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NONCROSSED PRODUCTS OVER $k_{\mathfrak{p}}(t)$

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Abstract. Noncrossed product division algebras are constructed over rational function fields $k(t)$ over number fields k by lifting from arithmetic completions $k(t)_{\mathfrak{p}}$. The existence of noncrossed products over $\mathfrak{p}$-adic rational function fields $k_{\mathfrak{p}}(t)$ is proved as a corollary.

Introduction

A division algebra over a field is a ring with division whose center is the field. In this paper a division algebra is also finite dimensional over its center. An algebra presentation consists of a vector space basis over the center along with a multiplication rule. A division algebra is called a *crossed product* if it has a presentation with multiplication rule given by a Galois 2-cocycle.

As one of the first steps toward a theory over a particular field one would like to say something intelligent about the possible presentations. Currently, however, a division algebra is either a crossed product, or no one really knows what it is. Thus there are crossed products and *noncrossed products*. Noncrossed products were discovered by Amitsur in 1972 ([1]), long after crossed products.

As a classifying scheme, the crossed product/noncrossed product dichotomy seems almost "good enough". Over many of the fields most often found in number theory, the division algebras are all crossed products. Notable and irritating exceptions include the function field of any curve over a number field, as well as any discretely henselian field whose residue field is a number field.

To a division algebra specialist, the lack of an alternative presentation is unsatisfactory. Finding a noncrossed product over a field is like discovering an uncharted island by plane. The territory remains unexplored even after it is put on the map. Still it is an interesting find, indicating the theory of division algebras over the field is nontrivial. Moreover, the fact that existence is provable hints that the theory is accessible.

In [13], Saltman proved that division algebras over the function fields of $\mathfrak{p}$-adic curves have index bounded by the square of their periods. This result suggests that here is an elemental function field setting with an interesting theory, and that some fine structure is computable. At the same time, whether there exist (yes or no) noncrossed products for the fields in the following table is known in every case but

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one:

$\mathbb{C}$ − no!	$\mathbb{C}((t))$ − no!	$\mathbb{C}(t)$ − no!
$\mathbb{F}$ − no!	$\mathbb{F}((t))$ − no	$\mathbb{F}(t)$ − no
$\mathbb{R}$ − no	$\mathbb{R}((t))$ − no	$\mathbb{R}(t)$ − no
$k_{\mathfrak{p}}$ − no	$k_{\mathfrak{p}}((t))$ − no	$k_{\mathfrak{p}}(t)$ − $\boxed{?}$
k − no	$k((t))$ − yes	$k(t)$ − yes

Here $\mathbb{C}$ and $\mathbb{R}$ are the complex and real numbers, and $\mathbb{F}$, $k_{\mathfrak{p}}$, and k are finite, $\mathfrak{p}$-adic, and number fields, respectively. Double parentheses denote henselian with respect to a discrete parameter t. From a Brauer group viewpoint this array grows roughly more complicated as it moves away from the northwest corner (the Brauer group is actually trivial in cases where the "no" is exclaimed). The block of six in the southeast corner all have (tame) cohomological dimension 3, those exclaimed at most 1, the rest 2.

This paper fills the gap in the table by proving the existence of noncrossed products over $k_{\mathfrak{p}}(t)$. Let $v_{\mathfrak{p}}$ be the additive $\mathfrak{p}$-adic valuation on k with trivial action on t, and let $k(t)_{\mathfrak{p}}$ be a $v_{\mathfrak{p}}$-henselian extension of $k(t)$, with the same residue field $\mathbb{F}(t)$. The idea is to construct a noncrossed product division algebra $D_{\mathfrak{p}}$ over $k(t)_{\mathfrak{p}}$ using the method of [6], lift it to a $k(t)$-division algebra D, and then prove that D is a noncrossed product. Since $k_{\mathfrak{p}}(t)$ is *en route*, the result follows.

Finding a lift D of the noncrossed product $D_{\mathfrak{p}}$ is easy, but proving it is a noncrossed product is hard unless D and $D_{\mathfrak{p}}$ have the same index. Then it is easy. Two circumstances conspire to make the index computation difficult: The residue field $\mathbb{F}(t)$ is not perfect, and $D_{\mathfrak{p}}$ is a noncrossed product.

Though the $k(t)$-noncrossed products here are constructed using the method of [6], the lift takes place over the embedding $k(t) \hookrightarrow k(t)_{\mathfrak{p}}$, whereas in [6] the lift is over $k(t) \hookrightarrow k(t)_t$, where $k(t)_t$ is t-henselian. As a result, the noncrossed products here form a distinct class from those in [6], which necessarily become crossed products over $k(t)_{\mathfrak{p}}$.

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Overview

Constructing the noncrossed product $D_{\mathfrak{p}}$ of index some n^2 over $k(t)_{\mathfrak{p}}$ requires having an $\mathbb{F}(t)$-division algebra $\overline{A}$ of index n^2, local index n^2 at two places q_1 and q_2 of $\mathbb{F}(t)$, and such that $\overline{A}$ contains the cyclic extension $\mathbb{F}(t)(\overline{\theta})/\mathbb{F}(t)$ defined by a certain character $\overline{\theta}$. This $\overline{\theta}$ must have order n and not be an n-multiple of any character, which means it must have specific ramification behavior.

Lifting the noncrossed product $D_{\mathfrak{p}}$ to a noncrossed product D over $k(t)$ requires lifting $\overline{A}$, $\overline{\theta}$, and the property "$\mathbb{F}(t)(\overline{\theta}) \subset \overline{A}$" to $k(t)$. The latter ensures that D's index equals that of $D_{\mathfrak{p}}$, making it possible to prove that D is itself a noncrossed product.

Technology for lifting central simple algebras over henselian fields starts with Galois 2-cocycles. It is most successful when the residue field has a multiplicative system of representatives, e.g., is perfect. Then the cocycle condition lifts *automatically* once the defining Galois extension lifts, so the whole problem reduces to field theory, for which there is [12]. Once a lift is found, determining its index is a problem if the cocycle doesn't already describe the underlying division algebra, e.g., if the division algebra is a *noncrossed product*.

Since the present residue field $\mathbb{F}(t)$ is not perfect and $D_{\mathfrak{p}}$ is a noncrossed product, the problem calls for new methods. The built in properties of $\overline{A}$ that make $D_{\mathfrak{p}}$ a noncrossed product are precisely the properties that make the condition $\mathbb{F}(t)(\overline{\theta}) \subset \overline{A}$ hard to lift. Since noncrossed products are the "non-Galois" objects in division algebras, it isn't surprising that they don't lift easily.

Three plausible approaches turn out to be naive. The first, a lift of $\overline{A}$ as a cyclic cocycle defined by some character $\overline{\theta}'$ whose n-multiple is $\overline{\theta}$, fails since $\overline{\theta}$ is not an n-multiple. The next attempt, a lift of $\overline{A}$ as an abelian crossed product of type (n,n) involving $\overline{\theta}$, also fails. For even though a trick involving a theorem of Albert allows the (n,n)-type cocycle to lift in principle, such a cocycle has been denied existence in order to establish the noncrossed product $D_{\mathfrak{p}}$.

A third approach is based on the fact that $D_{\mathfrak{p}}$ becomes a crossed product over a prime-to-n extension $K_{\mathfrak{p}}$ of $k(t)_{\mathfrak{p}}$, and the desired (n,n)-type cocycle can be found there. The idea then is to construct the algebra and perform the lift over $K_{\mathfrak{p}}$, then corestrict back down to $k(t)_{\mathfrak{p}}$. The problem is that the delicately constructed cocycle is not a restriction from $k(t)_{\mathfrak{p}}$, and the corestriction seems likely to wreck some of its delicate properties.

The solution is to build the desired crossed product algebra over $K_{\mathfrak{p}}$ as an *n-th root* of the restriction from $k(t)_{\mathfrak{p}}$ of a smaller crossed product algebra $\overline{C}$ that serves to carry a "blueprint" for the noncrossed product through the corestriction. Since the corestriction of restriction is multiplication by a harmless prime-to-n number m, the information survives. The resulting algebra is an n-th root of the m-th power of the "carrier" $\overline{C}$, and its lift is a noncrossed product $k(t)$-division algebra. This is the content of Lemmas 6 and 7.

Background

The following is background on crossed products and characters. For general reference, see [4], [10], [11] and [15]. Some notation:

- E is a field of characteristic zero.
- E^{sep} is a fixed separable closure of E.
- $E^{ab} \subset E^{\text{sep}}$ is the maximal abelian extension.
- $\text{Gal}(E) = \text{Gal}(E^{\text{sep}}/E)$.
- $\text{Gal}(E)_{\text{ab}} = \text{Gal}(E^{ab}/E)$.
- $\mu(\,-\,)$ is the group of roots of unity of $\,-\,$.
- $\mu = \mu(E^{\text{sep}}) \cong \mathbb{Q}/\mathbb{Z}$ is a fixed identification.
- μ_n is the group of n-th roots of unity.
- ζ_n is the primitive n-th root associated to $1/n$.
- $\text{X}(E) = \text{Hom}(\text{Gal}(E), \mathbb{Q}/\mathbb{Z})$ is the character group.
- For $\xi \in \text{X}(E)$, $E(\xi)/E$ is the cyclic extension (of degree $|\xi|$) corresponding to $\text{Gal}(E)/\ker(\xi)$.
- If D is an E-division algebra, $D^{\otimes n}$ is the E-division algebra underlying $D \otimes_E \cdots \otimes_E D$ (n times).

For any $\theta \in \text{X}(E)$ and $a \in E^{\bullet}$, let (θ, a) denote either the *crossed product algebra* $(E(\theta)/E, s, a)$, where $\theta(s) = 1/|\theta|$, or the corresponding element of $\text{Br}(E)$. For any $a, b \in E$, let $(a, b; \zeta_n)_n$ denote either the *symbol algebra* with E-basis $\{u^i v^j\}$ $(0 \leq$

$i, j \leq n-1$), relations $u^n = a$, $v^n = b$, and commutator $[u,v] := uvu^{-1}v^{-1} = \zeta_n$, or the corresponding element of $\mathrm{Br}(E)$.

For each $n \in \mathbb{N}$ let

$$\langle\ ,\ \rangle_n : E^{\mathrm{sep}^\bullet} \times \mathrm{Gal}(E) \ \to\ \mu$$

denote the *Kummer pairing*, given by

$$\langle a, t\rangle_n = \frac{t(a^{1/n})}{a^{1/n}}\ .$$

If $E \subset K \subset E^{\mathrm{sep}}$, call a character $\chi \in \mathrm{X}(K)$ *Kummer* if $|\chi| = n$ and $\zeta_n \in \mu(K)$; then $\chi = \langle a, -\rangle_n$ for some $a \in K$, unique (mod K^n), given the fixed $\mu \cong \mathbb{Q}/\mathbb{Z}$.

Theorem ("Albert's Theorem"). *Let K be any field containing the ℓ^r-th roots of unity μ_{ℓ^r}, and let $\lambda \in \mathrm{X}(K)$ be a character. Then λ is an ℓ^r-multiple in $\mathrm{X}(K)$ if and only if $\mu_{\ell^r} \in N(K(\lambda)/K)$.*

Proof. This was proved by Albert in the case $r = 1$, and more generally by Arason, Fein, Schacher, and Sonn in [3]. □

Theorem ("Saltman Lifting")**.** *Let E be a field with inequivalent real-valued valuations $v_{p_1}, \ldots, v_{p_m}$, and let E_i be the completion of E with respect to v_{p_i}. Let G be a finite abelian group of order prime to* char$E$. *Suppose $E(\zeta_{2^s})/E$ is cyclic, where 2^s is the highest power of 2 dividing the exponent of G. Let H_i be subgroups that together generate G, and suppose L_i/E_i is an H_i-Galois field extension for each i. Then there is a G-Galois field extension L/E whose completion with respect to each v_{p_i} is L_i/E_i, with accompanying decomposition group H_i.*

Proof. This generalization of the Grunwald-Wang theorem is part of Theorem 5.10 in [12]. □

Theorem 1. *Let L/K be Galois of degree n^2 with rank two abelian Galois group $\langle s\rangle \times \langle t\rangle$ of type (n, n). Then any crossed product A/K split by L has an L-basis $\{u^i v^j\}$ $(0 \leq i, j \leq n-1)$ such that*

$$u^n = a\ ; \quad v^n = b\ ; \quad [u,v] = c\ ; \quad u^i v^j x = (s^i t^j x) u^i v^j \quad \forall x \in L \tag{1.1}$$

for $a \in L^{\langle s\rangle}$, $b \in L^{\langle t\rangle}$, and $c \in L$ satisfying

$$\begin{aligned} N_{L|L^{\langle s\rangle}}(c) &= \frac{a}{ta}, \\ N_{L|L^{\langle t\rangle}}(c) &= \frac{sb}{b}\ . \end{aligned} \tag{1.2}$$

Conversely any nonzero $a \in L^{\langle s\rangle}$, $b \in L^{\langle t\rangle}$, and $c \in L$ satisfying (1.2) determine a unique crossed product A/K split by L with relations (1.1). Write

$$A \sim (L/K; s, t; a, b, c) = (a, b, c)$$

the latter when L/K, s, t are implicit. Then $(a, b, c) \sim (a', b', c')$ if and only if there exists a transformation

$$u \mapsto xu \qquad v \mapsto yv \quad (x, y \in L^\bullet) \tag{1.3}$$

that sends a, b, and c to a′, b′, and c′, respectively. The transformation (1.3) *sends*

$$a \mapsto N_{L|L^{\langle s\rangle}}(x) \cdot a,$$
$$b \mapsto N_{L|L^{\langle t\rangle}}(y) \cdot b, \quad (1.4)$$
$$c \mapsto \frac{sy}{y}\frac{x}{tx} \cdot c\ .$$

Proof. The derivation of $a \in L^{\langle s\rangle}$, $b \in L^{\langle t\rangle}$ and $c \in L$ from a given cocycle for L/K is standard, as is the effect of the transformation (1.3) on (a, b, c). The rest follows from [10], Theorem 4.6.29. The original source is [2]. □

The 2-cocycle f corresponding to the basis $\{u^i v^j\}$ above is given by

$$(u^i v^j)(u^k v^l) = f_{ij,kl} u^{i+k} v^{j+l}\ ,$$

where

$$f_{ij,kl} = \begin{cases} \gamma & \text{if } i+k < n \text{ and } j+l < n, \\ \gamma \cdot a & \text{if } i+k \geq n \text{ and } j+l < n, \\ \gamma \cdot (s^{i+k} b) & \text{if } i+k < n \text{ and } j+l \geq n, \\ \gamma \cdot a(s^{i+k} b) & \text{if } i+k \geq n \text{ and } j+l \geq n, \end{cases}$$

and

$$\gamma = s^i[(t^{j-1}(s^{k-1}c \cdots c)) \cdots (s^{k-1}c \cdots c)]^{-1}.$$

This is just a matter of working out the implications of the relations (1.1). If a, b and c define $(L/K, f)$ and a', b', and c' define $(L/K, g)$, it follows by [11] 29.9 that aa', bb', and cc' define $(L/K, fg)$ up to similarity. In particular,

$$A^{\otimes n} \sim (L/K; s, t; a^n, b^n, c^n)\ .$$

Preliminary Results

Lemma 2. *Suppose K contains a primitive n-th root of unity ζ_n. In the situation of Theorem 1, if $A \sim (a, b, c)$ with $c \in L^{\langle t\rangle}$, then $ta/a = \zeta_n^z \in K$ for some z, and for any Kummer elements x and y such that $L^{\langle s\rangle} = K(x)$, $L^{\langle t\rangle} = K(y)$, and $tx/x = sy/y = \zeta_n^z$,*

$$A^{\otimes n} \sim (L/K; s, t; x^n, y^n, \zeta_n^z) \sim (L^{\langle t\rangle}/K, s, x^n) \sim (L^{\langle s\rangle}/K, t^{-1}, y^n)\ .$$

Proof. The hypothesis $c \in L^{\langle t\rangle}$ implies $a/ta = N_{L|L^{\langle s\rangle}}(c)$ is fixed by both s and t, hence $a/ta \in L^{\langle s\rangle} \cap L^{\langle t\rangle} = K$. Since $N_{L|K}(c) = 1$, $(a/ta)^n = 1$, hence

$$ta/a = \zeta_n^z$$

for some z.

Since $\zeta_n \in K$, by Kummer theory there exist simple elements x and y for $L^{\langle s\rangle}$ and $L^{\langle t\rangle}$ over K whose n-th powers are in K. Replacing x and y by their own prime-to-ℓ powers if necessary, arrange for $\langle x^n, t\rangle_n = \langle y^n, s\rangle_n = \zeta_n^z$. Then

$$\frac{tx}{x} = \frac{sy}{y} = \zeta_n^z\ .$$

Let u and v give the standard basis for $(a^n, b^n, c^n) \sim A^{\otimes n}$ as in (1.1). Then by (1.4) and the invariance of a, x under s and b, y under t, the transformation $u \mapsto \frac{x}{a}u$ and $v \mapsto \frac{y}{b}v$ results in

$$\begin{aligned} a^n &\mapsto \left(N_{L|L^{\langle s\rangle}}\frac{x}{a}\right) a^n = x^n, \\ b^n &\mapsto \left(N_{L|L^{\langle t\rangle}}\frac{y}{b}\right) b^n = y^n, \\ c^n &\mapsto \left(\frac{x}{tx}\frac{sy}{y}\right)\left(\frac{ta}{a}\frac{b}{sb}\right) c^n \\ &= \left(\frac{ta}{a}\frac{b}{sb}\right) c^n \ . \end{aligned}$$

Since $c \in L^{\langle t\rangle}$, $N_{L|L^{\langle t\rangle}}(c) = sb/b = c^n$. Therefore in this case a^n, b^n, and c^n have been replaced by x^n, y^n, and $ta/a = \zeta_n^z$. Since $A^{\otimes n} \sim (a^n, b^n, c^n)$, now

$$A^{\otimes n} \sim (x^n, y^n, \zeta_n^z) \ .$$

Let u and v now give the new standard basis elements for (x^n, y^n, ζ_n^z), and let $S = (x^n, y^n; \zeta_n^z)_n$ be the *symbol* algebra generated by u and v over K, with commutator

$$[u, v] = \zeta_n^z \ .$$

Claim. $A^{\otimes n} \sim S$. It is clear that S is a K-subalgebra of $A^{\otimes n}$. By the double centralizer theorem and the description of S, the claim is true if the algebra centralized by u and v in $A^{\otimes n}$ is trivial.

Using the relations in (1.1), compute

$$u(wu^iv^j)u^{-1} = s(w)u^{i+1}v^ju^{-1} = \zeta_n^{zj}s(w)u^iv^j \quad (w \in L) \ .$$

Therefore the relation $u(wu^iv^j)u^{-1} = wu^iv^j$ holds if and only if $w = \zeta_n^{zj}s(w)$. Similarly the relation $v(wu^iv^j)v^{-1} = wu^iv^j$ holds if and only if $w = \zeta_n^{-zi}t(w)$. Since $x = s(x) = \zeta_n^{-z}t(x)$, setting $i = 1$, $j = 0$, and $w = x$ shows that S centralizes xu. Since $y = t(y) = \zeta_n^{-z}s(y)$, setting $i = 0$, $j = 1$, and $w = y^{-1}$ shows that S centralizes $y^{-1}v$.

Let $\hat{u} = xu$ and $\hat{v} = y^{-1}v$. Compute the commutator

$$\begin{aligned} [\hat{u}, \hat{v}] &= xuy^{-1}vu^{-1}x^{-1}v^{-1}y = xs(y^{-1})[u, v]t(x^{-1})y \\ &= \frac{x}{t(x)}\frac{y}{s(y)}\zeta_n^z = \zeta_n^{-z} \ . \end{aligned}$$

Conclude by degree count the algebra centralized by S in $A^{\otimes n}$ is the symbol algebra $(\hat{u}^n, \hat{v}^n; \zeta_n^{-z})_n$. But since y is fixed by t, $\hat{v}^n = v^n/y^n = 1$, so this algebra is trivial. This proves the claim.

S can be written as a cyclic crossed product in two ways. It is clear that $K(u) \cong K(x) = L^{\langle s\rangle}$, and since $vu = \zeta_n^{-z}uv = t^{-1}(u)v$ with $v^n = y^n \in K$, $S \cong (L^{\langle s\rangle}/K, t^{-1}, y^n)$. On the other hand, $K(v) \cong K(y) = L^{\langle t\rangle}$, and since $uv = \zeta_n^z vu = s(v)u$ with $u^n = x^n \in K$, $S \cong (L^{\langle t\rangle}/K, s, x^n)$. Therefore

$$A^{\otimes n} \sim (L^{\langle t\rangle}/K, s, x^n) \sim (L^{\langle s\rangle}/K, t^{-1}, y^n) \ .$$

This completes the proof. □

Remark. The author thanks A. R. Wadsworth for this argument for the triviality of S's centralizer.

For all $s \in \mathrm{Gal}(E)$ the automorphism

$$s : E^{\mathrm{sep}} \to E^{\mathrm{sep}}$$

induces on every Galois layer L/K a homomorphism

$$\begin{aligned} s_* : \mathrm{Gal}(sL/sK) &\to \mathrm{Gal}(L/K) \\ t &\mapsto s^{-1}ts\,. \end{aligned}$$

Applying $\mathrm{Hom}(\,-\,,\mathbb{Q}/\mathbb{Z})$ induces a homomorphism

$$s^* : \mathrm{X}(L/K) \to \mathrm{X}(sL/sK) \tag{3.1}$$

where $\mathrm{X}(L/K)$ is the subset of $\mathrm{X}(K)$ split by L. Explicitly, $s^*(\chi)(t) = \chi(s^{-1}ts)$ for each $\chi \in \mathrm{X}(L/K)$. Conjugation is trivial modulo commutators, so s_* is trivial on $\mathrm{Gal}(E)_{\mathrm{ab}}$, and since canonically $\mathrm{X}(E) = \mathrm{X}(E^{ab}/E)$, s^* is trivial on $\mathrm{X}(E)$.

It is clear that s^* is *functorial* on the Galois groups of layers in E^{sep}, since after all s_* is an automorphism of $\mathrm{Gal}(E)$.

Each $s \in \mathrm{Gal}(E)$ acts on both arguments of $\mathrm{Hom}(\mathrm{Gal}(\,-\,), \mu(\,-\,))$. Explicitly, if $\chi \in \mathrm{Hom}(\mathrm{Gal}(K), \mu(K))$ and $t \in \mathrm{Gal}(sK)$, then $s^*(\chi)(t) = s(\chi(s^{-1}ts))$. Whenever $\mu(E) \neq \mu(K)$, the action of s on $\mu(K)$ may be nontrivial, in which case $s^*\,\mathrm{X}(K) \neq s^*\,\mathrm{Hom}(\mathrm{Gal}(K), \mu(K))$. The correction is given by

Lemma 3. *Suppose $K : E \subset K \subset E^{\mathrm{sep}}$. For each $s \in \mathrm{Gal}(E)$ and n dividing $|\mu(K)|$,*

$$\begin{array}{ccc} K^{\bullet} & \xrightarrow{\langle\,,-\rangle_n} & \mathrm{X}(K) \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \phi(s)s^*} \\ sK^{\bullet} & \xrightarrow{\langle\,,-\rangle_n} & \mathrm{X}(sK) \end{array}$$

commutes, where $\phi : \mathrm{Gal}(E) \to (\mathbb{Z}/n\mathbb{Z})^{\bullet}$ is naturally defined by the action of $\mathrm{Gal}(E)$ on $\mu_n \subset K$.

Proof. An identification of $\mu_n \subset K$ with $\frac{1}{n}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$ has been fixed. Using the explicit description of the Kummer pairing it is easy to see that the diagram

$$\begin{array}{ccc} K^{\bullet} \times \mathrm{Gal}(K) & \longrightarrow & \mu_n = \dfrac{\frac{1}{n}\mathbb{Z}}{\mathbb{Z}} \\ {\scriptstyle (s,s_*^{-1})}\downarrow & & \downarrow{\scriptstyle s} \\ sK^{\bullet} \times \mathrm{Gal}(sK) & \longrightarrow & s\,\mu_n \end{array}$$

commutes. Fix $a \in K^{\bullet}$. By definition $s^*\langle a, t\rangle = \langle a, s_*t\rangle$ for all $t \in \mathrm{Gal}(K)$. By the diagram, $\langle a, s_*t\rangle = s^{-1}\langle sa, t\rangle$, hence $\langle sa, t\rangle = s(s^*\langle a, t\rangle) = (s^*\langle a, t\rangle)^{\phi(s)}$ for all $t \in \mathrm{Gal}(K)$. Therefore $\langle sa, -\,\rangle = \phi(s)s^*\langle a, -\,\rangle$, as claimed. □

The following structure theorem reduces the tame Brauer and character groups over a discretely henselian field to groups defined over the residue field. All field extensions are tame. More notation:

- K is a discretely henselian field.
- π is a uniformizer of K.
- $\overline{K}$ is the residue field of K.
- p is the characteristic of $\overline{K}$.
- $\mu(K)'$ is the subgroup of prime-to-p roots of unity in K.
- T/K is the maximal tamely ramified abelian extension.

- $\overline{T}$ is the residue field of T.
- $R \subset T/K$ is the maximal tame totally ramified abelian extension obtained by adjoining $\pi^{1/n}$ for each n such that $\zeta_n \in \mu(K)'$.
- $N \subset T/K$ is the maximal unramified abelian extension.
- $\mathrm{Br}(K)_{\mathrm{nr}}$ is the subgroup of classes split by unramified extensions.
- $\mathrm{X}(K)' = \mathrm{Hom}(\mathrm{Gal}(T/K), \mathbb{Q}/\mathbb{Z}) \subset \mathrm{X}(K)$ are the tame characters.

Theorem 4. *There exist split exact sequences*

$$0 \to \mathrm{Br}(\overline{K}) \to \mathrm{Br}(K)_{\mathrm{nr}} \to \mathrm{X}(\overline{K}) \to 0, \tag{4.1}$$

split by $\theta \mapsto (\theta, \pi)$,

$$0 \to \mathrm{Gal}(T/N) \to \mathrm{Gal}(T/K) \to \mathrm{Gal}(\overline{T}/\overline{K}) \to 0, \tag{4.2}$$

split by $\iota_\pi : \mathrm{Gal}(\overline{T}/\overline{K}) \to \mathrm{Gal}(T/K)$ *(defined below) and dually*

$$0 \to \mathrm{X}(\overline{K}) \to \mathrm{X}(K)' \to \mu(K)' \to 0, \tag{4.3}$$

split by $\zeta_n^d \mapsto \langle \pi^d, - \rangle_n$ *for all* n *such that* $(n,p) = 1$.

Proof. These results are well known. The first statement is Witt's Theorem ([15], Chapter XII). By [7], Chapter 1, Section 8, the maximal abelian tame totally ramified extension of N is obtained by adjoining $\pi^{1/n}$ for all n such that $(n,p) = 1$ and $\zeta_n \in \mu(N)$. Each $N(\pi^{1/n})$ is abelian over K if and only if $\zeta_n \in \mu(K)'$, and in this way T splits:

$$T \cong N \otimes_K R\,.$$

Correspondingly $\mathrm{Gal}(T/K)$ decomposes:

$$\mathrm{Gal}(T/K) \cong \mathrm{Gal}(T/N) \times \mathrm{Gal}(T/R)\,.$$

Restriction of the second factor to N gives an isomorphism onto $\mathrm{Gal}(\overline{T}/\overline{K})$. Let the composition

$$\iota_\pi : \mathrm{Gal}(\overline{T}/\overline{K}) \xrightarrow{\sim} \mathrm{Gal}(T/R) \hookrightarrow \mathrm{Gal}(T/K)$$

denote the inverse of this restriction, which depends on π. Since the cokernel is $\mathrm{Gal}(T/N)$ it splits the residue map of (4.2).

To prove (4.3) apply $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ to (4.2), then use the isomorphism

$$\mathrm{X}(T/N) \cong \mu(K)'$$

that sends the character $\langle \pi^d, - \rangle_n$ to ζ_n^d. This is split by sending ζ_n^d back to the Kummer character $\langle \pi^d, - \rangle_n$ over K. □

Lemma 5. *Suppose* $E \subset K \subset E^{\mathrm{sep}}$, *and* $s \in \mathrm{Gal}(E)$. *Then* s *induces an isomorphism* $\overline{s} : \overline{K} \to \overline{sK}$, *an injection* $\iota_{s\pi} = s_*^{-1}(\iota_\pi)$, *and an induced map* $\iota_{s\pi}^* = s^*(\iota_\pi^*)$.

Proof. Each s transports all of the structure of K^{ab}/K to sK^{ab}/sK, including the valuation theory ([7], Chapter 7). Restriction to the maximal unramified extension defines an isomorphism $\overline{s}$ on residue fields. By functoriality the preimage of ι_π under the induced map s_* on Galois groups is determined by the image $s\pi$ of the uniformizer π,

$$\iota_{s\pi} : \mathrm{Gal}(\overline{s}\overline{T}/\overline{s}\overline{K}) \to \mathrm{Gal}(sT/sK)\,.$$

Dually $\iota_\pi^* : \mathrm{X}(K) \to \mathrm{X}(\overline{K})$, $\iota_{s\pi}^* : \mathrm{X}(sK) \to \mathrm{X}(\overline{s}\overline{K})$, and $\iota_{s\pi}^* = s^*(\iota_\pi^*)$. □

Main Construction

For the rest of the paper,

- p is a prime number;
- $k_{\mathfrak{p}}/\mathbb{Q}_p$ is a $\mathfrak{p}$-adic number field;
- $\ell \nmid p$ is an odd prime such that
$$[k_{\mathfrak{p}}(\zeta_\ell) : k_{\mathfrak{p}}] = m > 1;$$
- $r \in \mathbb{N}$ is maximal such that $\zeta_{\ell^r} \in k_{\mathfrak{p}}(\zeta_\ell)$;
- k is any number field with completion $k_{\mathfrak{p}}$ such that
$$[k(\zeta_{\ell^{2r}}) : k] = [k_{\mathfrak{p}}(\zeta_{\ell^{2r}}) : k_{\mathfrak{p}}] = \ell^r m;$$
- $E = k(t)$, the rational function field;
- $v_{\mathfrak{p}}$ is the $\mathfrak{p}$-adic valuation on k, extended to E by trivial action on t;
- π is a $v_{\mathfrak{p}}$-uniformizer of E;
- "$-$" denotes passage to residue field with respect to $v_{\mathfrak{p}}$;
- $E_{\mathfrak{p}} \supset E$ is henselian with respect to $v_{\mathfrak{p}}$ with residue field $\overline{k_{\mathfrak{p}}}(t)$;
- $K = E(\zeta_\ell)$;
- $K_{\mathfrak{p}} = E_{\mathfrak{p}}(\zeta_\ell)$.

For example, $E_{\mathfrak{p}}$ could be the completion of E with respect to $v_{\mathfrak{p}}$. To see that this $E_{\mathfrak{p}}$ indeed has residue field $\overline{k_{\mathfrak{p}}}(t)$, let $f = \pi^m(a_0 + a_1\pi + \cdots + a_r\pi^r)$ and $g = \pi^n(b_0 + b_1\pi + \cdots + b_s\pi^s)$ be the π-adic expansions of $f, g \in k[t]$, with $a_i, b_j \in E$ all $v_{\mathfrak{p}}$-units, and $m, n \in \mathbb{Z}$. Then $f/g \in E$ is a $v_{\mathfrak{p}}$-integer if and only if $m \geq n$. If $m = n$, the residue of f/g is a_0/b_0, a quotient of polynomials whose coefficients are all $v_{\mathfrak{p}}$-units. It follows immediately that $\overline{k_{\mathfrak{p}}}(t)$ is the residue field of E, hence it is the residue field of $E_{\mathfrak{p}}$.

The odd prime ℓ exists since $\mu(k_{\mathfrak{p}})$ is finite. The author thanks Wadsworth for pointing out it is not clear that an ℓ exists with $r = 1$, that is, with $\zeta_{\ell^2} \notin k_{\mathfrak{p}}(\zeta_\ell)$, but that the existence of such an ℓ would follow from a conjecture of E. Artin.

Choose two places q_1 and q_2 of $\overline{E}$ of degrees ℓ^r and m, respectively. For example, let q_2 be the minimum polynomial of ζ_ℓ. Neither place ramifies in the constant extension $\overline{E}(\zeta_{\ell^{2r}})$. Since $(\ell, m) = 1$ and $[\overline{K} : \overline{E}] = m$, q_1 extends uniquely and q_2 splits completely in $\overline{K}$. Since $[\overline{K}(\zeta_{\ell^{2r}}) : \overline{K}] = \ell^r$, q_1 splits completely from $\overline{K}$ to $\overline{K}(\zeta_{\ell^{2r}})$. Since $\overline{K}(\zeta_{\ell^{2r}})/\overline{K}$ is constant, each extension of q_2 to $\overline{K}$ extends uniquely to $\overline{K}(\zeta_{\ell^{2r}})$. Thus

$$\begin{aligned} &\zeta_\ell \notin \overline{E}_{q_1}, \quad \zeta_{\ell^{2r}} \in \overline{K}_{q_1} = \overline{E}(\zeta_\ell)_{q_1}, \\ &\zeta_{\ell^{r+1}} \notin \overline{E}_{q_2}, \quad \zeta_{\ell^r} \in \overline{K}_{q_{2i}} = \overline{E}_{q_2}, \end{aligned} \tag{6.1}$$

where $\{q_{2i} \mid i = 1, \ldots, m\}$ are the extensions of q_2 to $\overline{K}$, and the subscript q_i denotes completion at q_i.

The following lemma constructs a Galois extension $\overline{L}/\overline{E}$ containing $\overline{K}$ and lifts it to L/E. Both $\overline{L}$ and L are designed to contain the cyclic extensions associated to $\overline{\theta}$ and its lift θ. They are going to split $\overline{A}$ and its lift A, thus controlling the index of these algebras as well as their index *reduction* by $\overline{\theta}$ and θ. This is the key to the main result. This lemma also defines and lifts a new character $\overline{\xi}'$ of order ℓ^{2r}; its role will be to lift a norm relation on $\overline{L}$ using Albert's Theorem, leading to the crucial lift of $\overline{A}$'s cocycle condition in Lemma 7.

Lemma 6. *There exists $\overline{\theta} \in \mathrm{X}(\overline{E})$ of order ℓ^r, $\overline{\xi}' \in \mathrm{X}(\overline{K})$ of order ℓ^{2r}, and $\overline{\xi} = \ell^r\overline{\xi}' \in \mathrm{X}(\overline{K})$ of order ℓ^r such that:*

(*i*) *$\overline{\theta}$ is unramified of order ℓ^r at q_1 and totally ramified at q_2.*
(*ii*) *$\overline{\xi}$ is totally ramified at q_1 and unramified of order ℓ^r at each q_{2i}.*
(*iii*) *$\overline{\xi} = \langle \overline{y}^{\ell^r}, -\rangle_{\ell^r}$ for $\overline{y} \in \overline{E}^{sep}$ such that $\overline{y}^{\ell^r} \in \overline{E}$ with values $v_{q_1}(\overline{y}^{\ell^r}) = m$, $v_{q_2}(\overline{y}^{\ell^r}) \equiv 0 (\mathrm{mod}\ \ell^r)$.*
(*iv*) *$\overline{\theta}$ and $\overline{\xi}'$ lift faithfully to $\theta \in \mathrm{X}(E)$ and $\xi' \in \mathrm{X}(K)$, respectively.*
(*v*) *The extension $\overline{L} = \overline{K}(\overline{\theta}, \overline{\xi})/\overline{E}$ is Galois and lifts faithfully to the extension $L = K(\theta, \xi)/E$ which is also Galois.*

Proof. Thanks to the ℓ^r-th root of unity at q_2 (6.1), by Saltman lifting there exists $\overline{\theta} \in \mathrm{X}(\overline{E})$ of order ℓ^r unramified of order ℓ^r at q_1 and totally ramified at q_2. This takes care of (i). By Theorem 4, $\overline{\theta}$ lifts faithfully to $\theta \in \mathrm{X}(E_{\mathfrak{p}})$, and by Saltman lifting again the lift may be defined over E. This proves the first part of (iv).

Similarly, let $\overline{\lambda}' \in \mathrm{X}(\overline{K})$ be of order ℓ^{2r} with the following local behavior. At the unique extension of q_1 to $\overline{K}_{q_1}$ let $\overline{\lambda}'_{q_1}$ be the totally ramified Kummer character $\langle q_1, -\rangle_{\ell^{2r}} \in \mathrm{X}(\overline{K}_{q_1})$. It exists since $\zeta_{\ell^{2r}} \in \overline{K}_{q_1}$ by (6.1). For the m extensions $\{q_{2i}\}$ of q_2 to $\overline{K}$, let $\overline{\lambda}'_{q_{21}}$ be unramified of order ℓ^{2r}, and set $\overline{\lambda}'_{q_{2i}} = 0$ for $i \neq 1$. Of course, $\overline{\lambda}'_{q_{21}}$ is the restriction of an unramified $\overline{\psi}'_{q_2} \in \mathrm{X}(E_{q_2})$.

Lift $\overline{\lambda}'$ to a $\lambda' \in \mathrm{X}(K)$, also of order ℓ^{2r}. Let

$$\xi' = \sum_{\mathrm{Gal}(K/E)} \phi(s) s^*(\lambda')$$

where here $\phi(s) \in \mathbb{Z}$ is defined by the action of s on μ_{ℓ^r}, and s^* is as in (3.1). The sum is well defined since $\mathrm{Gal}(K)$ acts trivially on $\mathrm{X}(K)$, $\mu_{\ell^r} \subset K$, and $\phi(s)s^*$ is a homomorphism.

The residue $\overline{\xi}'$ of ξ' is the same sum defined over $\overline{K}$. For

$$\iota_\pi^* \cdot \mathrm{res}_{K_{\mathfrak{p}}|K} : \mathrm{X}(K)(\ell) \to \mathrm{X}(\overline{K})(\ell) \tag{6.2}$$

is the residue homomorphism, where ι_π^* is from Theorem 4. By functoriality $s^* \cdot \mathrm{res}_{K_{\mathfrak{p}}|K} = \mathrm{res}_{K_{\mathfrak{p}}|K} \cdot s^*$, and $\overline{s}^* \cdot \iota_\pi^* = \iota_{s\pi}^* \cdot s^*$ by Lemma 5. Thus s^* commutes with (6.2). Moreover, the sum's coefficients $\phi(s)$ are the same as the $\phi(\overline{s})$ since π is *inertial* in K. Thus

$$\overline{\xi}' = \sum_{\mathrm{Gal}(\overline{K}/\overline{E})} \phi(\overline{s}) \overline{s}^*(\overline{\lambda}') \, .$$

Now *claim* $|\xi'| = \ell^{2r}$. The order of each $\phi(s)s^*(\lambda')$ divides ℓ^{2r}, so $|\xi'|$ divides ℓ^{2r}. Set

$$\overline{\lambda} = \ell^r \overline{\lambda}', \quad \lambda = \ell^r \lambda',$$
$$\overline{\xi} = \ell^r \overline{\xi}', \quad \xi = \ell^r \xi' \, .$$

Already at $\overline{K}_{q_1}$ the residue

$$\overline{\lambda}_{q_1} = \langle q_1^{\ell^r}, -\rangle_{\ell^{2r}} = \langle q_1, -\rangle_{\ell^r}$$

is Kummer, where of course $q_1 \in \overline{E}$. Since q_1 is inertial in $\overline{K}$, $\mathrm{X}(\overline{K}_{q_1})$ is stable under $\mathrm{Gal}(\overline{K}/\overline{E})$, and by Lemma 3

$$\overline{\xi}_{q_1} = \sum_{\mathrm{Gal}(\overline{K}/\overline{E})} \phi(\overline{s})\overline{s}^*\overline{\lambda}_{q_1} = \sum_{\mathrm{Gal}(\overline{K}/\overline{E})} \langle \overline{s}q_1, -\rangle_{\ell^r} = \langle q_1^m, -\rangle_{\ell^r} = m\overline{\lambda}_{q_1} .$$

Since $(m, \ell) = 1$, $|\overline{\xi}_{q_1}| = |\overline{\lambda}_{q_1}| = \ell^r$. Thus $\overline{\xi}'_{q_1}$ has order ℓ^{2r}, hence $\overline{\xi}'$ has order ℓ^{2r}, hence the lift ξ' also has order ℓ^{2r}, as claimed. This proves the rest of (iv).

The behavior of $\overline{\xi}$ at q_1 is the same as that of $\overline{\lambda}$, since $\langle \overline{\xi}_{q_1}\rangle = \langle m\overline{\lambda}_{q_1}\rangle = \langle\overline{\lambda}\rangle$. In particular it is totally ramified of order ℓ^r.

Since q_2 splits completely in $\overline{K}$, $\mathrm{Gal}(\overline{K}/\overline{E})$ permutes the m extensions q_{2j} of q_2, and by (3.1)

$$\overline{s}^* : \mathrm{X}(\overline{K}_{q_{2j}}) \to \mathrm{X}(\overline{s}\overline{K}_{q_{2j}}) = \mathrm{X}(\overline{K}_{\overline{s}q_{2j}}) .$$

On Kummer elements Lemma 3 implies

$$\phi(\overline{s})\overline{s}^* \operatorname{res}_{\overline{K}_{q_{2j}}|\overline{E}_{q_2}} = \operatorname{res}_{\overline{K}_{\overline{s}q_{2j}}|\overline{E}_{q_2}} .$$

By definition $\overline{\lambda}_{q_{21}} = \operatorname{res}_{\overline{K}_{q_{21}}|\overline{E}_{q_2}}(\overline{\psi}_{q_2})$ for the unramified $\overline{\psi}_{q_2}$ of order ℓ^r, and $\overline{\lambda}_{q_{2j}} = 0$ for $j \neq 1$. Let $\overline{s}_i \in \mathrm{Gal}(\overline{K}/\overline{E})$ be such that $\overline{s}_i q_{21} = q_{2i}$. Then

$$\begin{aligned}\overline{\xi}_{q_{2i}} &= \Big(\sum_{\mathrm{Gal}(\overline{K}/\overline{E})} \phi(\overline{s})\overline{s}^*(\overline{\lambda})\Big)_{q_{2i}} \\ &= \sum_{\mathrm{Gal}(\overline{K}/\overline{E})} \phi(\overline{s})\overline{s}^*(\overline{\lambda}_{\overline{s}^{-1}q_{2i}}) \\ &= \phi(\overline{s}_i)\overline{s}_i^*(\overline{\lambda}_{q_{21}}) \\ &= \operatorname{res}_{\overline{K}_{q_{2i}}|\overline{E}_{q_2}}(\overline{\psi}_{q_2}).\end{aligned}$$

Thus at the set $\{q_{2i}\}$, $\overline{\xi}$ behaves as if it were a restriction from $\overline{E}$. In particular, each $\overline{\xi}_{q_{2i}}$ is unramified of order ℓ^r. This proves (ii).

As $\overline{\lambda}$ and λ are Kummer, $\overline{\lambda} = \langle \overline{w}, -\rangle_{\ell^r}$ and $\lambda = \langle w, -\rangle_{\ell^r}$ for some $w \in K$ with residue $\overline{w}$. By Lemma 3

$$\xi = \sum_{\mathrm{Gal}(K/E)} \phi(s)s^*\lambda = \sum_{\mathrm{Gal}(K/E)} \langle sw, -\rangle_{\ell^r} = \langle N_{K|E}(w), -\rangle_{\ell^r}$$

and similarly for $\overline{\xi}$. Hence

$$\begin{aligned}\xi &= \langle y^{\ell^r}, -\rangle_{\ell^r}, \qquad & y^{\ell^r} &= N_{K|E}(w) \in E, \\ \overline{\xi} &= \langle \overline{y}^{\ell^r}, -\rangle_{\ell^r}, \qquad & \overline{y}^{\ell^r} &= N_{\overline{K}|\overline{E}}(\overline{w}) \in \overline{E} .\end{aligned}$$

Conclude that $K(\xi) = E(\zeta_\ell, y)$ is *Galois* over E, of degree $m\ell^r$. Similarly its residue $\overline{K}(\overline{\xi}) = \overline{E}(\zeta_\ell, \overline{y})$ is Galois over $\overline{E}$ of degree $m\ell^r$. This proves the first part of (v).

Now set $L = K(\theta, \xi)$. Then L/E is Galois, as the composite of the Galois extensions $E(\theta)/E$ and $E(\zeta_\ell)(\xi)/E$. Moreover, its residue $\overline{L}/\overline{E}$ is Galois of the same degree, defined by the residues $\overline{\theta}$, ζ_ℓ, and $\overline{\xi}$. This proves the rest of (v).

Finally, since $\overline{w} \equiv q_1 (\mathrm{mod}\ \overline{K}_{q_1}^{\ell^r})$, $N_{\overline{K}|\overline{E}}(\overline{w}) \equiv q_1^m$, hence

$$v_{q_1}(\overline{y}^{\ell^r}) = v_{q_1}(q_1^m) = m .$$

Since $\overline{\xi}_{q_{2i}}$ is unramified for each i, $\ell^r \mid v_{q_2}(\overline{sw})$ for each $\overline{s} \in \mathrm{Gal}(\overline{K}/\overline{E})$, therefore

$$\sum_{\mathrm{Gal}(\overline{K}/\overline{E})} v_{q_2}(\overline{sw}) \; = \; v_{q_2}(\overline{y}^{\ell^r}) \; \equiv \; 0(\mathrm{mod}\ \ell^r)\ .$$

This proves (iii). □

The next lemma finds the algebra A needed for construction of the noncrossed product D over E. The technique is to define an algebra $\overline{C}/\overline{E}$ with the ramification information needed for $\overline{A}^{\otimes \ell^r}$, build an ℓ^r-th root $\overline{B}/\overline{K}$ of $\overline{C} \otimes_{\overline{E}} \overline{K}/\overline{K}$ using $\overline{L}$, then corestrict $\overline{B}$ down to $\overline{A}/\overline{E}$, which is now an ℓ^r-th root of $\overline{C}^{\otimes m}$. Since $\overline{L}/\overline{E}$ is Galois, $\overline{A}$ is split by $\overline{L}$. This can all be done in parallel over E thanks to Albert's Theorem and the construction of the Galois extension L/E, so everything lifts.

Lemma 7. *There exists a division algebra $\overline{A}/\overline{E}$ and a lift A/E such that*

(*i*) *L splits A and $\overline{L}$ splits $\overline{A}$;*

(*ii*) *the periods and indexes of A, $\overline{A}$, $\overline{A}_{q_1}$, and $\overline{A}_{q_2}$ all equal ℓ^{2r}, where $\overline{A}_{q_i}$ is the completion of $\overline{A}$ at q_i.*

Proof. Since $\zeta_{\ell^r} \in K$,

$$\theta = \langle a^{\ell^r}, - \rangle_{\ell^r}$$

for some $a^{\ell^r} \in K$. Then $K(\theta) = K(a)$. As in the proof of Lemma 6 let $y^{\ell^r} \in E$ be such that $\xi = \langle y^{\ell^r}, - \rangle_{\ell^r}$.

Let $t \in \mathrm{Gal}(L/K(\xi))$ and $s \in \mathrm{Gal}(L/K(\theta))$ be generators such that $\theta(t) = \xi(s) = \zeta_{\ell^r}$. That is, $ta/a = sy/y = \zeta_{\ell^r}$. Then

$$L^{\langle s \rangle} = K(\theta) = K(a); \qquad L^{\langle t \rangle} = K(\xi) = K(y)$$

and

$$\mathrm{Gal}(L/K) = \langle s \rangle \times \langle t \rangle\ .$$

Be careful not to confuse this t with the element of E.

Define E and $\overline{E}$-algebras

$$C = (E(\theta)/E, t^{-1}, y^{\ell^r}),$$
$$\overline{C} = (\overline{E}(\overline{t})/\overline{E}, t^{-1}, \overline{y}^{\ell^r})\ .$$

Clearly $\overline{C}$ is the residue of C, and both algebras have index dividing ℓ^r. By Lemma 6(i) and (iii), $\overline{E}(\overline{t})$ and $\overline{E}(\overline{y})$ have opposite ramification at both q_1 and q_2, so the completions $\overline{C}_{q_1}$ and $\overline{C}_{q_2}$ are division algebras by [11] 31.6. Therefore

$$\mathrm{ind}(C) = \mathrm{ind}(\overline{C}) = \mathrm{ind}(\overline{C}_{q_1}) = \mathrm{ind}(\overline{C}_{q_2}) = \ell^r\ . \tag{7.1}$$

Since C is cyclic,

$$\mathrm{per}(C) = \mathrm{per}(\overline{C}) = \mathrm{per}(\overline{C}_{q_1}) = \mathrm{per}(\overline{C}_{q_2}) = \ell^r\ . \tag{7.2}$$

By Lemma 6(iv) ξ is an ℓ^r-multiple of ξ', so by Albert's theorem there exists an element $c \in L^{\langle t \rangle}$ such that $N_{L^{\langle t \rangle}|K}(c) = \zeta_{\ell^r}^{-1}$. Since $N_{L|K}(c) = N_{L^{\langle t \rangle}|K}(c^{\ell^r}) = 1$, and naturally $\mathrm{Gal}(L^{\langle t \rangle}/K) \cong \langle s \rangle$, by Hilbert 90 there exists an element $b \in L^{\langle t \rangle}$ such that $c^{\ell^r} = sb/b$. Now

$$N_{L|L^{\langle s \rangle}}(c) = \frac{a}{ta}\ ; \qquad N_{L|L^{\langle t \rangle}}(c) = \frac{sb}{b}\ .$$

In the notation of Theorem 1, let

$$B \sim (L/K; s, t; a, b, c) .$$

Then B/K is split by L with relations (1.1). The residue algebra $\overline{B}/\overline{K}$ of B/K can be constructed independently by the residues $\overline{a} \in \overline{L}^{\langle s \rangle}$, $\overline{b} \in \overline{L}^{\langle t \rangle}$, and $\overline{c} \in \overline{L}^{\langle t \rangle}$ of a, b, and c. Of course, $\overline{B}$ is split by $\overline{L}$.

Now c, $x = a$, and y satisfy the hypotheses of Lemma 2, with $n = \ell^r$ and $z = 1$. Therefore

$$B^{\otimes \ell^r} \sim (L^{\langle s \rangle}/K, t^{-1}, y^{\ell^r}) .$$

In particular, $B^{\otimes \ell^r}$ is similar to the restriction

$$B^{\otimes \ell^r} \sim \mathrm{res}_{K|E}(C) .$$

Similarly

$$\overline{B}^{\otimes \ell^r} \sim \mathrm{res}_{\overline{K}|\overline{E}}(\overline{C}) .$$

Let A and $\overline{A}$ be division algebras underlying the corestrictions

$$A \sim \mathrm{cor}_{K|E}(B) \qquad \overline{A} \sim \mathrm{cor}_{\overline{K}|\overline{E}}(\overline{B}) .$$

The corestriction is often defined on cohomology; see [8] for an explicit and compatible algebra definition.

Since K/E is *inertial* with respect to $v_{\mathfrak{p}}$, the diagram

$$\begin{array}{ccc} \mathrm{Br}(K) & \overset{\mathrm{res}}{\to} & \mathrm{Br}(K_{\mathfrak{p}}) \\ \mathrm{cor} \downarrow & \square & \downarrow \mathrm{cor} \\ \mathrm{Br}(E) & \overset{\mathrm{res}}{\to} & \mathrm{Br}(E_{\mathfrak{p}}) \end{array}$$

commutes by [5], Chapter III, Proposition 9.5. Of course, the images of B and A are respectively in $\mathrm{Br}(\overline{K})$ and $\mathrm{Br}(\overline{E})$, viewed via Theorem 4 as subgroups of $\mathrm{Br}(K_{\mathfrak{p}})$ and $\mathrm{Br}(E_{\mathfrak{p}})$. It follows that since B is a lift of $\overline{B}$, A is a lift of $\overline{A}$.

By Lemma 6(v), $\overline{L}/\overline{E}$ and L/E are Galois, so the corestriction really maps $\mathrm{Br}(L/K)$ to $\mathrm{Br}(L/E)$ and $\mathrm{Br}(\overline{L}/\overline{K})$ to $\mathrm{Br}(\overline{L}/\overline{E})$. Thus L splits A and $\overline{L}$ splits $\overline{A}$, proving (i).

It remains to compute indexes. The corestriction is a homomorphism on the Brauer group, so

$$A^{\otimes \ell^r} \sim \mathrm{cor}_{K|E}(B^{\otimes \ell^r}) \sim \mathrm{cor}_{K|E} \cdot \mathrm{res}_{K|E}(C),$$
$$\overline{A}^{\otimes \ell^r} \sim \mathrm{cor}_{\overline{K}|\overline{E}}(\overline{B}^{\otimes \ell^r}) \sim \mathrm{cor}_{\overline{K}|\overline{E}} \cdot \mathrm{res}_{\overline{K}|\overline{E}}(\overline{C}) .$$

By [15], Chapter VII, Proposition 6, these compositions cor $\cdot$ res are exponentiation by m, hence

$$A^{\otimes \ell^r} \sim C^{\otimes m} \qquad \overline{A}^{\otimes \ell^r} \sim \overline{C}^{\otimes m} .$$

By construction B has ℓ-primary period and index, hence so do A and $\overline{A}$. Now $A^{\otimes \ell^r}$, its residue, and both completions at the q_i are the m-th tensor power of C, $\overline{C}$, and the $\overline{C}_{q_i}$. Furthermore, since $(m, \ell) = 1$, these algebras all have period and index ℓ^r by (7.1) and (7.2). Therefore A, its residue, and both completions all have period and index divisible by ℓ^{2r}. Since the splitting field L already has ℓ-primary degree ℓ^{2r}, conclude that the periods and indexes of these algebras equal ℓ^{2r}. This completes the proof. ∎

Noncrossed Products

It remains now to assemble $E_{\mathfrak{p}}$ and E-division algebras that do what they're supposed to do. Denote by $A_{\mathfrak{p}}$ and $\theta_{\mathfrak{p}}$ the inertial lifts of Lemma 7's $\overline{A}$ and $\overline{\theta}$ to $E_{\mathfrak{p}}$. Let $D_{\mathfrak{p}}$ be the $E_{\mathfrak{p}}$-division algebra

$$D_{\mathfrak{p}} \sim A_{\mathfrak{p}} \otimes_{E_{\mathfrak{p}}} (\theta_{\mathfrak{p}}, \pi)$$

as per Theorem 4.

Lemma 8. *$D_{\mathfrak{p}}$ is a noncrossed product of period and index ℓ^{2r}.*

Proof. Since $\mathrm{per}(A_{\mathfrak{p}}) > \mathrm{per}((\theta_{\mathfrak{p}}, \pi))$, the period of $D_{\mathfrak{p}}$ is the period of $A_{\mathfrak{p}}$, which is ℓ^{2r} by Lemma 7(ii). By [9], Theorem 5.15 the index is

$$\mathrm{ind}(A_{\mathfrak{p}} \otimes_{E_{\mathfrak{p}}} (\theta_{\mathfrak{p}}, \pi)) = |\theta_{\mathfrak{p}}| \, \mathrm{ind}(A_{\mathfrak{p}} \otimes_{E_{\mathfrak{p}}} E_{\mathfrak{p}}(\theta_{\mathfrak{p}})) \ .$$

By construction $E_{\mathfrak{p}}(\theta_{\mathfrak{p}}) \subset A_{\mathfrak{p}}$ is a subfield. Indeed, by Lemma 7(i), $A_{\mathfrak{p}} \otimes_{E_{\mathfrak{p}}} E_{\mathfrak{p}}(\theta_{\mathfrak{p}})$ is split by the extension $K_{\mathfrak{p}}(\theta_{\mathfrak{p}}, \xi_{\mathfrak{p}})/E_{\mathfrak{p}}(\theta_{\mathfrak{p}})$ of degree $\ell^r m$, which implies that

$$\mathrm{ind}(A_{\mathfrak{p}} \otimes_{E_{\mathfrak{p}}} E_{\mathfrak{p}}(\theta_{\mathfrak{p}})) = \ell^r .$$

Therefore $\mathrm{ind}(D_{\mathfrak{p}}) = \ell^{2r}$.

If $F_{\mathfrak{p}} \subset D_{\mathfrak{p}}$ is a Galois maximal subfield, then its unramified part composed with $E_{\mathfrak{p}}(\theta_{\mathfrak{p}})$ is an unramified Galois maximal subfield of degree ℓ^{2r} over $E_{\mathfrak{p}}$ that splits $A_{\mathfrak{p}}$ ([9], Theorem 5.15). Of course, the residue field $\overline{F}(\overline{\theta})/\overline{E}$ splits $\overline{A}$, and so by Lemma 7(ii) $\overline{A}$ has index ℓ^{2r} at both places q_i, $\overline{F}(\overline{\theta})$ necessarily has full degree at both places, that is,

$$[\overline{F}(\overline{\theta})_{q_1} : \overline{E}_{q_1}] = [\overline{F}(\overline{\theta})_{q_2} : \overline{E}_{q_2}] = \ell^{2r} \ .$$

In particular, the two extensions have the same Galois group G.

By [14] Chapter 3, Section 4, a finite tame Galois extension requires a root of unity in the base field of order equal to the extensions's ramification degree. Since $\zeta_\ell \notin \overline{E}_{q_1}$ (6.1), $\overline{F}(\overline{\theta})_{q_1}/\overline{E}_{q_1}$ must be unramified, and since $\overline{E}_{q_1}$ has finite residue field, this means G is cyclic. Similarly, since $\zeta_{\ell^{r+1}} \notin \overline{E}_{q_2}$ (6.1), and since $\overline{\theta}$ is already totally ramified of order ℓ^r, G is noncyclic, contradiction. Conclude $D_{\mathfrak{p}}$ is a noncrossed product. □

Finally, lift $D_{\mathfrak{p}}$ to the E-division algebra

$$D \sim A \otimes_E (\theta, \pi) \ .$$

Theorem 9. *D is a noncrossed product of period and index ℓ^{2r}.*

Proof. Since A and θ are lifts of $\overline{A}$ and $\overline{\theta}$, D lifts $D_{\mathfrak{p}}$ in the Brauer group. To show D actually lifts $D_{\mathfrak{p}}$ it must be shown that the two have the same index. Let L/E be the Galois extension constructed in Lemma 6. Recall the data on $\overline{\theta}$: It is unramified of order ℓ^r at q_1 and totally ramified at q_2.

By Lemma 7 L/E splits A and $\mathrm{ind}(A) = \ell^{2r} = \frac{1}{m}[L : E]$, hence $E(\theta) \subset L$ is a subfield of A, and $\mathrm{ind}(A \otimes_E E(\theta)) = \ell^r$. Plainly $D \otimes_E E(\theta) \sim A \otimes_E E(\theta)$, and so $\mathrm{ind}(D \otimes_E E(\theta)) = \ell^r$. But then $\mathrm{ind}(D) \leq \ell^r \cdot [E(\theta) : E] = \ell^{2r}$. Since D lifts $D_{\mathfrak{p}}$ in the Brauer group and $\mathrm{ind}(D_{\mathfrak{p}}) = \ell^{2r}$ by Lemma 8, there must be equality. Therefore $\mathrm{per}(D) = \mathrm{ind}(D) = \ell^{2r}$, and D lifts $D_{\mathfrak{p}}$.

Finally, D is a noncrossed product: For since D and $D_{\mathfrak{p}} = D \otimes_E E_{\mathfrak{p}}$ have the same index, any Galois maximal subfield of D would lead to a Galois maximal subfield of $D_{\mathfrak{p}}$. Since $D_{\mathfrak{p}}$ is a noncrossed product by Lemma 8, this is impossible. □

Corollary 10. *Let $k_{\mathfrak{p}}$ be a $\mathfrak{p}$-adic number field, and let $\ell \nmid p$ be a prime such that $\zeta_\ell \notin k_{\mathfrak{p}}$. Let r be maximal such that $\zeta_{\ell^r} \in k_{\mathfrak{p}}(\zeta_\ell)$. Then there exists a noncrossed product $k_{\mathfrak{p}}(t)$-division algebra of period and index ℓ^{2r}.*

Proof. The noncrossed product E-division algebra D restricts to the noncrossed product $E_{\mathfrak{p}}$-division algebra $D_{\mathfrak{p}}$. Since $E \subset k_{\mathfrak{p}}(t) \subset E_{\mathfrak{p}}$, the intermediate algebra $D \otimes_E k_{\mathfrak{p}}(t)$ is a noncrossed product $k_{\mathfrak{p}}(t)$-division algebra. □

Remark. The noncrossed products over $k(t)$ of [6] all become crossed products over $k_{\mathfrak{p}}(t)$. For they are constructed in the form $A \otimes_{k(t)} (\theta, t)$, where A is a k-algebra and $\theta \in \mathrm{X}(k)$. When scalar is extended to $k_{\mathfrak{p}}(t)$, they are easily seen to possess Galois maximal subfields, namely the suitable $\mathfrak{p}$-unramified inertial lifts of $k_{\mathfrak{p}}(\theta_{\mathfrak{p}})$.

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