mathfrak{L}$-embedding φ of $\mathcal{C}$ into the base $\mathcal{L}$ for the closed sets of $\mathbb{H}^*$ we construct a map $\psi : \mathcal{C} \to 2^{\mathbb{H}}$ that satisfies

1. if $F \neq \emptyset$, then $\psi(F)$ is not compact;
2. if $F \cup G = K$ then there is a t in $\mathbb{H}$ such that $[t, \infty) \subseteq \psi(F) \cup \psi(G)$; and
3. if $F_1 \cap \cdots \cap F_n = \emptyset$ then $\psi(F_1) \cap \cdots \cap \psi(F_n)$ is compact.

As in Remark 1.4 we then define $\varphi(F) = \psi(F)^*$.

We found it necessary to use elementary substructures to guide the induction; as a consequence we shall build the embedding countably many steps at a time, rather than one.

Elementarity. Consider two structures S and L for the same language (two fields, two groups, two ordered sets) and assume that S is a substructure of L. We call S an *elementary* substructure of L if, loosely speaking, every equation with constants in S that has a solution in L already has a solution in S. Thus, the field $\mathbb{Q}$ of rational numbers is *not* an elementary substructure of $\mathbb{R}$ — consider the equation $x^2 = 2$ — but the algebraic numbers do form an elementary subfield of C — this may be gleaned from [14, Theorem A.5.1]. As one can see from our applications below, an 'equation' can be a system of equations and a 'solution' can be an n-tuple. By way of example we show that an elementary sublattice S of a normal lattice L is normal as well: take disjoint elements a and b of S and consider the system $\{M_2(a, y), M_2(b, x), J(x, y)\}$; because L is normal this system has a solution in L. Elementarity guarantees that it must have a solution in S as well.

Elementary substructures abound because of the Löwenheim-Skolem theorem, which says that every subset A of a structure L can be enlarged to an elementary substructure S; moreover, if the language in question is countable, then one can ensure that $|S| \leq |A| \cdot \aleph_0$, see [14, Chapter 3].

The construction. We well-order $\mathcal{C}$ in type ω_1 and repeatedly apply the Löwenheim-Skolem theorem to obtain an increasing transfinite sequence $\langle \mathcal{C}_\alpha : \alpha \in \omega_1 \rangle$ of elementary sublattices of $\mathcal{C}$, such that $\mathcal{C}_\alpha = \bigcup_{\beta<\alpha} \mathcal{C}_\beta$ for limit α.

As noted above, elementarity guarantees that $\mathcal{C}_0$ is a normal distributive lattice, and so its Wallman representation K_0 is a metric continuum. We may therefore apply Lemma 1.3 and obtain a map $\psi_0 : \mathcal{C}_0 \to 2^{\mathbb{H}}$ such that

- $\psi_0(\emptyset) = \emptyset$, $\psi_0(K) = \mathbb{H}$, and if $F \neq \emptyset$ then $\psi_0(F)$ is not compact;
- $\psi_0(F \cup G) = \psi_0(F) \cup \psi_0(G)$;
- if $F_1 \cap \cdots \cap F_n = \emptyset$ then $\psi_0(F_1) \cap \cdots \cap \psi_0(F_n)$ is compact; and
- $\mathbb{N} \subseteq \psi_0(\{x\})$, where x is the point fixed above.

We shall construct for each α a map $\psi_\alpha : \mathcal{C}_\alpha \to 2^{\mathbb{H}}$ with the desired properties and such that $\psi_\beta = \psi_\alpha \restriction \mathcal{C}_\beta$ whenever $\beta < \alpha$. We already have ψ_0, and at limit stages we simply take unions, so we only have to show how to construct $\psi_{\alpha+1}$ from ψ_α.

To this end enumerate $\mathcal{C}_\alpha$ as $\{c_n : n \in \omega\}$ and $\mathcal{C}_{\alpha+1} \setminus \mathcal{C}_\alpha$ as $\{d_n : n \in \omega\}$. Now for each n we can find, by elementarity, an n-tuple $\langle c_i^n : i < n \rangle$ of nonempty elements of $\mathcal{C}_\alpha$ such that

- for all $i, j < n$: $c_i^n \cup c_j = K$ iff $d_i \cup c_j = K$;
- for all $i, j < n$: $c_i^n \cup c_j^n = K$ iff $d_i \cup d_j = K$;
- for all $i < n$: $x \in c_i^n$ iff $x \in d_i$; and
- for all $f, g \subseteq n$: $\bigcap_{i \in f} c_i^n \cap \bigcap_{j \in g} c_j = \emptyset$ iff $\bigcap_{i \in f} d_i \cap \bigcap_{j \in g} c_j = \emptyset$.

To apply elementarity, simply set up the right system of equations: add $x_i \cup c_j = K$ if $d_i \cup c_j = K$ and add $x_i \cup c_j \neq K$ otherwise; likewise add $x_i \cup x_j = K$ or $x_i \cup x_j \neq K$ et cetera, whenever appropriate.

Given this n-tuple, find $m_n \in \mathbb{N}$ such that

- if $c_i^n \cup c_j = K$ then $[m_n, \infty) \subseteq \psi_\alpha(c_i^n) \cup \psi_\alpha(c_j)$;
- if $c_i^n \cup c_j^n = K$ then $[m_n, \infty) \subseteq \psi_\alpha(c_i^n) \cup \psi_\alpha(c_j^n)$;
- if $\bigcap_{i\in f} c_i^n \cap \bigcap_{j\in g} c_j = \emptyset$ then $\bigcap_{i\in f} \psi_\alpha(c_i^n) \cap \bigcap_{j\in g} \psi_\alpha(c_j) \subseteq [0, m_n)$;
- if $x \in c_i^n$ then $\mathbb{N} \setminus \psi_\alpha(c_i^n) \subseteq [0, m_n)$; and
- if $x \notin c_i^n$ then $\mathbb{N} \cap \psi_\alpha(c_i^n) \subseteq [0, m_n)$.

We can and will assume in addition that $m_{n+1} > m_n$ for all n, and also that $\psi_\alpha(c_i^n) \cap [m_n, m_{n+1}) \neq \emptyset$ for all i; this is possible because $\psi_\alpha(c_i^n)$ is not compact.

Now we define, for all i,

$$\psi_{\alpha+1}(d_i) = \bigcup_{n>i} \psi_\alpha(c_i^n) \cap [m_n, m_{n+1}).$$

We show that $\psi_{\alpha+1}$ has all the required properties.

The first thing to check is that $\psi_{\alpha+1}(d_i)$ is indeed closed. Note that only the points m_{n+1} with $n > i$ can possibly be in $\operatorname{cl}\psi_{\alpha+1}(d_i)$ but not in $\psi_{\alpha+1}(d_i)$. But if $m_{n+1} \in \operatorname{cl}\psi_{\alpha+1}(d_i)$, then $m_{n+1} \in \operatorname{cl}\bigl(\psi_\alpha(c_i^n) \cap [m_n, m_{n+1})\bigr)$ or $m_{n+1} \in \psi_\alpha(c_i^{n+1})$; in the latter case we are done, and in the former case we have $m_{n+1} \in \psi_\alpha(c_i^n)$ and hence $x \in c_i^n$, by the choice of m_n, and so $x \in d_i$. It then follows that $x \in c_i^{n+1}$, and so again $m_{n+1} \in \psi_\alpha(c_i^{n+1}) \subseteq \psi_{\alpha+1}(d_i)$.

It is now a routine matter to check that $\psi_{\alpha+1}$ is as required. For example, if $d_i \cup d_j = K$ then $[m_n, \infty) \subseteq \psi_{\alpha+1}(d_i) \cup \psi_{\alpha+1}(d_j)$, where $n = \max(i, j) + 1$ and if $\bigcap_{i\in f} d_i \cap \bigcap_{j\in g} c_j = \emptyset$ then $\bigcap_{i\in f} \psi_{\alpha+1}(d_i) \cap \bigcap_{j\in g} \psi_\alpha(c_j) \subseteq [0, m_n)$, where $n = \max(f \cup g) + 1$.

In the end $\psi = \bigcup_{\alpha<\omega_1} \psi_\alpha$ is the map that we want.

3. A topological proof

In this section we show how to modify the argument of Błaszczyk and Szymański from [6] to obtain a topological proof of Theorem 1. Although this may seem to be overdoing things somewhat, we feel it is worthwhile to include such an argument, because it illustrates our point that arguments using elementarity are necessary in the context of connected spaces.

Indeed, consider the following statement; it seems at first sight a reasonable property that $\mathbb{H}^*$ should share with ω^*.

False Lemma 3.1. *If f maps $\mathbb{H}^*$ onto the metric continuum X and if g maps the metric continuum Y onto X, then there is a map h from $\mathbb{H}^*$ onto Y such that $f = g \circ h$.*

That this lemma is false is witnessed by the following example.

Example 3.2. Define $f : \mathbb{H} \to S^1$ and $g : \mathbb{I} \to S^1$ by $f(t) = g(t) = e^{2\pi i t}$. Now the Tietze-Urysohn theorem implies that any map from $\mathbb{H}^*$ onto $\mathbb{I}$ is induced by a map $h : \mathbb{H} \to \mathbb{I}$, and if we want to have $f^* = g^* \circ h^*$ then we must have

$$\lim_{t\to\infty} \bigl|f(t) - g\bigl(h(t)\bigr)\bigr| = 0.$$

Choose $N \in \mathbb{N}$ such that $\bigl|f(t) - g\bigl(h(t)\bigr)\bigr| < \frac{1}{4}$ for $t \geq N$, and let $n > N$. The image, under f, of $[n - \frac{1}{4}, n + \frac{1}{4}]$ is the arc from $-i$, via 1, to i. Therefore the image under $g \circ h$ of this interval does not contain -1, and so its image under h does not contain $\frac{1}{2}$; however this last image does contain points above and below $\frac{1}{2}$. This contradicts the continuity of h.

The interested reader can certainly formulate the appropriate elementary substructure version of Lemma 3.1 by examining the inductive step in our direct algebraic construction and the relation between $\mathcal{C}_\alpha$ and $\mathcal{C}_{\alpha+1}$ exploited therein. We choose a more direct approach.

Let K be the continuum of weight $\aleph_1$, and assume that K is a subspace of the Tychonoff cube $\mathbb{I}^{\omega_1}$. We shall construct, by induction, maps $f_\alpha : \mathbb{H} \to \mathbb{I}$ such that the Čech-Stone extension of the diagonal map $f = \triangle_\alpha f_\alpha$ maps $\mathbb{H}^*$ onto K.

For this it suffices to make sure that the following conditions are met:

1. for every open set U around K there is $n \in \mathbb{N}$ such that $f\big[[n,\infty)\big] \subseteq U$, and
2. for every open set U that meets K, the set of $t \in \mathbb{H}$ for which $f(t) \in U$ is cofinal in $\mathbb{H}$.

Again we shall be using a chain of elementary structures to guide our induction, but this time we need substructures of a structure for the language of set theory. Such a structure is $H(\kappa)$, the set of sets that are hereditarily of cardinality less than κ. This means that $x \in H(\kappa)$ iff $\big|\mathrm{trcl}(x)\big| < \kappa$, where

$$\mathrm{trcl}(x) = x \cup \bigcup x \cup \bigcup\bigcup x \cup \cdots$$

is the *transitive closure* of a set x.

Within such an $H(\kappa)$ one can do a lot of set theory, and this makes these sets reasonable substitutes for the whole universe of set theory, which itself cannot be handled as an individual — see [17] for the basic properties of the $H(\kappa)$.

For our construction we take $\kappa = (2^{\mathfrak{c}})^+$; this κ is big enough to guarantee that K, $\mathbb{H}$ and $\mathbb{I}^{\omega_1}$ belong to $H(\kappa)$. We also fix a continuous chain $\langle M_\alpha : \alpha < \omega_1\rangle$ of countable elementary substructures of $H(\kappa)$ such that

- K belongs to M_0, and
- for every α the chain $\langle M_\beta : \beta < \alpha\rangle$ belongs to M_α.

Denote $M_\alpha \cap \omega_1$ by δ_α, and denote the projection of K onto the first δ_α coordinates by K_α.

It would take us too far afield to develop all possible properties of elementary substructures of $H(\kappa)$, but many familiar objects belong automatically to M_0, because they can be seen as the unique solution to an equation with no constants whatsoever or with constants already obtained from such equations: $\emptyset$ is the unique solution to $(\forall y \in x)(y \neq y)$; ω is the smallest set that satisfies $(\emptyset \in x) \wedge \big((\forall y \in x)(y \cup \{y\} \in x)\big)$; ω_1 is the first ordinal for which there is no injective map into ω.

Likewise $\mathbb{H}$, $\mathbb{I}$ and $\mathbb{I}^{\omega_1}$ belong to M_0. The structure M_0 also contains the base $\mathcal{B}$ for $\mathbb{I}^{\omega_1}$ that is built using only open intervals in $\mathbb{I}$ with rational end points, again because this set is the unique solution to an equation involving previously identified elements of M_0; this equation is nothing but the defining sentence for $\mathcal{B}$ written out in full in the language of set theory with $\mathcal{B}$ replaced by x.

We shall construct the maps f_β for $\beta < \delta_\alpha$, by induction on α. To facilitate this we assume that the point $\mathbf{0}$ with all coordinates zero belongs to K; this entails no loss of generality because $\mathbb{I}^{\omega_1}$ is homogenous.

We use the proof of Theorem 3 to find a map $h : \mathbb{H} \to \mathbb{I}^{\delta_0}$ such that conditions 1 and 2 are met with K replaced by K_0. We can and will assume that $h(n) = \mathbf{0}$ for all $n \in \mathbb{N}$ and that $h \in M_1$. We let f_β be the βth coordinate of h, of course.

Now let $\alpha \in \omega_1$, and assume that the maps f_β have been found for $\beta < \delta_\alpha$ such that $e_\alpha = \triangle_{\beta<\delta_\alpha} f_\beta$ belongs to $M_{\alpha+1}$ and meets conditions 1 and 2 with K replaced by K_α, and such that $f_\beta(n) = 0$ for all β and n.

Denote the family of elements of $\mathcal{B}$ that are supported in $\delta_{\alpha+1}$ by $\mathcal{C}$ and observe that $\mathcal{C} \subseteq M_{\alpha+1}$ — every element is defined from finitely many elements of $M_{\alpha+1}$: finitely many ordinals and rational numbers.

Let $\{U_n : n \in \omega\}$ enumerate a decreasing neighbourhood base for $K_{\alpha+1}$, where each U_n is a finite union of elements of $\mathcal{C}$ that all meet $K_{\alpha+1}$ and are arcwise connected. Furthermore we enumerate the elements of $\mathcal{C}$ that meet $K_{\alpha+1}$ as $\{V_n : n \in \omega\}$.

Let $n \in \omega$, and consider U_n and the sets V_i for $i \leq n$. The set of ordinals determined by these open sets is finite and may be split into two parts: $F_{n,0}$, those below δ_α, and $F_{n,1}$, the rest. One can express the facts that $K_{\alpha+1} \subseteq U_n$ and that each V_i meets $K_{\alpha+1}$ in one formula that involves as constants the elements of $F_{n,1}$ and some elements of M_α, notably K, the elements of $F_{n,0}$ and some rational numbers. For example, if $V_0 = \pi^{\leftarrow}_{\eta_1}\big[(q_1, q_2)\big] \cap \pi^{\leftarrow}_{\eta_2}\big[(q_3, q_4)\big]$, then $V_0 \cap K \neq \emptyset$ abbreviates

$$(\exists x \in K)(q_1 < x_{\eta_1} < q_2 \wedge q_3 < x_{\eta_2} < q_4).$$

By elementarity one can find, in the interval $(\max F_{n,0}, \delta_\alpha)$, a finite set of ordinals G_n that can replace $F_{n,1}$ without changing the validity of the formula. Using G_n, we can build open sets U'_n and $V_{n,i}$ for $i \leq n$ that are reflections of U_n and the V_i. In the above example this means that if, say, $\eta_1 \in M_\alpha$ but $\eta_2 \notin M_\alpha$, there must be an ϵ in M_α (larger than η_1) such that

$$(\exists x \in K)(q_1 < x_{\eta_1} < q_2 \wedge q_3 < x_\epsilon < q_4).$$

If this were the case $n = 0$, then we would take $V_{0,0} = \pi^{\leftarrow}_{\eta_1}\big[(q_1, q_2)\big] \cap \pi^{\leftarrow}_{\epsilon}\big[(q_3, q_4)\big]$.

We can therefore find, given a natural number N, two natural numbers k and l bigger than N and such that $e_\alpha\big[[k, \infty)\big] \subseteq U'_n$ and every $V_{n,i}$ meets $e_\alpha\big[[k, l]\big]$.

Use this to find a strictly increasing sequence $\langle k_n \rangle_n$ in $\mathbb{N}$ such that for every n we have $e_\alpha\big[[k_n, \infty)\big] \subseteq U'_n$ and $V_{n,i} \cap e_\alpha\big[[k_n, k_{n+1}]\big] \neq \emptyset$ for $i \leq n$.

Observe that the sets $F_{n,1}$ are increasing and that their union is the interval $[\delta_\alpha, \delta_{\alpha+1})$. So, given β in this interval, the set of n with $\beta \in F_{n,1}$ is a final segment of ω. For each such n let β_n be the element of G_n that corresponds to β. Finally define

$$f_\beta(t) = \begin{cases} 0 & \text{if } t \in [k_n, k_{n+1}] \text{ and } \beta \notin F_{n,1}, \text{ and} \\ f_{\beta_n}(t) & \text{if } t \in [k_n, k_{n+1}] \text{ and } \beta \in F_{n,1}. \end{cases}$$

Elementarity considerations will tell us that these f_β are as required. We have $e_{\alpha+1}\big[[k_n, k_{n+1}]\big] \subseteq U_n$ *because* $e_\alpha\big[[k_n, k_{n+1}]\big] \subseteq U'_n$, and $V_i \cap e_{\alpha+1}\big[[k_n, k_{n+1}]\big] \neq \emptyset$ *because* $V_{n,i} \cap e_\alpha\big[[k_n, k_{n+1}]\big] \neq \emptyset$ whenever $i \leq n$.

Finally note that the whole construction can be considered to have taken place in $M_{\alpha+2}$, so that the result is in $M_{\alpha+2}$ as well.

In the end the full diagonal mapping $\triangle_{\beta<\omega_1} f_\beta$ is as required.

Remark 3.3. Let us note that in all three proofs it was necessary to map the set ω^* to a single point. In the first proof this was done to get an $\aleph_1$-saturated structure, and in the other proofs to make certain inductions work.

4. Other universal continua

We can exhibit several other natural universal continua: for every $n \in \mathbb{N}$ the remainder $(\mathbb{R}^n)^*$ maps onto $\mathbb{H}^*$ and hence onto every continuum of weight $\aleph_1$.

Thus we find, under CH, for every n a universal continuum of weight $\mathfrak{c}$ and of dimension n.

There is one more natural type of continuum that we want to deal with. These we find in the remainder of the space $\mathbb{M} = \omega \times \mathbb{I}$ considered above. For every $u \in \omega^*$ the set

$$\mathbb{I}_u = \bigcap_{U \in u} \mathrm{cl}(U \times \mathbb{I})$$

is a component of $\mathbb{M}^*$ — see [12, Section 2]. Each of our three proofs can be modified to show that every continuum $\mathbb{I}_u$ also maps onto every continuum of weight $\aleph_1$.

This is easiest for the model-theoretic approach. Indeed, it is very easy to show that $\mathbb{I}_u$ has an $\aleph_1$-saturated base for its closed sets: let $\mathcal{I}$ be the family of finite unions of closed intervals in $\mathbb{I}$ with rational end points. The ultrapower $\mathcal{I}^\omega/u$ is naturally isomorphic to a base for the closed subsets of $\mathbb{I}_u$. It is well-known that such ultrapowers are $\aleph_1$-saturated — see [14, Theorem 9.5.4]. The isomorphism between $\mathcal{I}^\omega/u$ and a base for the closed sets of $\mathbb{I}_u$ is obtained by sending $F \in \mathcal{I}^\omega$ to $F_u = \mathbb{I}_u \cap \big(\bigcup_{n\in\omega}(\{n\} \times F_n)\big)^*$; standard properties of the Čech-Stone compactification imply that $F_u = G_u$ iff $\{n : F_n = G_n\} \in u$, and that unions and intersections are preserved under this map.

To show how to adapt the other two proofs we use the map q from $\mathbb{M}^*$ onto $\mathbb{H}^*$ defined by $q\big(\langle n, x\rangle\big) = n + x$. The modifications will be such that the final map will map each of the continua $q[\mathbb{I}_u]$ onto the target continuum as well.

The first step is to adapt the proof of Theorem 3. This is quite straightforward: make sure that, for every n, the finite set $\{a_{n,1}, \dots, a_{n,k_n}\}$ contains the points a_0, $\dots$, a_{n-1}; it then follows easily that $A \subseteq \beta e[\mathbb{I}_u]$ and hence $\beta e[\mathbb{I}_u] = M$ for every u.

The direct algebraic construction. As a result of the adaptation of Theorem 3 we know that we can ensure that for every $F \in \mathcal{C}_0$ there is a natural number n such that $\psi_0(F)$ meets $[m, m+1]$ for every $m \geq n$.

All that is needed to do then is to choose the numbers m_n so large that for every $i < n$ and every $m \geq m_n$ the set $\psi_\alpha(c_i^n)$ meets the interval $[m, m+1]$.

The topological proof. In this case the fact that h maps every $q[\mathbb{I}_u]$ onto K_0 translates to: if V is an open set that meets K_0, then there is a natural number n such that V meets $h\big[[m, m+1]\big]$ for every $m \geq n$.

The modification needed now is to demand each time that $V_{n,i}$ meet $e_\alpha\big[[k, k+1]\big]$ for all $k > k_n$.

In [9] it is shown that, under CH, all continua $\mathbb{I}_u$ are mutually homeomorphic. This gives us a second one-dimensional universal continuum of weight $\mathfrak{c}$. It is different from $\mathbb{H}^*$ because $\mathbb{H}^*$ is indecomposable and $\mathbb{I}_u$ is not.

5. Extensions and limitations

In this section we find analogues of some standard results about continuous images of ω^*.

First we extend our main result by showing that under Martin's Axiom (MA) every continuum of weight less than $\mathfrak{c}$ is a continuous image of $\mathbb{H}^*$.

Next we give some limiting results. We show that in the Cohen model the long segment of length ω_2 is not a continuous image of $\mathbb{H}^*$. This shows that an extra assumption like MA is necessary to push Theorem 1 from $\aleph_1$ to $\aleph_2$. The Cohen model result also shows that in Theorem 1 we cannot replace $\aleph_1$ by $\mathfrak{c}$. We

improve this by showing that the Proper Forcing Axiom (PFA) implies that there is a continuum of weight $\mathfrak{c}$ that is not a continuous image of $\mathbb{H}^*$, so not even MA is strong enough to allow us to prove Theorem 1 for $\mathfrak{c}$ instead of $\aleph_1$.

Martin's Axiom. We show that Martin's Axiom (MA) implies that all continua of weight less than $\mathfrak{c}$ are continuous images of $\mathbb{H}^*$. We need the following result from [8].

Theorem 5.1 (MA)**.** *If X is a compact space of weight less than $\mathfrak{c}$, then there is a compactification of ω with X as its remainder.*

Using this theorem and the ideas from the proof of Theorem 3, it is now quite straightforward to prove the following theorem by an extra application of MA.

Theorem 5.2 (MA)**.** *Let K be a continuum of weight less than $\mathfrak{c}$. Then K is a continuous image of $\mathbb{H}^*$.*

Proof. It suffices to construct a compactification of $\mathbb{H}$ with K as its remainder. Let κ be the weight of K and let $\gamma\omega$ be a compactification of ω with K as its remainder. We embed $\gamma\omega$ into $\mathbb{I}^\kappa$, and we note the following two facts.

1. If U is a neighbourhood of K in $\mathbb{I}^\kappa$, then $\omega \setminus U$ is finite; and
2. K has arbitrarily small arcwise connected neighbourhoods.

The first fact is immediate; the second follows by noting that every basic open set in $\mathbb{I}^\kappa$ is arcwise connected and that every neighbourhood of K contains a neighbourhood that is both a finite union of basic open sets and connected.

There is a map $h : \mathbb{H} \to \mathbb{I}^\kappa$ that almost gives the desired compactification of $\mathbb{H}$: the piecewise linear map that connects the natural numbers in their natural order. This map is defined coordinatewise by $h_\alpha(n+t) = (1-t)n_\alpha + t(n+1)_\alpha$. Here n_α denotes the αth coordinate of the point of $\gamma\omega$ that corresponds to n. We need to rectify a few problems: 1) h need not be one-to-one, 2) the intersection $h[\mathbb{H}] \cap K$ may be nonempty, and 3) K may be a proper subset of $\bigcap_{n\in\omega} \operatorname{cl} h\big[[n,\infty)\big]$.

The first two may be solved by introducing one more coordinate: the map $t \mapsto \big\langle 2^{-t}, h(t)\big\rangle$ from $\mathbb{H}$ to $\mathbb{I} \times \mathbb{I}^\kappa$ does not suffer from problems 1 and 2.

To remedy 3 we introduce the following partial order $\mathbb{P}$: an element of $\mathbb{P}$ is of the form $p = \langle n_p, f_p, \epsilon_p, F_p, U_p\rangle$, where $n_p \in \omega$, U_p is an arcwise connected neighbourhood of K that contains $\omega \setminus n_p$, $f_p : [0, n_p] \to \mathbb{I}^\kappa$ is such that $f_p(n) = n$ for $n \le n_p$ — so in particular $f_p(n_p) \in U_p$, $\epsilon_p > 0$ and $F_p \in [\kappa]^{<\omega}$.

The ordering is defined as follows: $p \le q$ iff $n_p \ge n_q$, $\epsilon_p \le \epsilon_q$, $U_p \subseteq U_q$, $F_p \supseteq F_q$, $f_p\big[[n_q, n_p]\big] \subseteq U_q$ and for all $x \in [0, n_q]$ and for all $\alpha \in F_q$ we have $\big|\pi_\alpha(f_p(x)) - \pi_\alpha(f_q(x))\big| < \epsilon_q$.

It should be clear that by making F_p larger or ϵ_p and U_p smaller one automatically obtains a stronger condition in $\mathbb{P}$; it follows that the following sets are dense in $\mathbb{P}$:

- $D_{1,\alpha} = \{p : \alpha \in F_p\}$, where $\alpha \in \kappa$;
- $D_{2,n} = \{p : \epsilon_p < 2^{-n}\}$, where $n \in \omega$; and
- $D_{3,U} = \{p : U_p \subseteq U\}$, where U is a basic neighbourhood of K.

We must also consider $D_{4,n} = \{p : n_p \ge n\}$, where $n \in \omega$. To see that such a set is dense, let n be given. Now if $p \in \mathbb{P}$ and $n_p < n$, then, because U_p is arcwise connected and $\{n_p, \dots, n\} \subseteq U_p$, we can extend f_p to $g : [0, n] \to \mathbb{I}^\kappa$ with $g\big[[n_p, n]\big] \subseteq U_p$. It follows that $\langle n, g, \epsilon_p, F_p, U_p\rangle$ is in $D_{4,n}$ and below p.

If G is a filter that meets all of the aforementioned κ many dense sets, then the net $\langle f_p \rangle_{p \in G}$ converges uniformly on each compact subset of $\mathbb{H}$ to a map f, which is continuous. For this map f we do have $K = \bigcap_{n \in \omega} \operatorname{cl} f\big[[n, \infty)\big]$. If we add the extra coordinate to f, then we get an embedding $\bar{f} : t \mapsto \big\langle 2^{-t}, f(t) \big\rangle$ of $\mathbb{H}$ into $\mathbb{I} \times \mathbb{I}^\kappa$ that satisfies $\operatorname{cl} \bar{f}[\mathbb{H}] = \bar{f}[\mathbb{H}] \cup K$.

It remains to show that $\mathbb{P}$ satisfies the countable chain condition. So let Q be an uncountable subset of $\mathbb{P}$. We may and do assume without loss of generality that there are fixed n and ϵ such that $n_p = n$ and $\epsilon_p \geq \epsilon$ for all $p \in Q$. We also assume that $\{F_p : p \in Q\}$ is a Δ-system with root F.

With the uniform metric, the function space $C\big([0, n], \mathbb{R}^F\big)$ is separable; hence the set $\{\pi_F \circ f_p : p \in Q\}$ has a complete accumulation point f in it. We therefore assume, again without loss of generality, that for all $p \in Q$ we have $d(f, \pi_F \circ f_p) < \epsilon$.

We show that any two elements of Q are compatible. Indeed, given p and q, define $g : [0, n] \to \mathbb{I}^\kappa$ by

$$g_\alpha(x) = \begin{cases} f_\alpha(x) & \text{if } \alpha \in F, \\ f_{p,\alpha}(x) & \text{if } \alpha \in F_p \setminus F, \\ f_{q,\alpha}(x) & \text{if } \alpha \in F_q \setminus F, \text{ and} \\ h_\alpha(x) & \text{otherwise.} \end{cases}$$

The quintuple $\langle n, g, \epsilon, F_p \cup F_q, U_p \cap U_q \rangle$ is below both p and q. $\square$

The Cohen Model. In [16, Chapter 12] Kunen investigated for what cardinals κ the σ-algebra $\mathcal{R}(\kappa)$ generated by the family of rectangles equals the full power set of κ^2, where a *rectangle* is a set of the form $X \times Y$ with $X, Y \subseteq \kappa$.

The following lemma, from [16], gives a convenient sufficient condition for a set to belong to $\mathcal{R}(\kappa)$.

Lemma 5.3. *If $X \subseteq \kappa^2$ and if one can find, for all α, subsets x_α and y_α of ω such that $\langle \alpha, \beta \rangle \in X$ iff $x_\alpha \cap y_\beta$ is infinite, then X belongs to $\mathcal{R}(\kappa)$.*

Proof. Define $X_n = \{\alpha : n \in x_\alpha\}$ and $Y_n = \{\alpha : n \in y_\alpha\}$. Now observe that $\langle \alpha, \beta \rangle \in X$ iff $(\forall m)(\exists n \geq m)(n \in x_\alpha \cap y_\beta)$. Therefore $X = \bigcap_m \bigcup_{n \geq m} X_n \times Y_n$. $\square$

Using this lemma, Kunen showed that $\mathcal{R}(\omega_1) = \mathcal{P}(\omega_1^2)$ and that MA implies $\mathcal{R}(\kappa) = \mathcal{P}(\kappa^2)$ for every $\kappa < \mathfrak{c}$.

Kunen complemented this by showing that in the model obtained by adding (at least) $\aleph_2$ Cohen reals to a model of CH the set $L = \big\{\langle \alpha, \beta \rangle : \alpha < \beta < \omega_2\big\}$ does not belong to $\mathcal{R}(\omega_2)$.

Using the results mentioned above, it is nearly immediate that in the Cohen model the ordinal space $\omega_2 + 1$ is not a continuous image of ω^*; this is very likely the reason that several authors refer to [16] as the source for this result. Indeed, from a continuous onto mapping one immediately obtains a sequence $\langle x_\alpha : \alpha < \omega_2 \rangle$ of subsets of ω such that $x_\beta \subset^* x_\alpha$ whenever $\alpha < \beta$. For each α let y_α be the complement of x_α. It follows that $\alpha < \beta$ iff $x_\alpha \cap y_\beta$ is infinite, and so, by Lemma 5.3, $L \in \mathcal{R}(\omega_2)$.

We show that a similar result can be formulated and proved for $\mathbb{H}^*$.

Theorem 5.4. *In the model obtained by adding (at least) $\aleph_2$ Cohen reals to a model of CH, the long segment of length ω_2 is not a continuous image of $\mathbb{H}^*$.*

The *long segment* L_{ω_2} of length ω_2 is the set $\omega_2 \times [0,1) \cup \{\omega_2\}$ ordered lexicographically.

We need the following adaptation of Lemma 5.3.

Lemma 5.5. *If there is, with respect to the mod finite order, a strictly increasing ω_2-sequence $\langle f_\alpha : \alpha < \omega_2 \rangle$ in $\mathbb{R}^\omega$, then $L \in \mathcal{R}(\omega_2)$.*

Proof. Much as in the proof of Lemma 5.3, define, for $n \in \omega$ and $q \in \mathbb{Q}$, sets $X_{n,q}$ and $Y_{n,q}$ by $X_{n,q} = \{\alpha : q > f_\alpha(n)\}$ and $Y_{n,q} = \omega_2 \setminus X_{n,q}$. It is easily seen that $L = \bigcap_m \bigcup_{n \geq m, q \in \mathbb{Q}} X_{n,q} \times Y_{n,q}$. □

To show that in the Cohen model there is no map from $\mathbb{H}^*$ onto L_{ω_2} we show that the existence of such a map entails the existence of a sequence $\langle x_\alpha : \alpha < \omega_2 \rangle$ of subsets of ω such that $x_\beta \subset^* x_\alpha$ whenever $\alpha < \beta$, or of a sequence $\langle f_\alpha : \alpha < \omega_2 \rangle$ in $\mathbb{R}^\omega$ as in Lemma 5.5.

So assume $f : \mathbb{H}^* \to L_{\omega_2}$ is a continuous surjection. Choose, for each $\alpha < \omega_2$, a standard open set U_α in $\beta\mathbb{H}$ that contains $f^{\leftarrow}\big[[\alpha+1, \omega_2]\big]$ but whose closure is disjoint from $f^{\leftarrow}\big[[0,\alpha]\big]$. A *standard* open set is one that can be written as $\bigcup_{n\in\omega}(a_n, b_n)$, where $a_n < b_n < a_{n+1}$ for all n and $\lim_{n\to\infty} a_n = \infty$.

Let $\langle I_n : n \in \omega \rangle$ be the sequence of intervals that determines U_0. For each $\alpha < \omega_2$ we let $J_\alpha = \{n : I_{0,n} \cap U_\alpha \neq \emptyset\}$. Now either $\langle J_\alpha : \alpha < \omega_2 \rangle$ is decreasing with respect to $\subset^*$, in which case we are done, or it becomes constant modulo finite sets, and then we may as well assume that it is constant and that $J_\alpha = \omega$ for all α.

Now observe that if $\beta < \alpha$ we have $\operatorname{cl} U_\alpha \cap \mathbb{H}^* \subseteq U_\beta$, so there must be an m such that $\operatorname{cl} U_\alpha \setminus U_\beta \subseteq [0,m]$. We can therefore define functions $f_\alpha : \omega \to \mathbb{R}$ by $f_\alpha(n) = \inf U_\alpha \cap I_n$. The sequence $\langle f_\alpha : \alpha < \omega_2 \rangle$ is strictly increasing with respect to $<^*$, and we are done.

The Proper Forcing Axiom. The obvious question whether MA is strong enough to imply that every compact space of weight $\mathfrak{c}$ is a continuous image of ω^* has a negative answer; see for example the surveys by Baumgartner [4] and van Mill [20].

We show, by obvious modifications of the arguments from [4], that PFA implies there is a continuum of weight $\mathfrak{c}$ that is not a continuous image of $\mathbb{H}^*$.

As PFA implies $\mathsf{MA} + \mathfrak{c} = \aleph_2$ (this was proved by Todorčević, see [5]), we know that there is an $\aleph_2$-saturated ultrafilter u on ω, see [10]. Such an ultrafilter has the property that every ultrapower over it of any countable structure is $\aleph_2$-saturated. We apply this to the linearly ordered set $\mathbb{Q}$ of rational numbers and find an embedding of the lexicographically ordered tree $T = 2^{<\omega_2}$ into $\mathbb{Q}^\omega/u$. One can map $\mathbb{I}_u^\omega$ and hence $\mathbb{Q}_u^\omega$ in a one-to-one fashion into $\mathbb{I}_u$: send $x \in \mathbb{I}^\omega$ to the unique point x_u in the intersection of $\big\{\langle n, x_n \rangle : n \in \omega\big\}^*$ and $\mathbb{I}_u$. It is explained in [12] that this map is order-preserving with respect to a natural linear quasi-order $\prec$ in $\mathbb{I}_u$; it suffices to know that $x_u \prec y_u$ iff $\{n : x_n < y_n\} \in u$. All this gives us an embedding of the tree T into $\mathbb{I}_u$. We show that this standard continuum is not a continuous image of $\mathbb{H}^*$.

Indeed, assume there is a continuous map h from $\mathbb{H}^*$ onto $\mathbb{I}_u$. Let $x \in 2^{\omega_2}$ be such that both $x^{\leftarrow}(0)$ and $x^{\leftarrow}(1)$ are cofinal in ω_2; then x determines a $\langle \omega_2, \omega_2^* \rangle$-gap $\big\langle l(x), r(x) \big\rangle$ in T, where

$$l(x) = \big\{s \in T : (\exists\alpha)(\operatorname{dom} s = \alpha+1, s \restriction \alpha = x \restriction \alpha, s(\alpha) = 0, x(\alpha) = 1)\big\},$$
$$r(x) = \big\{s \in T : (\exists\alpha)(\operatorname{dom} s = \alpha+1, s \restriction \alpha = x \restriction \alpha, x(\alpha) = 0, s(\alpha) = 1)\big\}.$$

Via the embedding of T into $\mathbb{I}_u$ and the map h we can associate with this gap two sequences $\{U_s : s \in l(x)\}$ and $\{V_s : s \in r(x)\}$ of standard open sets with the properties that

1. if $s \in l(x)$ and $t \in r(x)$ then $U_s \cap V_t = \emptyset$;
2. if $s < t$ in $l(x)$ then $\operatorname{cl} U_s \subseteq U_t$; and
3. if $t < s$ in $r(x)$ then $\operatorname{cl} V_s \subseteq V_t$.

Now the proof, given in [4], that in the Boolean algebra $\mathbb{B}$ (the Boolean algebra of clopen subsets of ω^*) there are no $\langle \omega_2, \omega_2^* \rangle$-gaps (assuming PFA) works *mutadis mutandis* to show that no such gaps exist in the lattice $\mathcal{L}$ either. It follows that there must be $L_x \in \mathcal{L}$ such that $U_s \subseteq L_x \subseteq \mathbb{H}^* \setminus V_t$ for all $s \in l(x)$ and $t \in r(x)$.

The assignment $x \mapsto L_x$ is clearly one-to-one; this is impossible, however, as there are $2^{\aleph_2}$ possible x's and only $\aleph_2$ possible L_x's.

Yet another universal continuum. In the previous subsection we used a $\mathfrak{c}$-saturated ultrafilter u to find a continuum of weight $\mathfrak{c}$, to wit $\mathbb{I}_u$, that is not a continuous image of $\mathbb{H}^*$. The methods used in this paper allow us to show that $\mathbb{I}_u$ is in fact a universal continuum of weight $\mathfrak{c}$.

As in Section 4, consider the family $\mathcal{I}$ of all finite unions of closed intervals in $\mathbb{I}$ with rational end points. We have seen that the ultrapower $\mathcal{I}_u = \mathcal{I}^\omega/u$ is a base for the closed sets of $\mathbb{I}_u$; this ultrapower is $\mathfrak{c}$-saturated and hence $\mathfrak{c}^+$-universal. But now the reasoning leading up to Theorem 2.1 can be used to show that $\mathbb{I}_u$ maps onto every continuum of weight $\mathfrak{c}$ or less.

6. Universal continua need not exist

The referee suggested that we address the very natural question of whether there exists, in ZFC, a universal compact space of weight $\mathfrak{c}$.

We have seen that from a $\mathfrak{c}$-saturated ultrafilter u we can construct a universal continuum of weight $\mathfrak{c}$. In a similar fashion one can construct a universal compact Hausdorff space of weight $\mathfrak{c}$: take the ultrapower of the Boolean algebra B of clopen subsets of the Cantor set by the ultrafilter u; the resulting Boolean algebra is $\mathfrak{c}$-saturated and hence $\mathfrak{c}^+$-universal. By Stone's duality this means that its Stone space maps onto every compact Hausdorff space of weight $\mathfrak{c}$. Unfortunately the existence of $\mathfrak{c}$-saturated ultrafilters is equivalent to the conjunction of $\mathsf{MA}_{\text{countable}}$ and $2^{<\mathfrak{c}} = \mathfrak{c}$ — see [11] — so $\mathfrak{c}$-saturated ultrafilters do not provide the definitive answer to the referee's question. In fact the answer to the question is negative: we shall exhibit a model of ZFC in which there is no compact space of weight $\mathfrak{c}$ that maps onto every continuum of weight $\mathfrak{c}$. This shows that there need not be a universal compact space of weight $\mathfrak{c}$, nor a universal continuum of weight $\mathfrak{c}$.

The partial order. We shall employ the notation of Kunen's book [17], except that we shall use dots over symbols to indicate names. We assume that our universe V satisfies GCH and force with the partial order $\mathbb{P} = \operatorname{Fn}(\omega_3 \times \omega_1, 2, \omega_1) \times \operatorname{Fn}(\omega_2, 2)$. We shall use the following properties of $\mathbb{P}$:

1. $\mathbb{P}$ preserves cofinalities and hence cardinals;
2. in the generic extension $V[G]$ we have $2^{\aleph_1} = \aleph_3$ and $\mathfrak{c} = \aleph_2$;
3. $\mathbb{P}$ satisfies the $\aleph_2$-cc and
4. $G = G_1 \times G_2$, where G_1 is V-generic on $\operatorname{Fn}(\omega_3 \times \omega_1, 2, \omega_1)$ and G_2 is $V[G_1]$-generic on $\operatorname{Fn}(\omega_2, 2)$.

The reader should consult [17], in particular Sections VII.6 and VIII.4, for proofs.

Preparations. Now assume that in $V[G]$ there is a universal compact space X of weight $\mathfrak{c}$; by Alexandroff's theorem, from [3], there is a compact zero-dimensional space Y of weight $\mathfrak{c}$ that maps onto X. This means that we can assume that our universal space is zero-dimensional, and hence that it is the Stone space of a Boolean algebra of cardinality $\mathfrak{c}$. We take ω_2 as the underlying set of this Boolean algebra, and denote its partial order by $\preccurlyeq$; for convenience we assume that the ordinal 0 is the 0 of the Boolean algebra.

Back in V we can take, in the terminology of [17], a nice name $\dot{\preccurlyeq}$ for $\preccurlyeq$ and assume that all of $\mathbb{P}$ forces "$\dot{\preccurlyeq}$ determines a Boolean algebra on ω_2 whose Stone space is a universal compact space of weight $\aleph_2$". Because of the $\aleph_2$-cc we can find an $\aleph_2$-sized subset I of ω_3 such that all conditions involved in $\dot{\preccurlyeq}$ have the domain of their first coordinates contained in $I \times \omega_1$; without loss of generality and for notational convenience we take $I = \omega_2$.

A nonimage. We first show how to find a zero-dimensional compact space of weight $\mathfrak{c}$ that is not a continuous image of our space X; in the next subsection we show how to modify the construction so as to obtain a continuum of weight $\mathfrak{c}$.

Consider the tree $T = 2^{<\omega_1}$ from V. In $V[G]$ we consider the following branches of T: for each $\alpha < \omega_2$ define B_α by $B_\alpha(\xi) = (\bigcup G_1)(\omega_2 + \alpha, \xi)$. Put $\mathcal{B} = \{B_\alpha : \alpha \in \omega_2\}$ and consider the tree $T \cup \mathcal{B}$. We turn this tree upside-down to make it generate a Boolean algebra $\mathbb{T}$ of cardinality $\mathfrak{c} = \aleph_2$. We shall show that this Boolean algebra cannot be embedded into $\langle \omega_2, \preccurlyeq \rangle$ or, dually, that its Stone space Y is not a continuous image of X. So we assume $\varphi : T \cup \mathcal{B} \to \omega_2$ is the restriction of an embedding of $\mathbb{T}$ into $\langle \omega_2, \preccurlyeq \rangle$, and proceed to reach a contradiction.

Back in V we take a nice name $\dot{\varphi}$ for φ. We apply the $\aleph_2$-cc once more to find a set J of size $\aleph_1$ such that for all $\langle p, q \rangle$ involved in $\dot{\varphi} \restriction T$ we have $\mathrm{dom}(p) \subseteq J \times \omega_1$. Put $S = \omega_2 \cup J$ and set $R = \omega_3 \setminus S$; apply the product lemma to write $G_1 = G_s \times G_r$, where G_s is V-generic on $\mathrm{Fn}(S \times \omega_1, 2, \omega_1)$ and G_r is $V[G_s]$-generic on $\mathrm{Fn}(R \times \omega_1, 2, \omega_1)$. In fact, the results from [17, Section VIII.1] allow us to conclude that G_r is $V[G_s \times G_1]$-generic on $\mathrm{Fn}(R \times \omega_1, 2, \omega_1)$.

We reach our final contradiction by taking $\delta \in [\omega_2, \omega_2 \cdot 2) \cap R$ and proving that $B_\delta \in V[G_s \times G_1]$. Indeed, let $\gamma = \varphi(B_\delta)$ and observe that B_δ is definable from γ, $\varphi \restriction T$ and $\preccurlyeq$:

$$B_\delta = \bigl\{ s \in T : \gamma \preccurlyeq \varphi(s) \bigr\};$$

this is because T is a tree and $0 \prec \gamma$. The three parameters belong to $V[G_s \times G_1]$; hence so does B_δ.

A continuum. We now show how to modify the nonimage from the previous subsection so as to get a continuum; basically we replace 2 by $\mathbb{Z}$, the set of integers. So we force with $\mathrm{Fn}(\omega_3 \times \omega_2, \mathbb{Z}, \omega_1) \times \mathrm{Fn}(\omega_2, 2)$ and consider the tree $T = \mathbb{Z}^{<\omega_1}$. We give T the lexicographic order:

- if $s \restriction \alpha = t \restriction \alpha$ and $s(\alpha) < t(\alpha)$, then $s \lhd t$; and
- if $s = t \restriction \alpha$, then $t \lhd s$ if $t(\alpha) < 0$, and $s \lhd t$ if $t(\alpha) \geq 0$.

Observe that $\lhd$ is a dense linear order on T.

Now, as before, assume $\preccurlyeq$ is a Boolean partial order on ω_2 and let $\dot{\preccurlyeq}$ be a nice name for it, which we assume to be determined by $\mathrm{Fn}(\omega_2 \times \omega_1, \mathbb{Z}, \omega_1) \times \mathrm{Fn}(\omega_2, 2)$. We use G_1 to define branches $\{B_\alpha : \alpha < \omega_2\}$ of T and use each of these branches to insert a copy of $\mathbb{Q}$ in $\langle T, \lhd \rangle$. Our continuum K is the Dedekind completion of this expanded linearly ordered set.

We show that K is not a continuous image of the Stone space, call it X, of our Boolean algebra. So assume there is a continuous surjection h of X onto K. Every $s \in T \cup \mathcal{B}$ determines a closed interval I_s in K, namely the closure of $\{t : s \subseteq t\}$. Because we used $\mathbb{Z}$ rather than 2, we know that I_t is contained in the interior of I_s whenever $s \subset t$. Therefore, if s is on a successor level, say with immediate predecessor s^-, we can take $\varphi(s) \in \omega_2$ so that $h^{\leftarrow}[I_s] \subseteq \varphi(s) \subseteq h^{\leftarrow}[I_{s^-}]$ (we identify $\varphi(s)$ with the clopen subset of X that it represents). We also choose, for each $\alpha < \omega_2$, one element γ_α whose nonempty clopen set is contained in $h^{\leftarrow}[I_{B_\alpha}]$.

The contradiction is reached exactly as before: one can recover B_δ from γ_δ, φ, T and $\preccurlyeq$ using almost the same formula:

$$B_\delta = \Big\{ s \in T : (\exists t \in T)\big(s \subseteq t \text{ and } \gamma_\delta \preccurlyeq \varphi(t)\big) \Big\}.$$

7. Some questions

Of course every result about continuous images of ω^* suggests the possibility of a corresponding one about continuous images of $\mathbb{H}^*$. The more obvious questions are.

Question 7.1 (compare [22]). Is every perfectly normal continuum a continuous image of $\mathbb{H}^*$?

Question 7.2. Is it consistent with $\neg\mathsf{CH}$ (or better still with $\mathsf{MA} + \neg\mathsf{CH}$) that every continuum of weight $\mathfrak{c}$ is a continuous image of $\mathbb{H}^*$?

A more interesting question, raised by G. D. Faulkner and also suggested by the proofs of Theorems 3 and 5.2, is.

Question 7.3. Is every continuum that is a continuous image of ω^* also a continuous image of $\mathbb{H}^*$?

A still more interesting problem is to generalize Parovičenko's characterization of ω^* to the connected case.

Problem 7.4. Find, assuming CH, topological characterizations of the continua $\mathbb{H}^*$ and $\mathbb{I}_u$.

This problem could have quite interesting consequences, since such a characterization would use some specific base for the closed sets of $\mathbb{H}^*$, and, as we see by the methods in this paper, it is likely that it is first necessary to find a natural description of the first-order theory of the lattice of discrete unions of intervals, or of some other natural base for the closed sets of $\mathbb{H}$.

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