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VECTOR A_2 WEIGHTS AND A HARDY-LITTLEWOOD MAXIMAL FUNCTION

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Abstract. An analogue of the Hardy-Littlewood maximal function is introduced, for functions taking values in finite-dimensional Hilbert spaces. It is shown to be L^2 bounded with respect to weights in the class A_2 of Treil, thereby extending a theorem of Muckenhoupt from the scalar to the vector case.

A basic chapter of the subject of singular integral operators is the weighted norm theory, which provides a necessary and sufficient condition on a nonnegative function w for such operators, and for the Hardy-Littlewood maximal function M, to be bounded in L^p with respect to the measure $w(x)\,dx$. See [3], [6], [12]. More recently, aspects of this theory have been extended first by Treil and Volberg [17, 18], then by other authors, to functions taking values in finite-dimensional Hilbert spaces, with weights taking values in the corresponding spaces of Hermitian forms. These extensions have relied on new ideas, quite different from those employed in the scalar case by earlier authors, and have shed new light on the scalar theory.

Nonetheless, some aspects of the scalar theory had apparently not been successfully generalized, including M itself. Indeed, doubts have been expressed [20, 17] as to whether there could exist any useful analogue of M.

In this note we introduce a vector analogue of the Hardy-Littlewood maximal function, and prove its boundedness, in the simplest case, $p = 2$. In doing so, we seek firstly, to clarify the relationships between the new vector theory, and the more familiar scalar case, and secondly, to open up an alternative approach to the subject, which might lead to vector analogues of other features of the scalar theory. The second author [5] has carried this program further by demonstrating that the analogue of our theorem holds for all $1 < p < \infty$, and that the boundedness of singular integral operators on $L^p(\mathbb{R}^n, \mathcal{H}, w)$ can be deduced from our theorem, in the spirit of [3].

When specialized to the scalar case, our analysis differs from that in [3] in that we study the operator $f \mapsto w^{1/2}M(w^{-1/2}f)$ on unweighted L^2, rather than M on w–weighted L^2. Although the direct analysis in this conjugated framework is slightly more intricate than [3], its generalization from the scalar to the vector setting is transparent.

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1. A Maximal Operator

Let $\mathcal{H}$ be a separable Hilbert space, possibly of infinite dimension. Denote by $\mathcal{M}$ the space of all nonnegative Hermitian quadratic forms on $\mathcal{H}$ that are bounded with respect to the norm of $\mathcal{H}$. The same symbol $\|\cdot\|$ will denote the norm either of an element of $\mathcal{H}$, or of an element of $\mathcal{M}$, considered as a linear operator from $\mathcal{H}$ to $\mathcal{H}$. If $w:\mathbb{R}^n\mapsto\mathcal{M}$ is measurable and locally integrable in the sense that the function $x\mapsto\|w(x)\|$ is locally integrable, define $L^p(\mathbb{R}^n,\mathcal{H},w(x)\,dx)$ to be the Banach space of all equivalence classes of measurable functions $f:\mathbb{R}^n\mapsto\mathcal{H}$ for which $\int_{\mathbb{R}^n}\langle w(x)f(x),\ f(x)\rangle^{p/2}\,dx$ is finite.

To simplify notation we will systematically write $g_E=\frac{1}{|E|}\int_E g$, where $E\subset\mathbb{R}^n$, $|E|$ denotes the Lebesgue measure of E, g takes values in $\mathbb{R},\mathcal{H}$, or $\mathcal{M}$, and the integral is taken with respect to Lebesgue measure. For $v\in\mathcal{M}$ or $\mathbb{R}^+$ and $r\in\mathbb{R}$, the notation v_E^r means $(v_E)^r$.

$A_2=A_2(\mathbb{R}^n,\mathcal{H})$ is by definition the class of all $w:\mathbb{R}^n\mapsto\mathcal{M}$ such that w,w^{-1} are locally integrable and there exists $C<\infty$ such that

$$\|w_Q^{1/2}(w^{-1})_Q^{1/2}\|\le C \qquad \text{for every cube } Q\subset\mathbb{R}^n. \tag{1.1}$$

The smallest constant C satisfying this inequality will be called the A_2 "norm" of w. Treil and Volberg [17] have proved that if $\mathcal{H}$ has finite dimension, then the Hilbert transform H maps $L^2(\mathbb{R},\mathcal{H},w(x)\,dx)$ boundedly to itself, if and only if $w\in A_2$.

When $\mathcal{H}$ has finite dimension, (1.1) is equivalent[1] to

$$\frac{1}{|Q|}\int_Q\|w^{1/2}(y)(w^{-1})_Q^{1/2}\|^2\,dy\le C; \tag{1.2}$$

another equivalent formulation is that $\frac{1}{|Q|}\int_Q\|w_Q^{1/2}w^{-1/2}(y)\|^2\,dy\le C$.

Now H preserves $L^2(\mathbb{R},\mathcal{H},w(x)\,dx)$, if and only if the operator

$$f\mapsto w^{1/2}(x)H(w^{-1/2}f)(x)$$

maps $L^2(\mathbb{R},\mathcal{H},dx)$ boundedly to itself. Our analogue of the Hardy-Littlewood maximal function likewise acts on $L^2(\mathbb{R}^n,\mathcal{H},dx)$ rather than on a weighted space, but $M_wf(x)$ is scalar-valued, rather than $\mathcal{H}$-valued.

Definition.

$$M_wf(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q\|w^{1/2}(x)w^{-1/2}(y)f(y)\|\,dy. \tag{1.3}$$

In the scalar case, the unweighted L^2 boundedness of M_w is equivalent to the weighted boundedness of the Hardy-Littlewood maximal function.

Maximal Theorem. *Let $\mathcal{H}$ be a finite-dimensional Hilbert space, and let $w\in A_2(\mathbb{R}^n,\mathcal{H})$. Then there exists $\delta>0$, depending only on the A_2 constant of w and the dimension of $\mathcal{H}$, such that the maximal operator M_w is bounded from $L^p(\mathbb{R}^n,\mathcal{H},dx)$ to $L^p(\mathbb{R}^n,\mathbb{R},dx)$ whenever $|p-2|<\delta$. Conversely, if M_w is bounded for $p=2$, then $w\in A_2$.*

[1] To see this, use the equivalence of norm with trace in finite dimensions.

2. Preparations

In this section we outline several preliminary results, deferring the proofs to a later section. The linchpin of the scalar theory, as developed in [3], is the reverse Hölder inequality. Our first ingredient is an analogue for vector A_2 weights, when $\mathcal{H}$ has finite dimension; this is the only step in our analysis that requires finite-dimensionality.

Proposition 2.1. *Let $\mathcal{H}$ have finite dimension, and let $w \in A_2(\mathcal{H})$. Then there exist $r > 2$ and $C < \infty$ such that for every Q,*

$$\int_Q \|w^{1/2}(y)(w^{-1})_Q^{1/2}\|^r \, dy \le C|Q|. \tag{2.1}$$

The same holds for $\|w_Q^{1/2} w^{-1/2}(y)\|$. The constants r, C can be taken to depend only on an upper bound for the A_2 norm of w, and on the dimension of $\mathcal{H}$.

While we are unaware of any explicit statement of this result in the literature, it is a rather direct consequence of the corresponding result from the scalar theory. However, one should note that whereas this inequality for $r \le 2$ is trivial for finite-dimensional $\mathcal{H}$, it is not automatic in infinite dimensions. We will refer to it as a "reverse Hölder" inequality, even when $r \le 2$.

By standard reasoning, involving the utilization of two incompatible dyadic grids, it suffices to consider the supremum only over dyadic cubes Q in defining M_w. Throughout the remainder of the paper, all cubes will hence be assumed to be dyadic.

We introduce the auxiliary maximal function

$$M'_w f(x) = \sup_{x \in Q} \tfrac{1}{|Q|} \int_Q \|w_Q^{1/2} w^{-1/2}(y) f(y)\| \, dy. \tag{2.2}$$

Denote also by m the Hardy-Littlewood maximal function in $\mathbb{R}^n$, acting on scalar-valued functions.

Lemma 2.2. *Let $w \in A_2(\mathcal{H})$. Suppose that w satisfies a reverse Hölder inequality (2.1) for some exponent $r > 2$. Then there exist $C < \infty$ and $\delta > 0$, depending only on the constants in (2.1), such that M'_w is bounded from $L^p(\mathbb{R}^n, \mathcal{H}, dx)$ to $L^p(\mathbb{R}^n, \mathbb{R}, dx)$, for every $p > 2 - \delta$. In particular, the conclusion holds whenever $\mathcal{H}$ has finite dimension.*

Proof. A direct consequence of the reverse Hölder inequality (2.1) is that $M'_w f(x) \le C(m(\|f\|^{r'}))^{1/r'}$ for all f, x, where $r' < 2$ is conjugate to the exponent r of (2.1). Consequently, M'_w is bounded from $L^p(\mathcal{H})$ to L^p, for all $p > r'$. $\square$

The new twist in our analysis is the presence of the variable factor $w^{1/2}(x)$, rather than the constant $w_Q^{1/2}$, in the definition of M_w. We will proceed by comparing M_w to M'_w via a sort of conditional expectation argument.

The following quite simple result will be proved below.

Lemma 2.3. *For any measurable $\mathcal{M}$–valued function v and any cube $Q \subset \mathbb{R}^n$,*

$$\|(v^{-1})_Q^{-1/2} v_Q^{-1/2}\| \le 1. \tag{2.3}$$

We assume here that $\|v\|, \|v^{-1}\|$ belong to $L^1(Q)$, and that $v_Q, (v^{-1})_Q$ are invertible elements of $\mathcal{M}$; both these hypotheses hold for all $v \in A_2$.

To any cube Q associate the quantity

$$N(x,Q) = \sup_{x\in S\subset Q} \|w^{1/2}(x)(w^{-1})_S^{1/2}\|. \tag{2.4}$$

The function $x \mapsto N(x,Q)$ will be denoted by N_Q.

The next lemma is a refinement of the reverse Hölder inequality, whose proof will be given later.

Lemma 2.4. *Let $w \in A_2$ satisfy a reverse Hölder inequality* (2.1) *with exponent $r \geq 1$. Then there exists $C < \infty$ such that for every dyadic cube $Q \subset \mathbb{R}^n$, with the same exponent r,*

$$\int_Q N_Q(x)^r \, dx \leq C|Q|. \tag{2.5}$$

In particular, if $\mathcal{H}$ has finite dimension, then this conclusion holds for some $r > 2$.

3. Proof of Theorem

Let $w \in A_2$, and let $p \geq 1$ be any finite exponent. To prove the main theorem, it suffices to show that if M'_w is bounded from $L^p(\mathbb{R}^n, \mathcal{H}, dx)$ to $L^p(\mathbb{R}^n, \mathbb{R}, dx)$, and if the functions N_Q satisfy the conclusion of Lemma 2.4 with exponent $r = p$, then M_w is likewise bounded.

Consider an arbitrary function $f \in L^p(\mathbb{R}^n, \mathcal{H})$, not identically zero. Let $\{Q_\imath^\jmath\}$ be the collection of all maximal dyadic cubes Q satisfying

$$2^j \leq \tfrac{1}{|Q|} \int_Q \|w_Q^{1/2} w^{-1/2}(y) f(y)\| \, dy < 2^{j+1}. \tag{3.1}$$

Define

$$E(\jmath) = \bigcup_\imath Q_\imath^\jmath \setminus \big(\bigcup_k Q_k^{\jmath+1}\big). \tag{3.2}$$

The sets $Q_\imath^\jmath \cap E(\jmath)$ are pairwise disjoint, as $\jmath, \imath$ range over all possible values. Moreover, modulo Lebesgue null sets, $E_\jmath = \{x : 2^\jmath \leq M'_w(x) < 2^{\jmath+1}\}$.

If $M_w f(x) = 0$, then x may be disregarded in our analysis. To each $x \in \mathbb{R}^n$ for which $M_w f(x) > 0$, associate a dyadic cube R_x satisfying

$$\tfrac{1}{|R_x|} \int_{R_x} \|w^{1/2}(x) w^{-1/2}(y) f(y)\| \, dy \geq \tfrac{1}{2} M_w f(x). \tag{3.3}$$

Fix $\jmath, \imath$ and consider any $x \in E(\jmath) \cap Q_\imath^\jmath$.

(3.4)

$$\begin{aligned} M_w f(x) &\leq 2N(x, Q_\imath^\jmath) \cdot \|(w^{-1})_{R_x}^{-1/2} w_{R_x}^{-1/2}\| \cdot \tfrac{1}{|R_x|} \int_{R_x} \|(w_{R_x})^{1/2} w^{-1/2}(y) f(y)\| \, dy \\ &\leq 2N(x, Q_\imath^\jmath) \tfrac{1}{|R_x|} \int_{R_x} \|(w_{R_x})^{1/2} w^{-1/2}(y) f(y)\| \, dy \leq 4N(x, Q_\imath^\jmath) 2^\jmath, \end{aligned}$$

first by Lemma 2.3, then by the definitions of $N(x, Q_\imath^\jmath)$ and of $E(\jmath)$. Thus

$$M_w f \leq 4 \sum_{j=-\infty}^{\infty} 2^\jmath \sum_\imath N(x, Q_\imath^\jmath) \chi_{Q_\imath^\jmath \cap E(\jmath)}, \tag{3.5}$$

where χ_S denotes the characteristic function of a set S. Thus by Lemma 2.4,

$$\begin{aligned}\|M_w f\|_p^p &\leq 4^p \sum_{j\in\mathbb{Z}} 2^{pj} \sum_i \int_{Q_i^j} N(\cdot, Q_i^j)^p \\ &\leq C \sum_j 2^{pj} \sum_i |Q_i^j| \leq C \sum_j 2^{pj} |\{x : M'_w f(x) \geq 2^j\}| \leq C\|M'_w f\|_p^p.\end{aligned}$$

Remark. We have implicitly proved certain implications for the case of infinite-dimensional Hilbert spaces. Assume that $w \in A_2$.

1. If w satisfies the reverse Hölder inequality for some exponent $r > 2$, then M_w is bounded on L^p for all $|p-2| < \delta$, and M'_w is bounded for all $p > 2-\delta$, for some $\delta > 0$.
2. If M'_w is bounded on L^2, and if w satisfies (2.1) with $r = 2$, then M_w is bounded on L^2.

4. Proofs of Lemmas

Proof of reverse Hölder inequality. It suffices to show

$$\tfrac{1}{|Q|}\int_Q \|w\|^r \leq C\big(\tfrac{1}{|Q|}\int_Q \|w\|\big)^r,$$

where $r > 1, C < \infty$ depend only on the dimension n of the ambient space, the dimension of $\mathcal{H}$, and the A_2 norm of w. For then the conclusion stated follows by applying this inequality to BwB, where $B = (w^{-1})_Q^{1/2}$. The A_2 norm of BwB equals that of w; this holds for any $B \in \mathcal{M}$, by Lemma 3.5 of [17]. And $\frac{1}{|Q|}\int_Q \|BwB\|$ is bounded by the A_2 norm squared of w.

It is known that for any nonnegative, locally integrable function u defined on $\mathbb{R}^n$, the set of all exponents $q > 1$ for which there exists $C < \infty$ for which

$$\big(\tfrac{1}{|Q|}\int_Q u^q\big)^{1/q} \leq C\tfrac{1}{|Q|}\int_Q u \qquad \text{for all cubes } Q \tag{4.1}$$

is open. Volberg and Treil [17], Lemma 3.6, have proved that

$$\tfrac{1}{|Q|}\int_Q \|w\| \leq C\big(\tfrac{1}{|Q|}\int_Q \|w\|^{1/2}\big)^2,$$

that is, $u(x) = \|w(x)\|^{1/2}$ satisfies (4.1) with $q = 2$, and the lemma follows. □

An alternative proof uses an easy consequence of the A_2 condition: For any nonzero $v \in \mathcal{H}$, for any $w \in A_2(\mathcal{H})$, the scalar function $\langle w(x)v, v\rangle$ is an A_2 weight, with A_2 constant bounded uniformly in v. Hence these scalar weights satisfy uniform reverse Hölder inequalities. Therefore the scalar case implies the finite-dimensional vector case, by taking the trace of $w_Q^{-1/2} w(x) w_Q^{-1/2}$.

Proof of Lemma 2.3. Let $\xi \in \mathcal{H}$ be arbitrary. For any $\mathcal{M}$–valued function u,

$$\|\xi\|^2 = \tfrac{1}{|Q|}\int_Q \langle u^{1/2}(x)\xi,\, u^{-1/2}(x)\xi\rangle\, dx \leq \langle \xi,\, u_Q\xi\rangle^{1/2} \cdot \langle \xi,\, (u^{-1})_Q\xi\rangle^{1/2}.$$

Set $u = (v^{-1})_Q^{1/2} \circ v \circ (v^{-1})_Q^{1/2}$, apply Cauchy-Schwarz, and eliminate a common factor of $\|\xi\|^{3/2}$ from both sides of the inequality, then square, to deduce that $\|\xi\| \leq \|v_Q^{1/2}(v^{-1})_Q^{1/2}\xi\|$. Inverting the matrix on the right yields the lemma. □

Proof of Lemma 2.4. We argue informally, assuming that $\int_Q N_Q^r \le B|Q|$ for all Q for some finite B, and deriving an *a priori* upper bound on B. To turn this argument into a legitimate proof is a routine exercise.

Let $A < \infty$ be a large constant to be specified later. Decompose $w^{1/2}(x)(w^{-1})_S^{1/2} = \big[w^{1/2}(x)(w^{-1})_Q^{1/2}\big] \circ \big[(w^{-1})_Q^{-1/2}(w^{-1})_S^{1/2}\big]$. Consider all maximal dyadic cubes $S_\jmath \subset Q$ satisfying $\|(w^{-1})_Q^{-1/2}(w^{-1})_S^{1/2}\| > A$.

For any $x \notin \bigcup_\jmath S_\jmath$, $N(x,Q) \le A\|[w^{1/2}(x)(w^{-1})_Q^{1/2}\|$. Thus $\int_{Q\setminus\bigcup_\jmath S_\jmath} N(x,Q)^r\,dx \le C|Q|$, by virtue of the reverse Hölder inequality (2.1).

We claim that $|\bigcup_\jmath S_\jmath| \le |Q|/2$, provided that A is sufficiently large, relative to the A_2 constant of w. Granting this, we conclude that

$$\int_{\bigcup_\jmath S_\jmath} N_Q^r = \sum_\jmath \int_{S_\jmath} N_Q^r \le B \sum_\jmath |S_\jmath| \le \tfrac{1}{2}B|Q|. \tag{4.2}$$

All together, we would find that $B \le C + (B/2)$, where $C < \infty$ depends only on the A_2 norm of w.

To prove the claim note first that for any index $\jmath$,

$$\begin{aligned} A^2 < \|(w^{-1})_Q^{-1/2}(w^{-1})_{S_\jmath}^{1/2}\|^2 &= \|(w^{-1})_Q^{-1/2}(w^{-1})_{S_\jmath}(w^{-1})_Q^{-1/2}\| \\ &\le \tfrac{1}{|S_\jmath|}\int_{S_\jmath} \|w_Q^{1/2}w^{-1}(y)w_Q^{1/2}\|\,dy = \tfrac{1}{|S_\jmath|}\int_{S_\jmath}\|w_Q^{1/2}w^{-1/2}\|^2. \end{aligned}$$

Therefore by the pairwise disjointness of the cubes $S_\jmath$ and the A_2 condition, summing the preceding inequality over $\jmath$ yields

$$A^2\sum_\jmath |S_\jmath| \le \int_Q \|w_Q^{1/2}w^{-1/2}(y)\|^2\,dy \le C|Q|, \tag{4.3}$$

justifying the claim. □

Denote by A_Q the averaging operators

$$A_Q f(x) = \chi_Q(x)\tfrac{1}{|Q|}\int_Q w^{1/2}(x)w^{-1/2}(y)f(y)\,dy.$$

Lemma 4.1. *Let $\mathcal{H}$ be any Hilbert space. Then the operators A_Q are bounded from $L^2(\mathbb{R}^n,\mathcal{H},dx)$ to itself uniformly for all cubes $Q \subset \mathbb{R}^n$, if and only if $w \in A_2$.*

Of course, boundedness of M'_w , or of M_w, directly implies uniform boundedness of $\{A_Q\}$.

For the proof, let $f \in L^2(\mathbb{R}^n,\mathcal{H})$ be supported in Q. Then

$$\begin{aligned} \tfrac{1}{|Q|}\|A_Q f\|_2^2 &= \tfrac{1}{|Q|}\int_Q \|\tfrac{1}{|Q|}\int_Q w^{1/2}(x)w^{-1/2}(y)f(y)\,dy\|^2\,dx \\ &= \|\tfrac{1}{|Q|}\int_Q w_Q^{1/2}w^{-1/2}(y)f(y)\,dy\|^2. \end{aligned}$$

By duality and Cauchy-Schwarz, the supremum of this last expression over all $f \in L^2$ having norm ≤ 1 equals

$$\begin{aligned}
&\sup_{\xi\in\mathcal{H},\ \|\xi\|\leq 1} \tfrac{1}{|Q|^2}\int_Q \|w^{-1/2}(y)w_Q^{1/2}\xi\|^2\,dy\\
&\quad= \tfrac{1}{|Q|}\sup_{\xi\in\mathcal{H},\ \|\xi\|\leq 1}\tfrac{1}{|Q|}\int_Q\langle\xi,\ w_Q^{1/2}w^{-1}(y)w_Q^{1/2}\xi\rangle\,dy\\
&\quad= \tfrac{1}{|Q|}\sup_{\xi\in\mathcal{H},\ \|\xi\|\leq 1}\|(w^{-1})_Q^{1/2}w_Q^{1/2}\xi\|^2\\
&\quad= \tfrac{1}{|Q|}\|w_Q^{1/2}(w^{-1})_Q^{1/2}\|^2.
\end{aligned}$$

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