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ESSENTIAL COHOMOLOGY AND EXTRASPECIAL p-GROUPS

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ABSTRACT. Let p be an odd prime number and let G be an extraspecial p-group. The purpose of the paper is to show that G has no non-zero essential mod-p cohomology (and in fact that $H^*(G, \mathbb{F}_p)$ is Cohen-Macaulay) if and only if $|G| = 27$ and $exp(G) = 3$.

1. INTRODUCTION

Let p be a prime number. For every p-group K, denote by $H^*(K)$ the mod-p cohomology ring of K. A mod-p cohomology class of K is called essential if it vanishes on restriction to every proper subgroup of K. Let $Ess(K)$ be the ideal of $H^*(K)$ consisting of such classes of $H^*(K)$. As observed in [3], the study of $Ess(K)$ could provide interesting information for $H^*(K)$ (but, in contrast, it seems in general rather difficult to obtain elements of $Ess(K)$). For instance, $Ess(K) \neq \{0\}$ implies that the depth of $H^*(K)$ is just the rank of the center of K (see [3] and [5]); furthermore, with the condition that $H^*(K)$ is Cohen-Macaulay, it follows from [1] that $Ess(K) \neq \{0\}$ if and only if every element of order p of K is central (a way to obtain some element of $Ess(K)$ in this case was shown there).

We are now interested in extraspecial p-groups G. For $p = 2$, it was proved by Quillen ([17]) that $H^*(G)$ is Cohen-Macaulay and $Ess(G) = \{0\}$, except for the case $G = Q_8$, the quaternion group of order 8 (this fact also follows from Adem and Karagueuzian's result, as Q_8 is the unique group in which every element of order 2 is central). However, the situation is quite different for the case $p > 2$—*which is assumed from now on.* Consider first the case $|G| = p^3$; it follows from [8], [9], [10], [16] that $Ess(G) \neq \{0\}$ (so $H^*(G)$ is not Cohen-Macaulay) if and only if $exp(G) > 3$. In order to generalize this fact, in this note, we prove

Theorem. *If G is an extraspecial p-group, then $Ess(G) = \{0\}$ iff $exp(G) = 3$ and $|G| = 3^3$.*

It follows that the unique extraspecial p-group which has no non-zero essential cohomology is the one of order 27 and of exponent 3. In each of the remaining cases, $H^*(G)$ is not Cohen-Macaulay and the depth of $H^*(G)$ is just 1; we also point out some non-zero essential classes of G (it turns out that, if $|G| = p^5$ or $exp(G) = p^2$, there exists such a class of G belonging to $\mathrm{Im}\ \mathrm{Inf}_G^{G/Z}$ with Z the center of G).

The note is organized as follows. In Section 2, given an extraspecial p-group G of order p^{2n+1}, we shall consider G as a subgroup of the central product $\Gamma_n = C_{p^2} \bullet G$ and give a sufficient and necessary condition for the fact that $\mathrm{Res}_G^{\Gamma_n}(\xi) \neq 0$ with

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$\xi \in H^*(\Gamma_n)$. The proofs of the theorem for the cases $exp(G) > p$ or $|G| = p^5$, which are rather simple, will be given in Section 3. Section 4 is devoted to the case $exp(G) = p$.

2. The group Γ_n

Let us recall that an extraspecial p-group G is of order p^{2n+1} ($n \in \mathbb{N}$) and is isomorphic to one of the following central products of groups:

$$\mathbb{E}_n = \mathbb{E} \bullet \cdots \bullet \mathbb{E} \ (n \text{ times}),$$
$$\mathbb{M}_n = \mathbb{M} \bullet \mathbb{E}_{n-1},$$

where

$$\mathbb{M} = \langle a, b | a^{p^2} = b^p = 1, b^{-1}ab = a^{1+p} \rangle,$$
$$\mathbb{E} = \langle a, b | a^p = b^p = [a,b]^p = [a,[a,b]] = [b,[a,b]] = 1 \rangle$$

are extraspecial p-groups of order p^3. Note that

$$exp(G) = \begin{cases} p^2, & \text{for } G = \mathbb{M}_n, \\ p, & \text{for } G = \mathbb{E}_n, \end{cases}$$

and $\mathbb{M}_n = \mathbb{M}_{n-1} \bullet \mathbb{M}$.

These groups can be obtained cohomologically as follows. Let V be a vector space of dimension $2n+1$ over the prime field $\mathbb{F}_p$ with basis $e, a_1, \ldots, a_{2n}$. Let $x, x_1, \ldots, x_{2n}$ be a basis of $H^1(V)$, dual to that of V, and let $y = \beta x, y_i = \beta x_i$ with β the Bockstein homomorphism, so

$$H = H^*(V) = E[x, x_1, \ldots, x_{2n}] \otimes \mathbb{F}_p[y, y_1, \ldots, y_{2n}]$$

with $E[u, v, \ldots]$ (resp. $\mathbb{F}_p[u, v, \ldots]$) the exterior (resp. polynomial) algebra over $\mathbb{F}_p$ with generators $u, v, \ldots$ of degree 1 (resp. 2). Consider the central extension

$$(\Gamma_n) \qquad 0 \to \mathbb{F}_p \xrightarrow{i} \Gamma_n \to V \to 0,$$

with factor set $z = z_n = y + x_1x_2 + \cdots + x_{2n-1}x_{2n}$. Via the inflation map, x and the x_i's can be considered as elements of $H^1(\Gamma_n)$. Given a subgroup K of Γ_n, with some abuse of notation, we also denote by x (resp. x_i) the element $\mathrm{Res}_K^{\Gamma_n}(x)$ (resp. $\mathrm{Res}_K^{\Gamma_n}(x_i)$).

It is easy to show

Lemma 1. (i) $\Gamma_n = C_{p^2} \bullet \mathbb{M}_n = C_{p^2} \bullet \mathbb{E}_n = \Gamma_{n-1} \bullet \mathbb{M}$.

(ii) $\mathbb{M}_n = \mathrm{Ker}\ (x + \alpha)$, $\mathbb{E}_n = \mathrm{Ker}\ x$ *and* $\Gamma_{n-1} \times C_p = \mathrm{Ker}\ \alpha$, *with* α *a non-zero linear combination of* $x_1, \ldots, x_{2n}$. □

Then $C_{p^2} = \bigcap_{i=1}^{2n} \mathrm{Ker}\ x_i$ is a subgroup of Γ_n. Let w be a generator of $H^2(C_{p^2})$, so

$$H^*(C_{p^2}) = E[x] \otimes \mathbb{F}_p[w].$$

Set $\mathcal{G}_n = C_{p^2} \times \mathbb{E}_n$. By the Künneth formula, we have

$$H^*(\mathcal{G}_n) = H^*(\mathbb{E}_n) \otimes E[x] \otimes \mathbb{F}_p[w].$$

As Γ_n is the central product of C_{p^2} and $\mathbb{E}_n$, there exists a central subgroup U_n of order p of $\mathcal{G}_n$ such that $\mathcal{G}_n/U_n = \Gamma_n$ and the factor set of the central extension

$$1 \to U_n \to \mathcal{G}_n \to \Gamma_n \to 1$$

is just y. Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U_n & \longrightarrow & U_n \times \mathbb{E}_n & \longrightarrow & \mathbb{E}_n & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & U_n & \longrightarrow & \mathcal{G}_n & \longrightarrow & \Gamma_n & \longrightarrow & 1 \end{array} \tag{1}$$

whose rows are central extensions and whose vertical arrows are inclusion maps. Pick elements s, t of $H^*(U_n)$ satisfying $H^*(U_n) = E[s] \otimes P[t]$. It follows from [11] (see also [2]) that t can be chosen so that $\mathrm{Res}_{U_n}^{\mathcal{G}_n}(w \times 1) = t$.

We now use the following notation. Given a ring R and elements $r, s, \dots \in R$, $(r, s, \dots)$ will denote the ideal of R generated by $r, s, \dots$. The main result of this section is the following.

Proposition 1. *If $\xi \in H^*(\Gamma_n)$, then $\mathrm{Res}_{\mathbb{E}_n}^{\Gamma_n}(\xi) \neq 0$ iff $x\xi \notin (y)$.*

Proof. Set $X = \mathrm{Inf}_{\mathcal{G}_n}^{\Gamma_n}(\xi)$. As $\mathrm{Ker}\ \mathrm{Inf}_{\mathcal{G}_n}^{\Gamma_n} = (y)$, it follows that $xX = 0$ iff $x\xi \in (y)$. Write $X = \sum w^i \otimes s_i + \sum w^i x \otimes t_i$ with $s_i, t_i \in H^*(\mathbb{E}_n)$. It is clear that $\mathrm{Res}_{\mathbb{E}_n}^{\Gamma_n}(\xi) \neq 0$ iff $\mathrm{Inf}_{U_n \times \mathbb{E}_n}^{\mathbb{E}_n} \mathrm{Res}_{\mathbb{E}_n}^{\Gamma_n}(\xi) \neq 0$. So, by the commutative diagram (1), $\mathrm{Res}_{\mathbb{E}_n}^{\Gamma_n}(\xi) \neq 0$ iff $\mathrm{Res}_{U_n \times \mathbb{E}_n}^{\mathcal{G}_n}(X) \neq 0$, which is equivalent to the fact that the s_i's are not all equal to zero, or equivalently, $x\xi \notin (y)$. The proposition follows. □

For convenience, given a central extension of groups

$$1 \to A \to K \to C \to 1, \tag{K}$$

denote by $\{E_r(K), d_r\}$ the Hochschild-Serre spectral sequence corresponding to the extension (K). We now recall some results given in [12], [13] (see also [2] for $n = 1$) concerning $\{E_r(\Gamma_n), d_r\}$. As usual, denote by $\mathcal{P}^i$ the Steenrod operations. Set $Z = i(\mathbb{F}_p) \subset C_{p^2} \subset \Gamma_n$. So $v = \mathrm{Res}_Z^{C_{p^2}}(w)$ is a generator of $H^2(Z)$. Let

$$\begin{aligned} X_n &= x_1 x_2 \dots x_{2n-1} x_{2n}, \\ \eta_i &= \mathcal{P}^{p^{i-2}} \dots \mathcal{P}^1 \beta z \\ &= \sum_{j=1}^{n} (x_{2j-1} y_{2j}^{p^{i-1}} - x_{2j} y_{2j-1}^{p^{i-1}}), \\ \xi_m &= \beta \mathcal{P}^{p^{m-1}} \dots \mathcal{P}^1 \beta z \\ &= \sum_{j=1}^{n} (y_{2j-1} y_{2j}^{p^m} - y_{2j} y_{2j-1}^{p^m}), \end{aligned}$$

$1 \leq i \leq n+1, 1 \leq m \leq n$, be elements of $H^*(V)$. We have

Theorem 1 ([2], as corrected in [13, Rk. 2.11(ii)], [12])**.** (i) *We have*

$$\begin{aligned} E_\infty(\Gamma_1) = & H^*(V)/(z, \eta_1, \eta_2, \xi_1) \otimes \mathbb{F}_p[v^p] \\ & \oplus (\mathbb{F}_p X_1 \oplus \mathbb{F}_p x X_1) \otimes \sum_{i=1}^{p-2} \mathbb{F}_p[v^p] v^i. \end{aligned}$$

(ii) *For* $n \geq 2$,

$$\begin{aligned} E_{2p+1}(\Gamma_n) =& H^*(V)/(z, \eta_1, \xi_1, A_n\eta_2) \otimes \mathbb{F}_p[v^p] \\ & \oplus (\mathbb{F}_p X_n \oplus \mathbb{F}_p x X_n) \otimes \sum_{i=1}^{p-2} \mathbb{F}_p[v^p]v^i \end{aligned}$$

with $A_n = \sum_{i=1}^{n} x_1x_2 \dots \hat{x}_{2i-1}\hat{x}_{2i} \dots x_{2n-1}x_{2n}$. □

Let W be the vector subspace of V given by $W = \mathrm{Ker}(x - x_1)$. We then have the central extension

$$(\mathbb{M}_n) \qquad 1 \to Z \to \mathbb{M}_n \to W \to 0$$

with factor set $z' = z'_n = y_1 + x_1x_2 + \dots + x_{2n-1}x_{2n}$. Following [10], [12], we also have

Proposition 2 ([10], [12])**.** (i) *We have*

$$\begin{aligned} E_\infty(\mathbb{M}_1) = E_{2p+1}(\mathbb{M}_1) =& H^*(W)/(z', \beta z') \otimes \mathbb{F}_p[v^p] \\ & \oplus (\mathbb{F}_p[y_2]x_1 \oplus \mathbb{F}_p[y_2]x_1x_2) \otimes \mathbb{F}_p[v^p]v^{p-1} \\ & \oplus (\mathbb{F}_p x_1 \oplus \mathbb{F}_p x_1x_2) \otimes \sum_{i=1}^{p-2} \mathbb{F}_p[v^p]v^i. \end{aligned}$$

(ii) *For* $n \geq 2$,

$$\begin{aligned} E_{2p+1}(\mathbb{M}_n) =& H^*(W)/(z', \beta z', \beta\mathcal{P}^1\beta z', B_n \cdot \mathcal{P}^1\beta z') \otimes \mathbb{F}_p[v^p] \\ & \oplus (\mathbb{F}_p x_1x_3x_4 \dots x_{2n-1}x_{2n} \oplus \mathbb{F}_p X_n) \otimes \sum_{i=1}^{p-2} \mathbb{F}_p[v^p]v^i \end{aligned}$$

with $B_n = \sum_{i=2}^{n} x_1x_3x_4 \dots \hat{x}_{2i-1}\hat{x}_{2i} \dots x_{2n-1}x_{2n}$. □

We also prove

Proposition 3. *If* $\xi \in H^*(\Gamma_n)$ *and* $|\xi| < 2n+2$, *then* $\xi \in \mathrm{Im}\ \mathrm{Inf}^V_{\Gamma_n}$.

The proof of the proposition is divided into the following lemmas. Set

$$\mathcal{R} = E[x_1, \dots, x_m] \otimes \mathbb{F}_p[t_1, \dots, t_m],$$

and let

$$\alpha_i = \sum_{j=1}^{m} x_j t_j^{p^{i-1}}, \qquad 1 \leq i \leq m,$$

be elements of $\mathcal{R}$. Denote by $\mathcal{I}_{k,m}$ the set consisting of subsets of k elements of $\{1, \dots, m\}$. For every element $I = \{i_1, \dots, i_k\}$ of $\mathcal{I}_{k,m}$ with $i_1 < \dots < i_k$, set $x_I = x_{i_1} \dots x_{i_k}$ and $x_\emptyset = 1$.

Lemma 2. *For* $X \in \mathcal{R}$ *and* $1 \leq k \leq m$,

(i) *if* $X \cdot \alpha_1 \dots \alpha_k = 0$, *then* $X \in (\alpha_1, \dots, \alpha_k, x_I | I \in \mathcal{I}_{m-k+1,m})$;
(ii) *if* $X \cdot \alpha_k = 0$, *then* $X \in (\alpha_k, x_1 \dots x_m)$.

Proof. (i) We argue by induction on m. The case $m = 2$ is obvious. Assume that (i) holds for $m - 1$.

If $k = m$, then

$$\alpha_1 \dots \alpha_m = \begin{vmatrix} t_1 & \dots & t_m \\ t_1^p & \dots & t_m^p \\ \cdot & \dots & \cdot \\ t_1^{p^{m-1}} & \dots & t_m^{p^{m-1}} \end{vmatrix} x_1 \dots x_m;$$

so $X \in (x_i | 1 \le i \le m)$. Suppose that $k < m$. Write

$$\alpha_i = \alpha_i' + x_m t_m^{p^{i-1}}$$

with $\alpha_i' = \sum_{j=1}^{m-1} x_j t_j^{p^{i-1}}, 1 \le i \le m$, and

$$X = X' + X'' x_m,$$

with X', X'' free of x_m. Since $X\alpha_1 \dots \alpha_k = 0$, we have

$$0 = X'\alpha_1' \dots \alpha_k',$$

$$0 = (-1)^k X''\alpha_1' \dots \alpha_k' + X' \sum_{i=1}^{k} (-1)^{k-i} t_m^{p^{i-1}} \alpha_1' \dots \widehat{\alpha}_i' \dots \alpha_k'.$$

By writing

$$X' = t_m^{r_1} f_1 + \dots + t_m^{r_j} f_j,$$
$$X'' = t_m^{s_1} g_1 + \dots + t_m^{s_i} g_i$$

with f_i, g_j free of $t_m, r_1 < \dots < r_j, s_1 < \dots < s_i$, we have

$$(-1)^k t_m^{s_i} g_i \alpha_1' \dots \alpha_k' + t_m^{p^{k-1}+r_j} f_j \alpha_1' \dots \alpha_{k-1}' = 0. \tag{2}$$

Consider the following cases:

- $r_j + p^{k-1} > s_i$: from (2), $f_j \alpha_1' \dots \alpha_{k-1}' = 0$. By the inductive hypothesis, $f_j \in (\alpha_1', \dots, \alpha_{k-1}', \mathcal{I}_{m-k+1,m-1})$. Since $\alpha_i' = \alpha_i - x_m t_m^{p^{i-1}}$, we have $X = t_m^{r_1} f_1 + \dots + t_m^{r_{j-1}} f_{j-1} \bmod (x_m, \alpha_1, \dots, \alpha_k, \mathcal{I}_{m-k+1,m})$. So we may suppose that $f_j = 0$.
- $r_j + p^{k-1} < s_i$: from (2), $g_i \alpha_1' \dots \alpha_k' = 0$. By the inductive hypothesis, $g_i \in (\alpha_1', \dots, \alpha_k', \mathcal{I}_{m-k,m-1})$. So $x_m g_i \in (\alpha_1, \dots, \alpha_k, \mathcal{I}_{m-k+1,m})$.
- $r_j + p^{k-1} = s_i$: from (2), $((-1)^{k+k-1} g_i \alpha_k' + f_j)\alpha_1' \dots \alpha_{k-1}' = 0$. By the inductive hypothesis, $f_j = g_i \alpha_k' \bmod (\alpha_1', \dots, \alpha_{k-1}', \mathcal{I}_{m-k+1,m-1})$. Since $\alpha_i' = \alpha_i - x_m t_m^{p^{i-1}}$, we have $f_j = g_i \alpha_k' \bmod (x_m, \alpha_1, \dots, \alpha_{k-1}, \mathcal{I}_{m-k+1,m})$. So we may suppose that $f_j = g_i \alpha_k'$. Since

$$\begin{aligned} t_m^{r_j} f_j + t_m^{s_i} g_i x_m &= t_m^{r_j}(f_j + t_m^{p^{k-1}} g_i x_m) \\ &= t_m^{r_j} g_i (\alpha_k' + t_m^{p^{k-1}} x_m) \\ &= t_m^{r_j} g_i \alpha_k, \end{aligned}$$

we may then suppose that $f_j = 0$ and $g_i = 0$.

The above arguments show that we may reduce to the case $X' = 0$. It follows that $X''\alpha_1' \dots \alpha_k' = 0$. By the inductive hypothesis, $X'' \in (\alpha_1', \dots, \alpha_k', \mathcal{I}_{m-k,m-1})$. Hence $x_m X'' \in (\alpha_1, \dots, \alpha_k, \mathcal{I}_{m-k+1,m})$. (i) is proved.

(ii) We again use induction on m. The case $m = 1$ is trivial. Assume that (ii) holds for $m - 1$. As above, write

$$X = X' + X''x_m,$$

with X', X'' free of x_m. Arguing as above, we may reduce to the case $X' = 0$. It follows that $X''\alpha'_k = 0$. Bu the inductive hypothesis, $X'' \in (\alpha'_k, x_1 \dots x_{m-1})$. Hence $x_m X'' \in (\alpha_k, x_1 \dots x_m)$.

The lemma is proved. □

Lemma 3. *Let $1 \le k \le n$ and let $Y_1, \dots, Y_k$ be elements of $H^*(V)$.*

(i) *If $Y_1\xi_1 + \cdots + Y_k\xi_k = 0$, then $Y_k \in (\xi_1, \dots, \xi_{k-1})$.*

(ii) *Assume that*

$$Y_k = \sum_{\substack{I \subset \{1,\dots,2n\} \\ \#(I) < 2n-k+1}} x_I f_I(y, y_1, \dots, y_{2n}).$$

We have:

(iia) *if $Y_1\eta_1 + \cdots + Y_k\eta_k = 0$, then $Y_k \in (\eta_1, \dots, \eta_k)$;*

(iib) *if $Y_k \in \bigcap_{i=1}^k (\eta_i)$, then $Y_k \in (\eta_1 \dots \eta_k)$;*

(iic) *if $Y_1\xi_1 + \cdots + Y_{k-1}\xi_{k-1} + Y_k\eta_\ell = 0$ with $1 \le \ell \le n$, then*

$$Y_k \in (\eta_\ell, \xi_1, \dots, \xi_{k-1}).$$

Proof. (i) For $1 \le i \le k$, write

$$Y_i = \sum_{I \subset \{1,\dots,2n\}} x_I f_I^{(i)}(y, y_1, \dots, y_{2n}).$$

Then, for every I, we have

$$\sum_{i=1}^k f_I^{(i)} \xi_i = 0.$$

According to [18], $\xi_1, \dots, \xi_k$ is a regular sequence in P. So the above equality implies $f_I^{(k)} \in (\xi_1, \dots, \xi_{k-1})$. Therefore $Y_k \in (\xi_1, \dots, \xi_{k-1})$.

(iia) It follows that $Y_k\eta_1 \dots \eta_k = 0$. By Lemma 2, $Y_k \in (\eta_1, \dots, \eta_k, \mathcal{I}_{2n-k+1,2n})$. So $Y_k \in (\eta_1, \dots, \eta_k)$.

(iib) We use induction on k. For $k = 2$, $X\eta_1 + Y\eta_2 = 0$ implies $Y\eta_1\eta_2 = 0$. By Lemma 2, $Y \in (\eta_1, \eta_2, \mathcal{I}_{2n-1,2n})$. So $Y \in (\eta_1, \eta_2)$. Write $Y = a\eta_1 + b\eta_2$. Then $Y_2 = Y\eta_2 = a\eta_1\eta_2$.

Assume that (iib) holds for $k - 1 \ge 2$. As $Y_k \in \bigcap_{i<k}(\eta_i)$, it follows from the inductive hypothesis that $Y_k = Y\eta_1 \dots \eta_{k-1}$. Write $Y_k = X\eta_k$. Then $Y\eta_1 \dots \eta_k = 0$. By Lemma 2, $Y = c_1\eta_1 + \cdots + c_k\eta_k$. So $Y_k = (-1)^{k-1}c_k\eta_1 \dots \eta_k$.

(iic) Again, we use induction on k. $Y_1\xi_1 + Y_2\eta_\ell = 0$ implies $Y_1\eta_\ell = 0$. By Lemma 2, $Y_1 \in (\eta_\ell)$. Write $Y_1 = c\eta_\ell$. Then $(c\xi_1 + Y_2)\eta_\ell = 0$. By Lemma 2, $c\xi_1 + Y_2 \in (\eta_\ell)$; hence $Y_2 \in (\eta_\ell, \xi_1)$.

Assume that (iic) holds for $k - 1 \ge 2$. As $Y_1\eta_\ell\xi_1 + \cdots + Y_{k-1}\eta_\ell\xi_{k-1} = 0$, it follows from (i) that $Y_{k-1}\eta_\ell \in (\xi_1, \dots, \xi_{k-2})$. By the inductive hypotheses, we may write $Y_{k-1} = c_1\xi_1 + \cdots + c_{k-2}\xi_{k-2} + c_{k-1}\eta_\ell$. Hence

$$(Y_1 + c_1\xi_{k-1})\xi_1 + \cdots + (Y_{k-2} + c_{k-2}\xi_{k-1})\xi_{k-2} + (Y_k + c_{k-1}\xi_{k-1})\eta_\ell = 0.$$

By the inductive hypothesis, $Y_k + c_{k-1}\xi_{k-1} \in (\xi_1, \dots, \xi_{k-2}, \eta_\ell)$, and hence $Y_k \in (\xi_1, \dots, \xi_{k-1}, \eta_\ell)$. □

For $1 \le i \le n+1, 0 \le k \le n$, denote by $\Delta_{i,k}$ the ideal of $H^*(V)$ given by

$$\Delta_{i,k} = \begin{cases} (z, \eta_j, \xi_m | 1 \le j \le i, 1 \le m \le k) & \text{if } k \ge 1, \\ (z, \eta_j | 1 \le j \le i) & \text{if } k = 0. \end{cases}$$

Lemma 4. *If $X = \sum_{\#(I)<2n-2k+1} x_I X_I(y_1, \dots, y_{2n})$ and $X\xi_j \in \Delta_{k,j-1}$ with $1 \le j \le k \le n$, then $X \in \Delta_{k,j-1}$.*

Proof. Write

$$X\xi_j = a_0 z + \sum_{i=1}^{k} a_i \eta_i + \sum_{i=1}^{j-1} b_i \xi_i$$

with $a_i, b_\ell \in H^*(V)$. Since $y = z - x_1x_2 - \cdots - x_{2n-1}x_{2n}$, we may suppose that $a_i, b_\ell, 1 \le i \le k, 1 \le \ell \le j-1$, are free of y. It follows that $a_0 = 0$ and

$$X\xi_j\eta_1 \dots \eta_k = \sum_{i=1}^{j-1} b_i \xi_i \eta_1 \dots \eta_k. \tag{3}$$

We now argue by induction on j. For $j = 1$, it follows that $X\xi_1\eta_1 \dots \eta_k = 0$. Hence $X\eta_1 \dots \eta_k = 0$. By Lemma 2, $X \in (\eta_1, \dots, \eta_k)$.

Assume that the lemma holds for $j - 1 \ge 1$. By Lemma 3 (i) and by (3), there exists $c_i \in H^*(V)$ such that

$$X\eta_1 \dots \eta_k = c_1\xi_1 + \cdots + c_{j-1}\xi_{j-1}.$$

Therefore, by Lemma 3 (i), $c_{j-1}\eta_i \in (\xi_1, \dots, \xi_{j-2})$, for every $1 \le i \le k$; by Lemma 3 (iic), $c_{j-1} \in \bigcap_{i \le k}(\xi_1, \dots, \xi_{j-2}, \eta_i)$. By writing

$$\begin{aligned} c_{j-1} &= d_1\xi_1 + \cdots + d_{j-2}\xi_{j-2} + d\eta_1 \dots \eta_{i-1} \\ &= e_1\xi_1 + \cdots + e_{j-2}\xi_{j-2} + e\eta_i, \end{aligned}$$

we get

$$[(e_1 - d_1)\xi_1 + \cdots + (e_{j-2} - d_{j-2})\xi_{j-2}]\eta_1 \dots \eta_i = 0.$$

By Lemma 2, $(e_1 - d_1)\xi_1 + \cdots + (e_{j-2} - d_{j-2})\xi_{j-2}$ contains $\eta_1 \dots \eta_i$ as a factor. Hence $e\eta_i \in \bigcap_{\ell \le i}(\eta_\ell)$. By Lemma 3 (iib), $c_{j-1} \in (\xi_1, \dots, \xi_{j-2}, \eta_1 \dots \eta_i)$. So we may suppose that $c_{j-1} \in (\eta_1 \dots \eta_k)$. By writing $c_{j-1} = c\eta_1 \dots \eta_k$, we have

$$(X - c\xi_{j-1})\eta_1 \dots \eta_k = c_1\xi_1 + \cdots + c_{j-2}\xi_{j-2}.$$

By the inductive hypothesis, this implies $X - c\xi_{j-1} \in \Delta_{k,j-2}$. So $X \in \Delta_{k,j-1}$. The lemma follows. □

Lemma 5. *If $X = \sum_{\#(I)<2n-2k} x_I X_I(y_1, \dots, y_{2n})$ and $X\eta_k \in \Delta_{k-1,k-1}$ with $1 \le k \le n+1$, then $X \in \Delta_{k,k-1}$.*

Proof. Write

$$X\eta_k = a_0 z + \sum_{i=1}^{k-1} (a_i\eta_i + b_i\xi_i).$$

Arguing as in the proof of Lemma 4, we may suppose that a_i, b_i, with $1 \le i \le k-1$, are free of y. It follows that $a_0 = 0$ and

$$X\eta_k = \sum_{i=1}^{k-1} (a_i\eta_i + b_i\xi_i).$$

Furthermore, we may suppose that every b_i is of form

$$b_i = \sum_{\#(I)<2n-2k+1} x_I b_I^{(i)}.$$

Therefore, applying Lemma 4 yields $b_{k-1} \in \Delta_{k,k-2}$. Hence, by induction, we need only consider the case

$$X\eta_k = b_1\xi_1 + \sum_{i=1}^{k-1} a_i\eta_i.$$

This implies $b_1\xi_1\eta_1\dots\eta_k = 0$. So $b_1\eta_1\dots\eta_k = 0$. By Lemma 2, $b_1 \in (\eta_1,\dots,\eta_k)$. The lemma follows. □

Let us now consider the Hochschild-Serre spectral sequence $\{E_r(\Gamma_n), d_r\}$. It follows that, for $k < 2n+2$,

$$\sum_{i+j=k} E_{2p+1}^{i,j} \subset E_{2p+1}^{k,0} \oplus \bigoplus_{r\geq 1} E_{2p+1}^{*,2pr}.$$

By Kudo's transgression theorem, for $m \leq n$, $1 \otimes v^{p^m}$ (resp. $\eta_m \otimes v^{p^{m-1}(p-1)}$) survives to E_{2p^m+1} (resp. $E_{2p^{m-1}(p-1)+1}$) and

$$d_{2p^m+1}(1 \otimes v^{p^m}) = \eta_{m+1},$$
$$d_{2p^{m-1}(p-1)+1}(\eta_m \otimes v^{p^{m-1}(p-1)}) = -\xi_m.$$

Lemma 6. *For $k < 2n+2$ and $1 \leq m \leq n$, we have*

$$\sum_{i+j=k} E_{2p^m+1}^{i,j} \subset E_{2p^m+1}^{k,0} \oplus \bigoplus_{r\geq 1} E_{2p^m+1}^{*,2p^m r}.$$

Proof. By the structure of $E_{2p+1}(\Gamma_n)$, the lemma holds for $m = 1$. Suppose that the lemma holds for $m = s \geq 1$. Let $\psi = X \otimes v^{\ell p^s}$ be an element of $E_2(\Gamma_n)$ surviving to $E_{2p^{s+1}+1}$, with $1 \leq \ell \leq p-1$ and $|X| + 2\ell p^s = k < 2n+2$. So $d_{2p^s+1}(\psi) = \ell X\eta_{s+1} \otimes v^{(\ell-1)p^s}$ must be hit by images under the differentials of elements of degrees less than $2n+2$. By the inductive hypothesis and by Kudo's theorem, these images belong to the ideal $\Delta_{s,s}$; hence so does $X\eta_{s+1}$. Since in $E_3(\Gamma_n)$ we have $y = -(x_1x_2 + \dots + x_{2n-1}x_{2n})$, we may suppose that X is free of y. As $|X| < 2n+2-2p^s < 2n-2s-2$, by Lemma 5, this means that $X \in \Delta_{s+1,s}$. So $\psi = 0$ in E_{2p^s+2} if $\ell < p-1$. If $\ell = p-1$, write $\psi = Y\eta_{s+1} \otimes v^{p^s(p-1)}$. Then $d_{2p^s(p-1)+1}(\psi) = -Y\xi_{s+1} \in \Delta_{s+1,s}$. Arguing as above, we may suppose that Y is free of y. By Lemma 4, as $|\psi| < 2n+2$, we have $Y \in \Delta_{s+1,s}$. So $\psi = 0$ in $E_{2p^s(p-1)+2}$. The lemma follows. □

Proof of Proposition 3. It follows from Lemma 6 that ξ either belongs to Im $\mathrm{Inf}_{\Gamma_n}^V$ or represents an element of $E_\infty^{*,2p^n r}$. As $2p^n > 2n+2$, the fact that $|\xi| < 2n+2$ implies $\xi \in$ Im $\mathrm{Inf}_{\Gamma_n}^V$. The proposition follows. □

3. The case $exp(G) > p$ or $|G| = p^5$

We first consider the case $G = \mathbb{M}_n$. Consider G as a subgroup of Γ_n by setting $G = \mathrm{Ker}\ (x - x_1)$. If $n = 1$, it follows from [10] (see also [4]) that $H^*(\mathbb{M})$ contains

a non-zero essential element, namely X_1. Assume inductively that $0 \neq X_{n-1} \in Ess(\mathbb{M}_{n-1})$. As $\mathbb{M}_n = \mathbb{M}_{n-1} \bullet \mathbb{M}$, we have the following central extension:

$$0 \to \mathbb{F}_p \to \mathbb{M}_{n-1} \times \mathbb{M} \to \mathbb{M}_n \to 1.$$

The fact that $H^*(\mathbb{M}_n)$ contains non-zero essential elements follows from

Proposition 4. $0 \neq X_n \in Ess(\mathbb{M}_n)$.

Proof. Let K be a maximal subgroup of $\mathbb{M}_n$. As $\dim_{\mathbb{F}_p} H^1(K) = 2n-1$, it follows that the product of any $2n$ elements of $H^1(K)$ vanishes. Hence $\mathrm{Res}_K^{\mathbb{M}_n}(X_n) = 0$, which implies that $X_n \in Ess(\mathbb{M}_n)$. Furthermore, as $\mathrm{Inf}_{\mathbb{M}_{n-1}\times\mathbb{M}}^{\mathbb{M}_n}(X_n) = X_{n-1} \times x_{2n-1}x_{2n} \neq 0$ in $H^*(\mathbb{M}_{n-1}\times\mathbb{M})$ by the inductive hypothesis, it follows that $X_n \neq 0$. The proposition is proved □

By Theorem 1 (i), X_1 and xX_1 are non-zero elements of $H^*(\Gamma_1)$. By considering the central extension $0 \to \mathbb{F}_p \to \Gamma_{n-1} \times \mathbb{M} \to \Gamma_n \to 1$, and by using the same argument given in the proof of Proposition 4, we also have

Proposition 5. *The elements xX_n and X_n are non-zero elements of $H^*(\Gamma_n)$.* □

Our next task is to prove that the theorem holds for the extraspecial p-group $G = \mathbb{E}_2$. Consider $\mathbb{E}_2$ as a subgroup of Γ_2 as in Lemma 1. Let Q be the element of $H^*(V)$ defined by $Q = Q_{2,1}^{1,2} - Q_{2,1}^{3,4}$ with

$$Q_{2,1}^{i,j} = Q_{2,1}(y_i, y_j) = \frac{y_i^{p^2} y_j - y_j^{p^2} y_i}{y_i^p y_j - y_j^p y_i}$$

(so $Q_{2,1}^{i,j}$ is nothing but the Dickson invariant of order $2(p^2-p)$ with variables y_i, y_j), and set $\eta = x_1x_2Q$. It follows from [19, Th. 8.25] that $0 \neq \eta \in H^*(\mathbb{E}_2)$. The case $G = \mathbb{E}_2$ is then proved by the following:

Proposition 6. $\eta \in Ess(\mathbb{E}_2)$.

Proof. Let K be a maximal subgroup of $\mathbb{E}_2$, so $K \cong \mathbb{E} \times C_p$. If $\mathrm{Res}_K^{\mathbb{E}_2}(x_3x_4) = 0$, it is clear that $\mathrm{Res}_K^{\mathbb{E}_2}(x_1x_2Q_{2,1}^{1,2}) = 0$; we can then assume that $\mathrm{Res}_K^{\mathbb{E}_2}(x_3x_4) \neq 0$. Choose a basis u_1, u_2, u_3, u_4 of $H^1(\mathbb{E}_2/Z)$ such that $K = \mathrm{Ker}\ u_4$, $x_1x_2 + x_3x_4 = u_1u_2 + u_3u_4$, $\mathrm{Res}_K^{\mathbb{E}_2}(x_1x_2) = u_1u_2 + u_1u_3$ and $\mathrm{Res}_K^{\mathbb{E}_2}(x_3x_4) = -u_1u_3$. This implies that $u_1u_2 = 0$ in $H^*(K)$. By setting $v_i = \beta u_i$, we have

$$\begin{aligned}\mathrm{Res}_K^{\mathbb{E}_2}(\eta) &= (u_1u_2 + u_1u_3)(Q_{2,1}(v_1, v_2+v_3) - Q_{2,1}(v_3, v_1)) \\ &= u_1u_3(Q_{2,1}(v_1, v_2+v_3) - Q_{2,1}(v_3, v_1)) \qquad \text{as } u_1u_2 = 0 \text{ in } H^*(K).\end{aligned}$$

Set $Y = Q_{2,1}(v_1, v_2+v_3) - Q_{2,1}(v_3, v_1)$. Following [13, Proof of Lemma 1.10], Y contains $v_2^p - v_2v_1^{p-1}$ as a factor. As $u_1(v_2^p - v_2v_1^{p-1}) = -\mathcal{P}^1\beta(u_1u_2) + v_1^{p-1}\beta(u_1u_2)$, we have $\mathrm{Res}_K^{\mathbb{E}_2}(\eta) = 0$. So $\eta \in Ess(\mathbb{E}_2)$. The proposition is proved. □

For $exp(G) > p$ or $|G| = p^5$, Propositions 4 and 6 tell us that there exist non-zero essential cohomology classes of G which belong to $\mathrm{Im}\ \mathrm{Inf}_G^V$. Furthermore, if $G = \mathbb{M}_2$, then [12, Proposition 1.9] and [13, Theorem 3.10] tell us that

$$x_3x_4N \quad \text{and} \quad (y_3x_4 - y_4x_3)N$$

are also non-zero elements of $Ess(\mathbb{M}_2)$ with $N = (y_2^{p-1} - y_3^{p-1})(y_2^{p-1} - y_4^{p-1})$. We can then end the section by the following

Question. For $G \ncong \mathbb{E}$, is it true that $Ess(G) \cap \mathrm{Im}\ \mathrm{Inf}_G^V \neq \{0\}$?

4. The case $exp(G) = p$

We first point out some mod-p cohomology classes of Γ_n, by using the following argument given by D.J. Green [6]. Let K be a p-group containing C as a central subgroup. We have the central extension

$$(K) \qquad 1 \to C \to K \xrightarrow{\mathrm{pr}} K/C \to 1,$$

On the other hand, by considering the extension

$$(K \times C) \qquad 1 \to C \xrightarrow{\ell} K \times C \xrightarrow{j} K \to 1$$

with $\ell(c) = (1, c), j(k, c) = k, c \in C, k \in K$, we have the commutative diagram

$$(4) \qquad \begin{array}{ccccccccc} 1 & \longrightarrow & C & \xrightarrow{\ell} & K \times C & \xrightarrow{j} & K & \longrightarrow & 1 \\ & & \| & & \mu\downarrow & & \downarrow\mathrm{pr} & & \\ 1 & \longrightarrow & C & \longrightarrow & K & \xrightarrow{\mathrm{pr}} & K/C & \longrightarrow & 1 \end{array}$$

with $\mu(k, c) = kc, k \in K, c \in C$. The Hochschild-Serre spectral sequences corresponding to these extensions are of the forms

$$E_2(K) = H^*(K/C) \otimes H^*(C) \Rightarrow H^*(K),$$
$$E_2(K \times C) = E_\infty(K \times C) = H^*(K) \otimes H^*(C).$$

Furthermore, vertical arrows in (4) also induce a map $\{\mu_r : E_r(K) \to E_r(K \times C)\}$ between spectral sequences with $\mu_2 = (\mathrm{Inf}_K^{K/C}, 1_{H^*(C)})$.

The following is due to D.J. Green.

Proposition 7. *For $r \geq 2$,*

$$\mathrm{Im}(d_r : E_r(K) \to E_r(K)) \subset \mathrm{Ker}\ \mathrm{Inf}_K^{K/C} \otimes H^*(C).$$

Proof. Let $\xi \in E_r(K)$ and write $d_r(\xi) = \sum \phi_j \otimes \psi_j, \phi_j \in H^*(K/C), \psi_j \in H^*(C)$. We can suppose that the ψ_j's are linearly independent in $H^*(C)$. From the commutative diagram (4) and from the fact that $d_r : E_r(K \times C) \to E_r(K \times C)$ vanishes, we have

$$\sum \mathrm{Inf}_K^{K/C}(\phi_j) \otimes \psi_j = \mu_r(d_r(\xi)) = d_r(\mu_r(\xi)) = 0.$$

So $\phi_j \in \mathrm{Ker}\ \mathrm{Inf}_K^{K/C}$. The proposition follows. □

Since $d_{2p+1}(v^p) = \eta_2$ in $E_{2p+1}(\Gamma_n)$ (resp. $\mathcal{P}^1\beta z'$ in $E_{2p+1}(\mathbb{M}_n)$), it follows from Theorem 1 and Proposition 2 that $A_n \otimes v^p \in E_{2p+2}(\Gamma_n)$ and $B_n \otimes v^p \in E_{2p+2}(\mathbb{M}_n)$. We then get

Proposition 8. *For $1 \leq i \leq p-2$,*

(i) *if $n \geq 2$ then $x_1x_3x_4 \ldots x_{2n-1}x_{2n} \otimes v^i, X_n \otimes v^i, x_3x_4 \ldots x_{2n-1}x_{2n} \otimes v^p$ and $B_n \otimes v^p$ represent non-zero elements of $E_\infty(\mathbb{M}_n)$;*

(ii) *$X_n \otimes v^i, xX_n \otimes v^i$ and $A_n \otimes v^p$ represent non-zero elements of $E_\infty(\Gamma_n)$.*

Proof. Note that, in $H^*(W)$, we have

$$\begin{aligned} x_3x_4 \ldots x_{2n-1}x_{2n} \cdot \mathcal{P}^1\beta z' &= (x_3x_4 \ldots x_{2n-1}x_{2n})(y_1^p x_2 - y_2^p x_1) \\ &= (x_3x_4 \ldots x_{2n-1}x_{2n})(z'^p x_2 + y_2^{p-1}\beta z' - y_2^{p-1} y_1 x_2) \\ &= (x_3x_4 \ldots x_{2n-1}x_{2n})(z'^p x_2 + y_2^{p-1}\beta z' - y_2^{p-1} z' x_2) \\ &\in (z', \beta z'). \end{aligned}$$

So $d_{2p+1}(x_3x_4\dots x_{2n-1}x_{2n}\otimes v^p)=0$. Therefore $x_3x_4\dots x_{2n-1}x_{2n}\otimes v^p$ survives to $E_\infty(\mathbb{M}_n)$.

By Proposition 4, $X_n\neq 0$ in $H^*(\mathbb{M}_n)$ implies that $X_n, x_1x_3x_4\dots x_{2n-1}x_{2n}$ and B_n are not elements of Ker $\mathrm{Inf}^W_{\mathbb{M}_n}$. Similarly, Proposition 5 shows that X_n, xX_n and A_n are not elements of Ker $\mathrm{Inf}^V_{\Gamma_n}$. The proposition follows from Proposition 7. □

For $n\geq 1$ and for $1\leq i\leq p-2$, let us pick elements $X_{n,i}\in H^{2(n+i)-1}(\mathbb{M}_n)$ and $Y_{n,i}\in H^{2(n+i)}(\Gamma_n)$ which represent respectively $x_1x_3x_4\dots x_{2n-1}x_{2n}\otimes v^i\in E_\infty(\mathbb{M}_n)$ and $X_n\otimes v^i\in E_\infty(\Gamma_n)$; for $n\geq 2$, pick elements $X_{n,p-1}\in H^{2(n+p)-3}(\mathbb{M}_n)$, $Z_{n,p-1}\in H^{2(n+p)-2}(\mathbb{M}_n)$ and $Y_{n,p-1}\in H^{2(n+p)-2}(\Gamma_n)$ which represent respectively $B_n\otimes v^p\in E_\infty(\mathbb{M}_n), x_3x_4\dots x_{2n-1}x_{2n}\otimes v^p\in E_\infty(\mathbb{M}_n)$ and $A_n\otimes v^p\in E_\infty(\Gamma_n)$ (the existence of such elements follows from Propositions 2 and 8). In particular, define $Y_{1,p-1}$ by

$$Y_{1,p-1}=\mathcal{N}_{\mathrm{Ker}\ x_2\to\Gamma_1}(w)$$

with $\mathcal{N}$ the Evens norm map (note that $\mathrm{Ker}\,x_2\cong C_{p^2}\times C_p\subset\Gamma_1$, so, by the Künneth formula, w can be considered as an element of $H^2(\mathrm{Ker}\,x_2)$).

We now define the following subgroups of Γ_n:

$$\begin{aligned}
\mathbb{M}_n &= \mathrm{Ker}(x-x_1),\\
\Gamma'_{n-1} &= \mathrm{Ker}\,x_{2n}\cong\Gamma_{n-1}\times C_p,\\
\mathbb{M}'_{n-1} &= \mathrm{Ker}\,x_{2n}\cap\mathrm{Ker}(x-x_1)\qquad(\text{so } \mathbb{M}'_{n-1}\cong\mathbb{M}_{n-1}\times C_p \text{ for } n>1),\\
{}^{\text{`}}\Gamma_{n-1} &= \mathrm{Ker}\,x_2\cap\mathrm{Ker}(x-x_1)\cong\Gamma_{n-1},\\
{}'\Gamma_{n-2} &= \mathrm{Ker}\,x_2\cap\mathrm{Ker}\,x_{2n}\cap\mathrm{Ker}(x-x_1)\cong\Gamma_{n-2}\times C_p\ (\text{for } n\geq 2),
\end{aligned}$$

with the convention that $\Gamma_0=C_{p^2}$. Therefore ${}'\Gamma_{n-2}=\mathbb{M}'_{n-1}\cap{}^{\text{`}}\Gamma_{n-1}$ and ${}'\Gamma_0=C_{p^2}\times C_p$. If K is one of the above subgroups, then K contains Z as a central subgroup and we have the central extension

$$(K)\qquad 1\to Z\to K\to K/Z\to 1.$$

For convenience, we also define the elements $Y_{0,i}\in H^{2i}(\Gamma_0), 1\leq i\leq p-1$, by $Y_{0,i}=w^i$. With some abuse of notation, by the Künneth formula, the $Y_{n-1,i}$ (resp. $X_{n-1,i}, Z_{n-1,i}$)'s are considered as elements of $H^*(\Gamma'_{n-1}), H^*({}'\Gamma_{n-1})$ and $H^*({}^{\text{`}}\Gamma_{n-1})$ (resp. $H^*(\mathbb{M}'_{n-1})$). We have

Lemma 7. *For $n\geq 1$ and $1\leq i\leq p-1, 1\leq j\leq p-2$,*

(i) $\mathrm{Res}^{\Gamma_n}_{\mathbb{M}_n}(Y_{n,j})+x_2X_{n,j}\in\mathrm{Im}\ \mathrm{Inf}^W_{\mathbb{M}_n}$*; if $n>1$ then* $\mathrm{Res}^{\Gamma_n}_{\mathbb{M}_n}(Y_{n,p-1})+x_2X_{n,p-1}-Z_{n,p-1}\in\mathrm{Im}\ \mathrm{Inf}^W_{\mathbb{M}_n}$*;*

(ii) *if $n>1$ then* $\mathrm{Res}^{\mathbb{M}_n}_{{}^{\text{`}}\Gamma_{n-1}}(X_{n,i})-xY_{n-1,i}\in\mathrm{Im}\ \mathrm{Inf}^{{}^{\text{`}}\Gamma_{n-1}/Z}_{\Gamma_{n-1}}$*;*

(iii) $\mathrm{Res}^{\Gamma_n}_{\Gamma'_{n-1}}(Y_{n,j})$ *belongs to* $\mathrm{Im}\ \mathrm{Inf}^{\Gamma'_{n-1}/Z}_{\Gamma'_{n-1}}$*; and*

(iv) *$xY_{n,i}\notin(y)$; furthermore, there exists no element $\xi\in H^*(\Gamma_n)$ satisfying $xY_{n,i}=y\xi$* mod $\mathrm{Im}\ \mathrm{Inf}^V_{\Gamma_n}$.

Proof. (i) and (ii) follow by considering the restriction in spectral sequences and by the structures of $E_{2p+1}(\mathbb{M}_n)$ and $E_{2p+1}(\Gamma_n)$ given in Theorem 1 (ii) and Proposition 2.

(iii) Set $T_{n,i} = \mathrm{Res}^{\Gamma_n}_{\Gamma'_{n-1}}(Y_{n,i})$. Since $y_{2n-1}Y_{n,i} \in \mathrm{Im\ Inf}^V_{\Gamma_n}$, it follows that

$$y_{2n-1}T_{n,i} = \mathrm{Res}^{\Gamma_n}_{\Gamma'_{n-1}}(y_{2n-1}Y_{n,i})$$

belongs to $\mathrm{Im\ Inf}^{\Gamma'_{n-1}/Z}_{\Gamma'_{n-1}}$. As $H^*(\Gamma'_{n-1}) = H^*(\Gamma_{n-1}) \otimes E[x_{2n-1}] \otimes \mathbb{F}_p[y_{2n-1}], T_{n,i}$ also belongs to $\mathrm{Im\ Inf}^{\Gamma'_{n-1}/Z}_{\Gamma'_{n-1}}$.

(iv) Assume that there exists $\xi \in H^{2n+1}(\Gamma_n)$ such that

$$y\xi = xY_{n,1} \bmod \mathrm{Im\ Inf}^V_{\Gamma_n}.$$

By Proposition 3, $\xi \in \mathrm{Im\ Inf}^V_{\Gamma_n}$. So $y\xi \in \mathrm{Im\ Inf}^V_{\Gamma_n}$. Hence $xY_{n,1} \in \mathrm{Im\ Inf}^V_{\Gamma_n}$, a contradiction.

Assume inductively that (iv) holds for $i-1$. For $i \geq 2$, we will prove in Lemmas 10, 11, 16 and 17 that $\mathrm{tr}^{\Gamma'_n}_{\Gamma_{n+1}}(Y_{n,i}) = \lambda_i Y_{n+1,i-1} \bmod \mathrm{Im\ Inf}^V_{\Gamma_{n+1}}$ with $0 \neq \lambda_i \in \mathbb{F}_p$. Let ϕ be the element of $\mathrm{Im\ Inf}^V_{\Gamma_{n+1}}$ satisfying $\mathrm{tr}^{\Gamma'_n}_{\Gamma_{n+1}}(Y_{n,i}) = \lambda_i Y_{n+1,i-1} + \phi$. Suppose that $xY_{n,i} + \eta = y\xi$, with $\xi \in H^*(\Gamma_n)$ and $\eta \in \mathrm{Im\ Inf}^V_{\Gamma_n}$. So

$$\begin{aligned}
\lambda_i x Y_{n+1,i-1} + x\phi &= x\mathrm{tr}^{\Gamma'_n}_{\Gamma_{n+1}}(Y_{n,i}) \\
&= \mathrm{tr}^{\Gamma'_n}_{\Gamma_{n+1}}(xY_{n,i} + \eta) \quad \text{since } \mathrm{tr}^{\Gamma'_n}_{\Gamma_{n+1}}(\eta) = 0 \\
&= \mathrm{tr}^{\Gamma'_n}_{\Gamma_{n+1}}(y\xi) \\
&= y\mathrm{tr}^{\Gamma'_n}_{\Gamma_{n+1}}(\xi).
\end{aligned}$$

Hence $\lambda_i x Y_{n+1,i-1} = y\mathrm{tr}^{\Gamma'_n}_{\Gamma_{n+1}}(\xi) - x\phi$, which contradicts the inductive hypothesis. (iv) is then proved.

The lemma follows. □

Further properties of $X_{n,i}$ and $Y_{n,i}$ are given by the following lemmas. The first one follows from Theorem 1, Proposition 2 and [14, Theorem 1.1].

Lemma 8. $\mathrm{tr}^{\mathbb{M}'_{n-1}}_{\mathbb{M}_n}(X_{n-1,i})$ *(resp.* $\mathrm{tr}^{\Gamma'_{n-1}}_{\Gamma_n}(Y_{n-1,i})$*) represents an element of* $E^{*,2j}_\infty(\mathbb{M}_n)$ *(resp.* $E^{*,2j}_\infty(\Gamma_n)$*), with* $j < i$. □

Lemma 9. *For* $2 \leq i \leq p-1$ *we have* $\mathrm{tr}^{\Gamma'_0}_{\Gamma_1}(Y_{0,i}) = \lambda_i Y_{1,i-1} \bmod \mathrm{Im\ Inf}^V_{\Gamma_1}$*, with* $0 \neq \lambda_i \in \mathbb{F}_p$.

Proof. Consider the commutative diagram

$$\begin{array}{ccc}
H^*(C_{p^2} \times C_p) & \xrightarrow{\mathrm{tr}} & H^*(\Gamma_1) \\
\mathrm{Res}\big\downarrow & & \big\downarrow\mathrm{Res} \\
H^*(Z \times C_p) & \xrightarrow[\mathrm{tr}]{} & H^*(\mathbb{E})
\end{array},$$

We have $\mathrm{Res}^{\Gamma_1}_{\mathbb{E}}\mathrm{tr}^{\Gamma'_0}_{\Gamma_1}(Y_{0,i}) = \mathrm{tr}^{Z\times C_p}_{\mathbb{E}}(v^i)$. Following [8] (see also [16]), $\mathrm{tr}^{Z\times C_p}_{\mathbb{E}}(v^i)$ is a non-zero element of $H^*(\mathbb{E})\backslash\mathrm{Im\ Inf}^{\mathbb{E}/Z}_{\mathbb{E}}$. So, by Theorem 1, $\mathrm{tr}^{\Gamma'_0}_{\Gamma_1}(Y_{0,i})$ represents an element of the form $\lambda_i x_1 x_2 \otimes v^{i-1} \in E^{2,2(i-1)}_\infty(\Gamma_1)$, with $0 \neq \lambda_i \in \mathbb{F}_p$. The lemma follows. □

In the following two lemmas, p is assumed to be greater than 3.

Lemma 10. *For $2 \le i \le p-2$,*

$$\mathrm{tr}_{\mathbb{M}_2}^{\mathbb{M}_1'}(X_{1,i}) = \lambda_i X_{2,i-1} \bmod \mathrm{Im}\ \mathrm{Inf}_{\mathbb{M}_2}^{W}$$

and

$$\mathrm{tr}_{\Gamma_2}^{\Gamma_1'}(Y_{1,i}) = \lambda_i Y_{2,i-1} \bmod \mathrm{Im}\ \mathrm{Inf}_{\Gamma_2}^{V},$$

with λ_i given in Lemma 9.

Proof. Set $Z_i = \mathrm{Res}_{`\Gamma_1}^{\mathbb{M}_2}\mathrm{tr}_{\mathbb{M}_2}^{\mathbb{M}_1'}(X_{1,i})$. By the double coset formula and by Lemma 7 (ii), we have

$$Z_i = \mathrm{tr}_{`\Gamma_1}^{'\Gamma_0}\mathrm{Res}_{'\Gamma_0}^{\mathbb{M}_1'}(X_{1,i}) = \mathrm{tr}_{`\Gamma_1}^{'\Gamma_0}(x_1 Y_{0,i}) = x_1 \mathrm{tr}_{`\Gamma_1}^{'\Gamma_0}(Y_{0,i}).$$

By Lemma 9, Z_i represents

$$\lambda_i x_1 x_3 x_4 \otimes v^{i-1} \in E_\infty^{3,2(i-1)}(`\Gamma_1).$$

By Lemma 8 and Proposition 2, this means that $\mathrm{tr}_{\mathbb{M}_2}^{\mathbb{M}_1'}(X_{1,i})$ represents

$$\lambda_i x_1 x_3 x_4 \otimes v^{i-1} \in E_\infty^{3,2(i-1)}(\mathbb{M}_2).$$

The first part of the lemma follows from the definition of $X_{2,i-1}$.

On the other hand, by setting $Y_i = \mathrm{Res}_{\mathbb{M}_2}^{\Gamma_2}\mathrm{tr}_{\Gamma_2}^{\Gamma_1'}(Y_{1,i})$, by the double coset formula, we have

$$\begin{aligned} Y_i &= \mathrm{tr}_{\mathbb{M}_2}^{\mathbb{M}_1'}\mathrm{Res}_{\mathbb{M}_1'}^{\Gamma_1'}(Y_{1,i}) \\ &= \mathrm{tr}_{\mathbb{M}_2}^{\mathbb{M}_1'}(-x_2 X_{1,i}) \qquad \text{by Lemma 7 (i)} \\ &= -x_2 \mathrm{tr}_{\mathbb{M}_2}^{\mathbb{M}_1'}(X_{1,i}). \end{aligned}$$

As shown above, $\mathrm{tr}_{\mathbb{M}_2}^{\mathbb{M}_1'}(X_{1,i})$ represents $\lambda_i x_1 x_3 x_4 \otimes v^{i-1} \in E_\infty^{3,2(i-1)}(\mathbb{M}_2)$, so Y_i represents $\lambda_i x_1 x_2 x_3 x_4 \otimes v^{i-1} \in E_\infty^{4,2(i-1)}(\mathbb{M}_2)$. By Lemma 8 and Theorem 1 (ii), this means that $\mathrm{tr}_{\Gamma_2}^{\Gamma_1'}(Y_{1,i})$ represents $\lambda_i x_1 x_2 x_3 x_4 \otimes v^{i-1} \in E_\infty^{4,2(i-1)}(\Gamma_2)$. The last part follows from the definition of $Y_{2,i-1}$. The lemma is proved. □

In general, we have

Lemma 11. *For $2 \le i \le p-2$ and $n \ge 2$,*

$$\mathrm{tr}_{\mathbb{M}_n}^{\mathbb{M}_{n-1}'}(X_{n-1,i}) = \lambda_i X_{n,i-1} \bmod \mathrm{Im}\ \mathrm{Inf}_{\mathbb{M}_n}^{W}$$

and

$$\mathrm{tr}_{\Gamma_n}^{\Gamma_{n-1}'}(Y_{n-1,i}) = \lambda_i Y_{n,i-1} \bmod \mathrm{Im}\ \mathrm{Inf}_{\Gamma_n}^{V},$$

with λ_i given in Lemma 9.

Proof. We argue by induction on n. The case $n = 2$ follows from the above lemma. Assume that the lemma holds for $n-1$. Set $Z_i = \mathrm{Res}_{`\Gamma_{n-1}}^{\mathbb{M}_n}\mathrm{tr}_{\mathbb{M}_n}^{\mathbb{M}_{n-1}'}(X_{n-1,i})$. By the double coset formula, we have

$$\begin{aligned} Z_i &= \mathrm{tr}_{`\Gamma_{n-1}}^{'\Gamma_{n-2}}\mathrm{Res}_{'\Gamma_{n-2}}^{\mathbb{M}_{n-1}'}(X_{n-1,i}) \\ &= \mathrm{tr}_{`\Gamma_{n-1}}^{'\Gamma_{n-2}}(xY_{n-2,i}) \quad \text{by Lemma 7 (ii)} \\ &= x\mathrm{tr}_{`\Gamma_{n-1}}^{'\Gamma_{n-2}}(Y_{n-2,i}). \end{aligned}$$

By the inductive hypothesis, $\mathrm{tr}_{`\Gamma_{n-1}}^{`\Gamma_{n-2}}(Y_{n-2,i}) = \lambda_i Y_{n-1,i-1} \bmod \mathrm{Im}\,\mathrm{Inf}_{`\Gamma_{n-1}}^{`\Gamma_{n-1}/Z}$. So Z_i and $\lambda_i x Y_{n-1,i-1}$ represent the same element of $E_\infty^{2n+1,2(i-1)}(`\Gamma_{n-1})$. The first part follows from Lemma 8 and Proposition 2.

Finally, by setting $Y_i = \mathrm{Res}_{\mathbb{M}_n}^{\Gamma_n}\mathrm{tr}_{\Gamma_n}^{\Gamma'_{n-1}}(Y_{n-1,i})$, we have

$$\begin{aligned} Y_i &= \mathrm{tr}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}\mathrm{Res}_{\mathbb{M}'_{n-1}}^{\Gamma'_{n-1}}(Y_{n-1,i}) \\ &= \mathrm{tr}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(-x_2 X_{n-1,i}) \quad \text{by Lemma 7 (i)} \\ &= -x_2\mathrm{tr}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(X_{n-1,i}). \end{aligned}$$

As shown above, $\mathrm{tr}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(X_{n-1,i}) = \lambda_i X_{n,i-1} \bmod \mathrm{Im}\,\mathrm{Inf}_{\mathbb{M}_n}^{W}$. So Y_i and $-\lambda_i x_2 X_{n,i-1}$ represent the same element of $E_\infty^{2n,2(i-1)}(\mathbb{M}_n)$.The last part follows from Lemma 8 and Theorem 1 (ii). The lemma is proved. □

We now calculate $\mathrm{tr}_{\Gamma_n}^{\Gamma'_{n-1}}(Y_{n-1,1})$. In so doing, let us recall the determination of the transfer map on bar cochain levels. Let L, K be subgroups of Γ_n with $Z \subset L \subset K$ and let $D = \{d\}$ be the set of cosets of L in K. For each d, specify a representative $\overline{d}$ of d such that $\overline{L} = 1$ and $\overline{d}\,\overline{d'}\,\overline{dd'}^{\,-1} \in Z$. The transfer map $\tilde{\mathrm{tr}}_K^L : C^*(L) \to C^*(K)$ is determined in [20] as follows:

$$\begin{aligned} \tilde{\mathrm{tr}}_K^L f(\cdot_L) &= \sum_{d\in D} f(\cdot_K), \\ \tilde{\mathrm{tr}}_K^L f(\ell_1,\dots,\ell_n) &= \sum_{d\in D} f(\overline{d}\ell_1\overline{d\ell_1}^{\,-1},\dots,\overline{d\ell_1\dots\ell_{n-1}}\ell_n\overline{d\ell_1\dots\ell_{n-1}\ell_n}^{\,-1}) \end{aligned}$$

for $f \in C^*(L), \ell_i \in K$.

Some properties of $\tilde{\mathrm{tr}}_K^L$ were given in [14]. Note that, if L is a direct factor of K, then $\tilde{\mathrm{tr}}_K^L$ is the zero map. Furthermore, if M is also a subgroup of Γ_n containing Z, we can choose representatives of the cosets of M in KM, and those of $K \cap M$ in K, so that the double coset formula

$$\mathrm{Res}_K^{KM}\tilde{\mathrm{tr}}_{KM}^M = \tilde{\mathrm{tr}}_K^{K\cap M}\mathrm{Res}_{K\cap M}^M \tag{5}$$

holds at the cochain level.

Since $v \in E_2(\Gamma'_{n-1})$ is transgressive, there exists a 2-cochain $\tilde{v}$ of Γ'_{n-1} satisfying $\tilde{v}|_Z = v, \delta\tilde{v} = \beta z_{n-1}$ (see *e.g.* [15] for a determination of such a cochain). It follows from [14, Lemma 1.4] that $\tilde{\mathrm{tr}}_{\Gamma_n}^{\Gamma'_{n-1}}(\beta z_{n-1}) = 0$, hence $\delta\tilde{\mathrm{tr}}_{\Gamma_n}^{\Gamma'_{n-1}}(\tilde{v}) = \tilde{\mathrm{tr}}_{\Gamma_n}^{\Gamma'_{n-1}}(\delta\tilde{v}) = 0$; in other words, $\tilde{\mathrm{tr}}_{\Gamma_n}^{\Gamma'_{n-1}}(\tilde{v})$ is a 2-cocycle of Γ_n. Set $\overline{v} = [\tilde{\mathrm{tr}}_{\Gamma_n}^{\Gamma'_{n-1}}(\tilde{v})] \in H^2(\Gamma_n)$ and let $\tilde{e}, \tilde{a}_1, \dots, \tilde{a}_{2n}$ be elements of Γ_n satisfying $\tilde{e}Z = e, \tilde{a}_i Z = a_i$ (recall that $e, a_1, \dots, a_{2n}$ was defined in Section 2 as a basis of V of which the dual is $x, x_1, \dots, x_{2n}$). We have

Lemma 12. $\overline{v} = -x_{2n-1}x_{2n}$.

Proof. Write

$$\overline{v} = \sum_{1\le i\le 2n}\mu_i x x_i + \sum_{1\le i<j\le 2n}\mu_{ij}x_i x_j + \sum_{1\le i\le 2n}\nu_i y_i$$

with $\mu_i, \mu_{ij}, \nu_i \in \mathbb{F}_p$ (note that, in $H^2(\Gamma_n), y = -(x_1x_2 + \dots + x_{2n-1}x_{2n})$). Consider the double coset formula (5) with $M = \Gamma'_{n-1}$ and $KM = \Gamma_n$ (this means that

$\tilde{a}_{2n} \in K$). For $K = \langle \tilde{e}, \tilde{a}_i, \tilde{a}_j, \tilde{a}_{2n} \rangle$ with $1 \leq i, j \leq 2n-2$, as $\tilde{a}_{2n}$ commutes with every element of $K \cap M$, we have $\tilde{\mathrm{tr}}_K^{K\cap M} = 0$, so $\mu_i = \mu_{2n} = \nu_i = \nu_{2n} = \mu_{ij} = \mu_{i\,2n} = 0$. For $K = \langle \tilde{e}, \tilde{a}_i, a_{2n-1}, \tilde{a}_{2n} \rangle$ with $1 \leq i \leq 2n-2$, we have $K \cong \Gamma_1 \times C_p, K \cap M = C_{p^2} \times C_p^2$ and $\mathrm{Res}^M_{K\cap M}(\tilde{v}) = w$; by a direct verification, we can show that $\mathrm{tr}_K^{K\cap M}(w) = y$, therefore $[\tilde{\mathrm{tr}}_K^{K\cap M}(w)] = y = -x_{2n-1}x_{2n}$, so $\mu_{2n-1} = \mu_{i\,2n-1} = \nu_{2n-1} = 0$ and $\mu_{2n-1\,2n} = -1$. The lemma follows. □

Lemma 13. *For $n \geq 1$, $\mathrm{tr}_{\Gamma_n}^{\Gamma'_{n-1}}(Y_{n-1,1}) = -X_n$; hence $\mathrm{Res}^{\Gamma_n}_{\mathbb{E}_n}\mathrm{tr}_{\Gamma_n}^{\Gamma'_{n-1}}(Y_{n-1,1}) = 0$.*

Proof. A cocycle representing $Y_{n-1,1}$ can be chosen as follows. Since $x_1x_2\ldots x_{2n-3}\cdot x_{2n-2}\cdot\beta z_{n-1} = 0$ in $H^*(\Gamma'_{n-1}/Z)$, there exists a cochain f of Γ'_{n-1}/Z (considered as a cochain of Γ'_{n-1} via the inflation map on cochains) satisfying $\delta f = x_1x_2\ldots x_{2n-3}\cdot x_{2n-2}\cdot\beta z_{n-1}$. Furthermore, it follows from the definition of $\tilde{v}$ that

$$\delta(x_1x_2\ldots x_{2n-3}x_{2n-2}\cdot\tilde{v}) = x_1x_2\ldots x_{2n-3}x_{2n-2}\cdot\beta z_{n-1};$$

hence $\delta(x_1x_2\ldots x_{2n-3}x_{2n-2}\cdot\tilde{v} - f) = 0$. Clearly $g = x_1x_2\ldots x_{2n-3}x_{2n-2}\cdot\tilde{v} - f$ is a cocycle representing $X_n \otimes v \in E_\infty(\Gamma'_{n-1})$. Hence

$$Y_{n-1,1} - [g] \in \mathrm{Im}\ \mathrm{Inf}_{\Gamma_{n-1}}^{\Gamma'_{n-1}/Z},$$

which implies that $\mathrm{tr}_{\Gamma_n}^{\Gamma'_{n-1}}(Y_{n-1,1})$ is represented by $\tilde{\mathrm{tr}}_{\Gamma_n}^{\Gamma'_{n-1}}(g)$. By [14, Lemma 1.4], $\tilde{\mathrm{tr}}_{\Gamma_n}^{\Gamma'_{n-1}}(g) = x_1x_2\ldots x_{2n-3}x_{2n-2}\cdot\tilde{\mathrm{tr}}_{\Gamma_n}^{\Gamma'_{n-1}}(\tilde{v})$. So $[\tilde{\mathrm{tr}}_{\Gamma_n}^{\Gamma'_{n-1}}(g)] = x_1x_2\ldots x_{2n-3}x_{2n-2}\cdot\overline{v}$. The lemma now follows from Lemma 12. □

Arguing as in the above proof, we can also choose a cocycle representing $X_{n-1}\otimes v^p$ (which is non-zero in $E_\infty(\Gamma'_{n-1})$, by Theorem 1, Propositions 5 and 7), as follows. As $v^p \in E_2(\Gamma'_{n-1})$ is transgressive and $d_{2p+1}(v^p) = \mathcal{P}^1\beta z_{n-1}$, there exists a cochain $\widetilde{v^p}$ of Γ'_{n-1} such that $\widetilde{v^p}|_Z = v^p$, and $\delta\widetilde{v^p} = \mathcal{P}^1\beta z_{n-1}$. Let h be a cochain of Γ'_{n-1}/Z satisfying $\delta h = \mathcal{P}^1\beta z_{n-1}\cdot X_{n-1}$. We have

Lemma 14. *$k = k_n = \widetilde{v^p}\cdot x_1x_2\ldots x_{2n-3}x_{2n-2} - h$ is a cocycle representing $X_{n-1}\otimes v^p$ and $\mathrm{Res}^{\Gamma_n}_{\mathbb{E}_n}\mathrm{tr}_{\Gamma_n}^{\Gamma'_{n-1}}([k]) = 0$.*

Proof. It follows from the definitions of $\widetilde{v^p}$ and h that k is a cocycle representing $X_{n-1}\otimes v^p$. Set $X = \mathrm{Res}^{\Gamma_n}_{\mathbb{E}_n}\mathrm{tr}_{\Gamma_n}^{\Gamma'_{n-1}}([k])$; then $X = [\tilde{\mathrm{tr}}_{\mathbb{E}_n}^{\mathbb{E}_{n-1}\times C_p}\mathrm{Res}^{\Gamma'_{n-1}}_{\mathbb{E}_{n-1}\times C_p}(k)]$ by the double coset formula. Denote also by $\widetilde{v^p}$ (resp. h) the restriction of the cochain $\widetilde{v^p}$ (resp. h) to $\mathbb{E}_{n-1}\times C_p$. By [14, Lemma 1.4], $\tilde{\mathrm{tr}}_{\mathbb{E}_n}^{\mathbb{E}_{n-1}\times C_p}(h) = 0$; hence $X = [\tilde{\mathrm{tr}}_{\mathbb{E}_n}^{\mathbb{E}_{n-1}\times C_p}(\widetilde{v^p}\cdot x_1x_2\ldots x_{2n-3}x_{2n-2})]$. Note that, in $H^*(\mathbb{E}_{n-1}\times C_p)$ we have $X_{n-1} = X_{n-2}(x_1x_2 + \cdots + x_{2n-3}x_{2n-2})$ and $x_1x_2 + \cdots + x_{2n-3}x_{2n-2} = 0$, so there exist cochains c of $(\mathbb{E}_{n-1}\times C_p)/Z$ and b of $\mathbb{E}_{n-1}\times C_p$ satisfying

$$\delta b = x_1x_2 + \cdots + x_{2n-3}x_{2n-2},$$
$$x_1x_2\ldots x_{2n-3}x_{2n-2} = x_1x_2\ldots x_{2n-5}x_{2n-4}\cdot\delta b + \delta c.$$

Hence

$$\begin{aligned}\widetilde{v^p}\cdot x_1x_2\ldots x_{2n-3}x_{2n-2} &= \widetilde{v^p}\cdot x_1x_2\ldots x_{2n-5}x_{2n-4}\cdot\delta b + \widetilde{v^p}\cdot\delta c\\ &= -\delta\widetilde{v^p}\cdot x_1x_2\ldots x_{2n-5}x_{2n-4}\cdot b - \delta\widetilde{v^p}\cdot c \bmod \mathrm{Im}\ \delta.\end{aligned}$$

So $X = -[\tilde{\mathrm{tr}}_{\mathbb{E}_n}^{\mathbb{E}_{n-1}\times C_p}(\delta\widetilde{v^p}\cdot x_1x_2\dots x_{2n-5}x_{2n-4}\cdot b + \delta\widetilde{v^p}\cdot c)]$. Following [14, Lemma 1.4], $\tilde{\mathrm{tr}}_{\mathbb{E}_n}^{\mathbb{E}_{n-1}\times C_p}(\delta\widetilde{v^p}\cdot c) = 0$ and $\tilde{\mathrm{tr}}_{\mathbb{E}_n}^{\mathbb{E}_{n-1}\times C_p}(\delta b) = 0$. This implies that $\tilde{\mathrm{tr}}_{\mathbb{E}_n}^{\mathbb{E}_{n-1}\times C_p}(b)$ is a cocycle of $\mathbb{E}_n$ and

$$X = -\mathcal{P}^1\beta z_{n-1}\cdot X_{n-2}\cdot[\tilde{\mathrm{tr}}_{\mathbb{E}_n}^{\mathbb{E}_{n-1}\times C_p}(b)].$$

Arguing as in the proof of Lemma 12, we can show that $[\tilde{\mathrm{tr}}_{\mathbb{E}_n}^{\mathbb{E}_{n-1}\times C_p}(b)] = 0$. Hence $X = 0$. The lemma follows. □

With some abuse of notation, we also denote by $\tilde{v}$ (resp. $\widetilde{v^p}$) the restriction of $\tilde{v}$ (resp. $\widetilde{v^p}$) to $\mathbb{M}'_{n-1}$. So $\delta(\tilde{v}) = \beta z'_{n-1}$ and $\delta(\widetilde{v^p}) = \mathcal{P}^1\beta z'_{n-1}$ in $C^*(\mathbb{M}'_{n-1})$. Let $\tilde{u}$ be a 1-cochain of $\mathbb{M}'_{n-1}$ satisfying $\delta(\tilde{u}) = z'_{n-1}$. It follows from the proof of Proposition 8 that there exists a cochain d of $\mathbb{M}'_{n-1}/Z$ such that

$$\begin{aligned}\delta d &= x_3x_4\dots x_{2n-3}x_{2n-2}(\mathcal{P}^1\beta z'_{n-1} - x_2{z'}^p_{n-1} - y_2^{p-1}\beta z'_{n-1} + y_2^{p-1}x_2z'_{n-1})\\ &= \delta(x_3x_4\dots x_{2n-3}x_{2n-2}(\widetilde{v^p} + x_2{z'}^{p-1}_{n-1}\tilde{u} - y_2^{p-1}\tilde{v} - y_2^{p-1}x_2\tilde{u})).\end{aligned}$$

So, for $n \geq 3$, $q = x_3x_4\dots x_{2n-3}x_{2n-2}(\widetilde{v^p} + x_2{z'}^{p-1}_{n-1}\tilde{u} - y_2^{p-1}\tilde{v} - y_2^{p-1}x_2\tilde{u}) - d$ is a cocycle of $\mathbb{M}'_{n-1}$ representing $Z_{n-1,p-1}$. We have

Lemma 15. *For $n \geq 3$,*

$$\mathrm{tr}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(Z_{n-1,p-1}) \in \mathrm{Im}\mathrm{Inf}_{\mathbb{M}_n}^W.$$

Proof. It follows that $\mathrm{tr}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(Z_{n-1,p-1}) = [\tilde{\mathrm{tr}}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(q)]$. By [14, Lemma 1.4],

$$\begin{aligned}\mathrm{tr}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(Z_{n-1,p-1}) = x_3x_4\dots x_{2n-3}x_{2n-2}([\tilde{\mathrm{tr}}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(\widetilde{v^p})] + x_2{z'}^{p-1}_{n-1}[\tilde{\mathrm{tr}}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(\tilde{u})]\\ - y_2^{p-1}[\tilde{\mathrm{tr}}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(\tilde{v})] - y_2^{p-1}x_2[\tilde{\mathrm{tr}}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(\tilde{u})])\end{aligned}$$

(note that $\tilde{\mathrm{tr}}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}$ maps each of $\widetilde{v^p}, \tilde{u}, \tilde{v}$ to a cocycle). Since each of $\tilde{\mathrm{tr}}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(\widetilde{v^p})$, $\tilde{\mathrm{tr}}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(\tilde{v})$, $\tilde{\mathrm{tr}}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(\tilde{u})$ is of degree $\leq 2p$, it follows from the structure of $E_{2p+1}(\mathbb{M}_n)$ that $[\tilde{\mathrm{tr}}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(\tilde{v})], [\tilde{\mathrm{tr}}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(\tilde{u})]$ and the cup-product of $x_3x_4\dots x_{2n-3}x_{2n-2}$ with $[\tilde{\mathrm{tr}}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(\widetilde{v^p})]$ belong to $\mathrm{Im}\ \mathrm{Inf}_{\mathbb{M}_n}^W$. The lemma follows. □

Lemma 16. *There exists a non-zero $\lambda \in \mathbb{F}_p$ such that $\mathrm{tr}_{\Gamma_2}^{\Gamma'_1}(Y_{1,p-1}) - \lambda Y_{2,p-2} \in \mathrm{Im}\ \mathrm{Inf}_{\Gamma_2}^V$.*

Proof. Set $K = \mathrm{Ker}\, x_2 \cap \mathrm{Ker}(x - x_4) \subset \Gamma_2$ and $X = \mathrm{tr}_{\Gamma_2}^{\Gamma'_1}(Y_{1,p-1})$. So $K \cong \mathbb{M}\times C_p, K\cap\Gamma'_1 \cong C_p^3$ and $\mathrm{Res}_K^{\Gamma_2}(X) = \mathrm{tr}_K^{\Gamma'_1\cap K}\mathrm{Res}_{\Gamma'_1\cap K}^{\Gamma'_1}(Y_{1,p-1})$. As $\mathrm{Res}_{\Gamma'_1\cap K}^{\Gamma'_1}(Y_{1,p-1}) = v^p - vy_1^{p-1}$, we have

$$\mathrm{Res}_K^{\Gamma_2}(X) = \mathrm{tr}_K^{\Gamma'_1\cap K}(v^p - vy_1^{p-1}) = \mathrm{tr}_K^{\Gamma'_1\cap K}(v^p) - \mathrm{tr}_K^{\Gamma'_1\cap K}(vy_1^{p-1}).$$

A direct verification shows that $\mathrm{tr}_K^{\Gamma'_1\cap K}(v) = y_4$, so $\mathrm{Res}_K^{\Gamma_2}(X) = -y_4y_1^{p-1} \neq 0$. Hence $X \neq 0$.

Suppose that $X \in \mathrm{Im}\ \mathrm{Inf}_{\Gamma_2}^V$. Since $y_4X = 0$, y_4X must belong to $(z, \eta_1, \eta_2, \xi_1)$. Write

$$y_4X = az + b\eta_1 + c\eta_2 + \mu\xi_1 \tag{6}$$

with $a, b, c \in H^*(V)$ and $\mu \in \mathbb{F}_p$. Multiplying (6) by $x_1x_2x_3x_4$ yields $\mu\xi_1 \in (y, y_4)$. Hence $\mu = 0$. Multiplying (6) by η_2 yields $y_4X\eta_2 \in (z, \eta_1)$. So, by [13, Lemma 2.4], $X\eta_2 \in (z, \eta_1, X_2)$. Since $X\eta_2$ is of degree > 4, it follows that $X\eta_2 \in (z, \eta_1)$. By [13, Lemma 2.14], $X = ey \bmod (z, \eta_1)$ with $e \in H^{2p-2}(V)$. Write

$$eyy_4 = a_1z + b_1\eta_1 + c_1\eta_2. \tag{7}$$

Multiplying (7) by $\eta_1\eta_2$ yields

$$\begin{aligned} eyy_4\eta_1\eta_2 &= a_1z\eta_1\eta_2 \\ &= a_1y\eta_1\eta_2 - a_1X_2\xi_1. \end{aligned}$$

So $a_1 \in (y, x_1, \ldots, x_4)$. Therefore $b_1 \in (y, x_ix_j)$ and $c_1 = 0$. By [13, Lemma 2.4], we have $ey \in (z, \eta_1, X_2)$. Since ey is of degree > 4, it follows that $ey \in (z, \eta_1)$. So $X \in (z, \eta_1)$, and hence $X = 0$ in $H^*(\Gamma_2)$, a contradiction. The lemma follows. □

Lemma 17. *For $n \geq 3$,*

$$\mathrm{tr}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(X_{n-1,p-1}) = \lambda X_{n,p-2} \bmod \mathrm{Im}\ \mathrm{Inf}_{\mathbb{M}_n}^{W}$$

and

$$\mathrm{tr}_{\Gamma_n}^{\Gamma'_{n-1}}(Y_{n-1,p-1}) = \lambda Y_{n,p-2} \bmod \mathrm{Im}\ \mathrm{Inf}_{\Gamma_n}^{V},$$

with λ given in Lemma 16.

Proof. Consider the case $n = 3$. Set $X = \mathrm{Res}_{`\Gamma_2}^{\mathbb{M}_3}\mathrm{tr}_{\mathbb{M}_3}^{\mathbb{M}'_2}(X_{2,p-1})$. By the double coset formula, we have

$$\begin{aligned} X &= \mathrm{tr}_{`\Gamma_2}^{'\Gamma_1}\mathrm{Res}_{'\Gamma_1}^{\mathbb{M}'_2}(X_{2,p-1}) \\ &= \mathrm{tr}_{`\Gamma_2}^{'\Gamma_1}(xY_{1,p-1}) \quad \text{by Lemma 7 (ii)} \\ &= x\mathrm{tr}_{`\Gamma_2}^{'\Gamma_1}(Y_{1,p-1}). \end{aligned}$$

It follows from Lemma 16 that X and $\lambda xY_{2,p-2}$ represent the same element of $E_\infty^{5,2(p-2)}(`\Gamma_2)$. By Lemma 8 and Proposition 2, it follows that $\mathrm{tr}_{\mathbb{M}_3}^{\mathbb{M}'_2}(X_{2,p-1}) = \lambda X_{3,p-2} \bmod \mathrm{Im}\ \mathrm{Inf}_{\mathbb{M}_3}^{W}$. Similarly, by setting $Y = \mathrm{Res}_{\mathbb{M}_3}^{\Gamma_3}\mathrm{tr}_{\Gamma_3}^{\Gamma'_2}(Y_{2,p-1})$, we have

$$\begin{aligned} Y &= \mathrm{tr}_{\mathbb{M}_3}^{\mathbb{M}'_2}\mathrm{Res}_{\mathbb{M}'_2}^{\Gamma'_2}(Y_{2,p-1}) \\ &= \mathrm{tr}_{\mathbb{M}_3}^{\mathbb{M}'_2}(-x_2X_{2,p-1} + Z_{2,p-1}) \quad \text{by Lemma 7 (i)} \\ &= -x_2\mathrm{tr}_{\mathbb{M}_3}^{\mathbb{M}'_2}(X_{2,p-1}) + \mathrm{tr}_{\mathbb{M}_3}^{\mathbb{M}'_2}(Z_{2,p-1}). \end{aligned}$$

As shown above, $\mathrm{tr}_{\mathbb{M}_3}^{\mathbb{M}'_2}(X_{2,p-1}) = \lambda X_{3,p-2} \bmod \mathrm{Im}\ \mathrm{Inf}_{\mathbb{M}_3}^{W}$. So, by Lemma 15, Y and $-\lambda x_2X_{3,p-2}$ represent the same element of $E_\infty^{6,2(p-2)}(\mathbb{M}_3)$. By Lemma 8 and Proposition 2, it follows that $\mathrm{tr}_{\Gamma_3}^{\Gamma'_2}(Y_{2,p-1}) = \lambda Y_{3,p-2} \bmod \mathrm{Im}\ \mathrm{Inf}_{\Gamma_3}^{V}$.

Assume that the lemma holds for $n-1$. Set $Z = \mathrm{Res}_{`\Gamma_{n-1}}^{\mathbb{M}_n}\mathrm{tr}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(X_{n-1,p-1})$. By the double coset formula, we have

$$\begin{aligned} Z &= \mathrm{tr}_{`\Gamma_{n-1}}^{'\Gamma_{n-2}}\mathrm{Res}_{'\Gamma_{n-2}}^{\mathbb{M}'_{n-1}}(X_{n-1,p-1}) \\ &= \mathrm{tr}_{`\Gamma_{n-1}}^{'\Gamma_{n-2}}(xY_{n-2,p-1}) \quad \text{by Lemma 7 (ii)} \\ &= x\mathrm{tr}_{`\Gamma_{n-1}}^{'\Gamma_{n-2}}(Y_{n-2,p-1}). \end{aligned}$$

By the inductive hypothesis, $\mathrm{tr}_{`\Gamma_{n-1}}^{`\Gamma_{n-2}}(Y_{n-2,p-1}) = \lambda Y_{n-1,p-2} \bmod \mathrm{Im}\,\mathrm{Inf}_{`\Gamma_{n-1}}^{`\Gamma_{n-1}/Z}$. So Z and $\lambda x Y_{n-1,p-2}$ represent the same element of $E_\infty^{2n-1,2(p-2)}(`\Gamma_{n-1})$. The first part follows from Lemma 8 and Proposition 2.

Finally, by setting $Y = \mathrm{Res}_{\mathbb{M}_n}^{\Gamma_n}\mathrm{tr}_{\Gamma_n}^{\Gamma'_{n-1}}(Y_{n-1,p-1})$, we have

$$\begin{aligned} Y &= \mathrm{tr}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}\mathrm{Res}_{\mathbb{M}'_{n-1}}^{\Gamma'_{n-1}}(Y_{n-1,p-1}) \\ &= \mathrm{tr}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(-x_2X_{n-1,p-1}) + \mathrm{tr}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(Z_{n-1,p-1}) \quad \text{by Lemma 7 (i)} \\ &= -x_2\mathrm{tr}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(X_{n-1,p-1}) + \mathrm{tr}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(Z_{n-1,p-1}). \end{aligned}$$

As shown above, $\mathrm{tr}_{\mathbb{M}_n}^{\mathbb{M}'_{n-1}}(X_{n-1,p-1}) = \lambda X_{n,p-2} \bmod \mathrm{Im}\,\mathrm{Inf}_{\mathbb{M}_n}^{W}$. So, by Lemma 15, Y and $-\lambda x_2 X_{n,p-2}$ represent the same element of $E_\infty^{2n,2(p-2)}(\mathbb{M}_n)$. The last part follows from Lemma 8 and Theorem 1 (ii). The lemma is proved. □

Let

$$\cdots \supset F^iC^*(\Gamma'_{n-1}) \supset F^{i+1}C^*(\Gamma'_{n-1}) \supset \dots$$

be the filtration of $C^*(\Gamma'_{n-1})$ introduced by Hochschild and Serre ([7]) corresponding to the central extension (Γ'_{n-1}). Let us recall that

$$F^iC^*(\Gamma'_{n-1}) = \begin{cases} C^*(\Gamma'_{n-1}) & \text{for } i \le 0, \\ \sum_{m=0}^{\infty} F^iC^m(\Gamma'_{n-1}) & \text{for } i > 0, \end{cases}$$

where $F^iC^m(\Gamma'_{n-1}) = 0$ for $i > m$; and for $0 < i \le m$, $F^iC^m(\Gamma'_{n-1})$ is the group of all m-cochains f for which $f(g_1, \dots, g_m) = 0$ whenever $m-i+1$ of the arguments g_k belong to Z. It is clear that the conjugation by $a = a_{2n}$ on $C^*(\Gamma'_{n-1})$ is compatible with the Hochschild-Serre filtration. We then have the induced conjugation on the Hochschild-Serre spectral sequence $\{E_r(\Gamma'_{n-1})\}$. As the action of a on $E_2^{*,*}(\Gamma'_{n-1})$ satisfies ${}^ax_k = x_k, 1 \le k \le 2n-1$, and ${}^av = v + y_{2n-1}$, it follows from the structure of $E_{2p+1}(\Gamma'_{n-1})$ that $Y_{n-1,i}$ and ${}^aY_{n-1,i}$ represent the same element of $E_\infty(\Gamma'_{n-1})$. Hence

$$Y_{n-1,i} -^a Y_{n-1,i} = \sum_{0<j<i} \mu_j Y_{n-1,j} y_{2n-1}^{i-j} + \sum_{0<j<i} \nu_j Y_{n-1,j} y_{2n-1}^{i-j-1} x x_{2n-1} \bmod \mathrm{Im}\,\mathrm{Inf}_{\Gamma'_{n-1}}^{\Gamma'_{n-1}/Z}, \tag{8}$$

with $\mu_j, \nu_j \in \mathbb{F}_p$. We have

Lemma 18. *For $n \ge 2$ we have $Y_{n-1,1} -^a Y_{n-1,1} = 0$.*

Proof. Set $K = \mathrm{Ker}\, x_{2n-2} \cap \Gamma'_{n-1}$. Since the transfer commutes with the conjugation and $\mathrm{Im}\,\mathrm{Inf}_{\Gamma_{n-1}}^{\Gamma'_{n-1}/Z}$ is invariant under the action of a, by Lemmas 9, 10, 11, 16 and 17, we have

$$Y_{n-1,1} -^a Y_{n-1,1} = \mathrm{tr}_{\Gamma'_{n-1}}^{K}(Y_{n-2,2} -^a Y_{n-2,2})$$

up to a non-zero constant multiple. By Lemma 13, $\mathrm{tr}_{\Gamma'_{n-1}}^{K}(Y_{n-2,1}) = -X_{n-1}$; hence $\mathrm{tr}_{\Gamma'_{n-1}}^{K}(Y_{n-2,1}y_{2n-3}) = 0$ and $\mathrm{tr}_{\Gamma'_{n-1}}^{K}(Y_{n-2,1}x_{2n-3}) = 0$ in $H^*(\Gamma'_{n-1})$. The lemma follows from (8) and from the fact that $\mathrm{tr}_{\Gamma'_{n-1}}^{K}\mathrm{Inf}_K^{K/Z} = 0$. □

We now have

Lemma 19. *For $n \geq 2$ and $1 \leq i \leq p-1$,*

$$Y_{n-1,i} +^a Y_{n-1,i} + \cdots +^{a^{p-1}} Y_{n-1,i} = 0;$$

hence

$$\mathrm{Res}^{\Gamma_n}_{\Gamma'_{n-1}} \mathrm{tr}^{\Gamma'_{n-1}}_{\Gamma_n}(Y_{n-1,i}) = 0.$$

Proof. Since $1 + a + \cdots + a^{p-1} = (1-a)^{p-1}$, we need prove that ${}^{(1-a)^{p-1}}Y_{n-1,i} = 0$. For $1 \leq k \leq p-1$, by (8) and by Lemma 18, ${}^{(1-a)^k}Y_{n-1,k} = 0$. Since $\mathrm{Res}^{\Gamma_n}_{\Gamma'_{n-1}} \mathrm{tr}^{\Gamma'_{n-1}}_{\Gamma_n}(Y_{n-1,i}) = {}^{(1-a)^{p-1}} Y_{n-1,i}$, the lemma follows. □

For $n \geq 2$ and $1 \leq i \leq p-2$, set $\kappa_{n,i} = \mathrm{Res}^{\Gamma_n}_{\mathbb{E}_n} \mathrm{tr}^{\Gamma'_{n-1}}_{\Gamma_n}(Y_{n-1,i+1})$. The proof of the theorem is completed by the following fact.

Proposition 9. *$0 \neq \kappa_{n,i} \in Ess(\mathbb{E}_n)$ with $1 \leq i < p-2$ for $p > 3$, and $i = 1$ for $p = 3$.*

Proof. It follows from Proposition 1, Lemmas 7 (iv), 9, 11 and 17 that $\kappa_{n,i} \neq 0$ in $H^*(\mathbb{E}_n)$. Let K be a maximal subgroup of $\mathbb{E}_n$. K is then of the form $\mathbb{E}_{n-1} \times C_p$. Let L be the central product of K and $C_{p^2} = \bigcap_{j=1}^{2n} \mathrm{Ker}\, x_j$. It follows that L is a subgroup of Γ_n containing K and $L \cong \Gamma_{n-1} \times C_p$. Therefore

$$\begin{aligned}\mathrm{Res}^{\mathbb{E}_n}_K(\kappa_{n,i}) &= \mathrm{Res}^{\Gamma_n}_K \mathrm{tr}^{\Gamma'_{n-1}}_{\Gamma_n}(Y_{n-1,i+1}) \\ &= \mathrm{Res}^L_K \mathrm{Res}^{\Gamma_n}_L \mathrm{tr}^{\Gamma'_{n-1}}_{\Gamma_n}(Y_{n-1,i+1}).\end{aligned}$$

Hence, if $\Gamma_n = \Gamma'_{n-1}L$, it follows from the double coset formula that

$$\mathrm{Res}^{\mathbb{E}_n}_K(\kappa_{n,i}) = \mathrm{Res}^L_K \mathrm{tr}^{L\cap\Gamma'_{n-1}}_L \mathrm{Res}^{\Gamma'_{n-1}}_{L\cap\Gamma'_{n-1}}(Y_{n-1,i+1}).$$

By Lemma 1, there exists a non-zero linear combination α of $x_1, \ldots, x_{2n}$ such that $L = \mathrm{Ker}\,\alpha$. Consider the following cases:

• $\alpha = x_{2n-1} + \gamma$ with γ a linear combination of $x_1, \ldots, x_{2n-2}, x_{2n}$: it follows that $\Gamma_n = \Gamma'_{n-1}L$ and $L \cap \Gamma'_{n-1} \cong \Gamma_{n-1}$ is a direct factor of L. Hence $\mathrm{tr}^{L\cap\Gamma'_{n-1}}_L$ is the zero map. We have

$$\begin{aligned}\mathrm{Res}^{\mathbb{E}_n}_K(\kappa_{n,i}) &= \mathrm{Res}^L_K \mathrm{tr}^{L\cap\Gamma'_{n-1}}_L \mathrm{Res}^{\Gamma'_{n-1}}_{L\cap\Gamma'_{n-1}}(Y_{n-1,i+1}) \\ &= 0;\end{aligned}$$

• $\alpha = \mu x_{2n} + \gamma$ with γ a non-zero linear combination of $x_1, \ldots, x_{2n-2}$ and $\mu \in \mathbb{F}_p$: it follows that $L \cap \Gamma'_{n-1} = H \times \langle \tilde{a}_{2n-1} \rangle$ for a subgroup H of Γ_{n-1} with $H \cong \Gamma'_{n-2}$. If $p > 3$, it follows from the proof of Lemma 7(iii) that $\mathrm{Res}^{\Gamma'_{n-1}}_{L\cap\Gamma'_{n-1}}(Y_{n-1,i+1})$ belongs to the ideal generated by $\mathrm{Im}\ \mathrm{Inf}^{(L\cap\Gamma'_{n-1})/Z}_{L\cap\Gamma'_{n-1}}$; since

$$\mathrm{Im}\mathrm{Inf}^{(L\cap\Gamma'_{n-1})/Z}_{L\cap\Gamma'_{n-1}} \subset \mathrm{Ker}\mathrm{tr}^{L\cap\Gamma'_{n-1}}_L,$$

it follows that

$$\mathrm{Res}^{\mathbb{E}_n}_K(\kappa_{n,i}) = 0.$$

If $p = 3$, by Lemma 14, there exist $\phi \in H^2(L \cap \Gamma'_{n-1}), \psi \in H^1(L \cap \Gamma'_{n-1})$ such that $\mathrm{Res}^{\Gamma'_{n-1}}_{L\cap\Gamma'_{n-1}}(Y_{n-1,2})$ is a linear combination of

$$[k_{n-1}], \quad Y_{n-2,1} \cdot \phi, \quad Y_{n-2,1} \cdot x\psi$$

and an element of $\mathrm{Im}\ \mathrm{Inf}^{(L\cap\Gamma'_{n-1})/Z}_{L\cap\Gamma'_{n-1}}$; since ϕ, ψ belong to $\mathrm{Im}\ \mathrm{Inf}^{(L\cap\Gamma'_{n-1})/Z}_{L\cap\Gamma'_{n-1}}$, by Lemmas 13 and 14, it follows that

$$\mathrm{Res}^L_K\, \mathrm{tr}^{L\cap\Gamma'_{n-1}}_L \mathrm{Res}^{\Gamma'_{n-1}}_{L\cap\Gamma'_{n-1}}(Y_{n-1,2}) = 0,$$

so $\mathrm{Res}^{\mathbb{E}_n}_K(\kappa_{n,1}) = 0$.

Finally, the case $\alpha = x_{2n}$ follows from Lemma 19. The proposition is proved. □

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