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DERIVED EQUIVALENCE IN $SL_2(p^2)$

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ABSTRACT. We present a proof that Broué's Abelian Defect Group Conjecture is true for the principal p-block of the group $SL_2(p^2)$. Okuyama has independently obtained the same result using a different approach.

1. INTRODUCTION AND PRELIMINARIES

Broué has posed a remarkable conjecture involving the derived categories of blocks of finite groups with abelian defect groups [2]. In the case of principal blocks this conjecture is particularly easy to state.

Conjecture 1 (Broué)**.** Let K be an algebraically closed field of prime characteristic p, let G be a finite group with an abelian Sylow p-subgroup P, and let H be the normalizer of P in G. Then the principal blocks of KG and KH have equivalent derived categories.

For good introductions to this conjecture, we suggest [3] and [8].

The purpose of this paper is to show that Broué's conjecture is true for the group $G = SL_2(p^2)$. Okuyama has independently obtained this result using an extension of the method he developed in [7]. We should also mention that the cases $p = 2$ and $p = 3$ have already been handled in [13] and [7], respectively, and that the cases $p = 5$ and $p = 7$ have been settled independently by Holloway.

The proof given here is made possible by a new method for constructing derived equivalences due to Rickard (section 6 of [9]). We also rely heavily on calculations of cohomology in Carlson's paper [4].

We will be dealing with the following categories associated to a finite-dimensional K-algebra Λ: $\mathrm{mod}(\Lambda)$, the category of finitely generated Λ-modules; $\mathrm{stmod}(\Lambda)$, the stable category of finitely generated Λ-modules (in which the objects are the same as in $\mathrm{mod}(\Lambda)$ and the morphisms are Λ-homomorphisms modulo those which factor through projective modules); and $\mathrm{D^b}(\mathrm{mod}(\Lambda))$, the derived category of bounded complexes of finitely generated Λ-modules. Viewing a module as a complex concentrated in degree zero defines a fully faithful functor from $\mathrm{mod}(\Lambda)$ to $\mathrm{D^b}(\mathrm{mod}(\Lambda))$; we will often identify a Λ-module with its image in $\mathrm{D^b}(\mathrm{mod}(\Lambda))$.

If Λ is a symmetric algebra (e.g. a block of a finite group algebra), then $\mathrm{stmod}(\Lambda)$ is a triangulated category and may be identified with a quotient of $\mathrm{D^b}(\mathrm{mod}(\Lambda))$ in the following way [10]: the full subcategory $\mathcal{P}$ of $\mathrm{D^b}(\mathrm{mod}(\Lambda))$ consisting of objects isomorphic to bounded complexes of projective modules is a thick subcategory, and the composition of the embedding $\mathrm{mod}(\Lambda) \to \mathrm{D^b}(\mathrm{mod}(\Lambda))$ and the projection

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$\mathrm{D^b}(\mathrm{mod}(\Lambda)) \to \mathrm{D^b}(\mathrm{mod}(\Lambda))/\mathcal{P}$ factors through the functor $\mathrm{mod}(\Lambda) \to \mathrm{stmod}(\Lambda)$; the resulting functor $\mathrm{stmod}(\Lambda) \to \mathrm{D^b}(\mathrm{mod}(\Lambda))/\mathcal{P}$ is an equivalence of triangulated categories. We say that two objects of $\mathrm{D^b}(\mathrm{mod}(\Lambda))$ are *stably isomorphic* if they are isomorphic when viewed as objects of the quotient category $\mathrm{D^b}(\mathrm{mod}(\Lambda))/\mathcal{P}$. For example, two corners of a distinguished triangle in $\mathrm{D^b}(\mathrm{mod}(\Lambda))$ are stably isomorphic if the third corner of the triangle lies in $\mathcal{P}$.

If Λ and Γ are symmetric algebras we say they are *Morita equivalent* if their module categories are equivalent, *stably equivalent* if their stable categories are equivalent (as triangulated categories), and *derived equivalent* if their derived categories are equivalent (as triangulated categories). A stable equivalence $\mathrm{stmod}(\Lambda) \to \mathrm{stmod}(\Gamma)$ is *of Morita type* if it is induced by an exact functor between the corresponding module categories.

If we would like to prove that Λ and Γ are derived equivalent, one way to proceed is to find a tilting complex T for Λ (an object of $\mathcal{P}$ satisfying certain conditions) and show that Γ is isomorphic to the endomorphism ring of T; this is part of Rickard's Morita theory for derived categories [11]. This approach was used for example by Rickard to prove that Conjecture 1 holds for groups with cyclic Sylow p-subgroups [10]. But this may not work well in more complicated cases: even if we have constructed an appropriate tilting complex T, it may be very difficult to calculate its endomorphism ring. Furthermore, we may not even know the structure of Γ explicitly. Okuyama used a theorem of Linckelmann [6] to develop a way around these problems in certain situations where one already has a stable equivalence of Morita type between the algebras Λ and Γ; he was then able to verify Conjecture 1 in a number of cases [7]. Partly in order to exploit Okuyama's idea, Rickard extended his theory, proving the following theorem.

Theorem 2 (Rickard [9]). *Suppose Γ is a symmetric finite-dimensional K-algebra and let $X_1, \dots, X_r$ be objects of $\mathrm{D^b}(\mathrm{mod}(\Gamma))$ which generate $\mathrm{D^b}(\mathrm{mod}(\Gamma))$ as a triangulated category and such that, given $n \leq 0$, the space*

$$\mathrm{Hom}(X_i, X_j[n])$$

is zero unless $i = j$ and $n = 0$, in which case it is one-dimensional. Then there exist a K-algebra Γ' and an equivalence

$$\mathrm{D^b}(\mathrm{mod}(\Gamma)) \to \mathrm{D^b}(\mathrm{mod}(\Gamma'))$$

sending $X_1, \dots, X_r$ to the simple Γ'-modules.

The application, explained by Rickard in [9], which makes use of Okuyama's idea is stated here as a corollary.

Corollary (Rickard). *Suppose Λ and Γ are finite-dimensional K-algebras, with Γ symmetric, and suppose there is a functor $\mathcal{F} : \mathrm{mod}(\Lambda) \to \mathrm{mod}(\Gamma)$ inducing a stable equivalence. Let $S_1, \dots, S_r$ be the simple Λ-modules, and let $X_1, \dots, X_r$ be objects of $\mathrm{D^b}(\mathrm{mod}(\Gamma))$ satisfying the conditions of Theorem 2 and such that $\mathcal{F}(S_i)$ and X_i are stably isomorphic (for $i = 1, \dots, r$). Then Λ and Γ are derived equivalent.*

Proof. By Theorem 2, we have a K-algebra Γ' and an equivalence $\mathcal{G} \colon \mathrm{D^b}(\mathrm{mod}(\Gamma)) \to \mathrm{D^b}(\mathrm{mod}(\Gamma'))$ sending $X_1, \dots, X_r$ to the simple Γ'-modules. Composing the stable equivalence $\mathrm{stmod}(\Gamma) \to \mathrm{stmod}(\Gamma')$ induced by $\mathcal{G}$ (see Corollary 5.5 of [12]) with the equivalence $\mathrm{stmod}(\Lambda) \to \mathrm{stmod}(\Gamma)$ induced by $\mathcal{F}$ yields a stable equivalence of Morita type $\mathrm{stmod}(\Lambda) \to \mathrm{stmod}(\Gamma')$ sending the simple Λ-modules to the simple

Γ'-modules. Thus, by Theorem 2.1 of [6], Λ and Γ' are Morita equivalent, and it follows that Λ and Γ are derived equivalent. □

We now turn to the example under consideration. Let G be the group $SL_2(p^2)$, where we now assume that p is odd. We will follow closely the notation and methods of [4]. Let F be the field with p^2 elements and let α be a generator of the multiplicative group F^*. In G let P be the subgroup of upper unipotent matrices and let y be the diagonal matrix with diagonal entries α in the first row and α^{-1} in the second row. Then P is an elementary abelian p-group of order p^2 and is a Sylow p-subgroup of G. The normalizer of P in G is $H = \langle y \rangle P$.

Let σ be the Frobenius automorphism given by $\sigma(\beta) = \beta^p$ for all $\beta \in F$. The simple KG-modules are described as follows. Let V_1 be the standard two-dimensional (left) KG-module and let $V_2 = \sigma(V_1)$. For $i = 1, 2$ and $0 \le t \le p-1$, let $V_i^{(t)}$ be the t-th symmetric power of V_i. Let $b = (b_1, b_2)$ be a pair of integers with $0 \le b_i \le p-1$, and let

$$M_b = V_1^{(b_1)} \otimes V_2^{(b_2)}.$$

Each simple module is isomorphic to M_b for a unique b. For example, $M_{0,0}$ is the trivial module and $M_{p-1,p-1}$ is the Steinberg module, which is simple and projective. M_b lies in the principal block if and only if $b \in \mathcal{I}$, where

$$\mathcal{I} = \{b : \ 0 \le b_i \le p-1, \ b \ne (p-1, p-1), \text{ and } b_1 + b_2 \text{ is even}\}.$$

Note also that $\sigma(M_{b_1,b_2})$ is isomorphic to M_{b_2,b_1}.

For any integer j, let U_j be the one-dimensional KH-module on which y acts by multiplication by α^j. Note that U_j only depends on j modulo $p^2 - 1$. Each simple KH-module is isomorphic to U_j for some j, and U_j lies in the principal block if and only if j is even. The Frobenius automorphism σ preserves H, so it acts on KH-modules as well: $\sigma(U_j)$ is isomorphic to U_{pj}.

Regarded as a KH-module, M_b has composition factors

$$\{U_{m_1+pm_2} : \ -b_i \le m_i \le b_i \text{ and } m_i \equiv b_i \mod 2\},$$

counted with multiplicities, and it has a unique simple quotient, isomorphic to $U_{-b_1-pb_2}$. Note that $U_j \otimes M_b$ lies in the principal block of KH if and only if $j + b_1 + b_2$ is even. In particular, M_b lies in the principal block of KH for all $b \in \mathcal{I}$.

Let A and B be the principal blocks of KG and KH, respectively. Because the Sylow subgroup P is a trivial intersection subgroup of G, restriction from A to B induces a stable equivalence (see, e.g., Chapter 10 of [1]). Thus the following result, which will be proved in the course of this paper, together with the Corollary to Theorem 2 implies that Conjecture 1 is true for the group $G = SL_2(p^2)$.

Theorem 3. *There exist objects Z_b in $\mathrm{D^b}(\mathrm{mod}(B))$ indexed by $b \in \mathcal{I}$ such that*

(1) *for all $b \in \mathcal{I}$, we have that Z_b and M_b are stably isomorphic;*
(2) *for all $b, c \in \mathcal{I}$ and all $n \le 0$, the space*

$$\mathrm{Hom}(Z_b, Z_c[n])$$

is zero unless $b = c$ and $n = 0$, in which case it is one-dimensional;
(3) *the objects Z_b generate $\mathrm{D^b}(\mathrm{mod}(B))$ as a triangulated category.*

Corollary. *A and B are derived equivalent.*

Remark. For blocks of group algebras, there is an important strengthening of the notion of derived equivalence, which is now usually called splendid Rickard equivalence. This idea (at least for principal blocks) was introduced and argued for in [13], where it was conjectured that in the situation of Broué's conjecture, there should be not only a derived equivalence but a splendid Rickard equivalence.

Carlson and Rouquier have proved that in certain situations one can deduce the existence of a splendid Rickard equivalence from that of a derived equivalence. In this paper we are in such a situation: as $H/C_G(P)$ is a cyclic p'-group (of order $(p^2-1)/2$) acting freely on P, Corollary 4.4 of [5] applies, and we conclude that A and B are splendidly Rickard equivalent.

2. An example: $p = 5$

In this section we describe the objects Z_b in the case $p = 5$. We will not use this explicit description to give a proof of Theorem 3, but we hope that familiarity with the Z_b in this case will leave the reader better prepared to tackle the general case. The reader may also wish to make a comparison with the case $p = 3$ (a complete exposition is given in section 6 of [9]), but it is not complicated enough to illustrate some features of the construction given in the next section. For simplicity's sake, we will write $K, 2, 4, \dots, 22$ for the simple modules of B in place of $U_0, U_2, U_4, \dots, U_{22}$. The restrictions to B of the 12 simple modules of A are described by the following diagrams (in which the rows are Loewy layers).

$$M_{0,0} = K, \qquad M_{2,0} = \begin{array}{c} 22 \\ K \\ 2 \end{array}, \qquad M_{0,2} = \begin{array}{c} 14 \\ K \\ 10 \end{array}, \qquad M_{1,1} = \begin{array}{ccc} & 18 & \\ 20 & & 4 \\ & 6 & \end{array},$$

$$M_{4,0} = \begin{array}{c} 20 \\ 22 \\ K \\ 2 \\ 4 \end{array}, \qquad M_{0,4} = \begin{array}{c} 4 \\ 14 \\ K \\ 10 \\ 20 \end{array},$$

$$M_{4,2} = \begin{array}{ccccccc} & & & & 10 & & \\ & & & 12 & & 20 & \\ & & 14 & & 22 & & 6 \\ & 16 & & K & & 8 & \\ 18 & & 2 & & 10 & & \\ & 4 & & 12 & & & \\ & & 14 & & & & \end{array}, \qquad M_{2,4} = \begin{array}{ccccccc} & & 2 & & & & \\ & 4 & & 12 & & & \\ 6 & & 14 & & 22 & & \\ & 16 & & K & & 8 & \\ & & 2 & & 10 & & 18 \\ & & & 12 & & 20 & \\ & & & & 22 & & \end{array},$$

$$M_{2,2} = \begin{array}{ccccc} & & 12 & & \\ & 14 & & 22 & \\ 16 & & K & & 8 \\ & 2 & & 10 & \\ & & 12 & & \end{array}, \qquad M_{3,3} = \begin{array}{ccccccc} & & & 6 & & & \\ & & 8 & & 16 & & \\ & 10 & & 18 & & 2 & \\ 12 & & 20 & & 4 & & 12 \\ & 22 & & 6 & & 14 & \\ & & 8 & & 16 & & \\ & & & 18 & & & \end{array},$$

$$M_{3,1} = \begin{array}{ccccc} & & & 16 & \\ & & 18 & & 2 \\ & 20 & & 4 & \\ 22 & & 6 & & \\ & 8 & & & \end{array}, \qquad M_{1,3} = \begin{array}{ccccc} & 8 & & & \\ 10 & & 18 & & \\ & 20 & & 4 & \\ & & 6 & & 14 \\ & & & 16 & \end{array}.$$

To begin with, we set $Z_{0,0} = M_{0,0}$, $Z_{2,0} = M_{2,0}$, $Z_{0,2} = M_{0,2}$, $Z_{1,1} = M_{1,1}$, $Z_{4,0} = \Omega^2 M_{4,0}[2]$, $Z_{0,4} = \Omega^2 M_{0,4}[2]$, $Z_{4,2} = \Omega^3 M_{4,2}[3]$, $Z_{2,4} = \Omega^3 M_{2,4}[3]$, $Z_{2,2} = \Omega M_{2,2}[1]$, and $Z_{3,3} = \Omega M_{3,3}[1]$. It is easy to see that Z_b and M_b are stably isomorphic in each of these cases; for example, $\Omega^2 M_{4,0}[2]$ is quasi-isomorphic to the complex

$$\cdots \to 0 \to Q_1 \to Q_0 \to M_{4,0} \to 0 \to \cdots,$$

with $M_{4,0}$ in degree 0, obtained by truncating a minimal projective resolution of $M_{4,0}$ (as a B-module), and this complex is stably isomorphic to $M_{4,0}$. We have a good handle on these Z_b because we can explicitly describe the structure of the Heller translates involved, for example,

$$\Omega^2 M_{4,0} = \begin{array}{c} 6 \\ 8 \\ 10 \\ 12 \\ 14 \end{array}, \qquad \Omega M_{2,2} = \begin{array}{ccccccccc} & 18 & & & & & & 18 & \\ 20 & & 4 & & & & 10 & & 4 \\ & 6 & & 14 & & 22 & & 6 & \\ & & 16 & & K & & 8 & & \\ & & & 2 & & 10 & & & \\ & & & & 12 & & & & \end{array}.$$

The remaining two objects, $Z_{3,1}$ and $Z_{1,3}$, are more difficult to describe because each has nonzero homology in more than one degree (this does not happen in the case $p = 3$). Consider $\Omega M_{3,1}$; its structure is given by the diagram

$$\begin{array}{ccccccccc} & & & & & & 12 & & \\ & & & & & 14 & & 22 & \\ K & & & & 16 & & K & & 8 \\ & 10 & & 18 & & 2 & & 10 & \\ & & 20 & & 4 & & 12 & & \\ & & & 6 & & 14 & & & \\ & & & & 16 & & & & \end{array}.$$

Let $Q \to \Omega M_{3,1}$ be a projective cover of $\Omega M_{3,1}$. Then Q is isomorphic to the direct sum of a projective cover of K and a projective cover of 12. Define $Z_{3,1}$ to be the complex

$$\cdots \to 0 \to Q' \to \Omega M_{3,1} \to 0 \to \cdots,$$

with $\Omega M_{3,1}$ in degree -1, where $Q' \to \Omega M_{3,1}$ is the restriction of $Q \to \Omega M_{3,1}$ to a submodule isomorphic to a projective cover of 12. Clearly $Z_{3,1}$ is stably isomorphic to $\Omega M_{3,1}[1]$, and, reasoning as above, we see that $\Omega M_{3,1}[1]$ is in turn stably isomorphic to $M_{3,1}$.

To complete our description of the objects Z_b, we apply the Frobenius automorphism to $Z_{3,1}$ to get $Z_{1,3}$.

Let us now take a closer look at $Z_{3,1}$. Its degree -1 homology is isomorphic to $U_5 \otimes M_{0,1}$, a nonsplit extension of K by 10, and its degree -2 homology is isomorphic to $U_3 \otimes M_{4,1}$, the structure of which is described by the diagram

$$\begin{array}{cccccc} & & & & 18 & \\ & & & 20 & & 4 \\ & & 22 & & 6 & \\ & K & & 8 & & \\ 2 & & 10 & & & \\ & 12 & & & & \end{array}$$

We therefore have an exact sequence

$$0 \to U_3 \otimes M_{4,1}[2] \to Z_{3,1} \to U_5 \otimes M_{0,1}[1] \to 0$$

of complexes of B-modules. Consequently there exists a distinguished triangle

$$U_5 \otimes M_{0,1} \to U_3 \otimes M_{4,1}[2] \to Z_{3,1} \to$$

in $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}(B))$. This description of $Z_{3,1}$ as one corner of a distinguished triangle in which the other two corners are just shifts of modules will be important in understanding the general construction which will be presented in the next section.

3. Construction of complexes

We now turn to the construction of the objects Z_b in the general case (p any odd prime). We begin with a few technical lemmas.

Lemma 4. *Suppose i is 1 or 2, and suppose l and t are integers with $0 \leq t < p-1$. Let $j = l + p^{i-1}(p+1+t)$ and $j' = l + p^{i-1}(p-1-t)$. Then any non-zero KH-homomorphism from $U_j \otimes V_i^{(p-1)}$ to $U_{j'} \otimes V_i^{(p-1)}$ has cokernel isomorphic to $U_l \otimes V_i^{(t)}$.*

Proof. By Lemma 2.1 of [4], as a KH-module $V_i^{(p-1)}$ is uniserial with pairwise non-isomorphic composition factors. The same is true of the tensor product of any simple KH-module with $V_i^{(p-1)}$. Thus the space of KH-homomorphisms between any two such modules is at most one-dimensional. By Lemma 2.2 of [4], $U_l \otimes V_i^{(t)}$ is isomorphic to the cokernel of some non-zero homomorphism from $U_j \otimes V_i^{(p-1)}$ to $U_{j'} \otimes V_i^{(p-1)}$, and hence is isomorphic to the cokernel of any such homomorphism. □

Lemma 5. *Suppose t is a non-negative integer less than $p-1$ and suppose h is a positive integer. If h is even, then there is a distinguished triangle*

$$Y \longrightarrow M_{p-1,t} \longrightarrow U_h \otimes M_{p-1,t}[h] \longrightarrow$$

in $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}(KH))$, and if h is odd, then there is a distinguished triangle

$$Y \longrightarrow M_{p-1,t} \longrightarrow U_h \otimes M_{p-1,p-2-t}[h] \longrightarrow$$

in $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}(KH))$, where in both cases Y is a bounded complex of projective modules and $Y^{-i} = 0$ for $i \geq h$.

Proof. By Lemma 2.2 of [4] , there exists an exact sequence

$$\ldots \longrightarrow X^{-2} \xrightarrow{\delta^{-2}} X^{-1} \xrightarrow{\delta^{-1}} X^0 \xrightarrow{\epsilon} V_2^{(t)},$$

where for $s \geq 0$,

$$X^{-2s} = U_{p(p-1-t+2ps)} \otimes V_2^{(p-1)} \quad \text{and} \quad X^{-(2s+1)} = U_{p(p+1+t+2ps)} \otimes V_2^{(p-1)}.$$

Let C be the kernel of $\delta^{-(h-1)}$. Then

$$\begin{array}{ccccccccccccccc}
 & & \cdots & \longrightarrow & 0 & \longrightarrow & X^{-(h-1)} & \longrightarrow & \cdots & \longrightarrow & X^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & & \downarrow & & \| & & & & \| & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & C & \longrightarrow & X^{-(h-1)} & \longrightarrow & \cdots & \longrightarrow & X^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \| & & \downarrow & & & & & & & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & C & \longrightarrow & 0 & \longrightarrow & \cdots & & & & & &
\end{array}$$

is a short exact sequence of bounded complexes of KH-modules, and the middle term is quasi-isomorphic to $V_2^{(t)}$. After tensoring this sequence with the module $V_1^{(p-1)}$, the middle term is quasi-isomorphic to $M_{p-1,t}$, so in the derived category we have a distinguished triangle

$$Y \longrightarrow M_{p-1,t} \longrightarrow V_1^{(p-1)} \otimes C[h] \longrightarrow,$$

where Y is a bounded complex of projective modules (because $V_1^{(p-1)} \otimes V_2^{(p-1)}$ is projective) and $Y^{-i} = 0$ for $i \geq h$.

C is isomorphic to the cokernel of $\delta^{-(h+1)} : X^{-(h+1)} \longrightarrow X^{-h}$. If h is even, then

$$X^{-(h+1)} = U_j \otimes V_2^{(p-1)} \quad \text{and} \quad X^{-h} = U_{j'} \otimes V_2^{(p-1)},$$

where $j = p(p+1+t+ph)$ and $j' = p(p-1-t+ph)$. Writing $j = p^2h + p(p+1+t)$ and $j' = p^2h + p(p-1-t)$, and noting that $U_{p^2h} = U_h$, we see by Lemma 4 that C is isomorphic to $U_h \otimes V_2^{(t)}$. Thus the last term in the triangle above is isomorphic to $U_h \otimes M_{p-1,t}[h]$.

If on the other hand h is odd, then $X^{-(h+1)}$ and X^{-h} are as above, but with $j = p(p-1-t+p(h+1))$ and $j' = p(p+1+t+p(h-1))$. We may rewrite $j = p^2h + p(p+1+(p-2-t))$ and $j' = p^2h + p(p-1-(p-2-t))$, so by Lemma 4, C is isomorphic to $U_h \otimes V_2^{(p-2-t)}$. Thus the last term in the triangle above is isomorphic to $U_h \otimes M_{p-1,p-2-t}[h]$. □

Lemma 6. *Suppose b_1 and b_2 are non-negative integers less than $p-1$. Then there is an exact sequence of KH-modules*

$$0 \longrightarrow U_p \otimes M_{p-2-b_1,b_2} \longrightarrow U_{p-1-b_1} \otimes M_{p-1,b_2} \longrightarrow M_b \longrightarrow 0.$$

Proof. By Lemma 2.1 of [4], there is an exact sequence

$$U_j \otimes V_1^{(p-1)} \longrightarrow U_{j'} \otimes V_1^{(p-1)} \longrightarrow U_{p-1-b_1} \otimes V_1^{(p-1)} \longrightarrow V_1^{(b_1)} \longrightarrow 0,$$

where $j = p-1-b_1+2p$ and $j' = p+1+b_1$. Since $j = p+(p+1+(p-2-b_1))$ and $j' = p+(p-1-(p-2-b_1))$, the cokernel of the first homomorphism in the sequence is isomorphic to $U_p \otimes V_1^{(p-2-b_1)}$, by Lemma 4. Thus there is an exact sequence

$$0 \longrightarrow U_p \otimes V_1^{(p-2-b_1)} \longrightarrow U_{p-1-b_1} \otimes V_1^{(p-1)} \longrightarrow V_1^{(b_1)} \longrightarrow 0.$$

Tensoring this with $V_2^{(b_2)}$ gives the desired exact sequence. □

Now we construct the complexes Z_b and show that the first assertion of Theorem 3 holds. We begin by defining some subsets of the index set $\mathcal{I}$. Let

$$\begin{aligned} \mathcal{I}_< &= \{b \in \mathcal{I} : \ b_1 + b_2 < p-1\}, \\ \mathcal{I}_{p-1} &= \{b \in \mathcal{I} : \ b_1 = p-1 \text{ or } b_2 = p-1\}, \\ \mathcal{I}_\geq &= \{b \in \mathcal{I} : \ b_1 + b_2 \geq p-1, \ b_1 \neq p-1, \text{ and } b_2 \neq p-1\}. \end{aligned}$$

$\mathcal{I}$ is the disjoint union of $\mathcal{I}_<$, $\mathcal{I}_{p-1}$, and $\mathcal{I}_\geq$. Given $b \in \mathcal{I}$, we define Z_b as follows:

- $b \in \mathcal{I}_<$: In this case simply define $Z_b = M_b$. The condition that Z_b and M_b are stably isomorphic is trivially satisfied.
- $b \in \mathcal{I}_{p-1}$ and $b_1 = p-1$: If $b_2 < (p-1)/2$, define

  $$Z_{p-1,b_2} = U_{b_2+2} \otimes M_{p-1,b_2}[b_2+2];$$

  and if $b_2 \geq (p-1)/2$, define

  $$Z_{p-1,b_2} = U_{p-b_2} \otimes M_{p-1,p-2-b_2}[p-b_2].$$

  Because $b \in \mathcal{I}$, we have that b_2 is even, and thus by lemma 5, Z_{p-1,b_2} and M_{p-1,b_2} are stably isomorphic. We remark that as Z_{p-1,b_2} is concentrated in a single degree, it is nothing but a shift of a certain Heller translate of M_{p-1,b_2}; for example, if $b_2 < (p-1)/2$, then

  $$Z_{p-1,b_2} = \Omega^{b_2+2} M_{p-1,b_2}[b_2+2].$$

  Note that the set of complexes constructed here may be written collectively as

  $$\{U_{s+2} \otimes M_{p-1,s}[s+2] : \ 0 \leq s \leq (p-3)/2\}.$$

- $b \in \mathcal{I}_{p-1}$ and $b_2 = p-1$: Define $Z_{b_1,p-1} = \sigma(Z_{p-1,b_1})$, using the previous case. It is clear that $Z_{b_1,p-1}$ and $M_{b_1,p-1}$ are stably isomorphic. The set of complexes constructed here may be written collectively as

  $$\{U_{p(s+2)} \otimes M_{s,p-1}[s+2] : \ 0 \leq s \leq (p-3)/2\}.$$

- $b \in \mathcal{I}_\geq$ and $b_2 \leq b_1$: By lemma 6, there is a distinguished triangle

  $$U_p \otimes M_{p-2-b_1,b_2} \xrightarrow{f} U_{p-1-b_1} \otimes M_{p-1,b_2} \longrightarrow M_b \longrightarrow$$

  in $\mathrm{D^b}(\mathrm{mod}(B))$.

  Let $w = \min\{b_2, p-2-b_2\}$. Applying lemma 5 with $t = b_2$ and $h = w+b_1-p+3$ (noting that h is even when $w = b_2$ and odd when $w = p-2-b_2$) and then tensoring with U_{p-1-b_1}, we obtain a distinguished triangle

  $$Y \longrightarrow U_{p-1-b_1} \otimes M_{p-1,b_2} \xrightarrow{g} U_{w+2} \otimes M_{p-1,w}[h] \longrightarrow,$$

  where Y is a bounded complex of projective modules and $Y^{-i} = 0$ for $i \geq h$.

  Define Z_b to be the third object in a distinguished triangle which contains the composite of f and g:

  $$U_p \otimes M_{p-2-b_1,b_2} \xrightarrow{g \circ f} U_{w+2} \otimes M_{p-1,w}[h] \longrightarrow Z_b \longrightarrow .$$

Then by the octahedral axiom there is a commutative diagram

$$\begin{array}{ccccccc}
U_p \otimes M_{p-2-b_1,b_2} & \longrightarrow & U_{p-1-b_1} \otimes M_{p-1,b_2} & \longrightarrow & M_b & \longrightarrow & \\
\Vert & & \downarrow & & \downarrow & & \\
U_p \otimes M_{p-2-b_1,b_2} & \longrightarrow & U_{w+2} \otimes M_{p-1,w}[h] & \longrightarrow & Z_b & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \\
 & & Y[1] & = & Y[1] & & \\
 & & \downarrow & & \downarrow & &
\end{array}$$

where all the rows and columns are distinguished triangles. Since $Y[1]$ is a bounded complex of projective modules, the last column shows that Z_b and M_b are stably isomorphic.

- $b \in \mathcal{I}_{\geq}$ and $b_1 < b_2$: Define $Z_{b_1,b_2} = \sigma(Z_{b_2,b_1})$. Again it is clear that Z_b and M_b are stably isomorphic.

We wish to record for future reference some details in the next-to-last construction above.

Lemma 7. *Suppose $b \in \mathcal{I}_{\geq}$ and $b_2 \leq b_1$. Then there exist distinguished triangles*

$$U_p \otimes M_{p-2-b_1,b_2} \longrightarrow U_{w+2} \otimes M_{p-1,w}[h] \longrightarrow Z_b \longrightarrow$$

and

$$Y \longrightarrow M_b \longrightarrow Z_b \longrightarrow,$$

where $w = \min\{b_2, p-2-b_2\}$, $h = w + b_1 - p + 3$, and Y is a bounded complex of projective modules such that $Y^{-i} = 0$ for $i \geq h$.

Remark. It is actually true that there is a derived equivalence of A and B which respects the Frobenius actions on $\mathrm{D^b}(\mathrm{mod}(A))$ and $\mathrm{D^b}(\mathrm{mod}(B))$. To get this as a corollary of Theorem 3, we need an additional condition on the objects Z_b: that $\sigma(Z_{b_1,b_2})$ is isomorphic to Z_{b_2,b_1} for all $b = (b_1, b_2) \in \mathcal{I}$. This is clear in our construction of the Z_b's except when $b \in \mathcal{I}_{\geq}$ and $b_1 = b_2$. In this case, $h = 1$ in the first distinguished triangle in Lemma 7, and then from the associated long exact sequence of homology groups we see that Z_b has homology concentrated in degree -1. As Z_b and M_b are stably isomorphic, we may therefore take Z_b to be $\Omega(M_b)[1]$, which is stable under the Frobenius action.

4. Homomorphisms

We prepare for a proof of the second assertion of Theorem 3 by calculating some cohomology groups of B-modules. We will need a version of Theorem 2.6 of [4].

Theorem 8 (Carlson)**.** *Suppose $b, c \in \mathcal{I}$, and suppose j, j', and r are integers with r non-negative. Then the dimension of $\mathrm{Ext}^r_{KH}(U_j \otimes M_b, U_{j'} \otimes M_c)$ is equal to the number of triples (e, f, k) (where $e = (e_1, e_2)$, $f = (f_1, f_2)$, and $k = (k_1, k_2)$ are pairs of non-negative integers) satisfying the following conditions:*

(1) $2e_1 + 2e_2 + f_1 + f_2 = r$;
(2) *for every i, f_i is either 0 or 1;*
(3) $e_i = f_i = 0$ *whenever b_i or c_i is $p-1$;*

(4) *if* $f_i = 0$*, then*

$$\max\{0, c_i - b_i\} \le k_i \le \begin{cases} c_i & \text{if} \quad e_i = 0, \\ \min\{c_i, p-2-b_i\} & \text{if} \quad e_i > 0, \end{cases}$$

while if $f_i = 1$ *then*

$$\max\{0, b_i + c_i + 2 - p\} \le k_i \le \min\{b_i, c_i\}.$$

(5) $a(e,f,k) \equiv 0 \mod (p^2-1)$ *where*

$$a(e,f,k) = j' - j - 2(pe_1 + e_2) + \sum_{i=1}^{2} p^{i-1}(b_i - c_i + 2k_i - f_i(2b_i + 2)).$$

Proof. The proof given by Carlson can be adapted with little alteration. The restrictions of U_j and U_j' to P are trivial, so $\mathrm{Ext}^r_{KP}(U_j \otimes M_b, U_{j'} \otimes M_c)$ has K-basis $\{\theta(e,f,k)\}$ indexed on triples (e,f,k) which satisfy conditions (1)-(4). A spectral sequence argument identifies $\mathrm{Ext}^r_{KH}(U_j \otimes M_b, U_{j'} \otimes M_c)$ with the $K(H/P)$-fixed points of $\mathrm{Ext}^r_{KP}(U_j \otimes M_b, U_{j'} \otimes M_c)$. Because y acts on $\mathrm{Ext}^r_{KP}(U_j \otimes M_b, U_{j'} \otimes M_c)$ by scaling the basis vector $\theta(e,f,k)$ by $\alpha^{a(e,f,k)}$, a basis for $\mathrm{Ext}^r_{KH}(U_j \otimes M_b, U_{j'} \otimes M_c)$ is indexed by triples (e,f,k) which in addition satisfy condition (5). □

We will only provide proofs for a few parts of the following lemma, to give an idea of the type of arguments involved; similar reasoning can be used to prove the others.

Lemma 9. *Suppose* $b, c \in \mathcal{I}_<$ *and* $b', c' \in \mathcal{I}_\ge$ *with* $b'_2 \le b'_1$ *and* $c'_2 \le c'_1$*; and suppose* s *and* t *are non-negative integers less than or equal to* $(p-3)/2$*. Let* $w = \min\{b'_2, p-2-b'_2\}$ *and* $h = w + b'_1 - p + 3$*. Then:*

(1) $\mathrm{Hom}_B(M_b, M_c) = 0$ *if* $b \ne c$ *and is one-dimensional if* $b = c$.
(2) $\mathrm{Ext}^r_B(M_b, U_p \otimes M_{p-2-c'_1,c'_2}) = 0$ *for* $r = 0$ *or* $r = 1$.
(3) $\mathrm{Ext}^r_B(U_{s+2} \otimes M_{p-1,s}, U_{t+2} \otimes M_{p-1,t}) = 0$ *whenever* $r \le t - s$ *unless* $s = t$ *and* $r = 0$*, in which case it is one-dimensional.*
(4) $\mathrm{Ext}^r_B(U_{p(s+2)} \otimes M_{s,p-1}, U_{t+2} \otimes M_{p-1,t}) = 0$ *whenever* $r \le t - s$.
(5) $\mathrm{Ext}^r_B(M_b, U_{s+2} \otimes M_{p-1,s}) = 0$ *whenever* $r \le s + 2$.
(6) $\mathrm{Ext}^r_B(M_{b'}, U_{s+2} \otimes M_{p-1,s}) = 0$ *whenever* $h < r \le s + 2$.
(7) $\mathrm{Ext}^r_B(M_{b'}, U_{p(s+2)} \otimes M_{s,p-1}) = 0$ *whenever* $h < r \le s + 2$.
(8) $\mathrm{Ext}^r_B(U_p \otimes M_{p-2-b'_1,b'_2}, U_{s+2} \otimes M_{p-1,s}) = 0$ *whenever* $r \le \min\{h, s+2\}$ *unless* $s = w$ *and* $r = h$ *in which case it is one-dimensional.*
(9) $\mathrm{Ext}^r_B(U_p \otimes M_{p-2-b'_1,b'_2}, U_{p(s+2)} \otimes M_{s,p-1}) = 0$ *whenever* $r \le \min\{h, s+2\}$
(10) $\mathrm{Hom}_B(U_{s+2} \otimes M_{p-1,s}, U_p \otimes M_{p-2-c'_1,c'_2}) = 0$.
(11) $\mathrm{Hom}_B(U_1 \otimes M_{b'_1,p-2-b'_2}, U_p \otimes M_{p-2-c'_1,c'_2}) = 0$.
(12) $\mathrm{Hom}_B(U_p \otimes M_{p-2-b'_1,b'_2}, U_p \otimes M_{p-2-c'_1,c'_2}) = 0$ *if* $b' \ne c'$*, and is one-dimensional if* $b' = c'$.
(13) $\mathrm{Hom}_B(U_{p(s+2)} \otimes M_{s,p-1}, U_p \otimes M_{p-2-c'_1,c'_2}) = 0$.

Proof. (1) Suppose that (e,f,k) is a triple satisfying the conditions in Theorem 8 with $r = 0$ and $j = j' = 0$. Condition (1) implies that $e_1 = e_2 = f_1 = f_2 = 0$. Let $\gamma_i = b_i - c_i + 2k_i$. Then by condition (4) we have $0 = b_i - c_i + (0 + c_i - b_i) \le \gamma_i \le b_i - c_i + 2c_i = b_i + c_i$, where the first inequality is an equality if and only if $b_i = c_i$ and $k_i = 0$. We have $\gamma_1 + \gamma_2 \le b_1 + c_1 + b_2 + c_2 \le 2(p-2)$, so either $\gamma_1 \le p - 2$ or $\gamma_2 \le p - 2$. Suppose the latter. In order for condition (5) to be satisfied we need $\gamma_1 + p\gamma_2 \equiv 0 \mod (p^2-1)$. But $0 \le \gamma_1 + p\gamma_2 \le$

$2(p-2)+p(p-2) < p^2-1$, so we must have $\gamma_1 + p\gamma_2 = 0$, which implies that $b_1 = c_1$, $b_2 = c_2$, and $k_1 = k_2 = 0$.

If $\gamma_1 \leq p-2$, we note that $\gamma_2 + p\gamma_1 \equiv p(\gamma_1 + p\gamma_2) \mod (p^2-1)$ and that $p(\gamma_1 + p\gamma_2) \equiv 0 \mod (p^2-1)$ if and only if $\gamma_1 + p\gamma_2 \equiv 0 \mod (p^2-1)$. Thus we may use the argument above, interchanging the roles of γ_1 and γ_2.

(3) Suppose that (e,f,k) is a triple satisfying the conditions in Theorem 8 with $j = s+2$, $b = (p-1,s)$, $j' = t+2$, and $c = (p-1,t)$. Because $b_1 = p-1$, we have $e_1 = f_1 = 0$, by condition (3). We divide the argument into two cases according to the parity of r.

Suppose that r is even. Then by condition (1), we have $r = 2e_2 + f_2$, which implies that f_2 is even and therefore zero, by condition (2). Condition (4) tells us that $0 \leq k_1 \leq p-1$ and $t-s \leq k_2 \leq t$, and that $t-s = k_2$ if and only if $s = t$ and $k_2 = 0$. We have $a = a(e,f,k) = (t-s-r)+2k_1+p(s-t+2k_2) \equiv 0 \mod (p^2-1)$ by condition (5). Now $a \geq 0 + 2\cdot 0 + p(s-t+2(t-s)) \geq 0$, with equalities throughout if and only if $s = t$ and $r = e_2 = k_1 = k_2 = 0$. This is in fact the only possibility, because $a \leq t + 2(p-1) + p(s+t) \leq (p-3) + 2(p-1) + p(p-3) < p^2-1$.

Suppose on the other hand that r is odd. Then $r = 2e_2 + f_2$ implies that f_2 is odd and therefore that $f_2 = 1$. By condition (4), we have $0 \leq k_1 \leq p-1$ and $0 \leq k_2 \leq s$. Let $\gamma = (t-s-r) + 2k_1 + p(p-2-s-t+2k_2)$. Then $\gamma \equiv a(e,f,k) \mod (p^2-1)$, so by condition (5) we have $\gamma \equiv 0 \mod (p^2-1)$. Now $\gamma \geq 0 + 2\cdot 0 + p(p-2-(p-3)+2\cdot 0) = p > 0$, and, because r is odd, $s-t \leq -1$, so $\gamma \leq t+2(p-1)+p(p-2+s-t) \leq (p-3)+2(p-1)+p(p-3) < p^2-1$; we have arrived at a contradiction.

(5) Suppose that (e,f,k) is a triple satisfying the conditions in Theorem 8 with $j = 0$, $j = s+2$, and $c = (p-1,s)$. Because $c_1 = p-1$, we have $e_1 = f_1 = 0$, by condition (3). We divide the argument into two cases according to the parity of r.

Suppose r is even. Then $r = 2e_2$ and $f_2 = 0$ by conditions (1) and (2). Note that $h \geq 0$. Thus $r > 0$ and $e_2 > 0$ as well. By condition (4) we have $p-1-b_1 \leq k_1 \leq p-1$ and $\max\{0, s-b_2\} \leq k_2 \leq \min\{s, p-2-b_2\}$, and by condition (5) we have $a = a(e,f,k) = (s+2-r) + b_1 - (p-1) + 2k_1 + p(b_2 - s + 2k_2) \equiv 0 \mod (p^2-1)$. First note that

$$\begin{aligned} a &\geq 0 + b_1 - (p-1) + 2(p-1-b_1) \\ &\quad + p(b_2 - s + 0 + (s-b_2)) = (p-1) - b_1 > 0, \end{aligned}$$

so we must have $a \geq p^2-1$. Now suppose that either $b_2 + s \leq p-3$ or $b_2 - s + 2(p-2-b_2) \leq p-3$. Then $b_2 - s + 2k_2 \leq p-3$, so $a \leq (p-3)+2+(p-2)-(p-1)+2(p-1)+p(p-3) < p^2-1$, giving a contradiction. Hence we must have $b_2 + s \geq p-2$ and $b_2 - s + 2(p-2-b_2) \geq p-2$. These two inequalities together imply that $b_2 + s = p-2$. We then also have $b_1 \leq p-2-b_2 = s$. Thus $a \leq s+2+s-(p-1)+2(p-1)+p(b_2+s) = 2s+1-p+p^2 \leq (p-3)+1-p+p^2 < p^2-1$, which is again a contradiction.

Suppose instead that r is odd. Then $r = 2e_2+1$ and $f_2 = 1$ by conditions (1) and (2). By condition (4) we have $p-1-b_1 \leq k_1 \leq p-1$ and

$$\max\{0, b_2+s+2-p\} \leq k_2 \leq \min\{b_2, s\}.$$

Let $\gamma = (s+2-r) + b_1 - (p-1) + 2k_1 + p(-b_2 - s - 2 + p + 2k_2)$. Then $\gamma \equiv a(e,f,k) \mod (p^2-1)$, so by condition (5) we have $\gamma \equiv 0 \mod (p^2-1)$.

Now $\gamma \geq 0 + (p-1) - b_1 + p \cdot 0 > 0$, so we must have $\gamma \geq p^2 - 1$. If $b_2 = s$, then $b_1 + s = b_1 + b_2 \leq p-2$, so $\gamma \leq b_1 + s + 2 - r + (p-1) + p(p-2) \leq (p-2) + 2 - 1 + (p-1) + p(p-2) < p^2 - 1$, which gives a contradiction. So we may assume that $b_2 \neq s$. Then $2k_2 \leq b_2 + s - 1$ and $\gamma \leq (p-3) + 2 + (p-2) + (p-1) + p(p-3) < p^2 - 1$, which is also a contradiction. □

We now prove the second assertion of Theorem 3. Given $b, c \in \mathcal{I}$ and $n \leq 0$, we aim to show that the space $\mathrm{Hom}(Z_b, Z_c[n])$ is zero unless $b = c$ and $n = 0$, in which case it is one-dimensional. Recall that if M and N are B-modules then

$$\mathrm{Hom}_{\mathrm{D^b}(\mathrm{mod}(B))}(M[i], N[j]) = \begin{cases} 0 & \text{if } j < i, \\ \mathrm{Ext}_B^{j-i}(M, N) & \text{if } j \geq i. \end{cases}$$

If $b \in \mathcal{I}_<$ and $c \in \mathcal{I}_<$, then Z_b is concentrated in degree zero and $Z_c[n]$ is concentrated in degree $-n$. Hence if $n < 0$ then $\mathrm{Hom}(Z_b, Z_c[n]) = 0$, and if $n = 0$ we may appeal to Lemma 9.1.

If $b \in \mathcal{I}_{p-1}$ and $c \in \mathcal{I}_<$, then Z_b is concentrated in some negative degree, while $Z_c[n]$ is concentrated in degree $-n \geq 0$.

Suppose $b \in \mathcal{I}_\geq$ and $c \in \mathcal{I}_<$. We may assume that $b_2 \leq b_1$, for if not we can apply the Frobenius automorphism σ. By Lemma 7, there is a distinguished triangle

$$U_p \otimes M_{p-2-b_1, b_2} \longrightarrow U_{w+2} \otimes M_{p-1,w}[h] \longrightarrow Z_b \longrightarrow,$$

where $h > 0$. Applying the functor $\mathrm{Hom}(-, M_c)$ to this triangle gives rise to a long exact sequence, a segment of which is

$$\begin{aligned} \mathrm{Hom}(U_p \otimes M_{p-2-b_1,b_2}, M_c[n-1]) &\longrightarrow \mathrm{Hom}(Z_b, M_c[n]) \\ &\longrightarrow \mathrm{Hom}(U_{w+2} \otimes M_{p-1,w}[h], M_c[n]). \end{aligned}$$

Because $n - 1 < 0$ and $n < h$, the first and third terms of this segment are zero; thus the second term is zero as well.

If $b \in \mathcal{I}_<$ and $c \in \mathcal{I}_{p-1}$, then, applying σ if necessary, we may assume that $c_1 = p - 1$ and then use Lemma 9.5.

Suppose $b \in \mathcal{I}_{p-1}$ and $c \in \mathcal{I}_{p-1}$. We may assume that $b_1 = p-1$. If $c_1 = p-1$ we use Lemma 9.3, and if $c_2 = p - 1$ we use Lemma 9.4.

Suppose $b \in \mathcal{I}_\geq$ and $c \in \mathcal{I}_{p-1}$. Applying σ if necessary, we may assume that $b_2 \leq b_1$. Suppose in addition that $c_1 = p-1$, so that $Z_c = U_{s+2} \otimes M_{p-1,s}[s+2]$, for some $0 \leq s \leq (p-3)/2$. Let $w = \min\{b_2, p-2-b_2\}$ and $h = w + b_1 - p + 3$. Note that $0 \leq w \leq (p-3)/2$. We divide the argument into three cases:

- $s + 2 + n < h$: By Lemma 7 we have a distinguished triangle
$$U_p \otimes M_{p-2-b_1,b_2} \longrightarrow U_{w+2} \otimes M_{p-1,w}[h] \longrightarrow Z_b \longrightarrow .$$
Applying the functor $\mathrm{Hom}(-, U_{s+2} \otimes M_{p-1,s}[s+2])$ to this triangle gives rise to a long exact sequence, a segment of which is
$$\begin{aligned} &\mathrm{Hom}(U_p \otimes M_{p-2-b_1,b_2}, U_{s+2} \otimes M_{p-1,s}[s+2+n-1]) \\ &\quad\longrightarrow \mathrm{Hom}(Z_b, U_{s+2} \otimes M_{p-1,s}[s+2+n]) \\ &\quad\longrightarrow \mathrm{Hom}(U_{w+2} \otimes M_{p-1,w}[h], U_{s+2} \otimes M_{p-1,s}[s+2+n]). \end{aligned}$$
The first term of this segment is zero by Lemma 9.8 because $s + 2 + n - 1 \leq \min\{h, s+2\}$, and the third term is zero since $s + 2 + n < h$; thus the second term is zero as well.

- $s+2+n=h$: Arguing as in the previous case, we get an exact sequence

$$\begin{aligned}&\mathrm{Hom}(U_p\otimes M_{p-2-b_1,b_2},U_{s+2}\otimes M_{p-1,s}[h-1])\\&\quad\longrightarrow \mathrm{Hom}(Z_b,U_{s+2}\otimes M_{p-1,s}[h])\\&\quad\longrightarrow \mathrm{Hom}(U_{w+2}\otimes M_{p-1,w}[h],U_{s+2}\otimes M_{p-1,s}[h])\\&\quad\longrightarrow \mathrm{Hom}(U_p\otimes M_{p-2-b_1,b_2},U_{s+2}\otimes M_{p-1,s}[h])\\&\quad\longrightarrow \mathrm{Hom}(Z_b,U_{s+2}\otimes M_{p-1,s}[h+1]).\end{aligned}$$

  By Lemma 9.8, the first term is zero and the fourth term is zero unless $s=w$, in which case it is one-dimensional. By Lemma 9.3, the third term is zero unless $s=w$, in which case it is one-dimensional. The fifth term is zero by the previous case. We conclude that the second term is zero, as desired.
- $s+2+n>h$: By Lemma 7 we have a distinguished triangle

$$Y\longrightarrow M_b\longrightarrow Z_b\longrightarrow,$$

  where Y is a bounded complex of projective modules such that $Y^{-i}=0$ for $i\geq h$. Applying the functor $\mathrm{Hom}(-,U_{s+2}\otimes M_{p-1,s}[s+2])$ to this triangle gives rise to a long exact sequence, a segment of which is

$$\begin{aligned}&\mathrm{Hom}(Y,U_{s+2}\otimes M_{p-1,s}[s+2+n-1])\\&\quad\longrightarrow \mathrm{Hom}(Z_b,U_{s+2}\otimes M_{p-1,s}[s+2+n])\\&\quad\longrightarrow \mathrm{Hom}(M_b,U_{s+2}\otimes M_{p-1,s}[s+2+n]).\end{aligned}$$

  The first term is zero because Y is a bounded complex of projective modules, $U_{s+2}\otimes M_{p-1,s}[s+2+n-1]$ is concentrated in degree $-(s+2+n-1)$, and $Y^{-(s+2+n-1)}=0$. Since $h<s+2+n\leq s+2$, the third term is zero as well, by Lemma 9.6. Thus we conclude that the second term is zero.

If instead $c_2=p-1$, then an analogous argument works, using parts 9, 4, and 7 of Lemma 9 in place of 8, 3, and 6.

We are now left with the case $c\in\mathcal{I}_{\geq}$. As before, we may assume that $c_2\leq c_1$. By Lemma 7, we have a distinguished triangle

$$U_p\otimes M_{p-2-c_1,c_2}\longrightarrow U_{w+2}\otimes M_{p-1,w}[h]\longrightarrow Z_c\longrightarrow,$$

where $w=\min\{c_2,p-2-c_2\}$ and $h=w+c_1-p+3$. Note that $0\leq w\leq(p-3)/2$ and $0<h\leq w+1$.

Suppose first that $b\in\mathcal{I}_{<}$. Applying the functor $\mathrm{Hom}(M_b,-)$ to the triangle above gives rise to a long exact sequence, a segment of which is

$$\begin{aligned}\mathrm{Hom}(M_b,U_{w+2}\otimes M_{p-1,w}[h+n])&\longrightarrow \mathrm{Hom}(M_b,Z_c[n])\\&\longrightarrow \mathrm{Hom}(M_b,U_p\otimes M_{p-2-c_1,c_2}[n+1]).\end{aligned}$$

The first term is zero by Lemma 9.5 and the third term is zero by Lemma 9.2; hence the second term is zero, as desired.

Now suppose that $b\in\mathcal{I}_{p-1}$. Suppose further that $b_1=p-1$, so we have that $Z_b=U_{s+2}\otimes M_{p-1,s}[s+2]$ for some $0\leq s\leq(p-3)/2$. Applying the functor $\mathrm{Hom}(U_{s+2}\otimes M_{p-1,s},-)$ to the triangle above gives rise to a long exact sequence, a segment of which is

$$\begin{aligned}&\operatorname{Hom}(U_{s+2}\otimes M_{p-1,s}[s+2],U_{w+2}\otimes M_{p-1,w}[h+n])\\&\qquad\longrightarrow \operatorname{Hom}(U_{s+2}\otimes M_{p-1,s}[s+2],Z_c[n])\\&\qquad\longrightarrow \operatorname{Hom}(U_{s+2}\otimes M_{p-1,s}[s+2],U_p\otimes M_{p-2-c_1,c_2}[n+1]).\end{aligned}$$

The first term is zero by Lemma 9.3 because $h+n-(s+2)\leq(w+1)+n-(s+2)\leq w-s$, and the third term is zero because $n+1<t+2$; thus the second term is zero, as desired. If instead $b_2=p-1$, an analogous argument which uses Lemma 9.4 works.

Finally, suppose that $b\in\mathcal{I}_{\geq}$. Applying the functor $\operatorname{Hom}(Z_b,-)$ to the triangle above gives rise to a long exact sequence, a segment of which is

$$\begin{aligned}&\operatorname{Hom}(Z_b,U_{w+2}\otimes M_{p-1,w}[h+n])\longrightarrow \operatorname{Hom}(Z_b,Z_c[n])\\&\qquad\longrightarrow \operatorname{Hom}(Z_b,U_p\otimes M_{p-2-c_1,c_2}[n+1])\\&\qquad\longrightarrow \operatorname{Hom}(Z_b,U_{w+2}\otimes M_{p-1,w}[h+n+1]).\end{aligned}$$

The last term may be rewritten as

$$\operatorname{Hom}(Z_b,U_{w+2}\otimes M_{p-1,w}[w+2][h-(w+1)+n]),$$

which we see is zero by noting that $h-(w+1)+n\leq 0$ and applying a previous case ($b\in\mathcal{I}_{\geq}$ and $c\in\mathcal{I}_{p-1}$). The first term is zero by a similar argument. Hence it suffices to show that $\operatorname{Hom}(Z_b,U_p\otimes M_{p-2-c_1,c_2}[n+1])=0$ for all $n\leq 0$ unless $n=0$ and $b=c$, in which case it is one-dimensional. Suppose now that $b_2\leq b_1$. By Lemma 7, there is a distinguished triangle

$$U_p\otimes M_{p-2-b_1,b_2}\longrightarrow U_{w+2}\otimes M_{p-1,w}[h]\longrightarrow Z_b\longrightarrow$$

where $w=\min\{b_2,p-2-b_2\}$ and $h=w+b_1-p+3$. Applying the functor $\operatorname{Hom}(-,U_p\otimes M_{p-2-c_1,c_2})$ to this triangle gives rise to a long exact sequence, a segment of which is

$$\begin{aligned}&\operatorname{Hom}(U_{w+2}\otimes M_{p-1,w}[h],U_p\otimes M_{p-2-c_1,c_2}[n])\\&\qquad\longrightarrow \operatorname{Hom}(U_p\otimes M_{p-2-b_1,b_2},U_p\otimes M_{p-2-c_1,c_2}[n])\\&\qquad\longrightarrow \operatorname{Hom}(Z_b,U_p\otimes M_{p-2-c_1,c_2}[n+1])\\&\qquad\longrightarrow \operatorname{Hom}(U_{w+2}\otimes M_{p-1,w}[h],U_p\otimes M_{p-2-c_1,c_2}[n+1]).\end{aligned}$$

If $n<0$, then, remembering that $h>0$, it is clear that the second and fourth terms are zero, and thus that the third term is zero, as desired. Finally if $n=0$, then the first term is clearly zero, the fourth term is zero by Lemma 9.10, and by Lemma 9.12 the second term is zero unless $b=c$, in which case it is one-dimensional. It follows as desired that the third term is zero unless $b=c$, in which case it is one-dimensional.

If instead $b_1<b_2$, a similar argument using parts 11 and 13 of Lemma 9 works.

5. Generation

Our final task is to show that the last statement of Theorem 3 holds. We take $\mathcal{Z}$ to be the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\operatorname{mod}(B))$ generated by the complexes Z_b.

Lemma 10. *$\mathcal{Z}$ contains the following modules:*

(1) M_b, *whenever* $b \in \mathcal{I}_<$;
(2) $U_p \otimes M_{p-2-b_1,b_2}$, *whenever* $b \in \mathcal{I}_\geq$ *and* $b_2 \leq b_1$;
(3) $U_1 \otimes M_{b_1,p-2-b_2}$, *whenever* $b \in \mathcal{I}_\geq$ *and* $b_1 \leq b_2$;
(4) $U_{p+1} \otimes M_{s,s}$, *whenever* $0 \leq s \leq (p-3)/2$.

Proof. If $b \in \mathcal{I}_<$, then $M_b = Z_b$ is in $\mathcal{Z}$, so part (1) is proved. If $b \in \mathcal{I}_\geq$ and $b_2 \leq b_1$, then by Lemma 7 there is a distinguished triangle

$$U_p \otimes M_{p-2-b_1,b_2} \longrightarrow U_{w+2} \otimes M_{p-1,w}[h] \longrightarrow Z_b \longrightarrow,$$

where $0 \leq w \leq (p-3)/2$. The last term is in $\mathcal{Z}$ and the second term is as well, being a translate of Z_c for some $c \in \mathcal{I}_{p-1}$; hence the first term is in $\mathcal{Z}$, which proves part (2). Part (3) is proved similarly: apply σ to the triangle above.

Finally, we prove part (4). Applying part (2) with $b_1 = b_2 = p-2-s$, we have that $U_p \otimes M_{s,p-2-s}$ is in $\mathcal{Z}$. Next, we use Lemma 6 with $b_1 = s$ and $b_2 = p-2-s$. Applying σ to the resulting exact sequence and then tensoring with U_p, we get a distinguished triangle

$$U_{p+1} \otimes M_{s,s} \longrightarrow U_{p(s+2)} \otimes M_{s,p-1} \longrightarrow U_p \otimes M_{p-2-s,s} \longrightarrow .$$

We have just seen that the last term is in $\mathcal{Z}$, and the second term is in $\mathcal{Z}$ because it is a translate of Z_c for some $c \in \mathcal{I}_{p-1}$; hence the first term is in $\mathcal{Z}$, as desired. □

Let $\mathcal{J}$ be the set of pairs $d = (d_1, d_2)$ of integers satisfying the following conditions:

1. $-(p-1) < d_1 + d_2 \leq p-1$;
2. $d_1 + d_2$ is even;
3. d_1 and d_2 are either both non-negative or both non-positive.

It is easy to see that for any even integer j there exists $d \in \mathcal{J}$ such that $j \equiv d_1 + pd_2$ modulo $p^2 - 1$; this implies that any simple B-module is isomorphic to $U_{d_1+pd_2}$ for some $d \in \mathcal{J}$.

For $d \in \mathcal{J}$, let $f(d) = \min\{|d_1|, |d_2|\}$. Define a partial order $\preceq$ on $\mathcal{J}$ as follows: if $d = (d_1, d_2)$ and $d' = (d'_1, d'_2)$ are distinct elements of $\mathcal{J}$, then set $d \prec d'$ if and only if one of the following conditions is met:

1. $f(d) < f(d')$;
2. $f(d) = f(d')$ and $|d_1 + d_2| < |d'_1 + d'_2|$;
3. $f(d) = f(d')$ and $|d_1 + d_2| = |d'_1 + d'_2|$ and d_1 and d_2 are both non-negative.

To prove that the complexes Z_b generate $\mathrm{D^b}(\mathrm{mod}(B))$ as a triangulated category, it suffices to show that every simple B-module is in $\mathcal{Z}$. We shall do this by proving that $U_{d_1+pd_2}$ is in $\mathcal{Z}$ for each $d \in \mathcal{J}$, inducting on the partial order $\preceq$. The only element of $\mathcal{J}$ minimal with respect to $\preceq$ is $(0,0)$, and $U_{0+p\cdot 0} = U_0 = M_{(0,0)}$ is in $\mathcal{Z}$, by Lemma 10.1, so we may assume that $d \in \mathcal{J}$ and $d \neq (0,0)$. The argument divides into four cases:

- $d_1, d_2 < 0$: We have $(-d_1, -d_2) \in \mathcal{I}_<$, so by Lemma 10.1, $\mathcal{Z}$ contains $M_{-d_1,-d_2}$. The composition factors of $M_{-d_1,-d_2}$ consist of the simple modules $U_{m_1+pm_2}$, where (m_1, m_2) runs over pairs of integers satisfying $d_i \leq m_i \leq -d_i$ and $m_i \equiv d_i \mod 2$. So in particular $U_{d_1+pd_2}$ is a composition factor of $M_{-d_1,-d_2}$. We will now show that every other composition factor of $M_{-d_1,-d_2}$ is in $\mathcal{Z}$; it will follow that $U_{d_1+pd_2}$ is in $\mathcal{Z}$. To that end, let m be a pair (m_1, m_2) of integers such that $d_i \leq m_i \leq -d_i$ and $m_i \equiv d_i \mod 2$ for $i = 1, 2$, and $m \neq (d_1, d_2)$. We aim to find an $m' = (m'_1, m'_2)$ in $\mathcal{J}$ such that $m_1' + pm_2' = m_1 + pm_2$ and $m' \prec d$, for then by induction we will have that $U_{m_1+pm_2}$ is in $\mathcal{Z}$.

If m_1 and m_2 are either both non-negative or both non-positive, then $m \in \mathcal{J}$. Moreover, $f(m) \leq f(d)$ and $|m_1 + m_2| \leq |d_1 + d_2|$, with equalities occurring only if $m_1 = -d_1$ and $m_2 = -d_2$. Hence $m \prec d$.

If m_1 is positive, m_2 is negative, and $|m_1| > |m_2|$, let $m_1' = m_1 - p$ and $m_2' = m_2 + 1$. Then $m_1' + pm_2' = m_1 + pm_2$; in addition $m_1' \leq 0$, $m_2' \leq 0$, and $m_1' + m_2' = m_1 + m_2 - (p-1) > -(p-1)$, so $m' = (m_1', m_2') \in \mathcal{J}$. Furthermore $f(m') \leq |m_2 + 1| < |m_2| \leq f(d)$, so $m' \prec d$.

If m_1 is positive, m_2 is negative, and $|m_1| \leq |m_2|$, let $m_1' = m_1 - 1$ and $m_2' = m_2 + p$. Then $m_1' + pm_2' = m_1 + pm_2$; in addition $m_1' \geq 0$, $m_2' \geq 0$, and $m_1' + m_2' = m_1 + m_2 + p - 1 \leq p - 1$, so $m' = (m_1', m_2') \in \mathcal{J}$. Furthermore $f(m') \leq |m_1 - 1| < |m_1| \leq f(d)$, so $m' \prec d$.

If m_1 is negative and m_2 is positive, we split the argument into the cases $|m_1| \geq |m_2|$ and $|m_1| < |m_2|$. These may be handled as above.

- $d_1, d_2 \geq 0$ and $d_1 = d_2 > 0$: Let $s = d_1 - 1$. Then $d_1 + d_2 \leq p - 1$ implies that $0 \leq s \leq (p-3)/2$. By Lemma 10.4, $\mathcal{Z}$ contains $U_{p+1} \otimes M_{s,s}$. The composition factors of $U_{p+1} \otimes M_{s,s}$ consist of the simple modules $U_{m_1+pm_2}$, where (m_1, m_2) runs over pairs of integers satisfying $-s+1 \leq m_i \leq s+1$ and $m_i \equiv s+1$ for $i = 1, 2$. Thus $U_{d_1+pd_2}$ is a composition factor of $U_{p+1} \otimes M_{s,s}$, and by an argument similar to that for the previous case, one can show that every other composition factor is in $\mathcal{Z}$.
- $d_1, d_2 \geq 0$ and $d_1 < d_2$: By Lemma 10.2 with $b_1 = p - 2 - d_1$ and $b_2 = d_2 - 1$, we have that $U_p \otimes M_{d_1, d_2-1}$ is in $\mathcal{Z}$. One can show, as above, that $U_{m_1+pm_2}$ is a composition factor of this module, while any other composition factor is in $\mathcal{Z}$.
- $d_1, d_2 \geq 0$ and $d_1 > d_2$: Using Lemma 10.3, one can show that $\mathcal{Z}$ contains $U_1 \otimes M_{d_1-1, d_2}$, and then argue as in previous cases.

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References

[1] J. L. Alperin, *Local representation theory*, Cambridge Studies in Advanced Mathematics, vol. 11, Cambridge University Press, Cambridge, 1986. MR **87i:**20002

[2] M. Broué, *Isométries parfaites, types de blocs, catégories dérivées*, Astérisque (1990), no. 181-182, 61–92. MR **91i:**20006

[3] ———, *Rickard equivalences and block theory*, Groups '93 Galway/St. Andrews, Vol. 1 (Galway, 1993), London Math. Soc. Lecture Note Ser., vol. 211, Cambridge Univ. Press, Cambridge, 1995, pp. 58–79. MR **96d:**20011

[4] J. Carlson, *The cohomology of irreducible modules over $SL(2, p^n)$*, Proc. London Math. Soc. (3) **47** (1983), no. 3, 480–492. MR **85a:**20025

[5] J. Carlson and R. Rouquier, *Self-equivalences of stable module categories*, Math. Z. **233** (2000), 165–178. CMP 2000:07

[6] M. Linckelmann, *Stable equivalences of Morita type for self-injective algebras and p-groups*, Math. Z. **223** (1996), 87–100. MR **97i:**20011

[7] T. Okuyama, *Some examples of derived equivalent blocks of finite groups*, preprint.

[8] J. Rickard, *The abelian defect group conjecture*, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998), Doc. Math. 1998, extra vol. II, 121–128 (electronic). MR **99f:**20014

[9] ———, *Bousfield localization for representation theorists*, preprint.

[10] ———, *Derived categories and stable equivalence*, J. Pure Appl. Algebra **61** (1989), 303–317. MR **91a:**16004

[11] ———, *Morita theory for derived categories*, J. London Math. Soc. (2) **39** (1989), 436–456. MR **91b:**18012

[12] ———, *Derived equivalences as derived functors*, J. London Math. Soc. (2) **43** (1991), 37–48. MR **92b:**16043

[13] ———, *Splendid equivalences: derived categories and permutation modules*, Proc. London Math. Soc. (3) **72** (1996), 331–358. MR **97b:**20011

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