

THE LIMITS OF REFINABLE FUNCTIONS

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ABSTRACT. A function ϕ is refinable ($\phi \in S$) if it is in the closed span of $\{\phi(2x - k)\}$. This set S is not closed in $L_2(\mathbb{R})$, and we characterize its closure. A necessary and sufficient condition for a function to be refinable is presented without any information on the refinement mask. The Fourier transform of every $f \in \overline{S} \setminus S$ vanishes on a set of positive measure. As an example, we show that all functions with Fourier transform supported in $[-\frac{4}{3}\pi, \frac{4}{3}\pi]$ are the limits of refinable functions. The relation between a refinable function and its mask is studied, and nonuniqueness is proved. For inhomogeneous refinement equations we determine when a solution is refinable. This result is used to investigate refinable components of multiple refinable functions. Finally, we investigate fully refinable functions for which all translates (by any real number) are refinable.

1. INTRODUCTION AND MAIN RESULTS

The central equation in wavelet analysis is the *refinement equation* for the scaling function ϕ :

$$(1.1) \quad \phi(x) = \sum_{k=0}^N a(k)\phi(2x - k).$$

In approximation theory, the sequence $\{a(k)\}$ is the *mask*. In signal processing these are the coefficients of a lowpass filter.

For a given mask $\{a(k)\}$, wavelet theory yields the properties of the family $\{\phi(x - k)\}$. We can determine whether these translates form a Riesz basis of a subspace in $L_2(\mathbb{R})$, whether this basis is orthogonal, and which polynomials $1, x, \dots, x^{p-1}$ are linear combinations of the translates. This theory is summarized in [3] and [14]. What we *do not know* is how to choose the mask $\{a(k)\}$ so that $\phi(x)$ is close to a given function $f(x)$.

This “inverse problem” arises naturally in applications. We want to recognize objects whose shape is indicated by $f(x)$. We hope that a scaling function of similar shape will allow us to identify a good match. The thesis of Chapa [2] made a start on this problem using band-limited scaling functions. In that case the Fourier transform $\hat{\phi}(\xi)$ has compact support and the sequence $\{a(k)\}$ is infinite.

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We want to start again, by answering this preliminary question: *What is the closure of the set S of all refinable functions in $L_2(\mathbb{R})$?* A solution to (1.1) is a refinable function. More generally, we say that

$$(1.2) \quad \phi \text{ is refinable } (\phi \in S) \quad \text{if and only if} \quad \phi(x) \in \overline{\text{span}}\{\phi(2x - k) : k \in \mathbb{Z}\}.$$

Thus an infinite mask is allowed. The refinability is better understood in the frequency domain. To see this, we need the following characterization of closed shift-invariant subspaces in $L_2(\mathbb{R})$ given explicitly by de Boor, DeVore, and Ron [1], based on doubly-invariant spaces discussed in [7]. Each such subspace is associated with a function ϕ in $L_2(\mathbb{R})$. The subspace $S_2(\phi) = \overline{\text{span}}\{\phi(x - k) : k \in \mathbb{Z}\}$ is

$$(1.3) \quad S_2(\phi) = \{f \in L_2(\mathbb{R}) : \hat{f}(\xi) = \tau(\xi)\hat{\phi}(\xi) \text{ for a } 2\pi\text{-periodic function } \tau(\xi)\}.$$

It follows that $\phi \in S$ if and only if, for some 2π -periodic function \tilde{a} ,

$$(1.4) \quad \hat{\phi}(2\xi) = \tilde{a}(\xi)\hat{\phi}(\xi), \quad \text{for almost every } \xi.$$

Let us turn to the set of all refinable functions. We wondered at first whether this set S is closed. We will show that it is not closed, and Theorem 1 will describe its closure \overline{S} . The crucial questions involve the *zeros of the Fourier transform*. Recall from (1.4) that $\phi \in S$ satisfies $\hat{\phi}(2\xi) = \tilde{a}(\xi)\hat{\phi}(\xi)$ for some 2π -periodic function. In the inverse direction, f will be refinable if $\hat{f}(2\xi)/\hat{f}(\xi)$ happens to be 2π -periodic, and $\hat{f}(\xi)$ is never zero. Then f will solve equation (1.1) with symbol of the mask given by

$$\tilde{a}(\xi) = \frac{\hat{f}(2\xi)}{\hat{f}(\xi)}.$$

But if $\hat{f}(\xi)$ has zeros (which is typical!), we have to consider their relation to the zeros of $\hat{f}(2\xi)$. This eventually leads to our characterization of the closure of S :

Theorem 1. *A function f lies in \overline{S} , the closure of S in $L_2(\mathbb{R})$, if and only if, for any positive integers j and k ,*

$$(1.5) \quad \hat{f}(2^j(\xi + 2k\pi))\hat{f}(\xi) = \hat{f}(2^j\xi)\hat{f}(\xi + 2k\pi) \quad \text{for almost every } \xi.$$

As an example, the function $f \in L_2(\mathbb{R})$ given by

$$\hat{f}(\xi) = \begin{cases} 1, & \text{if } \xi \in (-\frac{4}{3}\pi, -\frac{2}{3}\pi) \cup (\frac{2}{3}\pi, \frac{4}{3}\pi), \\ 0, & \text{otherwise,} \end{cases}$$

is in \overline{S} , but is not refinable. The condition (1.5) certainly holds in the band-limited case when $\hat{f}(\xi)$ is supported in $[-\frac{4}{3}\pi, \frac{4}{3}\pi]$, because for ξ in this interval we have

$$|2^j(\xi + 2k\pi)| \geq 2(2\pi - |\xi|) \geq \frac{4}{3}\pi.$$

Then both sides of (1.5) are identically zero and $f \in \overline{S}$. Section 3 will show that if $b > \frac{4}{3}\pi$, there are functions with $\hat{f}(\xi)$ supported on $[-b, b]$ for which (1.5) does not hold.

Our second main result is a lower bound on the distance $d(f, S)$ from f to S :

$$d(f, S) = \inf\{\|f - \phi\|_2 : \phi \in S\}.$$

From the characterization of Theorem 1, it is natural to measure this distance in terms of the family of functions

$$D_{j,k}(f)(\xi) := \hat{f}(2^j(\xi + 2k\pi))\hat{f}(\xi) - \hat{f}(2^j\xi)\hat{f}(\xi + 2k\pi).$$

Theorem 2. *Let f be a nonzero function in $L_2(\mathbb{R})$. Then*

$$(1.6) \quad d(f, S) \geq \frac{\sqrt{2}-1}{12\pi\|f\|_2} \sup_{k \in \mathbb{N}} \left\{ \sum_{j=1}^{\infty} \|D_{j,k}(f)\|_1 \right\}.$$

In the proof of Theorem 1, the family of sets $\{K_j(f)\}$ defined for functions as

$$(1.7) \quad K_j(f) := \{ \xi \in [-\pi, \pi) : \hat{f}(2^j(\xi + 2l\pi)) \neq 0 \text{ for some } l \in \mathbb{Z} \}$$

plays an essential role. In terms of these sets, the proof of Theorem 1 also provides a characterization of refinable functions.

Theorem 3. *Let $f \in \overline{S}$. Then f lies in S if and only if the set $\bigcup_{j=1}^{\infty} K_j(f) \setminus K_0(f)$ has measure zero.*

Corollary 1. *If $f \in \overline{S} \setminus S$, then $\hat{f}(\xi)$ vanishes on a set of positive measure.*

The final sections of the paper deal with smaller points in the theory of refinable functions:

- Section 4: Nonuniqueness of the mask.
- Section 5. Refinable solutions to inhomogeneous refinement equations.
- Section 6. Multiple refinable functions (leading to multiwavelets).
- Section 7. Fully refinable functions (all translates $\phi(x - t)$ are refinable).

2. PROOF OF THE MAIN RESULTS

In this section we prove the main results. Recall the characterization (1.3).

Proof of Theorem 1. Necessity of (1.5). Suppose that there is a sequence $\{\phi_n\} \subset S$ such that $\|\phi_n - f\|_2 \rightarrow 0$ as $n \rightarrow \infty$. Then $\|\hat{\phi}_n - \hat{f}\|_2 \rightarrow 0$. Hence there is a subsequence $\{\hat{\phi}_{n_k}(\xi)\}$ such that

$$\lim_{k \rightarrow \infty} \hat{\phi}_{n_k}(\xi) = \hat{f}(\xi)$$

almost everywhere. By replacing $\{\phi_n\}$ with this subsequence, we may assume that

$$(2.1) \quad \lim_{n \rightarrow \infty} \hat{\phi}_n(\xi) = \hat{f}(\xi), \quad \forall \xi \in \mathbb{R} \setminus T,$$

where T is a set of measure zero (null set).

Since ϕ_n is refinable, by (1.3) there is a 2π -periodic function $\tilde{a}_n(\xi)$ such that

$$(2.2) \quad \hat{\phi}_n(2\xi) = \tilde{a}_n(\xi)\hat{\phi}_n(\xi)$$

almost everywhere. By recursion this implies for all n and all $j = 1, 2, \dots$ that

$$(2.3) \quad \hat{\phi}_n(2^j\xi) = \tilde{a}_n(2^{j-1}\xi) \cdots \tilde{a}_n(\xi)\hat{\phi}_n(\xi), \quad \forall \xi \in \mathbb{R} \setminus T',$$

where T' is another null set. Then $T_j = (T' + 2\pi\mathbb{Z}) \cup (2^{-j}T' + 2\pi\mathbb{Z}) \cup (T' + 2\pi\mathbb{Z})$ is also a null set. Suppose $\xi \in \mathbb{R} \setminus T_j$.

Let $k \in \mathbb{N}$. If $\hat{f}(\xi + 2k\pi) = \hat{f}(\xi) = 0$, then (1.5) holds trivially.

If $\hat{f}(\xi + 2l\pi) \neq 0$ for $l = 0$ or $l = k$, then (2.1) and (2.3) imply that

$$\lim_{n \rightarrow \infty} \tilde{a}_n(2^{j-1}\xi) \cdots \tilde{a}_n(\xi) = \hat{f}(2^j(\xi + 2l\pi))/\hat{f}(\xi + 2l\pi).$$

By the 2π -periodicity, taking the limits in (2.1) and (2.3) again, we have

$$\hat{f}(2^j(\xi + 2p\pi)) = \frac{\hat{f}(2^j(\xi + 2l\pi))}{\hat{f}(\xi + 2l\pi)} \hat{f}(\xi + 2p\pi), \quad \forall p \in \mathbb{Z}.$$

In particular, the choice $p \in \{0, k\} \setminus \{l\}$ implies (1.5).

Thus (1.5) is true for every $k \in \mathbb{N}$ and this $\xi \in \mathbb{R} \setminus T_j$.

Since the set T_j has measure zero, (1.5) holds almost everywhere. This proves the necessity of (1.5).

Sufficiency. Suppose that (1.5) is true. It is still true if we replace ξ by $2^m(\xi + 2l\pi)$ and k by $2^m k$, for $m \in \mathbb{N}$ and $l \in \mathbb{Z}$. Now change $j + m$ back to j , and $k + l$ back to k . The result is

$$(2.4) \quad \hat{f}(2^j(\xi + 2k\pi))\hat{f}(2^m(\xi + 2l\pi)) = \hat{f}(2^j(\xi + 2l\pi))\hat{f}(2^m(\xi + 2k\pi))$$

for any $j, m \in \mathbb{N}, k, l \in \mathbb{Z}$ and every $\xi \in \mathbb{R} \setminus T$, where T is a null set.

Let us define a sequence $\{\phi_n\}$ of refinable functions tending to f .

Let $M_j(f)$ be the union of the sets $K_0(f), \dots, K_{j-1}(f)$ defined in (1.7):

$$M_j(f) := \bigcup_{i=0}^{j-1} K_i(f) \\ = \{\xi \in [-\pi, \pi) : \hat{f}(2^i(\xi + 2l\pi)) \neq 0 \text{ for some } 0 \leq i < j \text{ and } l \in \mathbb{Z}\}.$$

Naturally, we set $M_\infty(f)$ as

$$M_\infty(f) = \bigcup_{i=0}^{\infty} K_i(f) = \bigcup_{j=1}^{\infty} M_j(f).$$

For $\xi \in K_0(f)$, define

$$(2.5) \quad \hat{\phi}_n(\xi + 2k\pi) = \hat{f}(\xi + 2k\pi), \quad \forall k \in \mathbb{Z}.$$

For $j \in \mathbb{N}$ and $\xi \in K_j(f) \setminus M_j(f)$, define

$$(2.6) \quad \hat{\phi}_n(\xi + 2k\pi) = \hat{f}(2^j(\xi + 2k\pi))/n, \quad \forall k \in \mathbb{Z}.$$

For $\xi \in [-\pi, \pi) \setminus M_\infty(f)$, define

$$(2.7) \quad \hat{\phi}_n(\xi + 2k\pi) = 0, \quad \forall k \in \mathbb{Z}.$$

Thus, $\hat{\phi}_n(\xi)$ has been defined for all ξ .

We first show that $\hat{\phi}_n(\xi) \rightarrow \hat{f}(\xi)$ in $L_2(\mathbb{R})$. By equations (2.5) and (2.7),

$$\hat{\phi}_n(\xi + 2k\pi) - \hat{f}(\xi + 2k\pi) = 0, \quad \forall \xi \in K_0(f) \cup ([-\pi, \pi) \setminus M_\infty(f)), \quad k \in \mathbb{Z}.$$

Hence

$$\begin{aligned} \|\hat{\phi}_n - \hat{f}\|_2^2 &= \sum_{j=1}^{\infty} \int_{K_j(f) \setminus M_j(f)} \sum_{k \in \mathbb{Z}} |\hat{\phi}_n(\xi + 2k\pi) - \hat{f}(\xi + 2k\pi)|^2 d\xi \\ &= \sum_{j=1}^{\infty} \int_{K_j(f) \setminus M_j(f)} \sum_{k \in \mathbb{Z}} |\hat{f}(2^j(\xi + 2k\pi))/n|^2 d\xi \\ &\leq \sum_{j=1}^{\infty} \int_{\mathbb{R}} |\hat{f}(2^j \xi)|^2 d\xi / n^2 \\ &= \|\hat{f}\|_2^2 / n^2 \rightarrow 0. \end{aligned}$$

Therefore, $\phi_n \in L_2(\mathbb{R})$ and $\lim_{n \rightarrow \infty} \|\phi_n - f\|_2 = 0$.

Next we show that ϕ_n is refinable, by constructing $\tilde{a}_n(\xi)$ on $[-\pi, \pi)$ such that

$$(2.8) \quad \hat{\phi}_n(2(\xi + 2k\pi)) = \tilde{a}_n(\xi)\hat{\phi}_n(\xi + 2k\pi), \quad \forall k \in \mathbb{Z}, \quad \xi \in [-\pi, \pi) \setminus T.$$

Let $j \in \mathbb{N} \cup \{0\}$ and $\xi \in K_j(f) \setminus M_j(f)$, and furthermore choose $k_\xi \in \mathbb{Z}$ such that $\hat{f}(2^j(\xi + 2k_\xi\pi)) \neq 0$. Define

$$\tilde{a}_n(\xi) = \begin{cases} \hat{\phi}_n(2(\xi + 2k_\xi\pi))/\hat{f}(\xi + 2k_\xi\pi), & \text{if } j = 0, \\ n\hat{\phi}_n(2(\xi + 2k_\xi\pi))/\hat{f}(2^j(\xi + 2k_\xi\pi)), & \text{if } j \in \mathbb{N}. \end{cases}$$

For $\xi \in [-\pi, \pi) \setminus M_\infty(f)$, we can define $\tilde{a}_n(\xi)$ arbitrarily.

Let us now verify the refinement relation (2.8), first for $\xi \in M_\infty(f) \setminus T$.

Let $j \in \mathbb{N} \cup \{0\}$ and $\xi \in K_j(f) \setminus M_j(f) \setminus T$. For every $k \in \mathbb{Z}$,

$$(2.9) \quad \tilde{a}_n(\xi)\hat{\phi}_n(\xi + 2k\pi) = \frac{\hat{\phi}_n(2(\xi + 2k_\xi\pi))}{\hat{f}(2^j(\xi + 2k_\xi\pi))}\hat{f}(2^j(\xi + 2k\pi)).$$

To see that this equals $\hat{\phi}_n(2(\xi + 2k\pi))$, write 2ξ as $\eta + 2s\pi$ with $\eta \in [-\pi, \pi)$ and $s \in \mathbb{Z}$. Then, if $\eta \notin M_\infty(f)$,

$$\hat{\phi}_n(2(\xi + 2l\pi)) = \hat{\phi}_n(\eta + 2s\pi + 4l\pi) = 0, \quad \forall l \in \mathbb{Z}.$$

Hence the right-hand side of (2.9) equals $\hat{\phi}_n(2(\xi + 2k\pi))$ in this case.

If $\eta \in K_0(f)$, then

$$\hat{\phi}_n(2(\xi + 2l\pi)) = \hat{\phi}_n(\eta + 2s\pi + 4l\pi) = \hat{f}(\eta + 2s\pi + 4l\pi) = \hat{f}(2(\xi + 2l\pi)).$$

Hence the right-hand side of (2.9) equals $\hat{\phi}_n(2(\xi + 2k\pi))$ again by the condition (2.4).

If $\eta \in K_m(f) \setminus M_m(f)$ for some $m \in \mathbb{N}$, then

$$\begin{aligned} \hat{\phi}_n(2(\xi + 2l\pi)) &= \hat{\phi}_n(\eta + 2s\pi + 4l\pi) = \hat{f}(2^m(\eta + 2s\pi + 4l\pi))/n \\ &= \hat{f}(2^{m+1}(\xi + 2l\pi))/n. \end{aligned}$$

Hence the right-hand side of (2.9) equals $\hat{\phi}_n(2(\xi + 2k\pi))$ in this final case by the condition (2.4).

Thus, the refinement relation (2.8) has been proved for $\xi \in M_\infty(f) \setminus T$.

Next we consider $\xi \in [-\pi, \pi) \setminus M_\infty(f) \setminus T$. Here we have

$$\hat{\phi}_n(\xi + 2l\pi) = 0, \quad \forall l \in \mathbb{Z}.$$

Let us show that $\hat{\phi}_n(2(\xi + 2l\pi)) = 0$ for every $l \in \mathbb{Z}$. Write 2ξ as $\eta + 2s\pi$ again with $\eta \in [-\pi, \pi)$ and $s \in \mathbb{Z}$.

If $\eta \notin M_\infty(f)$, then

$$\hat{\phi}_n(2(\xi + 2l\pi)) = \hat{\phi}_n(\eta + 2s\pi + 4l\pi) = 0, \quad \forall l \in \mathbb{Z}.$$

If $\eta \in K_0(f)$, then $\xi \notin K_1(f)$ implies that for every $l \in \mathbb{Z}$,

$$\hat{\phi}_n(2(\xi + 2l\pi)) = \hat{f}(\eta + 2s\pi + 4l\pi) = \hat{f}(2(\xi + 2l\pi)) = 0.$$

If $\eta \in K_j(f) \setminus M_j(f)$ for some $j \in \mathbb{N}$, then

$$\begin{aligned} \hat{\phi}_n(2(\xi + 2l\pi)) &= \hat{\phi}_n(\eta + 2s\pi + 4l\pi) = \hat{f}(2^j(\eta + 2s\pi + 4l\pi))/n \\ &= \hat{f}(2^{j+1}(\xi + 2l\pi))/n = 0, \quad \forall l \in \mathbb{Z}, \end{aligned}$$

since $\xi \notin K_{j+1}(f)$.

Thus, in all three cases,

$$\hat{\phi}_n(2(\xi + 2l\pi)) = 0, \quad \forall l \in \mathbb{Z}.$$

Therefore, the refinement relation (2.8) holds true on $[\pi, \pi) \setminus T$. Hence ϕ_n is refinable; it lies in S . Then $\lim \|\phi_n - f\|_2 = 0$ tells us that f lies in the closure of S . □

Now that we have proved Theorem 1, the proof of Theorem 2 follows quickly.

Proof of Theorem 2. For $\phi \in S$, Theorem 1 gives $D_{j,k}(\phi)(\xi) = 0$ almost everywhere. Then

$$\begin{aligned} \|D_{j,k}(f)\|_1 &= \|D_{j,k}(f) - D_{j,k}(\phi)\|_1 \\ &= \int_{\mathbb{R}} |[\hat{f}(2^j(\xi + 2k\pi)) - \hat{\phi}(2^j(\xi + 2k\pi))] \hat{f}(\xi) \\ &\quad + \hat{\phi}(2^j(\xi + 2k\pi))[\hat{f}(\xi) - \hat{\phi}(\xi)] \\ &\quad + \hat{\phi}(2^j\xi)[\hat{\phi}(\xi + 2k\pi) - \hat{f}(\xi + 2k\pi)] \\ &\quad + \hat{f}(\xi + 2k\pi)[\hat{\phi}(2^j\xi) - \hat{f}(2^j\xi)]| d\xi. \end{aligned}$$

Applying the Schwarz inequality, we get

$$\begin{aligned} \|D_{j,k}(f)\|_1 &\leq 2 \left(\int |\hat{f}(2^j\xi) - \hat{\phi}(2^j\xi)|^2 d\xi \right)^{1/2} \|\hat{f}\|_2 + 2 \left(\int |\hat{\phi}(2^j\xi)|^2 d\xi \right)^{1/2} \|\hat{f} - \hat{\phi}\|_2 \\ &= 2^{1-j/2} \|\hat{f} - \hat{\phi}\|_2 (\|\hat{f}\|_2 + \|\hat{\phi}\|_2). \end{aligned}$$

For each k we sum over $j \in \mathbb{N}$:

$$(2.10) \quad \|f - \phi\|_2 (\|f\|_2 + \|\phi\|_2) \geq \frac{\sqrt{2} - 1}{4\pi} \sum_{j=1}^{\infty} \|D_{j,k}(f)\|_1.$$

In computing the distance $d(f, S)$ we may restrict to $\phi \in S$ with $\|\phi\|_2 \leq 2\|f\|_2$, since otherwise $\|\phi - f\| \geq \|0 - f\|$. Then (2.10), for each k , yields the lower bound on $d(f, S)$ in Theorem 2:

$$(2.11) \quad d(f, S) 3\|f\|_2 \geq \frac{\sqrt{2} - 1}{4\pi} \sum_{j=1}^{\infty} \|D_{j,k}(f)\|_1.$$

The proof of Theorem 2 is complete. □

Remark on condition (1.5). If the Fourier transform of a refinable function were never zero, division would be allowed and everything would become easy:

$$(2.12) \quad \frac{\hat{f}(2\xi)}{\hat{f}(\xi)} \quad \text{is periodic by the refinement equation, and}$$

$$(2.13) \quad \frac{\hat{f}(2^j\xi)}{\hat{f}(\xi)} = \frac{\hat{f}(2^j\xi)}{\hat{f}(2^{j-1}\xi)} \cdots \frac{\hat{f}(2\xi)}{\hat{f}(\xi)} \quad \text{is periodic by induction.}$$

Condition (1.5) is simply the periodicity of $\hat{f}(2^j\xi)/\hat{f}(\xi)$ after multiplication to clear out the (possibly zero!) denominators.

Since the periodicity (2.13) for all j follows from (2.12) for $j = 1$, it is natural to ask whether this is also true in condition (1.5). *Must we impose this condition for all $j \in \mathbb{N}$?*

The following example shows that we must.

Example 1. Let $\hat{f}(\xi) = 1$ for

$$\xi \in (-\pi, -\pi/2) \cup (-\pi/2^{m+1}, -\pi/2^{m+2}) \cup (2\pi - \pi/2^{m+1}, 2\pi - \pi/2^{m+2})$$

and zero elsewhere. Then (1.5) holds for $j = 1, \dots, m$ and all k , but not for $j = m + 1$ and $k = 1$.

Proof. If $\xi > 0$, then

$$\hat{f}(\xi + 2k\pi) = \hat{f}(2^j(\xi + 2k\pi)) = 0 \quad \text{for all } j, k \in \mathbb{N}.$$

If $\xi < -\pi$, then

$$\hat{f}(\xi) = \hat{f}(2^j\xi) = 0 \quad \text{for all } j \in \mathbb{N}.$$

If $-\pi < \xi < 0$, then

$$\hat{f}(2^j(\xi + 2k\pi)) = 0 \quad \text{for all } j, k \in \mathbb{N}.$$

If $-\pi < \xi < -\pi/2^{m+1}$ or $-\pi/2^{m+2} < \xi < 0$, then

$$\hat{f}(\xi + 2k\pi) = 0 \quad \text{for all } k \in \mathbb{N}.$$

If $-\pi < \xi < 0$, then

$$\hat{f}(\xi + 2k\pi) = 0 \quad \text{for all } k \geq 2.$$

Thus we only need to check condition (1.5) for $-\pi/2^{m+1} < \xi < -\pi/2^{m+2}$ and $k = 1$. In this case, $\hat{f}(\xi + 2\pi) = 1$. For $j = 1, \dots, m$ we have $\hat{f}(2^j\xi) = 0$, which implies (1.5). However, $\hat{f}(2^{m+1}\xi) = 1$, which contradicts (1.5) for $k = 1$. \square

3. BAND-LIMITED FUNCTIONS

Let X_b be the set of band-limited functions with frequencies ξ restricted to the band $[-b, b]$:

$$X_b := \{f \in L_2(\mathbb{R}) : \text{supp } \hat{f} \subset [-b, b]\}.$$

We observed in the introduction that $X_b \subset \overline{S}$ for $b \leq \frac{4}{3}\pi$. The converse is also true.

Theorem 4. $X_b \subset \overline{S}$ if and only if $b \leq \frac{4}{3}\pi$.

Proof. If $b > \frac{4}{3}\pi$, let $f \in L_2(\mathbb{R})$ be given by its Fourier transform as

$$\hat{f}(\xi) = \begin{cases} 1, & \text{if } |\xi| < B := \min\{b, 2\pi\}, \\ 0, & \text{otherwise.} \end{cases}$$

Then for $j = 1, k = 1, -B < \xi < -\frac{4}{3}\pi$, we have

$$\hat{f}(2(\xi + 2\pi)) = \hat{f}(\xi) = 1 \quad \text{but} \quad \hat{f}(2\xi) = 0.$$

Hence (1.5) does not hold on the interval $(-B, -\frac{4}{3}\pi)$ for $j = 1, k = 1$. Thus, $f \in X_b \setminus \overline{S}$. \square

Now we show that some functions are not refinable but are the limits of refinable functions. The example in the introduction was

$$\hat{f}(\xi) = \begin{cases} 1, & \text{if } \xi \in (-\frac{4}{3}\pi, -\frac{2}{3}\pi) \cup (\frac{2}{3}\pi, \frac{4}{3}\pi), \\ 0, & \text{otherwise.} \end{cases}$$

Corollary 2. *Let $f \in X_{\frac{4}{3}\pi}$, and let $K(f) := \{\xi \in \mathbb{R} : \hat{f}(\xi) \neq 0\}$ be the support of \hat{f} . Then f is refinable if and only if $\frac{1}{2}K(f) \subset K(f)$ up to a null set.*

Proof. By Theorem 4, $f \in \overline{S}$.

Observe that for $f \in X_{\frac{4}{3}\pi}$ and $j \in \mathbb{N}$,

$$K_j(f) = \{\xi \in [-\pi, \pi) : \hat{f}(2^j \xi) \neq 0\} = 2^{-j}K(f) \subset \left[-\frac{2}{3}\pi, \frac{2}{3}\pi\right].$$

Also,

$$K_0(f) \cap \left(-\frac{2}{3}\pi, \frac{2}{3}\pi\right) = K(f) \cap \left(-\frac{2}{3}\pi, \frac{2}{3}\pi\right).$$

Then our conclusion follows from Theorem 3. □

Combining Theorem 4 and Corollary 2, we know that every nonzero function in $X_{\frac{4}{3}\pi}$ whose Fourier transform vanishes on $[-\frac{2}{3}\pi, \frac{2}{3}\pi]$ lies in $\overline{S} \setminus S$.

4. REFINABLE FUNCTIONS AND MASKS: NONUNIQUENESS

We apply the characterization of refinable functions in Theorem 3 to show that the function may not determine the mask (and vice versa).

First, we show that the 2π -periodic symbol of the mask $\tilde{a}(\xi)$ is sometimes not unique.

Theorem 5. *Let ϕ be a nonzero refinable function in $L_2(\mathbb{R})$, and let $K_0(\phi)$ be defined by (1.7). Then the refinement mask $\tilde{a}(\xi)$ satisfying (1.4) is unique (up to a null set) if and only if $\text{meas}(K_0(\phi)) = 2\pi$, i.e., for almost every $\xi \in [-\pi, \pi)$ there is some $k_\xi \in \mathbb{Z}$ such that $\hat{\phi}(\xi + 2k_\xi\pi) \neq 0$.*

Proof. The sufficiency is clear, since $\tilde{a}(\xi)$ is determined for $\xi \in K_0(\phi)$ by

$$\tilde{a}(\xi) = \hat{\phi}(2(\xi + 2k_\xi\pi)) / \hat{\phi}(\xi + 2k_\xi\pi),$$

which defines $\tilde{a}(\xi)$ uniquely up to a null set.

For the necessity, suppose to the contrary that $\text{meas}(K_0(\phi)) < 2\pi$. Then the measure of the set $([-\pi, \pi) \setminus M_\infty(\phi))$ is positive by Theorem 3.

Let $\tilde{a}(\xi)$ be the symbol of a refinement mask satisfying (1.4). Choose $\tilde{b}(\xi)$ to be a 2π -periodic function satisfying

$$\tilde{b}(\xi) = \tilde{a}(\xi), \quad \forall \xi \in K_0(\phi).$$

Then we can see that for almost every $\xi \in [-\pi, \pi)$,

$$\hat{\phi}(2(\xi + 2k\pi)) = \tilde{b}(\xi)\hat{\phi}(\xi + 2k\pi), \quad \forall k \in \mathbb{Z}.$$

In fact, for $\xi \in [-\pi, \pi) \setminus M_\infty(\phi)$,

$$\hat{\phi}(2(\xi + 2k\pi)) = \hat{\phi}(\xi + 2k\pi) = 0, \quad \forall k \in \mathbb{Z},$$

while $\tilde{b}(\xi) = \tilde{a}(\xi)$ for $\xi \in K_0(\phi)$. Hence the refinement relation is reduced to (1.4).

Note that $\text{meas}(([-\pi, \pi] \setminus M_\infty(\phi)) \cup K_0(\phi)) = 2\pi$ by Theorem 3. The function $\tilde{b}(\xi)$ is also the symbol of a refinement mask for ϕ . Thus the mask is not unique. \square

Second, we show that the refinable function is never unique, given a refinement mask.

The classical approach begins with a sequence $\{a(k)\}$ satisfying

$$\sum_{k \in \mathbb{Z}} |a(k)||k|^\delta < \infty \quad \text{for some } \delta > 0.$$

Then the refinement equation (1.1) has at most one integrable solution up to a constant multiplication; see Daubechies and Lagarias [4].

When we consider solutions in $L_2(\mathbb{R})$, this uniqueness never holds. To see this, let $\phi \in L_2(\mathbb{R})$ satisfy the refinement equation (1.4) for some 2π -periodic function \tilde{a} . If $\tau(\xi)$ is an arbitrary measurable bounded function on $[-2\pi, 2\pi]$, then the function ψ defined by its Fourier transform as

$$\hat{\psi}(\xi) = \tau(2^j \xi) \hat{\phi}(\xi), \quad \xi \in [2^{-j}\pi, 2^{1-j}\pi] \cup (-2^{1-j}\pi, -2^{-j}\pi], \quad j \in \mathbb{Z},$$

also satisfies the refinement equation (1.4) with the same mask.

However, if we require that $\hat{\phi}(\xi)$ is continuous at the origin, which is the case when $\phi \in L_1(\mathbb{R})$ and $\hat{\phi}(0) \neq 0$, then the solution is unique up to a constant multiplication.

Corollary 3. *If $\phi \in L_2(\mathbb{R})$ satisfies (1.4) and its Fourier transform can be chosen to be continuous at the origin: $\lim_{\xi \rightarrow 0} \hat{\phi}(\xi) = \hat{\phi}(0) \neq 0$, then any solution $\psi \in L_2(\mathbb{R})$ of (1.4) with $\hat{\psi}$ continuous at the origin can be written as*

$$\psi(x) = \frac{\hat{\psi}(0)}{\hat{\phi}(0)} \phi(x).$$

Proof. By our assumption, for almost every ξ ,

$$\lim_{n \rightarrow \infty} \tilde{a}(\xi/2) \cdots \tilde{a}(\xi/2^n) = \frac{\hat{\phi}(\xi)}{\hat{\phi}(0)}.$$

Therefore, for almost every ξ ,

$$\hat{\psi}(\xi) = \lim_{n \rightarrow \infty} \tilde{a}(\xi/2) \cdots \tilde{a}(\xi/2^n) \lim_{n \rightarrow \infty} \hat{\psi}(\xi/2^n) = \frac{\hat{\psi}(0)}{\hat{\phi}(0)} \hat{\phi}(\xi).$$

The proof of Corollary 3 is complete. \square

As a consequence, if there is a solution $\phi \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ to (1.4) with $\hat{\phi}(0) \neq 0$, then all the other solutions in $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ are $c\phi(x)$. This extends the result of Daubechies and Lagarias [4].

5. INHOMOGENEOUS REFINEMENT EQUATIONS

In this section we study inhomogeneous refinement equations and characterize those solutions which are (homogeneously) refinable.

The inhomogeneous refinement equation was defined in [16] as

$$(5.1) \quad \phi(x) = \sum_{k \in \mathbb{Z}} a(k)\phi(2x - k) + F(2x).$$

Here we are interested in L_2 solutions, so we assume that F is a nonzero function in $L_2(\mathbb{R})$. Denote

$$\tilde{a}(\xi) = \frac{1}{2} \sum_{k \in \mathbb{Z}} a(k)e^{-ik\xi}, \quad \xi \in \mathbb{R}.$$

Then the inhomogeneous refinement equation (5.1) has an equivalent form in the Fourier domain:

$$(5.2) \quad \hat{\phi}(2\xi) = \tilde{a}(\xi)\hat{\phi}(\xi) + \hat{F}(\xi)/2.$$

If $\phi \in L_2$ is a solution of (5.1) and ϕ is (homogeneously) refinable, then (1.4) holds for some 2π -periodic function $\tilde{\tau}$. Hence

$$\hat{F}(\xi) = 2(\tilde{\tau}(\xi) - \tilde{a}(\xi))\hat{\phi}(\xi).$$

It follows from Theorem 1 that $F \in \overline{S}$.

Suppose now that F and the mask a are supported in $[0, N]$ for some $N \in \mathbb{N}$. Then $F \in S$ by Corollary 1. Also, $\phi \in S_2(F)$. Moreover, [16] tells us that ϕ is supported in $[0, N]$.

According to the analysis of Jia [8], for the function F there exists a unique function $\psi \in L_2(\mathbb{R})$ (up to a constant multiplication), compactly supported in $[0, N]$ but not in $[1, N]$, such that its integer translates are linearly independent and, for some sequence $\{c(k)\}$,

$$(5.3) \quad F(x) = \sum_{k=0}^{N-1} c(k)\psi(x - k).$$

By (1.4), $\phi \in S_2(F) = S_2(\psi)$. Corollary 1 implies again that ψ is refinable.

By the linear independence of ψ and the supports, there are sequences $\{b(k)\}$ and $\{d(k)\}$ such that

$$\begin{aligned} \phi(x) &= \sum_{k=0}^{N-1} b(k)\psi(x - k), \\ \psi(x) &= \sum_{k=0}^N d(k)\psi(2x - k). \end{aligned}$$

Taking Fourier transforms and using (5.1), (5.3), we have

$$2\tilde{b}(2\xi)\tilde{d}(\xi)\hat{\psi}(\xi) = 2\tilde{a}(\xi)\tilde{b}(\xi)\hat{\psi}(\xi) + \tilde{c}(\xi)\hat{\psi}(\xi),$$

which implies

$$(5.4) \quad \tilde{b}(2\xi)\tilde{d}(\xi) = \tilde{a}(\xi)\tilde{b}(\xi) + \tilde{c}(\xi)/2, \quad \forall \xi \in \mathbb{R}.$$

Moreover, the linear independence of ψ provides (see Theorem 2.4 in [9])

$$\hat{\psi}(0) \neq 0, \quad \tilde{d}(0) = 1, \quad \text{and} \quad \tilde{d}(\pi) = 0.$$

Conversely, we have

Theorem 6. *Assume that the sequence $\{a(k)\}$ and the function $F \in L_2(\mathbb{R})$ are supported in $[0, N]$. Let $\psi \in L_2(\mathbb{R})$ be a function, compactly supported in $[0, N]$ but not in $[1, N]$, such that its integer translates are linearly independent, and (5.3) holds. Then (5.1) has a refinable solution $\phi \in L_2(\mathbb{R})$ if and only if ψ is refinable, $\hat{\psi}(0) \neq 0$ and the equation (5.4) is solvable for some sequence $\{b(k)\}$ supported in*

$[0, N - 1]$, where $\tilde{d}(\xi)$ is the symbol of the refinement mask of the function ψ with $\tilde{d}(0) = 1$ and $\tilde{d}(\pi) = 0$.

Observe that (5.4) is a system of linear equations whose solvability can be easily checked.

6. APPLICATIONS TO MULTIPLE REFINABLE FUNCTIONS

In this section we apply Theorem 6 to a study of some examples of *multiple refinable functions*. For the general theory and more examples of multiple refinable functions, we refer the reader to [5], [6], [10], [11], [12], [15], [18].

The first example was introduced by Geronimo, Hardin and Massopust [5]. Consider the matrix refinement equation

$$(6.1) \quad \Phi(x) = \sum_{k \in \mathbb{Z}} a_k \Phi(2x - k).$$

Here $\Phi(x) = (\phi_1(x), \phi_2(x))^T$ and $\{a_k\}$ is supported in $[0, 3]$ with

$$a_0 = \begin{bmatrix} h_1 & 1 \\ h_2 & h_3 \end{bmatrix}, \quad a_1 = \begin{bmatrix} h_1 & 0 \\ h_4 & 1 \end{bmatrix},$$

$$a_2 = \begin{bmatrix} 0 & 0 \\ h_4 & h_3 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 0 & 0 \\ h_2 & 0 \end{bmatrix}.$$

The matrix entries involve a parameter s :

$$h_1 = -\frac{s^2 - 4s - 3}{2(s + 2)}, \quad h_2 = -\frac{3(s^2 - 1)(s^2 - 3s - 1)}{4(s + 2)^2},$$

$$h_3 = \frac{3s^2 + s - 1}{2(s + 2)}, \quad h_4 = -\frac{3(s^2 - 1)(s^2 - s + 3)}{4(s + 2)^2}.$$

When $|s| < 1$, the matrix refinement equation (6.1) has a continuous solution Φ with $\hat{\phi}_1(0) = 1$ and $\hat{\phi}_2(0) = (s - 1)^2 / (s + 2)$. Moreover, $\text{supp}\phi_1 = [0, 1]$ and $\text{supp}\phi_2 = [0, 2]$.

Applying Theorem 6, we conclude that neither ϕ_1 nor ϕ_2 is refinable.

Example 2. Let $|s| < 1$, and let $\Phi(x) = (\phi_1(x), \phi_2(x))^T$ be the continuous solution of (6.1) with $\hat{\phi}_1(0) = 1$ and $\hat{\phi}_2(0) = (s - 1)^2 / (s + 2)$. Then neither ϕ_1 nor ϕ_2 is refinable.

Proof. The first component of (6.1) is an inhomogeneous refinement equation:

$$(6.2) \quad \phi_1(x) = h_1\phi_1(2x) + h_1\phi_1(2x - 1) + \phi_2(2x).$$

It is proved in [12] that the integer translates of ϕ_2 are linearly independent. Hence we can take $N = 2, F = \psi = \phi_2$, and $\tilde{c} \equiv 1$. Under the restrictions $\tilde{d}(0) = 1$ and $\tilde{d}(\pi) = 0$, we find that the equation (5.4) is not solvable. Therefore, ϕ_1 is not refinable.

The second component of (6.1) is another inhomogeneous refinement equation,

$$\phi_2(x) = h_3\phi_2(2x) + \phi_2(2x - 1) + h_3\phi_2(2x - 2) + F(2x),$$

where

$$F(x) = h_2\phi_1(x) + h_4\phi_1(x - 1) + h_4\phi_1(x - 2) + h_2\phi_1(x - 3).$$

Take $N = 4$ and $\psi = \phi_1$, since the integer translates of ϕ_1 are linearly independent [12]. If ϕ_2 is refinable, then Theorem 6 shows that $\phi_1 = \psi$ is also refinable, which is a contradiction. Thus, ϕ_2 is not refinable, either. \square

Our second example is taken from [10], [11], [12]. Let $\{a_k\}$ be supported in $[0, 2]$ with

$$(6.3) \quad a_0 = \begin{bmatrix} \frac{1}{2} & \frac{s}{2} \\ t & \lambda \end{bmatrix}, \quad a_1 = \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix}, \quad \text{and} \quad a_2 = \begin{bmatrix} \frac{1}{2} & -\frac{s}{2} \\ -t & \lambda \end{bmatrix}.$$

Here s, t, λ, μ are real parameters. We assume that $|2\lambda + \mu| < 2$. Then there exists a unique distributional solution $\Phi(x) = (\phi_1(x), \phi_2(x))^T$ of (6.1) with $\hat{\Phi}(0) = (1, 0)^T$ supported in $[0, 2]$. The distribution $\phi_1(x)$ is symmetric about 1, and $\phi_2(x)$ is anti-symmetric about 1. It was proved in [10, Example 4.3] that the shifts of ϕ_1 and ϕ_2 reproduce all quadratic polynomials if and only if

$$(6.4) \quad t \neq 0, \quad \mu = 1/2, \quad \text{and} \quad \lambda = 1/4 + 2st.$$

In this case, the condition $|2\lambda + \mu| < 2$ reduces to $-3/4 < st < 1/4$, and it is verified in [11], [12] that the solution is continuous.

Example 3. Let $\{a_k\}$ be the mask given in (6.3) and (6.4) with $-3/4 < st < 1/4$. Let $\Phi(x) = (\phi_1(x), \phi_2(x))^T$ be the continuous solution of (6.1) with $\hat{\Phi}(0) = (1, 0)^T$. Then ϕ_1 is refinable if and only if $s = 0$, while ϕ_2 is never refinable.

Proof. First, we consider the case $s \neq 0$. In this case, it is proved in [12] that the integer translates of ϕ_1 and ϕ_2 are linearly independent.

For ϕ_1 , the first component of (6.1) is an inhomogeneous refinement equation,

$$\phi_1(x) = \phi_1(2x)/2 + \phi_1(2x - 1) + \phi_1(2x - 2)/2 + F(x),$$

where $F(x) = s\phi_2(x)/2 - s\phi_2(x - 2)/2$. Let $N = 4$ and $\psi = \phi_2$. If ϕ_1 is refinable, then Theorem 6 shows that $\hat{\phi}_2(0) = \hat{\psi}(0) \neq 0$, which is a contradiction.

The function ϕ_2 is trivially not refinable, since otherwise $\hat{\phi}_2(0) \neq 0$ by [9, Theorem 2.4].

Second, we investigate the case $s = 0$. Then ϕ_1 is refinable, since the first component of (6.1) reduces to a homogeneous equation for ϕ_1 . In fact, ϕ_1 is the hat function on $[0, 2]$.

To consider ϕ_2 , the second component of (6.1) is

$$\phi_2(x) = \phi_2(2x)/4 + \phi_2(2x - 1)/2 + \phi_2(2x - 2)/4 + F(2x),$$

where $F(x) = t\phi_1(x) - t\phi_1(x - 2)$. By Theorem 1 in [16], the solution to this equation is unique. Let $N = 4$ and $\psi = \phi_1$. Then $\tilde{d}(\xi) = (1 + e^{-i\xi})^2/4$ and $\tilde{c}(\xi) = t(1 - e^{-i2\xi})/2$, and the solvability of equation (5.4) would imply $t = 0$, which is a contradiction. Therefore, ϕ_2 is never refinable. \square

The explicit formula for the solution $\Phi(x)$ in the special case $s = 3/2, t = -1/8, \lambda = -1/8$, and $\mu = 1/2$ was given by Heil, Strang, and Strela [6]. In this case, $\Phi(x)$ is supported on $[0, 2]$:

$$\phi_1(x) = \begin{cases} x^2(-2x + 3) & \text{for } 0 \leq x \leq 1, \\ (2 - x)^2(2x - 1) & \text{for } 1 < x \leq 2, \end{cases}$$

and

$$\phi_2(x) = \begin{cases} x^2(x-1) & \text{for } 0 \leq x \leq 1, \\ (2-x)^2(x-1) & \text{for } 1 < x \leq 2. \end{cases}$$

7. FULLY REFINABLE FUNCTIONS

A function $\phi \in L_2(\mathbb{R})$ is *fully refinable* if for every $t \in \mathbb{R}$, the shifted function $\phi_t(x) := \phi(x-t)$ is refinable. It is shown in [17] that Meyer's well-known scaling function [13] is fully refinable.

Let ϕ be a refinable function in $L_2(\mathbb{R})$ and $t \in \mathbb{R}$. Then $K_j(\phi_t) = K_j(\phi)$ for $j \in \mathbb{N} \cup \{0\}$. Theorem 3 tells us that ϕ_t is refinable if and only if $\phi_t \in \overline{S}$. By Theorem 1, this is equivalent to

$$\hat{\phi}(2^j(\xi + 2k\pi))\hat{\phi}(\xi)(e^{-it(2^j-1)2k\pi} - 1) = 0$$

for any $j, k \in \mathbb{N}$, almost everywhere in ξ .

Thus, a translate ϕ_t of a compactly supported refinable function $\phi \in L_2(\mathbb{R})$ is not refinable unless t is an integer.

Moreover, if ϕ is fully refinable, then up to a null set $\hat{\phi}(\xi) \neq 0$ implies $\hat{\phi}(\xi/2) \neq 0$, and hence

$$\hat{\phi}(\xi + 4k\pi) = \hat{\phi}(2(\xi/2 + 2k\pi)) = 0, \quad \forall k \in \mathbb{Z} \setminus \{0\}.$$

This shows that the measure of the support of $\hat{\phi}$ is not greater than 4π .

However, Theorem 4 tells that every refinable function in $X_{\frac{4}{3}\pi}$ is fully refinable.

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