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HYPERBOLIC AUTOMORPHISMS AND ANOSOV DIFFEOMORPHISMS ON NILMANIFOLDS

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Abstract. We translate the problem of finding Anosov diffeomorphisms on a nilmanifold which is covered by a free nilpotent Lie group into a problem of constructing matrices in $\mathrm{GL}(n,\mathbb{Z})$ whose eigenvalues satisfy certain conditions. Afterwards, we show how this translation can then be solved in some specific situations. The paper starts with a section on polynomial permutations of $\mathbb{Q}^K$, a subject which is of interest on its own.

1. Introduction

In this paper we are concerned with Anosov diffeomorphisms on nilmanifolds. Let us recall the basic concepts: A nilmanifold will always be a closed manifold and is obtained as the quotient $N\backslash G$, where G is a simply connected and connected nilpotent Lie group and N is a discrete and uniform lattice of G. Such N are exactly the finitely generated torsion free nilpotent groups and G is called the Mal'cev completion of N. Any automorphism φ of N has a unique lift $\tilde{\varphi}$ to a continuous automorphism of G. As $\mathrm{Aut}(G) \cong \mathrm{Aut}(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G, by means of the differential map, $\tilde{\varphi}$ induces an automorphism $d\tilde{\varphi}$ of $\mathfrak{g}$. Since $\mathfrak{g}$ is a vector space, it makes sence to speak about the eigenvalues of $d\tilde{\varphi}$, these eigenvalues are also called the eigenvalues of $\tilde{\varphi}$ or the eigenvalues of φ. An automorphism φ of N is called a hyperbolic automorphism if all of its eigenvalues are of modulus different from one.

Note that the automorphism $\tilde{\varphi}$ of G, arising from an automorphism φ on N, induces a self-diffeomorphism f of the nilmanifold $N\backslash G$. Moreover, the induced map on the fundamental group $f_* : \pi_1(N\backslash G) \cong N \to \pi_1(N\backslash G) \cong N$ is exactly φ. If φ is hyperbolic, we will refer to the induced map on $N\backslash G$ as being a hyperbolic nilmanifold automorphism. Such hyperbolic nilmanifold automorphisms are standard examples of Anosov diffeomorphisms. Vaguely speaking, an Anosov diffeomorphism of a closed manifold M, is a diffeomorphism of M, for which the tangent bundle continuously splits into a contracting and an expanding part. For nilmanifolds, A. Manning showed that there are essentially no other Anosov diffeomorphisms than the hyperbolic nilmanifold automorphisms ([4]). Indeed he proved that any Anosov diffeomorphism on a nilmanifold is topologically conjugated to a hyperbolic nilmanifold automorphism.

So, the problem of finding the nilmanifolds which admit an Anosov diffeomorphism can be translated in the pure algebraic question of finding those finitely

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generated nilpotent groups admitting a hyperbolic automorphism. In spite of this translation, not too much is known on the question. In this paper we mainly concentrate on those nilmanifolds which are covered by a free nilpotent Lie group. In fact we translate the problem of finding hyperbolic automorphisms to a problem concerning matrices (and equivalently concerning polynomials). In the last section, we use this translation to prove the existence of Anosov diffeomorphisms on certain nilmanifolds.

Finally, we wish to remark that section 2, which is dealing with polynomial permutations of $\mathbb{Q}^K$, is used in the later sections on hyperbolic automorphisms but is not really related to Anosov diffeomorphisms and can also be of interest in other areas of mathematics.

2. Polynomial permutations of $\mathbb{Q}^K$ and $\mathbb{Z}^K$

This section is closely related to the section "Weighted groups of polynomial diffeomorphisms" of [1] (and we refer to [1] for more details). Nevertheless, we will be dealing with quite a different situation and we will prove and use the results in a very different setting. Throughout this section K will be a positive integer. We use $\mathbb{Z}$ (resp. $\mathbb{Q}$) to denote the set of integers (resp. rational numbers).

Definition 2.1. A map $\mu : \mathbb{Q}^K \to \mathbb{Q}^K$ is a polynomial permutation of $\mathbb{Q}^K$ if and only if

1. μ is a bijection;
2. μ and μ^{-1} are expressed by means of polynomials.

The set of all polynomial permutations of $\mathbb{Q}^K$ is denoted by $\mathrm{P}(\mathbb{Q}^K)$.

Example 2.2. Let

$$\mu : \mathbb{Q}^2 \to \mathbb{Q}^2 : \begin{pmatrix} y \\ x \end{pmatrix} \mapsto \begin{pmatrix} x + y^2 \\ y + 1 \end{pmatrix},$$

then

$$\mu^{-1} : \mathbb{Q}^2 \to \mathbb{Q}^2 : \begin{pmatrix} y \\ x \end{pmatrix} \mapsto \begin{pmatrix} x - 1 \\ y - x^2 + 2x - 1 \end{pmatrix}$$

showing that $\mu \in \mathrm{P}(\mathbb{Q}^2)$.

It is obvious that $\mathrm{P}(\mathbb{Q}^K)$ is a group under composition of maps. We will use $\mathrm{P}(\mathbb{Z}^K)$ to denote the subgroup of $\mathrm{P}(\mathbb{Q}^K)$ consisting of all $\mu \in \mathrm{P}(\mathbb{Q}^K)$ such that $\mu(\mathbb{Z}^K) = \mathbb{Z}^K$. Note that this does not mean that the polynomials expressing μ are polynomials with integral coefficients. For example, the map

$$\mu : \mathbb{Q}^2 \to \mathbb{Q}^2 : \begin{pmatrix} y \\ x \end{pmatrix} \mapsto \begin{pmatrix} y + \frac{x(x-1)}{2} \\ x \end{pmatrix}$$

belongs to $\mathrm{P}(\mathbb{Z}^K)$. In general, it is known ([5, Page 53]) that any integer valued polynomial (i.e. taking integer values, when integers are substituted for the indeterminates) in K variables $x_1, x_2, \dots, x_K$ can be written as a $\mathbb{Z}$–linear combination of the polynomials

$$\binom{x_1}{n_1}\binom{x_2}{n_2}\cdots\binom{x_K}{n_K}, \quad n_1, n_2, \dots, n_K \in \mathbb{N},$$

where $\binom{x}{n}$ stands for the polynomial $\frac{x(x-1)\cdots(x-n+1)}{n!}$.

Analogously we use $\mathrm{P}(\mathbb{Q}^K, \mathbb{Q}^k)$ to denote the group (under addition!) (even rational vector space) of all polynomial maps form $\mathbb{Q}^K$ to $\mathbb{Q}^k$. We will also refer to $\mathrm{P}(\mathbb{Z}^K, \mathbb{Z}^k)$, the group of all maps $p \in \mathrm{P}(\mathbb{Q}^K, \mathbb{Q}^k)$, with $p(\mathbb{Z}^K) \subseteq \mathbb{Z}^k$.

For the rest of this section we fix a chosen collection $k_1, k_2, \ldots, k_n$ of positive integers. We can view $\mathbb{Q}^K$, with $K = k_1 + k_2 + \cdots + k_n$ as being composed of n blocks:

$$\mathbb{Q}^K = \mathbb{Q}^{k_n} \oplus \cdots \oplus \mathbb{Q}^{k_2} \oplus \mathbb{Q}^{k_1}.$$

We will consider elements $\vec{a}$ (and analogously variables $\vec{x}$) of $\mathbb{Q}^K$ as being column vectors $(a_{n,1}, a_{n,2}, \ldots, a_{n,k_n}, \ldots, a_{2,k_2}, a_{1,1}, a_{1,2}, \ldots, a_{1,k_1})^T$.

For our purposes, we will not need the entire group of polynomial permutations of $\mathbb{Q}^K$, but only a very structured subgroup.

Definition 2.3. Let $k_1, k_2, \ldots, k_n$ be positive integers, The **blocked rational Jonquière group of type** $(k_n, k_{n-1}, \ldots, k_2, k_1)$ consists of all polynomial permutations p of $\mathbb{Q}^{k_1+k_2+\cdots+k_n}$ which are of the form

$$(1) \quad p(\vec{x}) = \begin{pmatrix} p_n(x_{n,1}, x_{n,2}, \ldots, x_{n,k_n}, x_{n-1,1}, \ldots, x_{2,k_2}, x_{1,1}, x_{1,2}, \ldots, x_{1,k_1}) \\ p_{n-1}(x_{n-1,1}, x_{n-1,2}, \ldots, x_{n-1,k_{n-1}}, \ldots, x_{2,k_2}, x_{1,1}, \ldots, x_{1,k_1}) \\ \vdots \\ p_2(x_{2,1}, x_{2,2}, \ldots, x_{2,k_2}, x_{1,1}, \ldots, x_{1,k_1}) \\ p_1(x_{1,1}, x_{1,2}, \ldots, x_{1,k_1}) \end{pmatrix}$$

where, for each i, the i-th last block is given by

$$(2) \quad \begin{aligned} & p_i(x_{i,1}, x_{i,2}, \ldots, x_{1,k_1}) \\ & = A_i \begin{pmatrix} x_{i,1} \\ x_{i,2} \\ \vdots \\ x_{i,k_i} \end{pmatrix} + q_i(x_{i-1,1}, x_{i-1,2}, \ldots, x_{2,k_2}, x_{1,1}, \ldots, x_{1,k_1}) \end{aligned}$$

with $A_i \in \mathrm{GL}(k_i, \mathbb{Q})$ and $q_i \in \mathrm{P}(\mathbb{Q}^{k_1+k_2+\cdots+k_{i-1}}, \mathbb{Q}^{k_i})$.

Remark 2.4. One can check that the blocked rational Jonquière group is indeed a subgroup of $\mathrm{P}(\mathbb{Q}^{k_1+k_2+\cdots+k_n})$.

To be able to describe the group structure of a blocked rational Jonquière group more precisely, we need the following lemma.

Lemma 2.5. *The group* $\mathrm{GL}(k, \mathbb{Q}) \times \mathrm{P}(\mathbb{Q}^K)$ *acts on* $\mathrm{P}(\mathbb{Q}^K, \mathbb{Q}^k)$ *as follows:*

$$\forall g \in \mathrm{GL}(k, \mathbb{Q}),\ \forall h \in \mathrm{P}(\mathbb{Q}^K),\ \forall \lambda \in \mathrm{P}(\mathbb{Q}^K, \mathbb{Q}^k) :\ {}^{(g,h)}\lambda = g \circ \lambda \circ h^{-1}.$$

Moreover, the resulting semi-direct product group $\mathrm{P}(\mathbb{Q}^K, \mathbb{Q}^k) \rtimes (\mathrm{GL}(k, \mathbb{Q}) \times \mathrm{P}(\mathbb{Q}^K))$ *embeds into* $\mathrm{P}(\mathbb{Q}^{K+k})$ *by defining*

$$\forall g \in \mathrm{GL}(k, \mathbb{Q}),\ \forall h \in \mathrm{P}(\mathbb{Q}^K),\ \forall \lambda \in \mathrm{P}(\mathbb{Q}^K, \mathbb{Q}^k),\ \forall x \in \mathbb{Q}^k,\ \forall y \in \mathbb{Q}^K :$$

$$(\lambda, g, h) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} g(x) + \lambda(h(y)) \\ h(y) \end{pmatrix}.$$

Analogously, the group $\mathrm{GL}(k, \mathbb{Z}) \times \mathrm{P}(\mathbb{Z}^K)$ *acts on* $\mathrm{P}(\mathbb{Z}^K, \mathbb{Z}^k)$ *and the resulting semi-direct product embeds into* $\mathrm{P}(\mathbb{Z}^{K+k})$.

Proposition 2.6. *Let J be the blocked rational Jonquière group of type $(k_n, k_{n-1}, \dots, k_2, k_1)$, then J can be identified with the semi-direct product group (under the identification of Lemma 2.5)*

$$\mathrm{P}(\mathbb{Q}^{k_1+k_2+\cdots+k_{n-1}}, \mathbb{Q}^{k_n}) \rtimes (\mathrm{GL}(k_n, \mathbb{Q}) \times J'),$$

where J' is the blocked Jonquière group of type $(k_{n-1}, \dots, k_2, k_1)$.
Moreover, under this identification $J_{\mathbb{Z}} = J \cap \mathrm{P}(\mathbb{Z})$ corresponds to

$$\mathrm{P}(\mathbb{Z}^{k_1+k_2+\cdots+k_{n-1}}, \mathbb{Z}^{k_n}) \rtimes (\mathrm{GL}(k_n, \mathbb{Z}) \times J'_{\mathbb{Z}}),$$

where $J'_{\mathbb{Z}} = J' \cap \mathrm{P}(\mathbb{Z}^{k_1+k_2+\cdots+k_{n-1}})$.

Proof. To see that, under the embedding of Lemma 2.5, the image of

$$\mathrm{P}(\mathbb{Q}^{k_1+k_2+\cdots+k_{n-1}}, \mathbb{Q}^{k_n}) \rtimes (\mathrm{GL}(k_n, \mathbb{Q}) \times J')$$

coincides with J, it is enough to realise that the polynomial permutation $p \in J$ of the form (1) corresponds to the triple $(q_n \circ \bar{p}^{-1}, A_n, \bar{p})$, where $\bar{p} \in \mathrm{P}(\mathbb{Q}^{k_1+k_2+\cdots+k_{n-1}})$ is given by

$$\bar{p}\begin{pmatrix} x_{n-1,1} \\ \vdots \\ x_{1,1} \\ \vdots \\ x_{1,k_1} \end{pmatrix} = \begin{pmatrix} p_{n-1}(x_{n-1,1}, x_{n-1,2}, \dots, x_{n-1,k_{n-1}}, \dots, x_{2,k_2}, x_{1,1}, \dots, x_{1,k_1}) \\ \vdots \\ p_2(x_{2,1}, x_{2,2}, \dots, x_{2,k_2}, x_{1,1}, \dots, x_{1,k_1}) \\ p_1(x_{1,1}, x_{1,2}, \dots, x_{1,k_1}) \end{pmatrix}.$$

Moreover, it is now easy to see that

$$\begin{aligned} p \in J_{\mathbb{Z}} \quad &\Leftrightarrow \quad \begin{cases} A_n \in \mathrm{GL}(k_n, \mathbb{Z}), \\ q_n \in \mathrm{P}(\mathbb{Z}^{k_1+k_2+\cdots+k_{n-1}}, \mathbb{Z}^{k_n}), \\ \bar{p} \in J'_{\mathbb{Z}} \end{cases} \\ &\Leftrightarrow \quad \begin{cases} A_n \in \mathrm{GL}(k_n, \mathbb{Z}), \\ q_n \circ \bar{p}^{-1} \in \mathrm{P}(\mathbb{Z}^{k_1+k_2+\cdots+k_{n-1}}, \mathbb{Z}^{k_n}), \\ \bar{p} \in J'_{\mathbb{Z}} \end{cases} \end{aligned}$$

which concludes the proof. □

Although the blocked rational Jonquière group is already much smaller and more structured than the group $\mathrm{P}(\mathbb{Q}^K)$, it is still too big for our purposes. In order to restrict this group even more, we recall the definition of the weight (or weighted degree) of a polynomial introduced in [1].

Definition 2.7. Write R for $\mathbb{Q}[x_{1,1}, x_{1,2}, \dots, x_{1,k_1}, x_{2,1}, \dots, x_{2,k_2}, x_{3,1}, \dots, x_{n,k_n}]$. Let $\kappa = (k_1, k_2, \dots, k_n)$ and $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ be two (fixed) n-tuples. The (κ, ω)**-weight of a monomial** $x_{1,1}^{\alpha_{1,1}} x_{1,2}^{\alpha_{1,2}} \dots x_{n,k_n}^{\alpha_{n,k_n}}$ in R is defined as

$$\deg_{\kappa,\omega}(x_{1,1}^{\alpha_{1,1}} x_{1,2}^{\alpha_{1,2}} \dots x_{n,k_n}^{\alpha_{n,k_n}}) = \sum_{i=1}^{n} \sum_{j=1}^{k_i} \omega_i \alpha_{i,j}.$$

More general, the (κ, ω)**-weight of a polynomial** $p(x_{n,1}, \dots, x_{1,1}, x_{1,2}, \dots, x_{1,k_1}) \in R$ is defined as the maximal weight of its terms. As in the case of the degree of a polynomial, we say that the (κ, ω)-weight of the zero polynomial is equal to $-\infty$. We denote the (κ, ω)-weight of the polynomial $p(\vec{x})$ by $\deg_{\kappa,\omega}(p(\vec{x}))$.

The weight of a polynomial behaves in many ways analogously to the degree of a polynomial. However, for our purposes it behaves even more nicely, since we can use it to define special subgroups of the blocked rational Jonquière group, a process which seems hopeless with the usual degree-notion.

Definition 2.8. Let $\kappa = (k_1, k_2, \ldots, k_n)$ and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ be two n-tuples of positive integers. We define $\mathrm{P}_{\kappa,\omega}(\mathbb{Q}^{k_1+k_2+\cdots+k_n})$ to be the subset of the blocked rational Jonquière group of type $(k_n, \ldots, k_2, k_1)$ consisting of those permutations p of $\mathbb{Q}^{k_1+k_2+\cdots+k_n}$ of the form (1) where for each $i \in \{1, 2, \ldots, n\}$, the polynomial $p_i(x_{i,1}, x_{i,2}, \ldots, x_{1,k_1})$ is of (κ, ω)–weight ω_i. Obviously, this is equivalent to saying that the polynomial q_i is of (κ, ω)–weight $\le \omega_i$.

Analogously, $\mathrm{P}_{\kappa,\omega}(\mathbb{Z}^{k_1+k_2+\cdots+k_n}) = \mathrm{P}_{\kappa,\omega}(\mathbb{Q}^{k_1+k_2+\cdots+k_n}) \cap \mathrm{P}(\mathbb{Z}^{k_1+k_2+\cdots+k_n})$.

Theorem 2.9. $\mathrm{P}_{\kappa,\omega}(\mathbb{Q}^{k_1+k_2+\cdots+k_n})$ *is a subgroup of the blocked rational Jonquière group of type* $(k_n, \ldots, k_2, k_1)$. *It follows that there is an iterative way of building up* $\mathrm{P}_{\kappa,\omega}(\mathbb{Q}^{k_1+k_2+\cdots+k_n})$ *as a semi-direct product group*

$$\mathrm{P}_{\kappa,\omega}(\mathbb{Q}^{k_1+k_2+\cdots+k_{n-1}}, \mathbb{Q}^{k_n}) \rtimes \left(\mathrm{GL}(k_n, \mathbb{Q}) \times \mathrm{P}_{\bar\kappa,\bar\omega}(\mathbb{Q}^{k_1+k_2+\cdots+k_{n-1}})\right),$$

where

1. $\bar\kappa = (k_1, k_2, \ldots, k_{n-1})$,
2. $\bar\omega = (\omega_1, \omega_2, \ldots, \omega_{n-1})$,
3. $\mathrm{P}_{\kappa,\omega}(\mathbb{Q}^{k_1+k_2+\cdots+k_{n-1}}, \mathbb{Q}^{k_n})$ *is the subset of* $\mathrm{P}(\mathbb{Q}^{k_1+k_2+\cdots+k_{n-1}}, \mathbb{Q}^{k_n})$ *of polynomial maps of* $(\bar\kappa, \bar\omega)$*–weight* $\le \omega_n$.

The analogous statement for $\mathrm{P}_{\kappa,\omega}(\mathbb{Z}^{k_1+k_2+\cdots+k_n})$ *is also valid.*

Proof. The proof is left to the reader, who can consult [1] if necessary. □

Another interesting property is the following theorem.

Theorem 2.10 ([1, Theorem 3.2])**.** *Let* Γ *be any finitely generated subgroup of a blocked rational Jonquière group of type* $(k_n, \ldots, k_2, k_1)$. *Then there exists a system of weights* $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$, *such that* $\Gamma \subseteq \mathrm{P}_{\kappa,\omega}(\mathbb{Q}^K)$, *with* $\kappa = (k_1, k_2, \ldots, k_n)$.

From now on we will diverge more from [1].

Let κ and ω be fixed as before. $\mathrm{P}_{\kappa,\omega}(\mathbb{Q}^{k_1+k_2+\cdots+k_{n-1}}, \mathbb{Q}^{k_n})$ is a rational vector space. Consider all maps

(3)

$$\mu : \mathbb{Q}^{k_1+k_2+\cdots+k_{n-1}} \to \mathbb{Q}^{k_n} : \begin{pmatrix} x_{n-1,1} \\ \vdots \\ x_{2,k_2} \\ x_{1,1} \\ \vdots \\ x_{1,k_1} \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \displaystyle\prod_{\substack{1 \le p \le n-1 \\ 1 \le q \le k_p}} \binom{x_{pq}}{n_{pq}} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \longleftarrow i\text{–th place},$$

where $1 \le i \le k_n$ and where the $n_{pq} \in \mathbb{N}$ are such that

$$\sum_{\substack{1\le p\le n-1\\ 1\le q\le k_p}} \omega_p n_{pq} \le \omega_n.$$

It is obvious that all such μ belong to $\mathrm{P}_{\kappa,\omega}(\mathbb{Q}^{k_1+\cdots+k_{n-1}},\mathbb{Q}^{k_n})$. In fact, the set B of all such maps μ (in some order) forms a basis of the (finite dimensional!) vector space $\mathrm{P}_{\kappa,\omega}(\mathbb{Q}^{k_1+\cdots+k_{n-1}},\mathbb{Q}^{k_n})$. Moreover, a map $\lambda:\mathbb{Q}^{k_1+\cdots+k_{n-1}}\to\mathbb{Q}^{k_n}$ belongs to $\mathrm{P}_{\kappa,\omega}(\mathbb{Z}^{k_1+\cdots+k_{n-1}},\mathbb{Z}^{k_n})$ if and only if λ can be written as a $\mathbb{Z}$–linear combination of the elements of B. Using the basis B, we can now prove the following embedding theorem.

Theorem 2.11. *Let (κ,ω) be two fixed n–tuples of positive integers. There exists an $m \in \mathbb{N}$ and a faithful representation*

$$\chi : \mathrm{P}_{\kappa,\omega}(\mathbb{Q}^{k_1+k_2+\cdots+k_n}) \to \mathrm{GL}(m,\mathbb{Q})$$

with

$$\chi(\mathrm{P}_{\kappa,\omega}(\mathbb{Z}^{k_1+k_2+\cdots+k_n})) = \chi(\mathrm{P}_{\kappa,\omega}(\mathbb{Q}^{k_1+k_2+\cdots+k_n})) \cap \mathrm{GL}(m,\mathbb{Z}).$$

Proof. We proceed by induction on n. For $n=0$, there is nothing to show. So assume that $n>0$ and that the theorem is valid for smaller values than n. By Theorem 2.9, we know that $\mathrm{P}_{\kappa,\omega}(\mathbb{Q}^{k_1+k_2+\cdots+k_n})$ is the semi-direct product group

$$\mathrm{P}_{\kappa,\omega}(\mathbb{Q}^{k_1+k_2+\cdots+k_{n-1}},\mathbb{Q}^{k_n}) \rtimes \left(\mathrm{GL}(k_n,\mathbb{Q}) \times \mathrm{P}_{\bar\kappa,\bar\omega}(\mathbb{Q}^{k_1+k_2+\cdots+k_{n-1}})\right).$$

By the induction hypothesis we know that there exists a natural number $\bar m \in \mathbb{N}$ and a faithful linear representation $\bar\chi : \mathrm{P}_{\bar\kappa,\bar\omega}(\mathbb{Q}^{k_1+k_2+\cdots+k_{n-1}}) \to \mathrm{GL}(\bar m,\mathbb{Q})$ with $\bar\chi(\mathrm{P}_{\bar\kappa,\bar\omega}(\mathbb{Z}^{k_1+k_2+\cdots+k_{n-1}})) = \bar\chi(\mathrm{P}_{\bar\kappa,\bar\omega}(\mathbb{Q}^{k_1+k_2+\cdots+k_{n-1}})) \cap \mathrm{GL}(\bar m,\mathbb{Z})$.

As $\mathrm{P}_{\kappa,\omega}(\mathbb{Q}^{k_1+k_2+\cdots+k_{n-1}},\mathbb{Q}^{k_n})$ is a finite dimensional rational vector space, we can identify it with $\mathbb{Q}^{m'}$, for some m', via the basis B given above. Using this identification, we can consider any element q of $\mathrm{P}_{\kappa,\omega}(\mathbb{Q}^{k_1+k_2+\cdots+k_{n-1}},\mathbb{Q}^{k_n})$ as a column vector with m' entries. Moreover, such a q belongs to $\mathrm{P}_{\kappa,\omega}(\mathbb{Z}^{k_1+k_2+\cdots+k_{n-1}},\mathbb{Z}^{k_n})$ iff the column vector belongs to $\mathbb{Z}^{m'}$. After, this identification, the action of an element $(A,\bar p)$ of $\mathrm{GL}(k_n,\mathbb{Q}) \times \mathrm{P}_{\bar\kappa,\bar\omega}(\mathbb{Q}^{k_1+k_2+\cdots+k_{n-1}})$ is given by a matrix $\zeta(A,\bar p) \in \mathrm{GL}(m',\mathbb{Q})$. Using this notation and identification it is now trivial to check that the map, with $m = k_n + m' + 1 + \bar m$:

$$\chi : \mathrm{P}_{\kappa,\omega}(\mathbb{Q}^{k_1+k_2+\cdots+k_n}) \to \mathrm{GL}(m,\mathbb{Q}) : (q,A,\bar p) \mapsto \begin{pmatrix} A & 0 & 0 & 0\\ 0 & \zeta(A,\bar p) & q & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & \bar\chi(\bar p)\end{pmatrix}$$

is a homomorphism of groups.

Moreover, if $\chi(q,A,\bar p) \in \mathrm{GL}(m,\mathbb{Z})$, then $A \in \mathrm{GL}(k_n,\mathbb{Z})$, $\bar\chi(p) \in \mathrm{GL}(\bar m,\mathbb{Z})$ and $q \in \mathbb{Z}^{m'}$, and so

$$(q,A,\bar p) \in \mathrm{P}_{\kappa,\omega}(\mathbb{Z}^{k_1+k_2+\cdots+k_{n-1}},\mathbb{Z}^{k_n}) \rtimes \left(\mathrm{GL}(k_n,\mathbb{Z}) \times \mathrm{P}_{\bar\kappa,\bar\omega}(\mathbb{Z}^{k_1+k_2+\cdots+k_{n-1}})\right).$$

Conversely, if

$$(q,A,\bar p) \in \mathrm{P}_{\kappa,\omega}(\mathbb{Z}^{k_1+k_2+\cdots+k_{n-1}},\mathbb{Z}^{k_n}) \rtimes \left(\mathrm{GL}(k_n,\mathbb{Z}) \times \mathrm{P}_{\bar\kappa,\bar\omega}(\mathbb{Z}^{k_1+k_2+\cdots+k_{n-1}})\right),$$

then also $\zeta(A,\bar p) \in \mathrm{GL}(m',\mathbb{Z})$. Indeed, as the action of the element $(A,\bar p)$ maps $\mathrm{P}_{\kappa,\omega}(\mathbb{Z}^{k_1+k_2+\cdots+k_{n-1}},\mathbb{Z}^{k_n})$ onto itself, it maps any element of the basis B onto a $\mathbb{Z}$–linear combination of such elements. This shows that the entries of $\zeta(A,\bar p)$ are integers. This also holds for $\zeta(A,\bar{p})^{-1}$, so we can conclude that $\zeta(A,\bar{p}) \in \mathrm{GL}(m',\mathbb{Z})$. This shows that

$$(q,A,\bar{p}) \in \mathrm{P}_{\kappa,\omega}(\mathbb{Z}^{k_1+k_2+\cdots+k_{n-1}},\mathbb{Z}^{k_n}) \rtimes \big(\mathrm{GL}(k_n,\mathbb{Z}) \times \mathrm{P}_{\bar{\kappa},\bar{\omega}}(\mathbb{Z}^{k_1+k_2+\cdots+k_{n-1}})\big)$$
$$\Updownarrow$$
$$\chi(q,A,\bar{p}) \in \mathrm{GL}(m,\mathbb{Z})$$

which finishes the proof. ☐

There is a morphism φ from the blocked rational Jonquière group of type $(k_n,\ldots,k_2,k_1)$ to the direct product group $\mathrm{GL}(k_n,\mathbb{Q}) \times \cdots \times \mathrm{GL}(k_2,\mathbb{Q}) \times \mathrm{GL}(k_1,\mathbb{Q})$ mapping any element p of the form (1) to $\varphi(p) = (A_n,\ldots,A_2,A_1)$. We will refer to the **characteristic polynomial of** p by which we mean the product of the characteristic polynomials of the A_i, i.e. the characteristic polynomial of p coincides with the characteristic polynomial of the matrix

$$\begin{pmatrix} A_n & \cdots & 0 & 0 \\ \vdots & \ddots & 0 & 0 \\ 0 & \cdots & A_2 & 0 \\ 0 & \cdots & 0 & A_1 \end{pmatrix}.$$

In [7] (see also [8]) the following lemma was proved.

Lemma 2.12. *Let $A \in \mathrm{GL}(m,\mathbb{Q})$, for some $m \in \mathbb{N}_0$ be a matrix whose characteristic polynomial has integer coefficients and unit constant term, then there exists some $k \in \mathbb{N}_0$ such that $A^k \in \mathrm{GL}(m,\mathbb{Z})$.*

This lemma generalizes to the blocked rational Jonquière groups and will fulfill a key role in the following sections.

Lemma 2.13. *Let $k_1,k_2,\ldots,k_n$ be positive integers and let p be an element of the blocked rational Jonquière group of type $(k_n,\ldots,k_2,k_1)$. If the characteristic polynomial of p has integer coefficients and unit constant term, then there is some $k \in \mathbb{N}$ such that $p^k \in \mathrm{P}(\mathbb{Z}^{k_1+k_2+\cdots+k_n})$.*

Proof. By Theorem 2.10, there is no loss of generality in assuming that

$$p \in \mathrm{P}_{\kappa,\omega}(\mathbb{Q}^{k_1+k_2+\cdots+k_n}) \quad \text{for some } \omega = (\omega_1,\omega_2,\ldots,\omega_n).$$

We will proceed by induction on n. For $n = 0$, there is nothing to show. Let $n > 0$ and suppose that the lemma is valid for values smaller than n. We will regard $\mathrm{P}_{\kappa,\omega}(\mathbb{Q}^{k_1+k_2+\cdots+k_n})$ as the semi-direct product

$$\mathrm{P}_{\kappa,\omega}(\mathbb{Q}^{k_1+k_2+\cdots+k_{n-1}},\mathbb{Q}^{k_n}) \rtimes (\mathrm{GL}(k_n,\mathbb{Q}) \times \mathrm{P}_{\bar{\kappa},\bar{\omega}}(\mathbb{Q}^{k_1+k_2+\ldots+k_{n-1}}))$$

as in Proposition 2.9. So p is a triple $(q,A_n,\bar{p})$. Let μ_1 (resp. μ_2) denote the characteristic polynomial of A_n (resp. $\bar{p}$). It follows that the characteristic polynomial μ of p is equal to the product $\mu_1\mu_2$. As μ has integer coefficients and unit constant term, the same must hold for μ_1 and μ_2. Therefore, using Lemma 2.12 and the induction hypothesis, there exists $l_1,l_2 \in \mathbb{N}_0$ for which $A_n^{l_1} \in \mathrm{GL}(k_n,\mathbb{Z})$ and $\bar{p}^{l_2} \in \mathrm{P}_{\bar{\kappa},\bar{\omega}}(\mathbb{Z}^{k_1+\cdots+k_{n-1}})$. So, by replacing p with $p^{l_1l_2}$ we can assume that

$$A_n \in \mathrm{GL}(k_n,\mathbb{Z}) \text{ and } \bar{p} \in \mathrm{P}_{\bar{\kappa},\bar{\omega}}(\mathbb{Z}^{k_1+\cdots+k_{n-1}}).$$

Note that $p = (q,1,1)(0,A_n,\bar{p})$. Now, consider the faithful linear representation χ of Theorem 2.11. Looking at the construction in the proof of Theorem 2.11, we see that

$$\chi(p) = \chi(q,1,1)\chi(0,A_n,\bar{p})$$
$$= \begin{pmatrix} I_{k_n} & 0 & 0 & 0 \\ 0 & I_{m'} & q & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & I_m \end{pmatrix} \begin{pmatrix} A_n & 0 & 0 & 0 \\ 0 & \zeta(A,\bar{p}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \chi'(\bar{p}) \end{pmatrix}.$$

It follows that the characteristic polynomial of $\chi(p)$ coincides with the characteristic polynomial of $\chi(0, A_n, \bar{p})$, which obviously has integer coefficients and unit constant term since $(0, A_n, \bar{p}) \in \mathrm{P}_{\kappa,\omega}(\mathbb{Z}^{k_1+\cdots+k_n})$ and thus $\chi(0, A_n, \bar{p}) \in \mathrm{GL}(m,\mathbb{Z})$ (Theorem 2.11). Using Lemma 2.12 again, we find that there exists a $k \in \mathbb{N}$ such that $(\chi(p))^k = \chi(p^k) \in \mathrm{GL}(m,\mathbb{Z})$ and consequently (Theorem 2.11 again) $p^k \in \mathrm{P}_{\kappa,\omega}(\mathbb{Z}^{k_1+k_2+\cdots+k_n}) \subset \mathrm{P}(\mathbb{Z}^{k_1+k_2+\cdots+k_n})$. □

3. Automorphisms of nilpotent groups and polynomial permutations

In this section we will show the link between polynomial permutations and the context of Anosov diffeomorphisms. As already mentioned in the introduction, any Anosov diffeomorphism on a nilmanifold can be seen as arising topologically from a hyperbolic automorphism on the covering Lie group.

Let N be a finitely generated torsion free nilpotent group. There are two (and more) "completions" which are canonically associated to N, namely the radicable hull, which we will denote by $N_{\mathbb{Q}}$ and the Mal'cev completion for which we will use $N_{\mathbb{R}}$.

The radicable hull of N is the unique minimal torsion free nilpotent and radicable (divisible) group containing N. It follows that $N_{\mathbb{Q}}$ is a radicable group containing N as a subgroup, such that every element of $N_{\mathbb{Q}}$ has some positive power lying in N.

$N_{\mathbb{R}}$ is the unique connected and simply connected nilpotent Lie group containing N as a uniform lattice.

There is a very easy construction of both completions using the lower central series of N (the upper central series works equally well). Recall that the lower central series of N is defined inductively via $\gamma_1(N) = N$ and $\gamma_{i+1}(N) = [\gamma_i(N), N]$. The group N is said to be c–step nilpotent if $\gamma_c(N) \neq 1$, but $\gamma_{c+1}(N) = 1$. The isolator of a subgroup H of N, denoted by $\sqrt[N]{H}$ is the set $\{n \in N \mid n^k \in H, \text{ for some } k \in \mathbb{N}_0\}$. It is well known ([6]) that the sequence (N is assumed to be nilpotent of class c)

$$N_{c+1} = 1 \subseteq N_c = \sqrt[N]{\gamma_c(N)} \subseteq N_{c-1} = \sqrt[N]{\gamma_{c-1}(N)} \subseteq \cdots$$
$$\subseteq N_2 = \sqrt[N]{\gamma_2(N)} \subseteq N_1 = \sqrt[N]{\gamma_1(N)} = N$$

forms a central series with $N_i/N_{i+1} \cong \mathbb{Z}^{k_i}$ for some $k_i \in \mathbb{N}_0$ $(1 \leq i \leq c)$.

We fix a set of generators

$$(4) \quad a_{c,1}, a_{c,2}, \ldots, a_{c,k_c}, a_{c-1,1}, \ldots, a_{3,k_3}, a_{2,1}, a_{2,2}, \ldots, a_{2,k_2}, a_{1,1}, a_{1,2}, \ldots, a_{1,k_1}$$

in such a way that $\bar{a}_{i,1}, \bar{a}_{i,2}, \ldots, \bar{a}_{i,k_i}$ freely generate the free abelian group N_i/N_{i+1} $(1 \leq i \leq c)$. Here, the $\bar{a}_{i,j}$ are used to denote the canonical images of the $a_{i,j}$ in the quotient group N_i/N_{i+1}. We will refer to (4) as a **Mal'cev set of generators**.

Any element $n \in N$ can now be uniquely expressed as a product

$$n = a_{c,1}^{z_{c,1}} a_{c,2}^{z_{c,2}} \dots a_{c,k_c}^{z_{c,k_c}} \dots a_{2,1}^{z_{2,1}} a_{2,2}^{z_{2,2}} \dots a_{2,k_2}^{z_{2,k_2}} a_{1,1}^{z_{1,1}} a_{1,2}^{z_{1,2}} \dots a_{1,k_1}^{z_{1,k_1}}, \text{ with } z_{i,j} \in \mathbb{Z}.$$

It follows that we can identify n with the column vector

$$\vec{z} = (z_{c,1}, \dots, z_{2,k_2}, z_{1,1}, z_{1,2}, \dots, z_{1,k_1})^T \in \mathbb{Z}^{k_1+k_2+\dots+k_c}.$$

Therefore, we sometimes write $n(\vec{z})$ to denote the element of N determined by the vector $\vec{z}$. By the work of Mal'cev it is known that the product in N is expressed by means of polynomial functions in the vectors $\vec{z}$, i.e. there exists a polynomial map $p \in \mathrm{P}(\mathbb{Z}^{2(k_1+k_2+\dots+k_c)}, \mathbb{Z}^{k_1+k_2+\dots+k_c})$, such that

$$n(\vec{x})n(\vec{y}) = n(p(\vec{x}, \vec{y})), \ \forall \vec{x}, \vec{y} \in \mathbb{Z}^{k_1+k_2+\dots+k_c}.$$

We can now define $N_{\mathbb{Q}}$ to consist of all formal products

$$n(\vec{q}) = a_{c,1}^{q_{c,1}} a_{c,2}^{q_{c,2}} \dots a_{2,k_2}^{q_{2,k_2}} a_{1,1}^{q_{1,1}} a_{1,2}^{q_{1,2}} \dots a_{1,k_1}^{q_{1,k_1}}$$
$$\text{with } \vec{q} = (q_{c,1}, \dots, q_{1,k_1})^T \in \mathbb{Q}^{k_1+k_2+\dots+k_c}$$

and where the product in $\mathbb{N}_{\mathbb{Q}}$ is given by the same polynomial p, i.e. $n(\vec{x})n(\vec{y}) = n(p(\vec{x}, \vec{y})), \ \forall \vec{x}, \vec{y} \in \mathbb{Q}^{k_1+k_2+\dots+k_c}$. The analogue is true for $N_{\mathbb{R}}$.

It is well known that any automorphism of N lifts uniquely to an automorphism of $N_{\mathbb{Q}}$ and a continuous automorphism of $N_{\mathbb{R}}$. We will now concentrate on the automorphisms of $N_{\mathbb{Q}}$. Let φ be any automorphism of $N_{\mathbb{Q}}$. Again by the work of Mal'cev, we know that there exists a polynomial map $p_\varphi \in \mathrm{P}(\mathbb{Q}^{k_1+k_2+\dots+k_c}, \mathbb{Q}^{k_1+k_2+\dots+k_c})$ such that

$$\varphi(n(\vec{q})) = n(p_\varphi(\vec{q})), \ \forall \vec{q} \in \mathbb{Q}^{k_1+k_2+\dots+k_c}.$$

Moreover, since φ is an automorphism, p_φ has to be a bijection and so $p_\varphi \in \mathrm{P}(\mathbb{Q}^{k_1+k_2+\dots+k_c})$. Let us call p_φ the **polynomial permutation representing** φ w.r.t. the Mal'cev generating set (4). The following proposition shows the reason for investigating blocked rational Jonquière groups.

Proposition 3.1. *Let N be any torsion free finitely generated c-step nilpotent group equipped with a Mal'cev set of generators* (4). *Let $N_{\mathbb{Q}}$ be the radicable hull of N and suppose that $\varphi \in \mathrm{Aut}(N_{\mathbb{Q}})$. Then, the polynomial permutation p_φ representing φ belongs to the rational blocked Jonquière group of type* $(k_c, \dots, k_2, k_1)$.

Proof. We proceed by induction on c. If $c = 1$, $p_\varphi \in \mathrm{GL}(k_c, \mathbb{Q})$ which proves the proposition. Now, assume that $c > 1$ and that the proposition is valid for smaller values.

Since p_φ is a polynomial permutation of $\mathbb{Q}^{k_1+k_2+\cdots+k_c}$ it is of the following form

$$p_\varphi : \mathbb{Q}^{k_1+k_2+\cdots+k_c} \to \mathbb{Q}^{k_1+k_2+\cdots+k_c} : \begin{pmatrix} q_{c,1} \\ \vdots \\ q_{c,k_c} \\ q_{c-1,1} \\ \vdots \\ q_{1,k_1} \end{pmatrix}$$
$$\mapsto \begin{pmatrix} p_{c,1}(q_{c,1}, q_{c,2}, \ldots, q_{2,k_2}, q_{1,1}, \ldots, q_{1,k_1}) \\ \vdots \\ p_{c,k_c}(q_{c,1}, q_{c,2}, \ldots, q_{2,k_2}, q_{1,1}, \ldots, q_{1,k_1}) \\ p_{c-1,1}(q_{c,1}, q_{c,2}, \ldots, q_{2,k_2}, q_{1,1}, \ldots, q_{1,k_1}) \\ \vdots \\ p_{1,k_1}(q_{c,1}, q_{c,2}, \ldots, q_{2,k_2}, q_{1,1}, \ldots, q_{1,k_1}) \end{pmatrix}$$

for some polynomials $p_{i,j}$ $(1 \leq i \leq c,\ 1 \leq j \leq k_i)$. We have to show that these polynomials $p_{i,j}$ are of the form (2).

φ induces an automorphism $\bar{\varphi}$ on $N_{\mathbb{Q}}/(\gamma_c(N_{\mathbb{Q}}))$. Using the canonical projections

$$\bar{a}_{c-1,1}, \bar{a}_{c-1,2}, \ldots, \bar{a}_{2,k_2}, \bar{a}_{1,1}, \ldots, \bar{a}_{1,k_1} \in N_{\mathbb{Q}}/(\gamma_c(N_{\mathbb{Q}}))$$

as the Mal'cev generating set for $N_{\mathbb{Q}}/(\gamma_c(N_{\mathbb{Q}}))$, we find that the polynomial permutation representing $\bar{\varphi}$ is

$$p_{\bar{\varphi}} : \mathbb{Q}^{k_1+k_2+\cdots+k_{c-1}} \to \mathbb{Q}^{k_1+k_2+\cdots+k_{c-1}} : \begin{pmatrix} q_{c-1,1} \\ q_{c-1,2} \\ \vdots \\ q_{1,k_1} \end{pmatrix}$$
$$\mapsto \begin{pmatrix} p_{c-1,1}(q_{c-1,1}, \ldots, q_{2,k_2}, q_{1,1}, \ldots, q_{1,k_1}) \\ p_{c-1,2}(q_{c-1,1}, \ldots, q_{2,k_2}, q_{1,1}, \ldots, q_{1,k_1}) \\ \vdots \\ p_{1,k_1}(q_{c-1,1}, \ldots, q_{2,k_2}, q_{1,1}, \ldots, q_{1,k_1}) \end{pmatrix}.$$

The induction hypothesis now implies that the polynomials $p_{i,j}$ are of the form (2), for $1 \leq i \leq c-1$ and $1 \leq j \leq k_i$.

From now on, we only need to concentrate on the polynomials $p_{c,j}$ $(1 \leq j \leq k_c)$. Note that

$$\gamma_c(N_{\mathbb{Q}}) = \{a_{c,1}^{q_{c,1}} a_{c,2}^{q_{c,2}} \ldots a_{c,k_c}^{q_{c,k_c}} \mid q_{c,1}, q_{c,2}, \ldots, q_{c,k_c} \in \mathbb{Q}\} \cong \mathbb{Q}^{k_c}.$$

Therefore, since

$$\varphi(\gamma_c(N_{\mathbb{Q}})) = \gamma_c(N_{\mathbb{Q}}),$$

there exists an element $A_c = (\alpha_{i,j})_{1\le i,j\le k_c} \in \mathrm{GL}(k_c,\mathbb{Q})$ such that

$$p_\varphi \begin{pmatrix} q_{c,1} \\ q_{c,2} \\ \vdots \\ q_{c,k_c} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} A_c \cdot (q_{c,1}, q_{c,2}, \dots, q_{c,k_c})^T \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{k_c} \alpha_{1,j} q_{c,j} \\ \sum_{j=1}^{k_c} \alpha_{2,j} q_{c,j} \\ \vdots \\ \sum_{j=1}^{k_c} \alpha_{k_c,j} q_{c,j} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We now make the following computation:

$$\begin{aligned}
&\varphi(a_{c,1}^{q_{c,1}} a_{c,2}^{q_{c,2}} \cdots a_{c,k_c}^{q_{c,k_c}} a_{c-1,1}^{q_{c-1,1}} a_{c-1,2}^{q_{c-1,2}} \cdots a_{1,k_1}^{q_{1,k_1}}) \\
&\quad = \varphi(a_{c,1}^{q_{c,1}} a_{c,2}^{q_{c,2}} \cdots a_{c,k_c}^{q_{c,k_c}}) \cdot \varphi(a_{c-1,1}^{q_{c-1,1}} a_{c-1,2}^{q_{c-1,2}} \cdots a_{1,k_1}^{q_{1,k_1}}) \\
&\quad = a_{c,1}^{\sum_{j=1}^{k_c} \alpha_{1,j} q_{c,j}} a_{c,2}^{\sum_{j=1}^{k_c} \alpha_{2,j} q_{c,j}} \cdots a_{c,k_c}^{\sum_{j=1}^{k_c} \alpha_{k_c,j} q_{c,j}} a_{c,1}^{\beta_1(q_{c-1,1},\dots,q_{1,k_1})} \cdots \\
&\qquad a_{c,k_c}^{\beta_{k_c}(q_{c-1,1},\dots,q_{1,k_1})} a_{c-1,1}^{p_{c-1,1}(q_{c-1,1},\dots,q_{1,k_1})} \cdots a_{1,k_1}^{p_{1,k_1}(q_{1,1},\dots,q_{1,k_1})} \\
&\qquad\qquad (\text{for some polynomials } \beta_i(q_{c-1,1}, q_{c-1,2}, \dots, q_{1,k_1})) \\
&\quad = a_{c,1}^{\sum_{j=1}^{k_c} \alpha_{1,j} q_{c,j} + \beta_1(q_{c-1,1},\dots,q_{1,k_1})} a_{c,2}^{\sum_{j=1}^{k_c} \alpha_{2,j} q_{c,j} + \beta_2(q_{c-1,1},\dots,q_{1,k_1})} \cdots \\
&\qquad a_{c,k_c}^{\sum_{j=1}^{k_c} \alpha_{k_c,j} q_{c,j} + \beta_{k_c}(q_{c-1,1},\dots,q_{1,k_1})} a_{c-1,1}^{p_{c-1,1}(q_{c-1,1},\dots,q_{1,k_1})} \cdots a_{1,k_1}^{p_{1,k_1}(q_{1,1},\dots,q_{1,k_1})}.
\end{aligned}$$

This shows that

$$p_{c,i}(q_{c,1}, \dots, q_{1,k_1}) = \sum_{j=1}^{k_c} \alpha_{i,j} q_{c,j} + \beta_i(q_{c-1,1}, \dots, q_{1,k_1})$$

which is clearly of the form (2). □

The following proposition, although very trivial, is very important.

Proposition 3.2. *Let N be any torsion free finitely generated c-step nilpotent group equipped with a Mal'cev set of generators* (4)*. Let $N_\mathbb{Q}$ be the radicable hull of N and suppose that $\varphi \in \mathrm{Aut}(N_\mathbb{Q})$. Then, φ restricts to an automorphism of N if and only if the polynomial permutation p_φ representing φ belongs to* $\mathrm{P}(\mathbb{Z}^{k_1+k_2+\cdots+k_c})$.

Proof. This is obvious, since φ restricts to an automorphism of N iff $\varphi(N) = N$ iff $p_\varphi(\mathbb{Z}^{k_1+k_2+\cdots+k_c}) = \mathbb{Z}^{k_1+k_2+\cdots+k_c}$ iff $p_\varphi \in \mathrm{P}(\mathbb{Z}^{k_1+k_2+\cdots+k_c})$. □

Any automorphism $\varphi \in \mathrm{Aut}(N_\mathbb{Q})$ induces c automorphisms φ_i on the quotient groups

$$\gamma_i(N_\mathbb{Q})/\gamma_{i+1}(N_\mathbb{Q}) \cong \mathbb{Q}^{k_i}, \ \ 1 \le i \le c.$$

So, after having chosen a Mal'cev generating set (4), we can associate with φ c matrices $A_i \in \mathrm{GL}(k_i,\mathbb{Q})$ representing the (linear) morphisms φ_i w.r.t. to the basis $\bar{a}_{i,1}, \bar{a}_{i,2}, \dots, \bar{a}_{i,k_i}$ of the $\mathbb{Q}$-vector space $\gamma_i(N_\mathbb{Q})/\gamma_{i+1}(N_\mathbb{Q})$ $(1 \le i \le c)$.

Let μ_i denote the characteristic polynomial of A_i, then we define **the characteristic polynomial of** φ to be $\mu = \mu_1\mu_2\cdots\mu_c$.

Remark 3.3. The characteristic polynomial is independent of the chosen set of generators for N. Indeed choosing another set of Mal'cev generators corresponds to a change of basis in the vector spaces $\gamma_i(N_{\mathbb{Q}})/\gamma_{i+1}(N_{\mathbb{Q}})$ $(1 \leq i \leq c)$, an operation leaving the characteristic polynomial invariant.

Let $\mathfrak{n}_{\mathbb{Q}}$ be the rational Lie algebra associated to $N_{\mathbb{Q}}$. The characteristic polynomial of φ coincides with the characteristic polynomial of the linear automorphism $d\varphi$ on the rational Lie algebra $\mathfrak{n}_{\mathbb{Q}}$, corresponding to φ.

We leave it to the reader to check that the characteristic polynomial of φ is exactly the same as the characteristic polynomial of p_φ. In fact, the matrices A_i are precisely those matrices appearing in the expression (2) of p_φ.

The main result of this section, another generalization of Lemma 2.12, can now be stated as follows:

Theorem 3.4. *Let N be any torsion free finitely generated nilpotent group of class c. Let φ be an automorphism of the radicable hull $N_{\mathbb{Q}}$ of N whose characteristic polynomial has integer coefficients and unit constant term. Then there exists some $k \in \mathbb{N}_0$ such that $\varphi^k \in \mathrm{Aut}(N)$.*

Proof. This is an immediate consequence of Proposition 3.1, Proposition 3.2 and Lemma 2.13. □

Corollary 3.5. *Let N be a finitely generated, torsion free nilpotent group. Then N admits a hyperbolic automorphism if and only if $N_{\mathbb{Q}}$ admits a hyperbolic automorphism whose characteristic polynomial has integer coefficients and unit constant term.*

Remark 3.6. By a hyperbolic automorphism of $N_{\mathbb{Q}}$, we mean an automorphism having no eigenvalues of modulus 1.

Corollary 3.7. *Let N_1 and N_2 be two finitely generated torsion free nilpotent groups which are commensurable. Then N_1 admits a hyperbolic automorphism if and only if N_2 admits a hyperbolic automorphism.*

Proof. N_1 and N_2 are commensurable if and only if $N_{1_{\mathbb{Q}}} \cong N_{2_{\mathbb{Q}}}$. □

Corollary 3.8. *Let M_1 and M_2 be two closed nilmanifolds, whose fundamental groups are commensurable, then*

$$M_1 \textit{ admits an Anosov diffeomorphism} \\ \Updownarrow \\ M_2 \textit{ admits an Anosov diffeomorphism}$$

4. Hyperbolic automorphisms on free c-step nilpotent Lie algebras

Definition 4.1. A Lie group G is said to be free c–step nilpotent on n generators if and only if the associated Lie algebra $\mathfrak{g}$ is the free c–step nilpotent (real) Lie algebra on n generators.

In [2], we studied those nilmanifolds G/N, where N was a uniform lattice of the free 2–step nilpotent Lie group G. In this paper, we will consider the much more general question, where G is free c–step nilpotent. To make things easier, we introduce the following terminology and notations.

Remark 4.2.
1. Let N be a uniform lattice of a Lie group G, then we will refer to G as the **covering Lie group** of the nilmanifold G/N.
2. $G^c_{n,\mathbb{R}}$ denotes the free c–step nilpotent Lie group on n generators.
3. $\mathfrak{g}^c_{n,\mathbb{R}}$ denotes the free c–step nilpotent Lie algebra over $\mathbb{R}$ on n generators.
4. $\mathfrak{g}^c_{n,\mathbb{Q}}$ denotes the free c–step nilpotent Lie algebra over $\mathbb{Q}$ on n generators.
5. $G^c_{n,\mathbb{Q}}$ denotes the radicable group corresponding to $\mathfrak{g}^c_{n,\mathbb{Q}}$.

The following lemma is easy to verify.

Lemma 4.3. *A finitely generated torsion free nilpotent group N is a uniform lattice of $G^c_{n,\mathbb{R}}$ if and only if $N_{\mathbb{Q}} \cong G^c_{n,\mathbb{Q}}$.*

Corollary 4.4. *Let M_1 and M_2 be two nilmanifolds with covering Lie group $G^c_{n,\mathbb{R}}$. Then, M_1 admits an Anosov diffeomorphism if and only if M_2 admits an Anosov diffeomorphism. Moreover, this is also equivalent to the requirement that $\mathfrak{g}^c_{n,\mathbb{Q}}$ admits a hyperbolic automorphism, whose characteristic polynomial has integer coefficients and unit constant term.*

From now on, we will investigate which of the free c–step nilpotent Lie algebras $\mathfrak{g}^c_{n,\mathbb{Q}}$ admit such hyperbolic automorphisms.

We recall a possible construction of the free c–step nilpotent Lie algebra $\mathfrak{g}^c_{n,\mathbb{Q}}$.

Let $V = \mathbb{Q}^n = \mathbb{Q}\,x_1 \oplus \mathbb{Q}\,x_2 \oplus \cdots \oplus \mathbb{Q}\,x_n$ the n–dimensional rational vector space with basis $\{x_1, x_2, \dots, x_n\}$.

The tensor algebra TV of V over $\mathbb{Q}$ is defined as the direct sum

$$TV = \bigoplus_{n=0}^{\infty} T^n V,$$

where $T^0 V = \mathbb{Q}$ and $T^n V = \underbrace{V \otimes V \otimes \cdots \otimes V}_{n \text{ times}} = \bigotimes^n V$, the n-fold tensor product over $\mathbb{Q}$. The tensor algebra is equipped with a bilinear product, where

$$\forall v_1 \otimes v_2 \otimes \cdots \otimes v_p \in T^p V,\ \forall w_1 \otimes w_2 \otimes \cdots \otimes w_q \in T^q V :$$

$$\begin{aligned}&(v_1 \otimes v_2 \otimes \cdots \otimes v_p)(w_1 \otimes w_2 \otimes \cdots \otimes w_q)\\ &\quad = v_1 \otimes v_2 \otimes \cdots \otimes v_p \otimes w_1 \otimes w_2 \otimes \cdots \otimes w_q \in T^{p+q} V\end{aligned}$$

Let $T^{\geq n} V = \bigoplus_{k=n}^{\infty} T^k V$. The free c–step nilpotent Lie algebra $\mathfrak{g}^c_{n,\mathbb{Q}}$ can now be identified with the Lie subalgebra of

$$TV/T^{\geq c+1} V = \mathbb{Q} \oplus V \oplus (V \otimes V) \oplus \cdots \oplus \otimes^c V$$

generated by V.

We now start the investigation of $\mathrm{Aut}(\mathfrak{g}^c_{n,\mathbb{Q}})$, since we want to find hyperbolic automorphisms of $\mathfrak{g}^c_{n,\mathbb{Q}}$.

Let f be an endomorphism of $\mathfrak{g}^c_{n,\mathbb{Q}}$, then f induces a linear transformation

$$\bar{f} : \mathfrak{g}^c_{n,\mathbb{Q}}/\gamma_2(\mathfrak{g}^c_{n,\mathbb{Q}}) \to \mathfrak{g}^c_{n,\mathbb{Q}}/\gamma_2(\mathfrak{g}^c_{n,\mathbb{Q}}) : x + \gamma_2(\mathfrak{g}^c_{n,\mathbb{Q}}) \mapsto f(x) + \gamma_2(\mathfrak{g}^c_{n,\mathbb{Q}}).$$

We will call any endomorphism g of $\mathfrak{g}^c_{n,\mathbb{Q}}$, which induces $\bar{f}$ (i.e. $\bar{g} = \bar{f}$) a lift of the linear transformation $\bar{f}$.

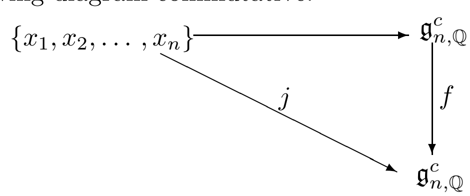
Suppose that

$$\bar{f}(x_i + \gamma_2(\mathfrak{g}^c_{n,\mathbb{Q}})) = f_{1,i}x_1 + f_{2,i}x_2 + \cdots + f_{n,i}x_n + \gamma_2(\mathfrak{g}^c_{n,\mathbb{Q}}), \tag{5}$$

then

$$f(x_i) = f_{1,i}x_1 + f_{2,i}x_2 + \cdots + f_{n,i}x_n + y_i, \text{ for some } y_i \in \gamma_2(\mathfrak{g}^c_{n,\mathbb{Q}}).$$

This implies that any lift f of $\bar{f}$ determines an n-tuple $(y_1, y_2, \ldots, y_n) \in (\gamma_2(\mathfrak{g}^c_{n,\mathbb{Q}}))^n$. Conversely, let $\bar{f}$ be any linear transformation of $\mathfrak{g}^c_{n,\mathbb{Q}}/\gamma_2(\mathfrak{g}^c_{n,\mathbb{Q}})$ which is given by (5) (for some rational numbers $f_{i,j}$) and let $(y_1, y_2, \ldots, y_n) \in (\gamma_2(\mathfrak{g}^c_{n,\mathbb{Q}}))^n$. Let $j : \{x_1, x_2, \ldots, x_n\} \to \mathfrak{g}^c_{n,\mathbb{Q}} : x_i \mapsto \sum_{k=1}^n f_{k,i}x_k + y_i$, then, by the universal property of the free c–step nilpotent Lie algebra, there is a unique Lie algebra homomorphism making the following diagram commutative:



We can conclude that there is a one-to-one correspondence between the lifts f of a given $\bar{f}$ and the n-tuples $(y_1, y_2, \ldots, y_n) \in (\gamma_2(\mathfrak{g}^c_{n,\mathbb{Q}}))^n$.

Lemma 4.5. *Let $\bar{f}$ be any linear transformation of $\mathfrak{g}^c_{n,\mathbb{Q}}/\gamma_2(\mathfrak{g}^c_{n,\mathbb{Q}})$. Then all lifts f of $\bar{f}$ induce the same linear transformation on $\gamma_i(\mathfrak{g}^c_{n,\mathbb{Q}})/\gamma_{i+1}(\mathfrak{g}^c_{n,\mathbb{Q}})$ $(1 \leq i \leq c)$.*

Proof. We proceed by induction on i. For $i = 1$, the induced map is exactly the linear transformation $\bar{f}$, by definition of "lift".

Now, assume that $i > 1$ and that the lemma is valid for smaller values. Let f_1, f_2 be two lifts of $\bar{f}$. As any element $x \in \gamma_i(\mathfrak{g}^c_{n,\mathbb{Q}})$ can be written as a linear combination of elements of the form $[z_1, z_2]$, with $z_1 \in \gamma_1(\mathfrak{g}^c_{n,\mathbb{Q}})$ and $z_2 \in \gamma_{i-1}(\mathfrak{g}^c_{n,\mathbb{Q}})$, it suffices to prove that

$$f[z_1, z_2] + \gamma_{i+1}(\mathfrak{g}^c_{n,\mathbb{Q}}) = g[z_1, z_2] + \gamma_{i+1}(\mathfrak{g}^c_{n,\mathbb{Q}}), \quad \forall z_1 \in \gamma_1(\mathfrak{g}^c_{n,\mathbb{Q}}), \; \forall z_2 \in \gamma_{i-1}(\mathfrak{g}^c_{n,\mathbb{Q}}).$$

By the induction hypothesis, we know that $g(z_1) = f(z_1) + w_1$, with $w_1 \in \gamma_2(\mathfrak{g}^c_{n,\mathbb{Q}})$ and $g(z_2) = f(z_2) + w_2$, with $w_2 \in \gamma_i(\mathfrak{g}^c_{n,\mathbb{Q}})$. It follows that

$$\begin{aligned} g[z_1, z_2] + \gamma_{i+1}(\mathfrak{g}^c_{n,\mathbb{Q}}) &= [g(z_1), g(z_2)] + \gamma_i(\mathfrak{g}^c_{n,\mathbb{Q}}) \\ &= [f(z_1) + w_1, f(z_2) + w_2] + \gamma_{i+1}(\mathfrak{g}^c_{n,\mathbb{Q}}) \\ &= [f(z_1), f(z_2)] + [w_1, f(z_2)] + [f(z_1), w_2] + [w_1, w_2] + \gamma_{i+1}(\mathfrak{g}^c_{n,\mathbb{Q}}) \\ &= f[z_1, z_2] + \gamma_{i+1}(\mathfrak{g}^c_{n,\mathbb{Q}}). \end{aligned}$$

□

This lemma shows that all automorphisms g of $\mathrm{Aut}(\mathfrak{g}^c_{n,\mathbb{Q}})$ inducing the same linear transformation on $\mathfrak{g}^c_{n,\mathbb{Q}}/\gamma_2(\mathfrak{g}^c_{n,\mathbb{Q}})$ have the same eigenvalues. Therefore, in the investigation of the eigenvalues of an automorphism f of $\mathrm{Aut}(\mathfrak{g}^c_{n,\mathbb{Q}})$ we can restrict to that lift of $\bar{f}$ which corresponds to the n-tuple $(0, 0, \ldots, 0) \in \gamma_2(\mathfrak{g}^c_{n,\mathbb{Q}})$. So, let's consider a linear transformation $\bar{f}$ of $\mathfrak{g}^c_{n,\mathbb{Q}}/\gamma_2(\mathfrak{g}^c_{n,\mathbb{Q}})$, given by (5). In order to describe the eigenvalues of any lift of this linear transformation, we construct the following commutative diagram (from the left to the right):

$$\begin{array}{ccccccc} V & \xrightarrow{i} & TV & \xrightarrow{p} & TV/T^{\geq c+1}V & \longleftarrow & \mathfrak{g}^c_{n,\mathbb{Q}} \\ & \searrow^{f_0} & \downarrow{\scriptstyle \tilde{f}_0} & & \downarrow{\scriptstyle \bar{\tilde{f}}_0} & & \downarrow{\scriptstyle f} \\ & & TV & \xrightarrow[p]{} & TV/T^{\geq c+1}V & \longleftarrow & \mathfrak{g}^c_{n,\mathbb{Q}} \end{array} \tag{6}$$

with

1. $f_0 : V \to TV : x_i \mapsto \sum_{j=1}^n f_{j,i}x_j$ (so f_0 is essentially the same as $\bar{f}$).
2. $\tilde{f}_0$ is the unique algebra morphism making the triangle on the left commutative (use the universal property of TV).
3. As $\tilde{f}_0(T^{\geq c+1}V) \subseteq T^{\geq c+1}V$, $\tilde{f}_0$ induces a map $\bar{\tilde{f}}_0$, making the middle square commutative.
4. Moreover, as $\bar{\tilde{f}}_0$ is an algebra map, it maps $\mathfrak{g}^c_{n,\mathbb{Q}}$ to itself, and consequently, by restricting to $\mathfrak{g}^c_{n,\mathbb{Q}}$, we obtain the Lie algebra map f.

Note that the constructed f is exactly the lift of $\bar{f}$ corresponding to the n-tuple $(0, 0, \cdots, 0) \in \gamma_2(\mathfrak{g}^c_{n,\mathbb{Q}})$. Moreover, $\tilde{f}_0$ restricts to an endomorphism of $T^nV = \bigotimes^n V$, for any $n \geq 1$ and this restriction is exactly

$$\bigotimes^n \bar{f} : T^nV \to T^nV : v_1 \otimes v_2 \otimes \cdots \otimes v_n \\ \mapsto f_0(v_1) \otimes f_0(v_2) \otimes \cdots \otimes f_0(v_n) = \bar{f}(v_1) \otimes \bar{f}(v_2) \otimes \cdots \otimes \bar{f}(v_n).$$

Having made these observations, we are now ready to prove the following theorem, which is a generalization of Theorem 3.3 of [2].

Theorem 4.6. $\mathfrak{g}^c_{n,\mathbb{Q}}$ *admits a hyperbolic automorphism whose characteristic polynomial has integer coefficients and unit constant term if and only if there exists a matrix* $A \in \mathrm{GL}(n, \mathbb{Z})$ *with eigenvalues* $\lambda_1,\ \lambda_2,\ \ldots, \lambda_n$ *such that*

$$|\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_r}| \neq 1, \quad \forall r \in \{1, 2, \ldots, c\},\ \forall i_1, i_2, \ldots, i_r \in \{1, 2, \ldots, n\}.$$

Proof. Let f be any automorphism of $\mathfrak{g}^c_{n,\mathbb{Q}}$. Without changing the eigenvalues of f, we can assume that f is the lift of $\bar{f}$ corresponding to the n-tuple $(0, 0, \ldots, 0) \in (\gamma_2(\mathfrak{g}^c_{n,\mathbb{Q}}))^n$. This allows us to use diagram (6). Assume the eigenvalues of $\bar{f}$ are

$$\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n$$

(we list each eigenvalue as many times as its multiplicity).

It is easy to prove (e.g. using complexification) that the eigenvalues of $\bigotimes^r \bar{f}$ are

$$\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_r} \quad (1 \leq i_1, i_2, \ldots, i_r \leq n). \tag{7}$$

(Again each eigenvalue is listed as many times as its multiplicity.)

We claim that the following 3 statements are equivalent (still using the notation of diagram (6)):

1. $\bar{f}$ has a characteristic polynomial with integral coefficients and unit constant term.
2. $\bar{\tilde{f}}_0$ has a characteristic polynomial with integral coefficients and unit constant term.

3. f has a characteristic polynomial with integral coefficients and unit constant term.

Indeed, assume that $\bar{\bar{f}}_0$ has a characteristic polynomial with integral coefficients and unit constant term. Since the characteristic polynomial of f is a quotient (in $\mathbb{Q}[x]$) of the characteristic polynomial of $\bar{\bar{f}}_0$, it is also a polynomial with integral coefficients and unit constant term, proving the implication "$2 \Rightarrow 3$". The implication "$3 \Rightarrow 1$" is proved in the same way. Now, assume that $\bar{f}$ has a characteristic polynomial in $\mathbb{Z}[x]$ with unit constant term. This means that

$$(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n) \in \mathbb{Z}[x] \quad \text{and} \quad \lambda_1\lambda_2\cdots\lambda_n = \pm 1,$$

showing that the elementary symmetric polynomials in $\lambda_1, \lambda_2, \ldots, \lambda_n$, i.e.

$$\begin{array}{l} \lambda_1 + \lambda_2 + \cdots + \lambda_n \\ \lambda_1\lambda_2 + \lambda_1\lambda_3 + \cdots + \lambda_{n-1}\lambda_n \\ \qquad\vdots \\ \lambda_1\lambda_2\cdots\lambda_n \end{array}$$

are in fact integers. It follows that all symmetric polynomials in the λ_i are integers. Using this it follows that the characteristic polynomial of each $\bigotimes^m \bar{f}$ belongs to $\mathbb{Z}[x]$ and has unit constant term. This is enough to conclude the last implication of the claim.

From (7), it follows that any eigenvalue of $\bar{\bar{f}}_0$ and so also of f is of the form

$$\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_r} \quad (1 \leq i_1, i_2, \ldots, i_r \leq n, \ \ 1 \leq r \leq c). \tag{8}$$

Moreover, it is important to note that any product of the form (8) in which not all λ_{i_j} are equal to each other do appear as an eigenvalue of f. To see this we look at the complexification of all spaces, algebras and maps. In the complex case, we can assume that for each eigenvalue λ_i, there exists an eigenvector $\tilde{x}_i$ of $V^{\mathbb{C}} = V \otimes \mathbb{C}$, the complexification of V. Now, assume that $\lambda_{i_1} \neq \lambda_{i_2}$, then the vector

$$[[[\tilde{x}_{i_1}, \tilde{x}_{i_2}], \tilde{x}_{i_3}], \ldots, \tilde{x}_{i_r}]$$

is a non-zero eigenvector with eigenvalue $\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_r}$ for $f^{\mathbb{C}}$, the induced map on the complexification of $\mathfrak{g}^c_{n,\mathbb{Q}}$, which is the free complex Lie algebra on n generators. This shows that $\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_r}$ is really an eigenvalue of f.

Now, we assume that f is a hyperbolic automorphism of $\mathfrak{g}^c_{n,\mathbb{Q}}$, whose characteristic polynomial has integer coefficients and unit constant term. By Lemma 2.12, we can assume that, with respect to a chosen basis of $\mathfrak{g}^c_{n,\mathbb{Q}}$ (and after replacing f with one of its powers), f is represented by a matrix in $\mathrm{GL}(m,\mathbb{Z})$ ($m = \dim(\mathfrak{g}^c_{n,\mathbb{Q}})$) and so also $\bar{f} \in \mathrm{GL}(n,\mathbb{Z})$. Moreover, as f is a hyperbolic automorphism, it follows that $|\lambda| \neq 1$ for all eigenvalues of f. This shows that

1. $|\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_r}| \neq 1$ for $(1 \leq i_1, i_2, \ldots, i_r \leq n, \ \ 1 \leq r \leq c)$ and not all λ_{i_j} equal to each other.
2. $|\underbrace{\lambda_i\lambda_i\cdots\lambda_i}_{r \text{ times}}| = |\lambda_i|^r \neq 1$ as $|\lambda_i| \neq 1$ $(1 \leq i \leq n,\ 1 \leq r)$.

These two facts together prove one direction of the theorem.

Conversely, let $\bar{f}$ be a matrix (linear transformation) in $\mathrm{GL}(n,\mathbb{Z})$ satisfying the criteria of the theorem. The reasoning above shows that any lift f of $\bar{f}$ is a hyperbolic automorphism satisfying the fact that its characteristic polynomial has integer coefficients and unit constant term. □

Corollary 4.7. *A nilmanifold whose covering Lie group is* $G^c_{n,\mathbb{R}}$ *admits an Anosov diffeomorphism if and only if there exists a matrix* $A \in \mathrm{GL}(n, \mathbb{Z})$ *with eigenvalues* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *such that*

$$|\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_r}| \neq 1, \quad \forall r \in \{1,2,\ldots,c\}, \ \forall i_1, i_2, \ldots, i_r \in \{1,2,\ldots,n\}.$$

Corollary 4.8. *Let* M *be an nilmanifold whose covering Lie group is* $G^c_{n,\mathbb{R}}$*, with* $n \leq c$*, then* M *does not admit any Anosov diffeomorphism.*

5. Examples

In this section, we show how the theory developed above can be used to prove the existence or non-existence of Anosov diffeomorphisms on certain nilmanifolds.

5.1. **Negative examples.**

Recall that a group N is said to be a $T(n,m)$ group iff N is finitely generated, torsion free, nilpotent of class 2 and

$$h(N/[N,N]) = n \text{ and } h([N,N]) = m$$

where $h(\Gamma)$ denotes the Hirsch length of a polycyclic group Γ. Note that for any $T(n,m)$ group $m \leq n(n-1)/2$.

In [3], it was already shown by means of direct computations that a $T(3,2)$-group does not allow any hyperbolic automorphism. Here we reprove this result in a more conceptual manner and see that the same holds for $T(4,5)$–groups. Note that any $T(3,3)$ (i.e. $T(3, \frac{3(3-1)}{2})$) and any $T(4,6)$ (i.e. $T(4, \frac{4(3-1)}{2})$) group do admit hyperbolic automorphisms ([2] and below).

Lemma 5.1. *Let* N *be a* $T(3,2)$ *or a* $T(4,5)$ *group. Then* N *does not admit a hyperbolic automorphism.*

Proof. Let $\mathfrak{n}$ be a rational Lie algebra satisfying

Case 1: $\dim(\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]) = 3$ and $\dim([\mathfrak{n},\mathfrak{n}]) = 2$ or
Case 2: $\dim(\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]) = 4$ and $\dim([\mathfrak{n},\mathfrak{n}]) = 5$.

In order to prove the lemma, we have to show that for any automorphism φ of $\mathfrak{n}$ whose characteristic polynomial has integer coefficients and unit constant term, there is at least one eigenvalue of φ of modulus 1. We can see $\mathfrak{n}$ as the quotient of $\mathfrak{g}^2_{n,\mathbb{Q}}$ (with $n = 3$ in Case 1 or $n = 4$ in Case 2) by a 1-dimensional subspace $\mathfrak{h}$ of $[\mathfrak{g}^2_{n,\mathbb{Q}}, \mathfrak{g}^2_{n,\mathbb{Q}}]$. The automorphism φ lifts to an automorphism $\tilde{\varphi}$ of $\mathfrak{g}^2_{n,\mathbb{Q}}$, mapping $\mathfrak{h}$ onto itself. This implies that $\mathfrak{h}$ is a 1-dimensional eigenspace of $\tilde{\varphi}$. Moreover, we know that the eigenvalues of $\tilde{\varphi}$ restricted to $[\mathfrak{g}^2_{n,\mathbb{Q}}, \mathfrak{g}^2_{n,\mathbb{Q}}]$ are $\lambda_i\lambda_j$ ($1 \leq i < j \leq n$), where the λ_i are the eigenvalues of the induced map on $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}] = \mathfrak{g}^2_{n,\mathbb{Q}}/[\mathfrak{g}^2_{n,\mathbb{Q}}, \mathfrak{g}^2_{n,\mathbb{Q}}]$. Without loss of generality we may assume that the eigenvalue corresponding to the space $\mathfrak{h}$ is $\lambda_1\lambda_2$. However, from the conditions on the characteristic polynomial of φ it follows that $\lambda_1\lambda_2 = \pm 1$.

Case 1: As $\lambda_1\lambda_2\lambda_3 = \pm 1$, it follows that $\lambda_3 = \pm 1$, proving the lemma.
Case 2: As $\lambda_1\lambda_2\lambda_3\lambda_4 = \pm 1$, it follows that $\lambda_3\lambda_4 = \pm 1$. Since $\lambda_3\lambda_4$ is an eigenvalue of φ, this finishes the proof.

∎

5.2. **Positive examples.** The theory can however also be applied to find positive answers. In this respect it is often convenient, in order to apply Theorem 4.6 and Corollary 4.7 to note the following equivalence:

Lemma 5.2. *There exists a matrix* $A \in \mathrm{GL}(n,\mathbb{Z})$ *with eigenvalues* $\lambda_1,\ \lambda_2,\ \ldots,\lambda_n$ *such that*

$$|\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_r}| \neq 1,\ \ \forall r \in \{1,2,\ldots,c\},\ \forall i_1,i_2,\ldots,i_r \in \{1,2,\ldots,n\}$$

if and only if there exists a monic polynomial of degree n, with integer coefficients and unit constant term whose roots $\lambda_1,\ \lambda_2,\ \ldots,\lambda_n$ *are such that*

$$|\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_r}| \neq 1,\ \ \forall r \in \{1,2,\ldots,c\},\ \forall i_1,i_2,\ldots,i_r \in \{1,2,\ldots,n\}.$$

Proof. It is well known that any monic polynomial is the characteristic polynomial of a matrix (the companion matrix of the polynomial), which has integer entries and unit determinant under the conditions of the lemma. □

Since it is easier to write down polynomials than matrices of $\mathrm{GL}(n,\mathbb{Z})$, we try to find polynomials satisfying the conditions of Lemma 5.2.

For creating families of monic polynomials with integer coefficients, we can use a recurrence relation. Let us therefore concentrate on the polynomials $P_{n,b}(x)$, where $b \in \mathbb{Z}$, satisfying the following recurrence

$$P_{n,b}(x) = xP_{n-1,b}(x) - (bx-1)P_{n-2,b}(x)$$

and initial conditions

$$P_{0,b}(x) = 1 \quad \text{and} \quad P_{1,b}(x) = x - 1.$$

Note that the initial conditions are such that the resulting polynomials are monic polynomials with unit constant term. Using these polynomials we now obtain

Computation 5.3. All Lie algebras

$$\mathfrak{g}^c_{n,\mathbb{Q}},\ \text{with } c = 1,2,3 \text{ or } 4 \text{ and } n > c$$

admit a hyperbolic automorphism whose characteristic polynomial has integer coefficients and unit constant term; it follows that the nilmanifolds M with covering Lie group $G^c_{n,\mathbb{R}}$ ($c = 1,2,3$ or 4 and $n > c$) always admit an Anosov diffeomorphism.

Proof. We explain how to check the situation for $c = 4$. Let $b = -2$ (many negative b's seem to work well) and let a computer calculate the roots of the polynomials

$$P_{5,-2}(x), P_{6,-2}(x), P_{7,-2}(x), P_{8,-2}(x) \text{ and } P_{9,-2}(x).$$

One finds 35 roots $\lambda_1,\lambda_2,\ldots,\lambda_{35}$ and for these roots, one checks that

$$|\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_r}| \neq 1,\quad 1 \leq i_1,i_2,\ldots,i_r \leq 35,\quad 1 \leq r \leq 4.$$

Now, for any $n \geq 5$, we can produce a polynomial $p_n(x)$ of degree n, which we obtain as a product of the polynomials $P_{5,-2}(x), P_{6,-2}(x), P_{7,-2}(x), P_{8,-2}(x)$ and $P_{9,-2}(x)$. It follows that $p_n(x)$ is a monic polynomial with unit constant term such that the roots $\mu_1,\mu_2,\ldots,\mu_n$ of $p_n(x)$ satisfy

$$|\mu_{i_1}\mu_{i_2}\ldots\mu_{i_r}| \neq 1,\ \ \forall r \in \{1,2,3,4\},\ \forall i_1,i_2,\ldots,i_r \in \{1,2,\ldots,n\}.$$

And this proves our claim, for $c = 4$. □

Note that one can easily continue with these kind of computations and prove analogous statements for bigger c. This leads to the following conjecture:

Conjecture 5.4. All Lie algebras

$$\mathfrak{g}^c_{n,\mathbb{Q}}, \text{ with } n > c$$

admit a hyperbolic automorphism whose characteristic polynomial has integer coefficients and unit constant term.

Remark once again that a Lie algebra $\mathfrak{g}^c_{n,\mathbb{Q}}$, with $n \leq c$ never admits hyperbolic automorphisms with a characteristic polynomial with integer coefficients and unit constant term.

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